On a Feynman-Kac type result for coupled elliptic systems-I-

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PAPIER RECUPERE ET RECYCLE

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ABSTRACT

In this paper we consider a class of second order elliptic systems which has the form

$$- \sum_{i,j} a_{ij}^k \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_i b_i^k \frac{\partial u_k}{\partial x_i} + \sum_\ell C^{k\ell} u_\ell = f_k \qquad \text{in } \mathcal{O}$$

Where a_{ij}^k, b_i^k, $c^{k\ell}$, f_k are given data and $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^N$.

For the Dirichlet condition we give an existence and uniqueness result. Also we obtain a Feynman-Kac type representation of the solution which appears as a generalization of the one, known for the scalar case.

KEY WORDS

Elliptic - Feynman-Kac representation.

RESUME

Dans ce papier on considère une classe de systèmes elliptiques du second ordre ayant la forme :

$$- \sum_{i,j} a_{i,j}^k \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_i b_i^k \frac{\partial u_k}{\partial x_i} + \sum_\ell c^{k\ell} u_\ell = f_k \qquad \text{dans } \mathcal{O}$$

où a_{ij}^k, b_i^k, $c^{k\ell}$, f_k sont des données, $\mathcal{O}$ est un ouvert borné de $\mathbb{R}^N$. Pour le problème de Dirichlet on obtient un résultat d'existence et d'unicité. Aussi nous obtenons un résultat de type Feynman-Kac pour la solution généralisant la représentation comme dans le cas scalaire.

MOTS-CLES

Ellitpique - Représentation du type Feynman-Kac.

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INTRODUCTION

Consider the following system of second order elliptic equations :

$$(*)\qquad \begin{cases} -\sum\limits_{i,j} a_{ij}^{k} \dfrac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum\limits_{i} b_i^k \dfrac{\partial u_k}{\partial x_i} + \sum\limits_{\ell} c^{k\ell}\, u_\ell = f_k \quad \text{in } \mathcal{O} \\ u_k\big|_{\partial \mathcal{O}} = 0 \quad , \qquad 1 \le k \le M \end{cases}$$

where a_{ij}^k, b_i^k, $c^{k\ell}$, f_k are given data and $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^N$.

We give in this note a function space representation for $u = (u_1,\ldots,u_M)$ in terms of f using random evolution method. This representation reduces to the Feynman-Kac formula in the case of one elliptic equation.

Let us give a class of fully coupled systems of elliptic equations which can be reduced to the form $(*)$:

$$(***)\qquad \begin{cases} -\sum\limits_{\ell}\sum\limits_{i,j} a_{ij}^{k\ell} \dfrac{\partial^2 u_\ell}{\partial x_i \partial x_j} + \sum\limits_{\ell}\sum\limits_{i} b_i^{k\ell} \dfrac{\partial u_\ell}{\partial x_i} + \sum\limits_{\ell} c^{k\ell}\, u_\ell = f_k \\ u_k\big|_{\partial \mathcal{O}} = 0 \ . \end{cases}$$

We assume that the coefficients $b_i^{k\ell}$, $a_{ij}^{k\ell}$ are constants and

i) the ellipticity condition : there exists a similarity transformation Q such that

$$Q^{-1} A_{ij} Q = \Lambda_{ij} \quad \forall\, i,j$$

Λ_{ij} are real diagonal matrices and

$$\sum_{i,j} \lambda_{ij}^{k} \xi_i \xi_j \geq \alpha\, |\xi|^2 \quad , \quad \alpha > 0 \ , \ \forall\, k \ , \ \forall\, \xi \in \mathbb{R}^N$$

with

$$\Lambda_{ij} = (\lambda_{ij}^{k} \delta_{k\ell}) \quad \forall\, i,j$$

$$\delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise} . \end{cases}$$

ii) $$Q^{-1} B_i Q = H_i \quad \forall\, i$$

H_i are real diagonale matrices.

Then problem (***) can be transformed to

$$\begin{cases} -\displaystyle\sum_{i,j} \lambda_{ij}^{k} \frac{\partial^2 v_k}{\partial x_i \partial x_j} + \sum_i h_i^k \frac{\partial v_k}{\partial x_i} + \sum_\ell d^{k\ell}\, v_\ell = g_k \\ v_k\big|_{\partial\mathcal{O}} = 0 \end{cases}$$

where $V = \begin{pmatrix} v_1 \\ \vdots \\ v_M \end{pmatrix} = Q^{-1} U \quad , \quad U = \begin{pmatrix} u_1 \\ \vdots \\ u_M \end{pmatrix}$

$$D = (d^{k\ell}) = Q^{-1} C Q \quad , \quad C = (c^{k\ell})$$

$$H_i = (h_i^k \delta_{k\ell}) \quad , \quad g = Q^{-1} f .$$

Our result can be applied directly to this last problem.

The assumption (i) can be satisfied if B_i, A_{ij} are commuting matrices. We remark that when B_i are diagonale matrices, then we can take them as functions of x.

Using the $c^{k\ell}$ coefficients we construct a Markov chain whose evolution is governed by a matrix depending on the state of the diffusion. Then we obtain a representation result for the solution of (*) using a process which is defined through the solution of a martingale problem.

As an application, we treat in [1] the game problem associated to the nonlinear partial differential equation :

$$(**)\quad \begin{cases} \inf\limits_{k} \sup\limits_{\ell} \left(-\sum\limits_{i,j} a_{ij}^{k\ell} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum\limits_{i} b_i^{k\ell} \dfrac{\partial u}{\partial x_i} + c^{k\ell} u - f^{k\ell}\right) = 0 \quad \text{in } \mathcal{O} \\ u_{|\partial\mathcal{O}} = 0 \end{cases}$$

amoung others.

The paper is organized as follows : section I is devoted to the study of the martingale problem. Section II is concerned with the representation result of the solution of (*). Section III contains results on the existence and uniqueness questions for (*) using the accretive operators method.

The probabilistic part is based on the well known result of D.W. Stroock [3] concerning the martingale problem for Levy generators.

I. THE MARTINGALE PROBLEM

I.1. Assumptions and notations

Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^N$, and $\varepsilon = \{1,2,\ldots,M\}$ endowed with the discrete topology.

