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RESUME

Nous considérons une file d'attente à temps discret composée d'un serveur unique et d'une salle d'attente de capacité infinie. Cette file est visitée par des clients appartenant à $K+1$ classes distinctes. Aux instants de décision $t = 1, 2, \ldots$, la politique de service indique la classe de clients seule autorisée à utiliser le serveur dans l'intervalle de temps $(t, t+1)$. Pour chaque classe de clients, la suite des instants d'arrivée forme une séquence de renouvellement, de fonction de renouvellement arbitraire. Le temps de service requis par un client quelconque suit une loi géométrique, de paramètre dépendant de sa classe. Le critère d'optimisation de type ergodique vise à minimiser une combinaison linéaire des nombres moyens de clients de classes $1, 2, \ldots, K$ dans le système, sous une contrainte globale portant sur les clients de classe 0. Nous montrons qu'il existe une politique de service optimale, stationnaire et non déterministe, consistant en une modification aléatoire de la politique déterministe classique dite *μc rule*. Le problème d'optimisation est alors réduit au calcul d'un facteur de biais. Ce calcul est effectué dans un cas particulier et sous les hypothèses que les processus des arrivées sont mutuellement indépendants et géométriques.

ABSTRACT

We consider a discrete-time queueing system consisting of $K+1$ classes of customers competing for a single server at an infinite capacity queue. At each decision epoch $t = 1, 2, \ldots$, the scheduling policy specifies the customer class to be served during the time slot $(t, t+1)$. For each customer class the arrival sequence forms a renewal sequence but is otherwise arbitrary. The service requirements are geometric with class dependent parameter. The optimization criterion is to minimise a linear combination of the average line lengths for classes 1 through K, while simultaneously subjecting the average line length of class-0 customers to a hard constraint. The optimal policy is shown to be a randomized modification of the *μc rule*. The optimization problem is thereby reduced to a problem of finding the optimal randomization factor. This is done in a particular case, when the arrival processes are independent and geometrically distributed.

KEYWORDS

Queueing Theory; Markov Chain; Dynamic Programming; Optimal Control; Generating Function; Boundary Value Problems.

I. INTRODUCTION

We consider a discrete-time queueing system consisting of $K+1$ classes of customers competing for a single server. Within time slots the customers arrive to an infinite capacity queue. For each customer class the arrivals form a renewal sequence with finite mean. The arrivals from the classes $k = 0, 1, \ldots, K$ are independent from slot to slot and possibly dependent within a particular slot. At the beginning of each time slot, the server's attention is given to a customer class on the basis of the control policy. Once a customer from class k, $k = 0, 1, \ldots, K$ has been selected to receive service, he either departs from the system during that slot with probability μ_k or remains in the system with probability $1 - \mu_k$, $0 < \mu_k \leq 1$. At most one customer departs from the system during a time slot. The service requirements are therefore geometrically distributed with class dependent rate μ_k and are independent from customer to customer. Note that a customer may be preempted as a new priority decision is made at the beginning of each slot.

The objective is to find an assignment policy u that minimizes the long-run average cost

$$C_n(u) := \limsup_t t^{-1} E_u\left[\sum_{s=1}^{t} \sum_{k=1}^{K} c_k N_k(s) \,|\, N(1) = n\right], \tag{1.1}$$

subject to the hard constraint

$$V_n(u) := \limsup_t t^{-1} E_u\left[\sum_{s=1}^{t} N_0(s) \,|\, N(1) = n\right] \leq \alpha, \tag{1.2}$$

for all initial states $n \in S := \mathbf{N}^{K+1}$, where $\mathbf{N}$ denotes the set of all nonnegative integers. Here, $c_k \geq 0$ are constants, $N_k(s)$ is the number of class-k customers in the system at the beginning of slot s, and $N(s) := \{N_0(s), \ldots, N_K(s)\}$.

The control policy u may be randomized, and may depend on the current line lengths as well as the past history of the system.

The problem studied here is closely related to that studied by Baras, Dorsey and Makowski [1] and by Buyukkoc, Varaiya and Walrand [4], in which the optimization criterion was to minimize (1.1) without regard to the hard constraint (1.2).

