



## Level number sequences for trees

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**LEVEL NUMBER  
SEQUENCES FOR TREES**

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# LEVEL NUMBER SEQUENCES FOR TREES

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**Abstract:** *We give explicit asymptotic expressions for the number of "level number sequences" (l.n.s.) associated to binary trees. The level number sequences describe the number of nodes present at each level of a tree.*

**Résumé:** Nous présentons des expressions explicites de dénombrement des "séquences de niveaux" relatives aux arbres binaires. Ces séquences décrivent le nombre de sommets présents à chaque niveau d'un arbre binaire.

# LEVEL NUMBER SEQUENCES FOR TREES

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## ABSTRACT

*We give explicit asymptotic expressions for the number of "level number sequences" (l.n.s.) associated to binary trees. The level number sequences describe the number of nodes present at each level of a tree.*

## 1. Introduction

This paper concerns some statistical properties of a parameter related to the *profiles* of binary trees.

Define the *level* of a node  $\nu$  of a rooted tree  $t$  as the number of nodes on the branch connecting  $\nu$  to the root of  $t$  (counting both end nodes). The *level number sequence* of a tree  $t$  is the infinite sequence of integers  $(n_1, n_2, \dots)$  such that  $n_j$  is the number of nodes at level  $j$  in tree  $t$ . With our previous definitions, a level number sequence starts with a 1 and consists of eventually null integers.

Figure 1 below displays a tree whose level number sequence is  $(1, 1, 3, 9, 25, 0, 0, \dots)$ , each node being labelled with its depth. (We are indebted to Ms C. Cabart for providing this computer generated diagram which is built after a natural botanical growth model for coffee and tree shrubs.)

Let  $t$  be a binary tree as defined for instance in Knuth's book [Kn68]: such trees are: (i) rooted, *i.e.* a certain node is distinguished as the root of the tree; (ii) binary, that is to say, each node has either 0 or 2 descendants; (iii) planar, *i.e.* subtrees hanging from a binary node are distinguished as left or right subtrees. (Note: planarity is not essential here).

In an unpublished paper [CMW77], Clowes, Mitrani and Wilson address the problem of determining the number of distinct level number sequences associated to all binary trees formed with  $n$  binary nodes. Without loss of generality, we shall define level number sequences for binary trees by taking here  $n_j$  to be the number of *binary nodes* at depth  $j$  in the tree. (In computer science terms, this amounts to considering nullary nodes as null pointers).

Let  $H_n$  be the set of all such sequences, and let  $H_n = \text{card}(H_n)$ . It is clear that an element of  $H_n$  is formed from a sequence  $(n_1, n_2, \dots, n_k)$  of non-zero integers followed by an infinite sequence of trailing zeroes:  $n_j = 0$  for  $j > k$ . The

$n_j$  with  $j \leq k$  satisfy the following characteristic conditions:

- C1.  $n_1=1$ ;
- C2. for all  $j$  such that  $1 < j \leq k: 1 \leq n_j \leq 2n_{j-1}$ ;
- C3.  $n_1+n_2+\dots+n_k = n$ .

The parameter  $k$  in the above definition is called the *height* of the level number sequence since it corresponds to the (usual) height of any associated tree.

In classical combinatorial analysis terms [An76], [GJ83],  $H_n$  is isomorphic to the set of all *compositions* of integer  $n$  such that the first summand is equal to 1 and such that each summand is at most twice the previous summand. Thus our result can be interpreted as a counting result for restricted compositions. (See [An76], [GJ83]).

The first few values of  $H_n$  for  $n=1 \dots 10$  are readily found to be:  
1, 1, 2, 3, 5, 9, 16, 28, 50, 89

Clowes, Mitrani and Wilson first observe the inequalities (with  $F_n$  the  $n$ -th Fibonacci number):

$$F_{n+1} \leq H_{n+1} \leq 2^{n-1} . \tag{1}$$

The upper bound results from the fact that  $2^{n-1}$  is the number of (unrestricted) compositions of  $n$ , while the lower bound counts the subset of  $H_n$  formed with summands only equal to 1 or 2.