We consider functions such that :

$$(1) \qquad b_i^k, c^{k\ell} \in L^\infty(\mathbb{R}^N) \; ; \; a_{ij}^k \in W^{1,\infty}(\mathbb{R}^N) \quad , \quad 1 \le i,j \le N; \; k,\ell \in \varepsilon.$$

We also assume the ellipticity condition :

$$(2) \qquad \sum_{i,j} a_{ij}^k \, \xi_i \xi_j \ge \gamma |\xi|^2 \quad , \quad \forall\, k \in \varepsilon, \; \forall\, \xi \in \mathbb{R}^N, \text{ with } \gamma > 0 \; .$$

We define the family of second order differential operators

$$(3) \qquad L_z = - \sum_{i,j} a_{ij}(x,z)(\partial^2/\partial x_i \partial x_j) + \sum_i b_i(x,z)(\partial/\partial x_i)$$

(it is clear that $\phi(x,z) = \phi_\ell(x)$ for $z = \ell$). We denote

$$(4) \qquad K\phi(x,z) = \sum_{\ell \neq z} c^{z\ell}(x)\phi_\ell(x) \qquad \forall\, \phi_\ell \in \mathcal{D}(\mathbb{R}^N).$$

I.2. The martingale problem

Let

$$\Omega = C([0,\infty);\mathbb{R}^N) \times D([0,\infty);\varepsilon)$$

where $C([0,\infty);\mathbb{R}^N)$ is the space of continuous functions on $[0,\infty)$ into $\mathbb{R}^N$ and $D([0,\infty);\varepsilon)$ is the space of right continuous functions on $[0,\infty)$ into ε having left limits.

Given $\omega \in \Omega$, let $(y(t,\omega),z(t,\omega))$ denotes the position of ω at time t.

For $0 \le s \le t$, we set $F_t^s = \mathcal{B}(y(\lambda),z(\lambda);s \le \lambda \le t)$ and $F^s = \sigma(\bigcup_{t\ge s} F_t^s)$.

We assume for instance that

(5) $$c^{k\ell} \le 0 \quad \text{for} \quad k \ne \ell.$$

We have :

A probability measure $P_{x,k,s}$ on (Ω,F^s) is a solution of the martingale problem for the operator L_z+K starting from (x,k,s) if

(6) $$P_{x,k,s}(y(s)=x,\ z(s)=k) = 1$$

(7) $$\forall\ \phi_\ell \in \mathcal{D}(\mathbb{R}^N)\ ;\ \ell \in \varepsilon$$

$$X_s(t)\phi_{z(t)}(y(t)) + \int_s^t X_s(\lambda)[L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) + K\phi(y(\lambda),z(\lambda))]d\lambda$$

is a $P_{x,k,s}$-martingale

where

(8) $$X_s(\theta) = \exp(-\sum_{\ell=1}^{N} \int_s^\theta c^{z(\lambda)\ell}(y(\lambda))d\lambda).$$

We have clearly that $X_s(\theta)$ is the solution of the stochastic integral equation

(9) $$X_s(\theta) = 1 + \int_s^\theta X_s(\lambda)\ [-\sum_{\ell=1}^{M} c^{z(\lambda)\ell}(y(\lambda))]d\lambda \quad , \quad \theta \ge s$$

a.s. $P_{x,k,s}$.

We have :

Theorem 1.1: Under assumptions (1), (2), (5), there exists a solution $P_{x,k,s}$ to the martingale problem (6), (7).

To prove this theorem we follow technics of [3]. Let

$$\tilde{\Omega} = \Omega \times [0,\infty)^{\mathbb{N}}$$

$$\tilde{\omega} = (\omega;\rho_0,\ldots,\rho_n,\ldots)$$

$$y(t,\tilde{\omega}) = y(t,\omega) \quad , \quad z(t,\tilde{\omega}) = z(t,\omega)$$

and $$\tau_n(\tilde{\omega}) = \rho_n \quad , \quad n \geq 0 \ .$$

Let us denote

$$\tilde{F}^s_{t,n} = \mathcal{B}(y(\lambda),\tau_r;\ s \leq \lambda \leq t,\ 0 \leq r \leq n)$$

$$\tilde{F}^s_n = \tilde{F}^s_{\infty,n} \ .$$

For x,k,s fixed, we know that there exists a unique solution $Q_{x,k,s}$ of the martingale problem for the operator L_k (see [2]).

Next let

$$(10) \qquad q^{k\ell} = - c^{k\ell} \quad \text{for} \quad \ell \neq k$$

$$q^{kk} = \sum_{r\neq k} c^{kr} \quad \forall\ k\ .$$

Then we consider

$$(11) \qquad \tilde{q}^{k\ell} = -q^{k\ell}/q^{kk} \quad , \quad \forall\ k,\ell : k \neq \ell.$$

For ω,t fixed, we define a probability measure on (Ω,F^t) by

$$(12) \qquad Q_{\omega,t} = \delta_\omega \sum_{\ell \neq z(t^-)} Q_{y(t),\ell,t}\, \tilde{q}^{z(t^-)\ell}(y(t))$$

thus

$$Q_{\omega,t}(\Gamma \cap \Delta) = \chi_\Gamma(\omega) \sum_{\ell \neq z(t^-)} Q_{y(t),\ell,t}(\Delta)\tilde{q}^{z(t^-),\ell}(y(t))$$

for $\Gamma \in \sigma(\bigcup_{s \leq \lambda < t} F^s_\lambda)$ and $\Delta \in F^t$.

Now, for $\tilde{\omega}, n$ fixed, we define the probability measure on $(\tilde{\Omega}, \tilde{F}^s_n)$ given by

$$(13) \qquad \tilde{Q}^n_{\tilde{\omega}} = Q_{\omega,\tau_n(\tilde{\omega})} \otimes \delta_{\tau_0(\tilde{\omega})} \otimes \dots \otimes \delta_{\tau_n(\tilde{\omega})}$$

and a probability measure on $(\tilde{\Omega}, \mathcal{B}(\tau_n))$ such that

$$(14) \qquad \mu_{n,\tilde{\omega}}([t,\infty)) = \exp \int_{t \wedge \tau_{n-1}(\tilde{\omega})}^{t} \sum_{\ell \neq z(\lambda)}^{M} c^{z(\lambda)\ell}(y(\lambda))d\lambda \ , n \geq 1.$$

Next, we define by induction a family of probability measure $P^n_{x,k,s}$ on $(\tilde{\Omega}, \tilde{F}^s_n)$, such that

$$(15) \qquad P^0_{x,k,s} = Q_{x,k,s} \otimes \delta_s \ .$$

Knowing $P^n_{x,k,s}$, we firstly define $\tilde{P}^n_{x,k,s}$ on $(\tilde{\Omega}, \tilde{F}^s_{n+1})$ by

$$(16) \qquad \tilde{P}^n_{x,k,s} = P^n_{x,k,s} \otimes \mu_{n+1,\tilde{\omega}}$$

thus we have

$$\tilde{P}^n_{x,k,s}(\Gamma \cap \Delta) = E^{P^n_{x,k,s}}[\chi_\Gamma(\tilde{\omega})\mu_{n+1,\tilde{\omega}}(\Delta)]$$

for $\Gamma \in \tilde{F}^s_n$ and $\Delta \in \mathcal{B}(\tau_{n+1})$.