In this paper we determine an optimal policy with a simple structure. To describe this policy, relabel the customer classes 1 through K so that

$$\mu_1 c_1 \leq \mu_2 c_2 \leq \cdots \leq \mu_K c_K,$$

and let g_0 be the static policy that gives priority to the highest label, i.e., class-K customers receive the highest priority and class-0 customers the lowest priority. In particular, customers from classes 1 through K are served according to the so-called *μc rule* (e.g. see [1], [4], [10, pp.198-200], [12, p.125]). Consider also the policy g_k that is identical to g_0 except for giving priority to class-0 customers over customer classes 1 through k, for all $k = 1, 2, \ldots, K$. In particular, g_K gives the highest priority to class-0 customers and then gives priority to the other customers according to the *μc rule*.

Using a Lagrange multiplier technique, we show that if there is a policy that meets the constraint (1.2), then there is also an *optimal policy* that randomizes between two static policies g_{j-1} and g_j, for some $j = 1, 2, \ldots, K$. The randomization mechanism involves repeated tosses of a coin with a bias factor that depends *neither* on the current line lengths *nor* on the past history. Note that the special structure of the static policies g_{j-1} and g_j implies that the randomization is only necessary at the decision epochs for which there is contention between customer classes 0 and j; otherwise one simply applies the μc *rule*.

One feature of this optimisation problem which is not characteristic for many controlled queueing problems with multidimensional state space [16], is that we are able to completely specify the optimal policy for two particular (but natural) cases. Indeed, once having established the aforementioned result, it only remains to determine the bias factor in order to completely specify the optimal policy.

When the arrival patterns are *independent* from class to class, and when the arrival distributions are *geometric* within each slot, then the *optimal bias factor* is calculated for two cases:

i) when the constraint is *tight* (to be made more precise),

ii) when $K = 1$.

In Section II, we collect some results regarding the equilibrium behaviour of the priority queue. The structure of the optimal policy is determined in Section III. In Section IV, the optimal bias factor is determined for the particular cases and numerical results are provided.

II. EQUILIBRIUM BEHAVIOR

Before constructing the optimal policy, we need to examine the behavior of the queueing model when governed by randomized stationary policies, i.e., policies that depend neither on the past history nor the slot index t. For any such policy f, the state process $\{ N(t),\, t \geq 1 \}$ is a homogeneous Markov chain. Write Q_f for the corresponding transition matrix. Define the traffic intensity

$$\rho := \sum_{k=0}^{K} \frac{\lambda_k}{\mu_k},$$

where λ_k is the expected number of class-k customers arriving in a given slot for $k = 0, 1, \ldots, K$.

We shall always assume that $\rho < 1$. For a fixed initial state $n \in S$, consider the hitting time to the state θ corresponding to an empty system. In [2] it is shown that the expected hitting time is finite whenever the policy g_0 is employed and $\rho < 1$. Now consider an arbitrary nonidling stationary policy f. Because such a policy is *work conserving* [10], [12], it can easily be established by sample path methods that Q_f has the same recurrence property. Therefore, for each nonidling stationary policy f, Q_f has one positive recurrent class (containing θ) which is reached by the corresponding Markov chain in finite expected time from any initial state. It therefore follows that there is a probability measure π_f [8, p.51] such that

$$\pi_f(n) = \lim_t t^{-1} \sum_{s=1}^{t} P_f(N(s) = n \,|\, N(1) = m),$$

for all m, $n \in S$; moreover, $V_n(f)$ and $C_n(f)$ given in (1.1), (1.2) are independent of the initial state $n \in S$.

For a fixed nonidling stationary policy f, write $N := \{N_0, \ldots, N_K\}$ for the equilibrium line length vector, i.e., N is a random vector satisfying for all $n \in S$

$$P_f(\, N = n \,) = \pi_f(n).$$

Let $A_k(t)$ be the number of class-k customers who joined the queue during the slot t, $k = 0, 1, \ldots, K$. The following theorem gives the equilibrium mean line lengths when a static priority policy is employed; it will be needed to construct the optimal policy in Section III.