The authors of [CMW77] then provide a succession of refinements of this simple combinatorial argument until their best bound which is of the form:

$$C_1 1.755^n < H_n < C_2 1.802^n ,$$

and from numerical evidence, they conjecture that  $H_n$  grows roughly like  $1.794^n$ . We derive here a precise asymptotic estimate of  $H_n$  in the form of:

**Theorem 1:** *The number of level number sequences  $H_n$  satisfies the asymptotic estimate:*

$$H_n \sim K \cdot \vartheta^n \tag{2}$$

where  $K = 0.25450\ 55235\ 65319$  and  $\vartheta = 1.79414\ 71875\ 41685$  is the inverse of the smallest positive root  $\rho$  of the transcendental equation:

$$\sum_{j \geq 1} (-1)^{j+1} \frac{\rho^{2^{j+1}-2-j}}{(1-\rho)(1-\rho^3)(1-\rho^7) \dots (1-\rho^{2^j-1})} = 1 .$$

Values given by formula (2) are fairly accurate; for instance,  $H_{10}=89$  while the integer truncation,  $H_{10}^*$ , of approximation (2) is 88; for  $n=15$ , corresponding values are  $H_{15}=1639$  and  $H_{15}^*=1635$ . For  $n=100$ , one finds:

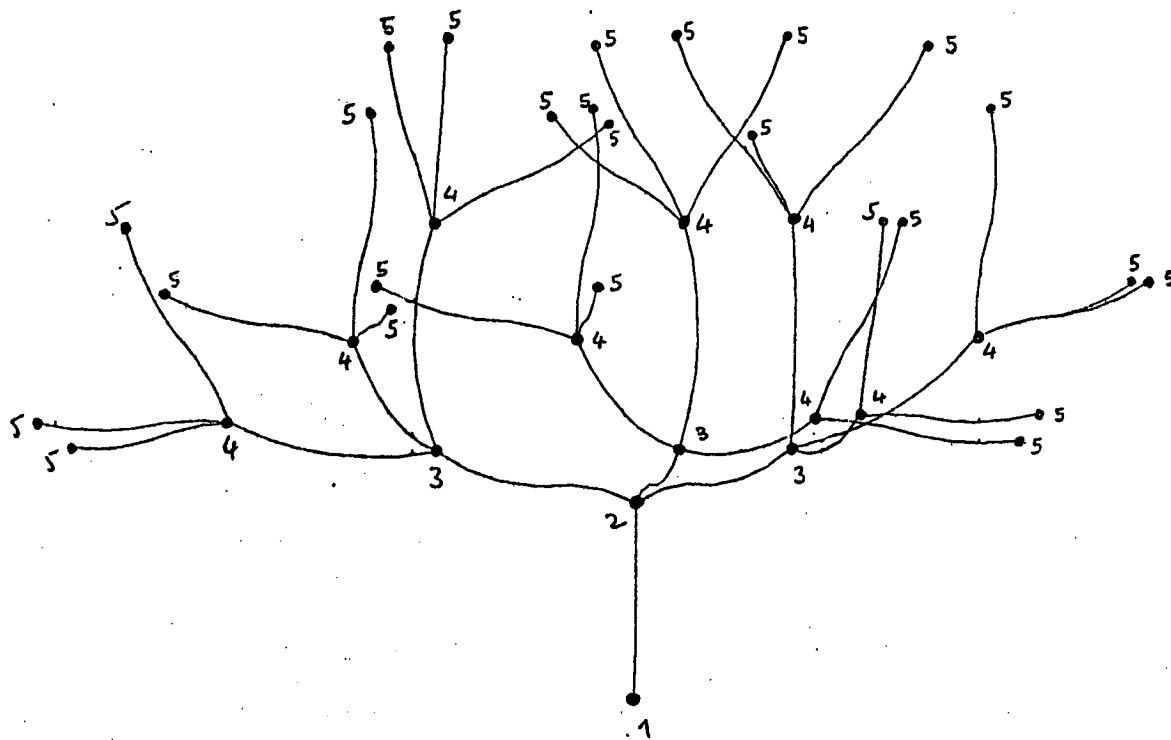


Figure 1: A tree with level number sequence: (1,1,3,9,25,0,0, ...).