For $\tilde{\Gamma} \in \tilde{F}^s_{n+1}$, we set

$$(17)\qquad P^{n+1}_{x,k,s}(\tilde{\Gamma}) = E^{\tilde{P}^n_{x,k,s}}[\tilde{Q}^{n+1}_{\tilde{\omega}}(\tilde{\Gamma})] \ .$$

We shall prove several lemma concerning the family $P^n_{x,k,s}$. We follow [3].

Lemma 1.1: Let $\Gamma \in F^s_{s_1}$, $s \le s_1 \le s_2$; we have

$$(18)\qquad E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \chi_{\tau_n > s_2}] = E^{P^n}[\phi_{z(s_1)}(y(s_1))\chi_\Gamma \chi_{\tau_n > s_1}] \quad (*)$$

$$- E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} (\chi_{\tau_n>\lambda} L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - \chi_{\tau_{n-1}>\lambda} \sum_{\ell=1}^{M} q^{z(\lambda)\ell}(y(\lambda)) \phi_\ell(y(\lambda)))d\lambda]$$

$$+ E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_{n-1}\le\lambda<\tau_n} q^{z(\lambda)z(\lambda)}(y(\lambda))\phi_{z(\lambda)}(y(\lambda))d\lambda] \ .$$

$$\forall\, \phi_\ell \in \mathcal{D}(\mathbb{R}^N),\ \ell \in \varepsilon,\ \forall\, n \ge 1 \ .$$

Proof : First we have for $n = 1$

$$(19)\qquad E^{P^1}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \chi_{\tau_1>s_2}] = E^{\tilde{P}^0} E^{\tilde{Q}^1_{\tilde{\omega}}}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \chi_{\tau_1>s_2}]$$

and

$$(20)\qquad E^{\tilde{Q}^1_{\tilde{\omega}}}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \chi_{\tau_1>s_2}] = \chi_{\tau_1>s_2} E^{Q_{\omega,\tau_1}}[\chi_\Gamma \phi_{z(s_2)}(y(s_2))]$$

$$= \chi_{\tau_1>s_2}\chi_\Gamma\, \phi_{z(s_2)}(y(s_2)) \ .$$

Hence,

(*) $P^n \equiv P^n_{s,x} \quad \forall\, n$.

$$(21)\qquad E^{\tilde{P}^0}\,\chi_{\tau_1>s_2}\chi_\Gamma\,\phi_{z(s_2)}(y(s_2)) = E^{P^0}\,\chi_\Gamma\,\phi_{z(s_2)}(y(s_2))\,\exp\int_{s_1\wedge\tau_0}^{s_2}\sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda$$

$$= E^{Q_{x,k,s}}\chi_\Gamma\,\phi_{z(s_2)}(y(s_2))\,\exp\int_s^{s_2}\sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda$$

On the other hand we know that

$$(22)\qquad \phi_{z(t)}(y(t))\exp\Big(\int_s^t\sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda\Big) + \int_s^t [L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - $$

$$- \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))\phi_{z(\lambda)}(y(\lambda))]\,\exp\Big(\int_s^\lambda\sum_{\ell\neq z(\theta)} c^{z(\theta)\ell}(y(\theta))d\theta\Big)d\lambda$$

is also a $Q_{x,k,s}$, F_t^s martingale. Thus we have that

$$(23)\qquad (19) = E^{Q_{x,k,s}}\,\chi_\Gamma\phi_{z(s_1)}(y(s_1))\exp\int_s^{s_1}\sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda -$$

$$- E^{Q_{x,k,s}}\{\chi_\Gamma\int_{s_1}^{s_2}[L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))\phi_{z(\lambda)}(y(\lambda))]$$

$$\exp\Big(\int_s^\lambda\sum_{\ell\neq z(\theta)} c^{z(\theta)\ell}(y(\theta))d\theta\Big)d\lambda\,\}$$

and by similar calculations, we get that

$$(24)\qquad (23) = E^{P^1}\,[\phi_{z(s_1)}(y(s_1))\chi_\Gamma\chi_{\tau_1>s_2}] -$$

$$- E^{P^1}[\chi_\Gamma\int_{s_1}^{s_2}\chi_{\tau_1>\lambda}\,(L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) -$$

$$- q^{z(\lambda)z(\lambda)}(y(\lambda))\phi_{z(\lambda)}(y(\lambda)))d\lambda]$$

which give us (18) for $n = 1$, since

(25) $$E^{P^1}(\tau_0 = s) = 1 .$$

Next, assume that (18) hold for step n, then

(26) $$E^{P^{n+1}}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma\chi_{\tau_{n+1}>s_2}] = E^{\tilde{P}^n} E^{\tilde{Q}_{\omega}^{n+1}} [\phi_{z(s_2)}(y(s_2))\chi_\Gamma\chi_{\tau_{n+1}>s_2}]$$

$$= E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \exp \int_{\tau_n\wedge s_2}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda]$$

$$= E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \exp \int_{\tau_n}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda \ \chi_{\tau_n\leq s_2<\tau_{n+1}}] +$$

$$+ E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma\chi_{\tau_n>s_2}] = I_1 + I_2 .$$

We shall estimate separately these two terms.

(27) $$I_2 = E^{\tilde{P}^{n-1}} E^{\tilde{Q}_{\omega}^{n}} [\phi_{z(s_2)}(y(s_2))\chi_\Gamma\chi_{\tau_n\leq s_2} \exp \int_{\tau_n}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda]$$

$$= E^{\tilde{P}^{n-1}} \chi_{\tau_n\leq s_2} E^{Q_{\omega,\tau_n}} [\phi_{z(s_2)}(y(s_2))\chi_\Gamma \exp \int_{\tau_n}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda]$$

$$= E^{\tilde{P}^{n-1}} \chi_{\tau_n\leq s_1} E^{Q_{\omega,\tau_n}}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \exp \int_{\tau_n}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda] +$$

$$+ E^{\tilde{P}^{n-1}} \chi_{s_1<\tau_n\leq s_2}\chi_\Gamma E^{Q_{\omega,\tau_n}}[\phi_{z(s_2)}(y(s_2)) \exp \int_{\tau_n}^{s_2} \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))d\lambda]$$

We have also :