Theorem 2.1. *Let $L_j := E_{g_0}[N_j]$, $j = 0, 1, \ldots K$. Then*

$$L_K = \frac{\sigma_{KK} + \lambda_K(1-\lambda_K)}{2(\mu_K - \lambda_K)}, \tag{2.1}$$

where $\sigma_{ij} := E[(A_i(t) - \lambda_i)(A_j(t) - \lambda_j)]$. The remaining L_j, $j = 0, 1, \ldots, K-1$ can be computed in the following recursive manner. Suppose $L_{j+1}, L_{j+2}, \ldots, L_K$ are known. Then L_j is obtained by solving the following system of $K - j + 1$ independent linear equations in the unknowns $W_{j+1}, W_{j+2}, \ldots, W_K, L_j$:

$$\begin{aligned} \mu_j W_{j+1} - \lambda_j L_j &= (\sigma_{jj} + \lambda_j(1-\lambda_j))/2, \\ -\mu_i W_i + \mu_i W_{i+1} - \lambda_i L_j &= \sigma_{ij} + \lambda_j(L_i - \lambda_i), \qquad j < i < K, \\ -\mu_K W_K + (\mu_K - \lambda_K)L_j &= \sigma_{jk} + \lambda_j(L_K - \lambda_K). \end{aligned} \tag{2.2}$$

Proof. Let the policy g_0 be in force and define for $j = 0, 1, \ldots, K$

$$Y_j(t) := \sum_{k=j}^{K} N_k(t), \quad Y_{K+1}(t) := 0.$$

Then $N_j(t)$ evolves according to

$$N_j(t+1) = N_j(t) + A_j(t) - B_j(t)\mathbf{1}(Y_{j+1}(t) = 0, N_j(t) > 0), \tag{2.3}$$

where $B_j(t)$ are independent $\{0,1\}$-valued random variables such that $P(B_j(t) = 1) = \mu_j$, for $j = 0, 1, \ldots, K$. We proceed as in [11, pp.181-184]. Take the expectation of both sides of (2.3) and observe that in equilibrium $E[N_j(t+1)] = E[N_j(t)]$. Writing Y_j for the equilibrium variable corresponding to $Y_j(t)$, we obtain

$$P_{g_0}(Y_{j+1} = 0, N_j > 0) = \frac{\lambda_j}{\mu_j}, \qquad j = 0, 1, \ldots, K. \tag{2.4}$$

With $j = K$, squaring both sides of (2.3), taking expectations, invoking (2.4) and finally observing that in equilibrium $E[N_K^2(t+1)] = E[N_K^2(t)]$, we obtain (2.1).

Now fix j, $j = 1, \ldots, K-1$ and suppose that $L_{j+1}, L_{j+2}, \ldots, L_K$ are known. Furthermore for $i = j+1, \ldots, K$ define

$$W_i := E[N_j \mathbf{1}(Y_i = 0)].$$

We first show that $L_j, W_{j+1}, \ldots, W_K$ satisfy (2.2). To this end, multiply $N_i(t+1)$ times $N_j(t+1)$ in (2.3), take expectation of both sides, invoke formula (2.4) and again observe that in equilibrium $E[N_i(t+1)N_j(t+1)] = E[N_i(t)N_j(t)]$. Carrying out the procedure for each $i = j, \ldots, K$ we obtain (2.2).

It remains to show that equations (2.2) are independent. Define

$$Z_j := [W_{j+1}, \ldots, W_K, L_j]^T.$$

Then (2.2) corresponds to a matrix equation of the form $\mathbf{C}_j Z_j = \mathbf{b}_j$. Consequently it suffices to show that $\det \mathbf{C}_j \neq 0$. It is easily seen that $\mathbf{C}_j$ can be expressed as

$$\mathbf{C}_j = \begin{pmatrix} \mu_j & 0 & \ldots & 0 & -\lambda_j \\ -\mu_{j+1} & & & & \\ 0 & & \mathbf{C}_{j+1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} \tag{2.5}$$

with $\mathbf{C}_K := \mu_K - \lambda_K$, $j = 0, 1, \ldots, K-1$. It follows from (2.5) that

$$\det \mathbf{C}_j = \mu_j \det \mathbf{C}_{j+1} - \lambda_j \prod_{i=j+1}^{K} \mu_i,$$

and therefore by induction

$$\det \mathbf{C}_j = (1 - \sum_{i=j}^{K} \frac{\lambda_i}{\mu_i}) \prod_{i=j}^{K} \mu_i. \tag{2.6}$$