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$$H_{100} = 6187341363780618339584784 \quad (3a)$$

$$H_{100}^* = 6187341363780614360373016 \quad (3b)$$

respectively, so that there:

$$1 < \frac{H_{100}}{H_{100}^*} < 1 + 7 \times 10^{-16}.$$

Our approach in this note consists in setting up a difference equation for a series closely related to the *generating function*  $H(q) = \sum_{n \geq 0} H_n q^n$ . The equa-

tion is then solved and the equation involves a non-standard form of so-called  $q$ -series (See [An76]). As a by-product of our analysis, we obtain:

**Theorem 2:** *The generating function of the quantities  $H_n$  is expressible as:*

$$H(q) = \frac{a(q)}{1-b(q)} \quad (4)$$

where:

$$a(q) = \sum_{j \geq 1} (-1)^{j+1} \frac{q^{2^{j+1}-2-j}}{(1-q)(1-q^3)(1-q^7) \cdots (1-q^{2^j-1})} \quad (5)$$

$$b(q) = \sum_{j \geq 1} (-1)^{j+1} \frac{q^{2^{j+1}-2-j}}{(1-q)(1-q^3)(1-q^7) \cdots (1-q^{2^j-1})} \quad (6)$$

The proof of Theorem 1 proceeds from there by noticing that  $H(q)$  has a meromorphic continuation from which  $H_n$  can be recovered using Cauchy's integral formula in combination with a suitable contour of integration.

We finally mention that somewhat related results on the profiles of trees have been obtained by Meir and Moon [MM78] ("thickness of layers"), Flajolet and Odlyzko [FO82] ("height"), Odlyzko and Wilf [OW84] ("width"). However the statistical models are usually different. Notice for instance that, under the uniform statistics, the expected width of a binary tree of size  $n$  (this corresponds to  $\max\{n_j\}$ ) has not been exactly determined although it is known to be of order  $O(\sqrt{n})$ .

## 2. Generating function Equations.

We let  $H_n^{[k]}$  denote the subset of  $H_n$  formed with level number sequences (l.n.s) of height  $k$ , whose last non-zero component has value  $j$ ; we also let  $H_{n,j}^{[k]} = \text{card} H_n^{[k]}$ . We introduce the corresponding bivariate generating functions:

$$H^{[k]}(q, u) = \sum_{n, j \geq 1} H_{n,j}^{[k]} q^n u^j \quad (7a)$$

$$H(q, u) = \sum_{k \geq 1} H^{[k]}(q, u). \quad (7b)$$

Thus  $H(q, 1)$  is the generating function of the  $H_n$ :  $H(q, 1) = H(q)$ .

**Lemma 1:** *The bivariate generating function  $H(q, u)$  satisfies the functional equation:*

$$H(q, u) = qu + \frac{uq}{1-uu} [H(q, 1) - H(q, q^2 u^2)]. \quad (8)$$

**Proof:** From the definition, we have:

$$H^{[0]}(q, u) = qu$$

corresponding to the unique l.n.s.  $(1, 0, 0, \dots)$  of height 1 whose last non-zero component is 1.

A recurrence relating  $H^{[k]}$  to  $H^{[k+1]}$  is easily obtained by the technique of "adding a new slice": consider the set  $H_n^{[k]}$ ; when adding a new non-zero component  $n_{k+1}$  to it, it will give rise to l.n.s. of height  $k+1$  with last element

equal to  $j^*$  where  $j^*$  can be any of the integers  $1, 2, \dots, 2j$ , the weight (i.e. the size of any representing tree) becoming  $n + j^*$ .

In terms of generating functions, this means that the process of going from  $H^{[k-1]}$  to  $H^{[k]}$  is achieved by the substitution

$$u^j \rightarrow uq + (uq)^2 + \dots + (uq)^{2j} = \frac{uq}{1-uq} (1 - (uq)^{2j}).$$

Whence the recurrence:

$$H^{[k+1]}(q, u) = \frac{uq}{1-uq} [H^{[k]}(q, 1) - H^{[k]}(q, q^2 u^2)],$$

and summing over all values of  $k$  yields the functional equation in the statement of the lemma. ■

We can now easily finish the proof of Theorem 2. A functional equation of the form ( $\Phi$  the unknown function):

$$\Phi(u) = \lambda(u) + \mu(u)\Phi(\sigma(u)) \quad (9)$$

admits, by iteration of (9), the formal solution:

$$\Phi(u) = \sum_{k \geq 0} \left[ \prod_{j=0}^{k-1} \mu(\sigma^{(j)}(u)) \right] \lambda(\sigma^{(k)}(u)), \quad (10)$$

where  $\sigma^{(k)}(u)$  denotes the  $k$ -th iterate of  $\sigma(u)$ .