(28) $$I_2 = E^{P^n}[\chi_\Gamma\chi_{\tau_n\leq s_1<\tau_{n+1}} \ \phi_{z(s_1)}(y(s_1))] -$$

$$- E^{P^n}[\chi_\Gamma \chi_{\tau_n \le s_1} \int_{s_1}^{s_2} \chi_{\lambda<\tau_{n+1}} (L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))\phi_{z(\lambda)}(y(\lambda)))d\lambda]$$

$$+ E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_{n-1}\le\lambda<\tau_n} \sum_{\ell\neq z(\lambda)} \phi_\ell(y(\lambda))c^{z(\lambda)\ell}(y(\lambda))d\lambda]$$

$$- E^{P^n}[\chi_\Gamma \chi_{s_1<\tau_n} \int_{s_1}^{s_2} \chi_{\tau_n\le\lambda<\tau_{n+1}} (L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))\phi_{z(\lambda)}(y(\lambda)))d\lambda]$$

(29) $$= E^{P^n}[\chi_\Gamma \chi_{\tau_n\le s_1<\tau_{n+1}} \phi_{z(s_1)}(y(s_1))] -$$

$$- E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_n\le\lambda<\tau_{n+1}} (L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) - \sum_{\ell\neq z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))\phi_{z(\lambda)}(y(\lambda)))d\lambda]$$

$$+ E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_{n-1}\le\lambda<\tau_n} \sum_{\ell\neq z(\lambda)} \phi_\ell(y(\lambda))c^{z(\lambda)\ell}(y(\lambda))d\lambda] .$$

Using the induction hypothesis , we get that (30) holds for $n+1$.

We obtain by similar methods that :

Lemma 1.2 : Let $\Gamma \in F^s_{s_1}$, $s \le s_1 < s_2$, then

(30) $$E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma \chi_{s_1<\tau_n\le s_2}] =$$

$$= -E^{P^n}[\chi_{\tau_n>s_1}\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_n\le\lambda} L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda))d\lambda] +$$

$$E^{P^n}[\chi_\Gamma \int_{s_1}^{s_2} \chi_{\tau_{n-1}\le\lambda<\tau_n} (\sum_{\ell\neq z(\lambda)} \phi_\ell(y(\lambda))q^{z(\lambda)\ell}(y(\lambda)))d\lambda].$$

$\forall\ \phi_\ell \in \mathcal{D}(\mathbb{R}^N)$, $\forall\ \ell \in \varepsilon$, $\forall\ n \geq 1$.

<u>Lemma 1.3</u> : Let $\Gamma \in F^s_{s_1}$, $s \le s_1 \le s_2$, we have

$$(31)\qquad E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma\chi_{\tau_n\le s_2}] = E^{P^n}[\phi_{z(s_1)}(y(s_1))\chi_\Gamma\chi_{\tau_n\le s_1}]$$

$$- E^{P^n}[\chi_{\tau_n\le s_1}\chi_\Gamma \int_{s_1}^{s_2} L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda))d\lambda],$$

$\forall\ \phi_\ell \in \mathcal{D}(\mathbb{R}^N)$, $\ell \in \varepsilon$, $\forall\ n \geq 1$.

Now, combining the results of these lemmas, we get that

$$(32)\qquad E^{P^n}[\phi_{z(s_2)}(y(s_2))\chi_\Gamma] + E^{P^n}[\int_s^{s_2} (L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) -$$

$$- \chi_{\tau_n>\lambda} \sum_{\ell=1}^{M} q^{z(\lambda)\ell}(y(\lambda))\phi_\ell(y(\lambda)))d\lambda\chi_\Gamma] =$$

$$E^{P^n}[\phi_{z(s_1)}(y(s_1))\chi_\Gamma] + E^{P^n}[\int_s^{s_1} (L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda))$$

$$- \chi_{\tau_n>\lambda} \sum_{\ell=1}^{M} q^{z(\lambda)\ell}(y(\lambda))\phi_\ell(y(\lambda)))d\lambda\chi_\Gamma]$$

$\forall\ \phi_\ell \in \mathcal{D}(\mathbb{R}^N)$, $\ell \in \varepsilon$, $\forall\ n \geq 1$.

On the other hand, for n fixed, we obtain only by induction on $r = 1,2,\ldots,n$ that

(33) $$P^n_{x,k,s}(\tau_n \le t) \le E^{P^{n-r}}\{[1-(\exp-\tilde{\gamma}(t-\tau_{n-r})) \sum_{i=0}^{r-1} (\tilde{\gamma}(t-\tau_{n-r}))^i/i!]\chi_{\tau_{n-r}\le t}\}$$

where $\tilde{\gamma} = \sup_{k,x}(-\sum_{\ell\neq k} c^{k\ell}(x))$.

Finally set

$$\tilde{\mathcal{B}}_n = \mathcal{B}[y(\lambda);\ s \le \lambda < \tau_n]$$

and as (33) yields

$$P^n_{x,k,s}(\tau_n \le t) \le 1-(\exp-\tilde{\gamma}(t-s)) \sum_{i=0}^{n-1} \frac{(\tilde{\gamma}(t-s))^i}{i!}$$

which goes to zero when $n \to \infty$.

By Tulcea's extension theorem, there exists a unique probability measure $P^\infty_{x,k,s}$ on $(\tilde{\Omega},\tilde{F}^s_0)$ such that

(34) $$P^\infty_{x,k,s} = P^n_{x,k,s} \quad \text{on } \tilde{\mathcal{B}}_n,\ n \ge 1 .$$

Let $P_{x,k,s}$ be the measure induced on (Ω,F^s) by $P^\infty_{x,k,s}$. This measure satisfies :

(35) $$P_{x,k,s}(y(s)=x,z(s)=k) = 1 .$$

(36) $$\phi_{z(t)}(y(t)) + \int_s^t [\, L_{z(\lambda)}\phi_{z(\lambda)}(y(\lambda)) + \sum_{\substack{\ell=1\\ \ell\neq z(\lambda)}}^{M} c^{z(\lambda)\ell}(y(\lambda))\phi_\ell(y(\lambda)) +$$
$$+ (-\sum_{\substack{\ell=1\\ \ell\neq z(\lambda)}}^{M} c^{z(\lambda)\ell}(y(\lambda)))\phi_{z(\lambda)}(y(\lambda))]d\lambda$$

is a $P_{x,k,s}$-martingale, $\forall\ \phi_\ell \in \mathcal{D}(\mathbb{R}^N), \ell \in \varepsilon$.

Hence, we have also (6) (7) with $X_s(\theta)$ given in (8) which completes the proof.

II. REPRESENTATION RESULTS

II.1. A linear semi-group

The purpose of this section is to study a family of linear operators $Q(t)$ on $C(\bar{\mathcal{O}})^M$,

(37) $$(Q(t)v)_k(x) = E_{x,k}X(t\wedge\tau)\,\mathrm{sign}\Big(\prod_{n=0}^{\nu(0,t\wedge\tau)} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n))\Big)v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

where the expectation is made with respect to the solution of the martingale problem (6), (7) with $|c^{k\ell}|$ instead of $(-c^{k\ell})$ for $\ell \neq k$ and

(38) $$X(t) = \exp[(\int_0^t \sum_{\ell\neq z(\lambda)} |c^{z(\lambda)\ell}(y(\lambda))|d\lambda - \int_0^t c^{z(\lambda)z(\lambda)}(y(\lambda)d\lambda)]$$

$t \geq 0;$

(39) $$\mathrm{sign}(r) = \begin{cases} 1 & \text{if} \quad r > 0 \\ 0 & \phantom{\text{if}} \quad r = 0 \\ -1 & \phantom{\text{if}} \quad r < 0 \end{cases}$$

(40) $\tau = \inf\{t \geq 0,\ y(t) \notin \mathcal{O}\}$ is the first exit time of $y(t)$ from $\mathcal{O}$;

(41) $\nu(0,t)$ is the number of jumps of the process $z(t)$ in the interval of time $(0,t]$,
$a \wedge b = \min(a,b)$, $\forall\ a,b \in \mathbb{R}$.

We shall prove that $Q(t)$ is a linear semi-group of contractions on $C(\bar{\mathcal{O}})^M$ and its generator is an extension of the operator introduced in problem $(*)$.

$(*)$ $\tau_0^- = \tau_0 \equiv 0$.

We assume for simplicity that

(42) $\quad c^{kk} \geq \alpha > 0 \qquad \forall\ k \in \varepsilon$ (α large enough).

More precisely, we have

Theorem 2.1 : Under assumptions (1), (2), (42), the family of linear operators $Q(t)$ introduced in (37) has the properties

1) $Q(t) : [C(\bar{\mathcal{O}})]^M \to [C(\bar{\mathcal{O}})]^M$

2) $Q(0) = I$, $Q(t+s) = Q(t)Q(s) = Q(s)Q(t)$

3) $||Q(t)v - v||_\infty \to 0$ as $t \to 0$

4) $||Q(t)v_1 - Q(t)v_2||_\infty \leq ||v_1 - v_2||_\infty$.

Thus $Q(t)$ is a linear semi-group of contractions on $C(\bar{\mathcal{O}})^M$.

Proof : 1) This assertion is a direct consequence of the continuous dependence of $P_{x,k}$ on x which can be proved using the same methods as in [3].

2) We have only to remark that

(43) $\quad X(t+s,\omega) = X(t,\omega)\ X(s,\theta_t\omega)$

where $\theta_t\omega$ is the shifted path; the strong Markov property of the family $\{P_{s,x,k}\}$ and

(44) $$\operatorname{sign}\Big(\prod_{n=1}^{\nu(0,t)} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n))\Big) = \prod_{n=1}^{\nu(0,t)} \operatorname{sign} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n))$$

3) and 4) follow by simple calculations.

II.2. The generator of $Q(t)$

Here we shall prove that the generator of $Q(t)$ is an extension of the operator

$$(45)\qquad \phi \in [C^2(\mathcal{O})]^M \to \Big(\sum_{i,j} a^k_{ij}(\partial^2/\partial x_i\partial x_j)\phi_k - \sum_i b^k_i(\partial/\partial x_i)\phi_k - \sum_\ell c^{k\ell}\,\phi_\ell\Big)_k .$$

We have

Theorem 2.2 : Under assumptions (1), (2), (42), we have that

$$(46)\qquad \frac{1}{t}\,[Q(t)v-v]_k(x) \underset{t\downarrow 0}{\to} \Big[\sum_{i,j} a^k_{ij}(\partial^2/\partial x_i\partial x_j)v_k - \sum_i b^k_i(\partial/\partial x_i)v_k - \sum_\ell c^{k\ell}v_\ell\Big](x)$$

uniformly on any compact subset of $\mathcal{O}$, $\forall\, v \in [C^2(\bar{\mathcal{O}})]^M$.

Proof :

$$(47)\qquad (Q(t)v)_k(x) = E_{x,k}X(t\wedge\tau)\,\mathrm{sign}\Big(\prod_{n=1}^{\nu(0,t\wedge\tau)} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n))\Big)v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

$$= E_{x,k}\Big(X(t\wedge\tau)\,\mathrm{sign}\Big(\prod_{n=1}^{\nu(0,t\wedge\tau)} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n))\Big)v_{z(t\wedge\tau)}(y(t\wedge\tau))\Big]\chi_{t\wedge\tau<\tau_1} +$$

$$+ E_{x,k}[\ ]\chi_{\tau_1\le t\wedge\tau<\tau_2} + E_{x,k}[\ ]\chi_{t\wedge\tau\ge\tau_2}$$

$$= I_1 + I_2 + I_3 .$$

Let us evaluate separately these terms. We have

$$(48)\qquad I_1 = E^{P^1}[X(t\wedge\tau)v_k(y(t\wedge\tau))\chi_{t\wedge\tau<\tau_1}] = E^{\tilde{P}^0}E^{\tilde{Q}^1_\varpi}[X(t\wedge\tau)v_k(y(t\wedge\tau))\chi_{t\wedge\tau<\tau_1}]$$

$$= E^{P^0}[X(t\wedge\tau)v_k(y(t\wedge\tau))\exp\int_0^{t\wedge\tau}\sum_{\ell\neq z(\lambda)}|c^{z(\lambda)\ell}(y(\lambda))|d\lambda]$$

$$= E^{Q_{x,k}}[v_k(y(t\wedge\tau))\exp -\int_0^{t\wedge\tau} c^{kk}(y(\lambda))d\lambda]$$

$$= v_k(x) - E^{Q_{x,k}}[\int_0^{t\wedge\tau}(L_kv_k + c^{kk}v_k)(y(\lambda)))\exp -\int_0^{\lambda} c^{kk}(y(\theta))d\theta d\lambda] .$$