Solution (10) applies to Equation (8) with:

$$\sigma(u) = q^2 u^2$$

$$\lambda(u) = uq + \frac{uq}{1-uq} H(q, 1)$$

$$\mu(u) = \frac{uq}{1-uq},$$

where the iterates of  $\sigma(\cdot)$  are given by  $\sigma^{(k)}(u) = q^{2^{k+1}-2} u^{2^k}$ . In this fashion, one obtains for  $H(q, u)$  a "solution" of the form:

$$H(q, u) = A(q, u) + B(q, u)H(q, 1). \quad (11)$$

Setting  $u=1$  in (11), and solving the resulting linear equation for  $H(q, 1) \equiv H(q)$ , we get:

$$H(q) = \frac{A(q, 1)}{1-B(q, 1)}. \quad (12)$$

A simple computation from the form of the iterates of  $\sigma(\cdot)$  and from the scheme (10) shows that  $A(q, 1) \equiv a(q)$  and  $B(q, 1) \equiv b(q)$  where  $a(q)$  and  $b(q)$  are given by (5), (6). This concludes the proof of Theorem 2.

### 3. Asymptotics.

In order to recover  $H_n$  from the expression of  $H(q)$  provided by Theorem 2 (Equation (4)), we use Cauchy's integral formula for coefficients of analytic functions. Since, by the bound (1),  $H(q)$  has a radius of convergence  $\rho$  larger than  $1/\varphi = 0.61803$ , we have:

$$H_n = \frac{1}{2i\pi} \int_{|q|=1/2} H(q) \frac{dq}{q^{n+1}}. \quad (13)$$



Classically (see e.g. [Fl85]), the asymptotic form of  $H_n$  is easy to predict, and is simply determined by the dominant singularities of  $H(q)$ .

We first observe:

**Lemma 2:** *Function  $H(q)$  is meromorphic for  $|q| \leq 7/10$  with a unique simple pole at  $q = \rho$ , where  $\rho$  is defined in the statement of Theorem 2.*

**Proof:** Since  $H(q)$  is the quotient of two functions analytic for  $|q| < 1$ , it is meromorphic for  $|q| < 1$ . Since  $H(q)$  represents a series with positive coefficients, it has at least a real positive singularity on its circle of convergence. Therefore its radius of convergence is equal to the smallest root of the equation  $1 - b(q) = 0$  if that root does not cancel  $a(q)$ . Let  $\rho$  denote the smallest positive root of  $1 - b(q) = 0$ . We find numerically that  $\rho = 0.57367\dots$ , and at that point  $a(\rho) = 0.34373\dots$  and  $b'(\rho) = 2.42320\dots$ . Thus  $\rho$  is a simple pole of  $H(q)$ .

We can check, by numerical analysis, that the equation  $1 - b(q) = 0$  has no other zero satisfying  $|q| \leq 7/10$  (This checking could if necessary be transformed into an unpleasingly formal proof). To that purpose, we use the principle of the argument [He77]:

The number of solutions to the equation  $f(q) = 0$  that lie inside a simple closed curve  $\Gamma$ , with  $f(q)$  analytic inside and on  $\Gamma$  is equal to the variation of the argument of  $f(q)$  along  $\Gamma$ , a quantity also equal to the winding number of the transformed curve  $f(\Gamma)$  around the origin.

Figure 2 below shows the shape of the curve  $f(\Gamma)$  when  $\Gamma$  is the circle  $|q| = 7/10$  and  $f(q) = 1 - b(q)$ . Its winding number is clearly equal to 1, so that  $f(q)$  has only  $q = \rho$  as a zero when  $|q| \leq 7/10$ . ■

We can now conclude the proof of Theorem 1. To that purpose we consider the integral:

$$I_n = \frac{1}{2i\pi} \int_{|q|=7/10} H(q) \frac{dq}{q^{n+1}}. \quad (14)$$

By the residue theorem, the quantity  $H_n - I_n$  which represents the integral of  $H(q)/q^{n+1}$  along two concentric circles is equal to the sum of the residues of the integrand taken with a minus sign.