For I_2 we get that

$$(49)\qquad I_2 = E^{P^2}[X(t\wedge\tau)\text{sign } c^{kz(\tau_1)}(y(\tau_1))v_{z(t\wedge\tau)}(y(t\wedge\tau))\chi_{\tau_1\le t\wedge\tau<\tau_2}]$$

$$= E^{\tilde{P}^1}E^{\tilde{Q}^2_\varpi}[X(t\wedge\tau)\text{sign } c^{kz(t\wedge\tau)}(y(\tau_1))v_{z(t\wedge\tau)}(y(t\wedge\tau))\chi_{\tau_1\le t\wedge\tau<\tau_2}]$$

$$= E^{P^1}[X(t\wedge\tau)\text{sign } c^{kz(t\wedge\tau)}(y(\tau_1))v_{z(t\wedge\tau)}(y(t\wedge\tau))\chi_{t\wedge\tau\ge\tau_1}$$

$$\exp\int_{\tau_1}^{t\wedge\tau}\sum_{\ell\neq z(\lambda)}|c^{z(\lambda)\ell}(y(\lambda))|d\lambda]$$

$$= E^{\tilde{P}^0}\chi_{\tau_1\le t\wedge\tau}E^{Q_{\omega,\tau_1}}[X(t\wedge\tau)\text{sign } c^{kz(t\wedge\tau)}(y(\tau_1))v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

$$\exp\int_{\tau_1}^{t\wedge\tau}\sum_{\ell\neq z(\lambda)}|c^{z(\lambda)\ell}(y(\lambda))|d\lambda]$$

$$= E^{\tilde{P}^0}\chi_{\tau_1\le t\wedge\tau}X(\tau_1)\,E^{Q_{\omega,\tau_1}}[\text{sign } c^{kz(t\wedge\tau)}(y(\tau_1))v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

$$\exp - \int_{\tau_1}^{t\wedge\tau} c^{z(\lambda)z(\lambda)}(y(\lambda))d\lambda]$$

$$= E^{P^0} \chi_{\tau_1 \le t\wedge\tau} X(\tau_1) [\sum_{\ell \ne z(\tau_1 -)} \text{sign } c^{k\ell}(y(\tau_1))v_\ell(y(\tau_1))\tilde{q}^{z(\tau_1 -)\ell} y(\tau_1))] -$$

$$- E^{\tilde{P}^0} \chi_{\tau_1 \le t\wedge\tau} X(\tau_1) \{ \int_{\tau_1}^{t\wedge\tau} \text{signc}^{kz(\lambda)}(y(\tau_1))$$

$$[L_{z(\lambda)} v_{z(\lambda)}(y(\lambda)) + c^{z(\lambda)z(\lambda)}(y(\lambda))v_{z(\lambda)}(y(\lambda))]$$

$$\exp(- \int_{\tau_1}^{\lambda} c^{z(\theta)z(\theta)} (y(\theta))d\theta)d\lambda \}$$

$$= I_{21} + I_{22} .$$

We have

$$I_{21} = E^{P^1} \chi_{0<\tau_1 \le t\wedge\tau} X(\tau_1) [\sum_{\ell \ne k} \text{sign } c^{k\ell}(y(\tau_1))v_\ell(y(\tau_1))\tilde{q}^{k\ell}(y(\tau_1))] \tag{50}$$

$$= E^{P^1} \int_0^{t\wedge\tau} \chi_{\lambda<\tau_1} X(\lambda) [- \sum_{\ell \ne k} \text{sign } c^{k\ell}(y(\lambda))v_\ell(y(\lambda))|c^{k\ell}(y(\lambda))|]d\lambda .$$

Hence

$$I_2 = E^{P^1} \int_0^{t\wedge\tau} \chi_{\lambda<\tau_1} X(\lambda) [- \sum_{\ell \ne z(\lambda)} c^{z(\lambda)\ell}(y(\lambda))v_\ell(y(\lambda))]d\lambda + I_{22} . \tag{51}$$

Now we evaluate I_3 :

(52)
$$I_3 = \sum_{m=2}^{\infty} [E_{x,k}\, X(t\wedge\tau)\,\mathrm{sign}\Big(\prod_{n=0}^{\nu(0,t\wedge\tau)} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))$$

$$v_{z(t\wedge\tau)}(y(t\wedge\tau))\chi_{\tau_m \le t\wedge\tau<\tau_{m+1}}]=$$

$$= \sum_{m=2}^{\infty} E^{P^{m+1}} [X(t\wedge\tau)\,\mathrm{sign}\Big(\prod_{n=0}^{m} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))\Big)$$

$$v_{z(t\wedge\tau)}(y(t\wedge\tau))\chi_{\tau_m \le t\wedge\tau<\tau_{m+1}}]$$

$$= \sum_{m=2}^{\infty} E^{P^{m}} [X(t\wedge\tau)\ \exp\Big(\int_{\tau_m}^{t\wedge\tau} \sum_{\ell\neq z(\lambda)} |c^{z(\lambda)\ell}(y(\lambda))|\,d\lambda\Big)v_{z(t\wedge\tau)}(y(t\wedge\tau)).$$

$$.\mathrm{sign}(c^{z(\tau_m -)z(t\wedge\tau)}(y(\tau_m)).$$

$$.\mathrm{sign}\Big(\prod_{n=0}^{m-1} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))\chi_{\tau_m\le t\wedge\tau}]$$

$$= \sum_{m=2}^{\infty} E^{P^{m}} [X(\tau_m)\,\mathrm{sign}\Big(\prod_{n=0}^{m-1} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))\chi_{\tau_m\le t\wedge\tau}$$

$$.\mathrm{sign}(c^{z(\tau_m -)z(t\wedge\tau)}(y(\tau_m))).v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

$$\exp- \int_{\tau_m}^{t\wedge\tau} c^{z(\lambda)z(\lambda)}(y(\lambda))\,d\lambda]$$

$$= \sum_{m=2}^{\infty} E^{\tilde{P}^{m-1}}\ X(\tau_m)\,\mathrm{sign}\Big(\prod_{n=0}^{m-1} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))\Big)\chi_{\tau_m\le t\wedge\tau}$$

$$E^{Q_{\omega,\tau_m}}[\mathrm{sign}(c^{z(\tau_m -)z(t\wedge\tau)}(y(\tau_m))v_{z(t\wedge\tau)}(y(t\wedge\tau))$$

$$\exp- \int_{\tau_m}^{t\wedge\tau} c^{z(\lambda)z(\lambda)}(y(\lambda))\,d\lambda]$$

$$= \sum_{m=2}^{\infty} E^{\tilde{P}^{m-1}}\ X(\tau_m)\,\mathrm{sign}\Big(\prod_{n=0}^{m-1} c^{z(\tau_n -)z(\tau_n)}(y(\tau_n))\Big)\chi_{\tau_m\le t\wedge\tau}$$

$$[\sum_{\ell\neq z(\tau_m -)} \mathrm{sign}(c^{z(\tau_m -)\ell}(y(\tau_m)))v_\ell(y(\tau_m))\tilde{q}^{z(\tau_m -)\ell}(y(\tau_m))]$$

$$- E^{\tilde{P}^{m-1}} X(\tau_m)\operatorname{sign}(\prod_{n=0}^{m-1} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n)))\chi_{\tau_m\le t\wedge\tau}$$

$$[\int_{\tau_m}^{t\wedge\tau} (L_{z(\lambda)}v_{z(\lambda)}(y(\lambda)) + c^{z(\lambda)z(\lambda)}(y(\lambda))v_{z(\lambda)}(y(\lambda)))$$

$$\exp(-\int_{\tau_m}^{\lambda} c^{z(\theta)z(\theta)}(y(\theta))d\theta).\operatorname{sign}c^{z(\tau_m-)z(\lambda)}(y(\tau_m)))d\lambda]$$

$$= I_{31} + I_{32}.$$

Proceeding as in the proof of (28) we obtain that

$$I_{31} = \sum_{m=2}^{\infty}\{E^{P^m}\int_0^{t\wedge\tau} X(\lambda)\operatorname{sign}(\prod_{n=0}^{m-1} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n)))\chi_{\tau_{m-1}\le\lambda<\tau_m} \tag{53}$$

$$[\sum_{\ell\neq z(\tau_{m-1})} v_\ell(y(\lambda))c^{z(\tau_{m-1})\ell}(y(\lambda))]d\lambda .$$

Hence, we have

$$I_3 = \sum_{m=2}^{\infty} E^{P^m}\int_0^{t\wedge\tau} X(\lambda)\operatorname{sign}(\prod_{n=0}^{m-1} c^{z(\tau_n-)z(\tau_n)}c(y(\tau_n))\chi_{\tau_{m-1}\le\lambda<\tau_m} \tag{54}$$

$$[\sum_{\ell\neq z(\tau_{m-1})} v(y(\lambda))c^{z(\tau_{m-1})\ell}(y(\lambda))]d\lambda + I_{32} .$$

Finally letting $t \to 0$ we get that

$$\frac{1}{t}(I_1 - v_k(x)) \to (-L_k v_k - c^{kk}v_k)(x) \tag{55}$$

$$\frac{1}{t}I_{21} \to -\sum_{\ell\neq k} c^{k\ell}v_\ell(x)$$

$$\frac{1}{t}[I_{22} + I_3] \to 0$$

uniformly on any compact subset of $\mathcal{O}$, which completes the proof.

II.3. The L^p-estimate

We have the following estimate for the measure $P_{x,k,s}$:

Theorem 2.3 : Under assumptions (1), (2) and for

(56) $$f_k \in L^p(\mathbb{R}^N \times (s,T)) \quad , \quad p > N+1 ,$$

we have

(57) $$\left|E_{x,k,s} \int_s^T f_{z(\lambda)}(y(\lambda))d\lambda\right| \leq C_{T,p}\cdot||f||_{L^p}$$

where the constant $C_{T,p}$ depends on T,p and γ.

$$||f||_{L^p} = \sum_{k=1}^{M} ||f_k||_{L^p} .$$

Proof : Indeed we have

(58) $$E_{x,k,s} \int_s^T f_{z(\lambda)}(y(\lambda))d\lambda = \sum_{m=0}^{\infty} E_{x,k,s} \int_{\tau_m}^{\tau_{m+1}} f_{z(\lambda)}(y(\lambda))d\lambda$$

with $\tau_{m+1} = \inf\{t | t \in [\tau_m,T], \nu(\tau_m,t) > 0\}$.

From the L^p-estimates on the measures $Q_{x,\ell,s}$, $\ell \in \varepsilon$ we get that

(59) $$\left|E_{x,k,s} [\int_{\tau_m}^{\tau_{m+1}} f_{z(\tau_m)}(y(\lambda))d\lambda | F^s_{\tau_m}]\right| \leq \tilde{C}_{T,p}\cdot||f||_{L^p} .$$

Thus,

(60) $$\left|E_{x,k,s} \int_s^T f_{z(\lambda)}(y(\lambda))d\lambda\right| \leq \tilde{C}_{T,p} \, ||f||_{L^p} \sum_{m=0}^{\infty} E_{x,k,s} \chi_{\tau_m<T} .$$

On the other hand we have

(61) $$\sum_{m=0}^{\infty} E_{x,k,s} \chi_{\tau_m<T} \leq 1 + E_{x,k,s} \nu(s,T) = 1 + \sum_{\substack{\ell,r \\ \ell\neq r}} E_{x,k,s} \nu^{\ell,r}(s,T)$$

where $\nu^{\ell,r}(s,T)$ is the number of jumps of the process $z(t)$ on $(s,T]$ from the state ℓ to the state r.

But from (7), we have that

$$E_{x,k,s}\, \nu^{\ell,r}(s,T) \le C_{\tilde{\gamma}}(T-s) \tag{62}$$

where $C_{\tilde{\gamma}}$ is a constant depending on $\tilde{\gamma}$. Therefore, we get (57) with $C_{T,p} = \tilde{C}_{T,p}.C_{\tilde{\gamma}}T$.

III. EXISTENCE AND UNIQUENESS RESULTS

In this section we shall prove existence and uniqueness results by analytic method. More precisely we shall use accretive operators method.

III.1. Existence result

We begin by recalling the definition of m-accretive operators in $C(\bar{O})$.

We define the pairing

$$[f,g]_+ = \max_{\substack{y\in\bar{O} \\ |f(y)|=||f||_\infty}} g(y)\operatorname{sign} f(y) \quad , \quad \forall\ f,g \in C(\bar{O}),\ f \not\equiv 0 \tag{63}$$

then a nonlinear operator A with domain $D(A)$ in $C(\bar{O})$ and range in $C(\bar{O})$ is called accretive if

$$[f_1-f_2, Af_1-Af_2]_+ \ge 0 \quad , \quad \forall\ f_1,f_2 \in D(A) \tag{64}$$

and m-accretive if, in addition,

$$R(I+\lambda A) = C(\bar{O}) \quad \text{for some} \quad \lambda > 0\ . \tag{65}$$