Since one has, when  $q \rightarrow \rho$ :

$$H(q) \sim -\frac{a(\rho)}{b'(\rho)} \frac{1}{q - \rho},$$

one gets:

$$H_n - I_n = \frac{a(\rho)}{b'(\rho)} \rho^{-n-1}. \quad (15)$$

We finally notice that since the integrand is analytic for  $|q| = 7/10$ , quantity  $I_n$  is  $O((\frac{10}{7})^n)$ , whence finally for  $H_n$  an expression of the form:

$$H_n = K \vartheta^n + O((\frac{10}{7})^n)$$

where  $K$  and  $\vartheta$  are obtained from (15), and  $\vartheta = 1/\rho = 1.794\dots$

This therefore completes the proof of Theorem 1. Notice that the error term in (15) is *exponentially smaller* than the dominant asymptotic equivalent: our proof shows that it is here at most a fraction  $O(0.8^n)$  of the

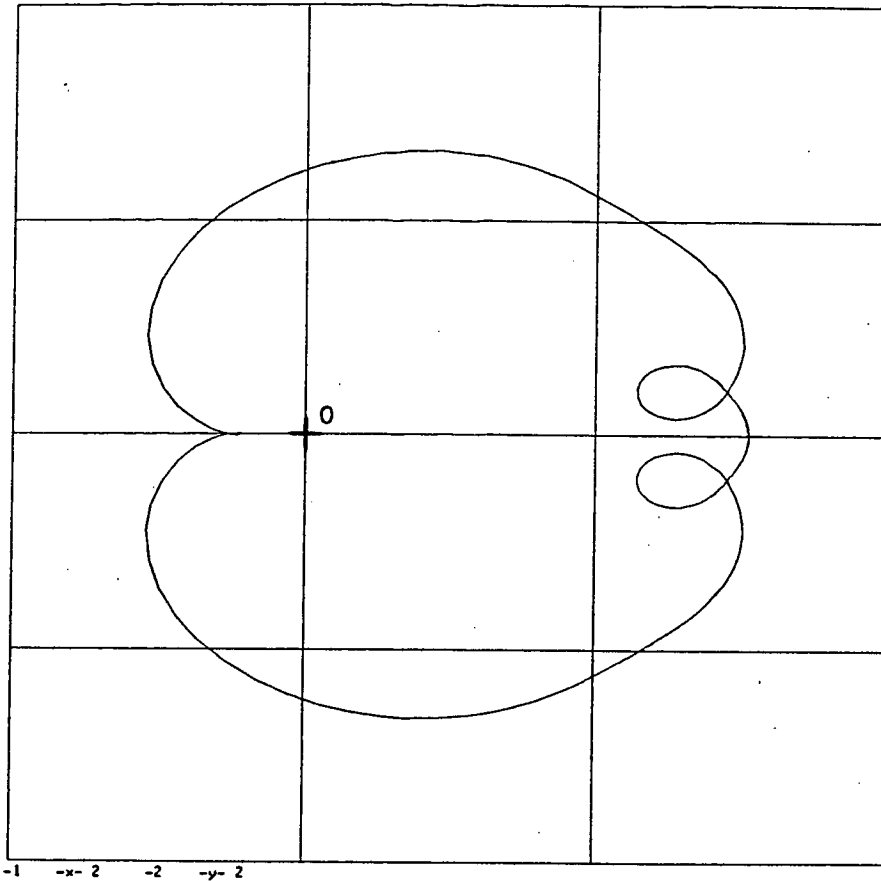
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**Figure 2:** The transform of the circle  $\Gamma = \{q \mid |q| = 7/10\}$  by function  $1-b(q)$ . (Function  $b(q)$  has been estimated from the first 50 terms of its Taylor expansion at  $q=0$ ; the curve is obtained by transforming 100 regularly spaced points on  $\Gamma$ ).

main term. This fact common to the asymptotic behaviour of coefficients of meromorphic functions accounts for the excellent numerical accuracy of the approximation (2), as is exemplified by (3a), (3b).

Notice that if  $t$ -ary trees were considered (a node in such a tree may have out-degree 0 or  $t$  only), then one could prove similarly that the associated l.n.s satisfy:

$$H_n(t) = K(t) \cdot \vartheta^n(t)$$

and we should expect  $\vartheta(t)$  to tend to 2 as  $t \rightarrow \infty$ , since as  $t$  increases, a larger fraction of integer compositions become level number sequences.

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