Let us denote

$$(66)\qquad L^k = -a^k_{ij}(\partial^2/\partial x_i \partial x_j) + b^k_i(\partial/\partial x_i)$$

and assume that

$$(67)\qquad \phi = a^k_{ij}\ ,\ b^k_i\ ,\ c^{k\ell}\ ,\ f_k \quad ; \quad \phi \in C^2(\bar{\mathcal{O}})$$

$$\partial\mathcal{O} \in C^{2+\beta} \quad \text{for some} \quad \beta > 0$$

$$(68)\qquad \sum_\ell c^{k\ell}u^k u^\ell \geq 0 \quad , \quad \forall\, k \in \{k \mid |u^k| = \max_{1\leq\ell\leq M} |u^\ell|\}, \quad \forall\, u \in \mathbb{R}^M .$$

Let us denote

$$Lu = (L^1u^1, L^2u^2, \ldots, L^Mu^M)^t$$

$$Cu = (\sum_\ell c^{1\ell}u^\ell, \sum_\ell c^{2\ell}u^\ell, \ldots, \sum_\ell c^{M\ell}u^\ell)^t .$$

Each operator L^k is defined on

$$(69)\qquad D(L^k) = \{v \in W_0^{1,\ell}(\mathcal{O}) \cap W^{2,\ell}(\mathcal{O}) \ ;\ L^k v \in C(\bar{\mathcal{O}})\}$$

for any fixed $p > N$. Then L is defined on

$$D(L) = \prod_{k=1}^{M} D(L^k)$$

and C is defined on $[C(\bar{\mathcal{O}})]^M$.

We have

<u>Lemma 3.1</u> : The operator $L+C$ with domain $D(L)$ is m-accretive in $[C(\bar{\mathcal{O}})]^M$.

<u>Proof</u> : Step 1 : L is accretive.

It is enough to prove that each L^k is accretive (see the appendix A).

By the maximum principle, if $v \in C(\bar{\mathcal{O}}) \cap W^{2,\ell}_{loc}(\bar{\mathcal{O}})$ and v takes a positive maximum at a point $x_0 \in \mathcal{O}$ then

$$(70) \qquad \operatorname{ess.lim\,inf}_{x \to x_0} (L^k v(x)) \geq 0$$

(see [5], [6]), and since $L^k v$ is continuous, we have also

$$(71) \qquad v(x_0)(L^k v(x_0)) \geq 0$$

which implies (64) for L^k.

Step 2 : L is m-accretive

By the general theory of elliptic equations, each L^k is m-accretive consequently L is also m-accretive.

Step 3 : C is accretive

By the appendix (A) we have

$$(72) \qquad [u-\tilde{u}, Cu-C\tilde{u}]_+ = \max_{\substack{k: \\ \|u^k-\tilde{u}^k\|_\infty = \|u-\tilde{u}\|_\infty}} \{[u^k-\tilde{u}^k, \sum_\ell c^{k\ell}u^\ell - \sum_\ell c^{k\ell}\tilde{u}^\ell]_+\}$$

$$= \max_{\substack{k: \\ \|u^k-\tilde{u}^k\|_\infty = \|u-\tilde{u}\|_\infty}} \max_y \left(\sum_\ell c^{k\ell}(y)(u^\ell-\tilde{u}^\ell)(y)\operatorname{sign}(u^k-\tilde{u}^k)(y) \,\Big|\, \|u^k-\tilde{u}^k\|_\infty = |u^k-\tilde{u}^k|(y)\right)$$

But we have also

$$(73) \qquad \sum_\ell c^{k\ell}(y)(u^\ell-\tilde{u}^\ell)(y)\operatorname{sign}(u^k-\tilde{u}^k)(y) = \frac{1}{\|u-\tilde{u}\|_\infty} \sum_\ell c^{k\ell}(y)(u^\ell-\tilde{u}^\ell)(y)(u^k-\tilde{u}^k)(y)$$

and using condition (68) yields

$$(74) \qquad [u-\tilde{u}, Cu-C\tilde{u}]_+ \geq 0 \quad , \quad \forall\, u,\tilde{u} \in [C(\bar{\mathcal{O}})]^M .$$

Step 4 : It is clear that C is also Lipschitz and according to a standard perturbation theory [7] L+C is then m-accretive in $[C(\bar{\mathcal{O}})]^M$.

Now from the m-accretiveness of L+C and standard regularity results for elliptic equations, we obtain :

Theorem 3.1 : Under conditions (2), (63), (64), the problem (*) has a unique solution u with components $u_k \in \mathcal{D}(L_k)$; further, u_k belongs to $C^{2,\beta}(\bar{\mathcal{O}}) \cap C^{3,\theta}(\mathcal{O}')$ for any $0 < \theta < 1, \bar{\mathcal{O}}' \subset \mathcal{O}$. Moreover, we have

$$u_k(x) = E_{x,k} \int_0^\tau X(\lambda) \prod_{n=0}^{\nu(0,\lambda)} \operatorname{sign} c^{z(\tau_n-)z(\tau_n)}(y(\tau_n)) f^{z(\lambda)}(y(\lambda))d\lambda .$$

The last result is a consequence of the Itô formula.

Remark 3.1 : One can treat also parabolic problems by the same methods.

APPENDIX A

Here we shall recall some known results about accretive operators. Proofs of those assertions may be found in [7], [8].

Let $\mathcal{V}$ be a real Banach space with norm $||.||$. A possibly nonlinear operator $B : \mathcal{D}(B) \subset \mathcal{V} \to \mathcal{V}$ is called accretive if

$$||v-\bar{v}|| \le ||v-\bar{v}+\lambda(Bv-B\bar{v})|| \quad \text{for all} \quad v,\bar{v} \in \mathcal{D}(B)$$

and $\lambda > 0$.

If in addition $R(I+\lambda B) = \mathcal{V}$ for all (equivalently for some) $\lambda > 0$, B is m-accretive.

We have that

1) B is accretive if and only if

$$[v-\bar{v}, Bv-B\bar{v}]_+ \ge 0 \quad \text{for all} \quad v,\bar{v} \in \mathcal{D}(B).$$

2) Let A be an operator Lipschitz continuous, everywhere defined accretive ; B m-accretive then $A+B$ is m-accretive (here $\mathcal{D}(A+B) = \mathcal{D}(B)$).

When $\mathcal{V} = \mathcal{V}_1 \times \dots \times \mathcal{V}_M$ is the product of real Banach spaces $\mathcal{V}_k$ with norm

$$||v|| = \max_{1\le k\le M} ||v_k||.$$

Denote $[\ ,\ ]_+^k$ and $[\ ,\ ]_+$ the brackets in $\mathcal{V}_k$ and $\mathcal{V}$ respectively. Then

$$[v,\bar{v}]_+ = \max_{1\le k\le M} \{[v_k,\bar{v}_k]_+^k ; ||v_k|| = ||v||\}.$$

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