The proof is completed upon observing that the ergodicity condition $\rho < 1$ implies that (2.6) is strictly positive. ∎

Remark 2.1. For $K = 1$ (there are only two customer classes) then Theorem 2.1 gives

$$L_0 = \frac{\frac{\lambda_0 \mu_0}{\mu_1 - \lambda_1}[\sigma_{11} + \lambda_1(1-\lambda_1)] + \mu_1\sigma_{00} + 2\mu_0\sigma_{01} + \mu_1\lambda_0(1-\lambda_0) - 2\lambda_0\lambda_1\mu_0}{2\mu_0\mu_1(1-\rho)}.$$

Define for $k = 1, 2, \ldots, K$ the *randomized μc rule*

$$f_{k,q} := [\, g_{k-1},\, g_k;\, q\,]$$

which at each decision epoch, employs either g_{k-1} or g_k, with probability q and $1-q$, respectively. We will also need the following result

Theorem 2.2. *The average line length for class-0 customers $V(f_{k,q})$ is a continuous function of q over the interval [0,1], for each $k = 1, 2, \ldots, K$.*

Proof. Fix k and write π_q for the equilibrium probability measure $\pi_{f_{k,q}}$. Due to the memoryless property of the geometric service requirements and to the *conservation law* [10, pp.172-181], [12, pp.113-118], we have

$$\lim_t t^{-1} E_{f_{k,q}}[\sum_{k=0}^{K} \mu_k^{-1} N_k(t) \mid N(1) = n] = \sum_{k=0}^{K} \mu_k^{-1} L_k := M, \tag{2.7}$$

for any $q \in [0,1]$ and any initial state $n \in S$. Because M is finite, (2.7) implies that the family of probability measures

$$\{\, \pi_q \; : \; q \in [0,1] \}$$

is *tight* [3], [15, pp.97-98]. Combining this fact with Theorem 3.1 of Malysev and Mensikov [13], we see that the function $q \longrightarrow \pi_q(n)$ is continuous over [0,1] for each $n \in S$.

It can be easily shown by *dynamic programming* [15, pp.130-131] that for any $l, t \in \mathbf{N}$ and $n \in S$, the policy $f_{k,1}$ maximizes

$$E_{f_{k,q}}[\sum_{s=1}^{t} \mathbf{1}(N_0(s) \geq l) \mid N(1) = n]$$

over the class of randomized μc *rules* $\{\, f_{k,q} \; : \; q \in [0,1]\}$ (which is intuitively obvious). This in turn implies

$$\sum_{n=(n_0,\ldots,n_K) \in S} \pi_q(n)\, \mathbf{1}(n_0 \geq l) \; \leq \sum_{n=(n_0,\ldots,n_K) \in S} \pi_1(n)\, \mathbf{1}(n_0 \geq l), \tag{2.8}$$

for each integer l and $q \in [0,1]$. Now for each $q \in [0,1]$ let X_q be a $\mathbf{N}$-valued random variable with

$$P(X_q = i) = \sum_{n=(n_0,\ldots,n_K)\in S} \pi_q(n)\,\mathbf{1}(n_0 = i), \qquad i \in \mathbf{N}.$$

By Scheffe's Theorem [3, pp.224-225] and the componentwise continuity of π_q it follows that for any $i \in \mathbf{N}$, $q \longrightarrow P(X_q = i)$ is continuous on [0,1]. Moreover, (2.8) implies that $\{X_q : q \in [0,1]\}$ is uniformly integrable [3, eq.5.2]. Since

$$V(f_{k,q}) = E[X_q],$$

the theorem follows from the continuity of $q \longrightarrow P(X_q = i)$ and the uniform integrability (see [3, Theorem 5.4]). ∎

III. THE STRUCTURE OF THE OPTIMAL POLICY

A constrained optimization problem can be reduced to one without constraints through the introduction of Lagrange multipliers. For each fixed multiplier $\omega \geq 0$, define the Lagrangian

$$J_n^{\omega}(u) := \limsup_t t^{-1} E_u\left[\sum_{s=1}^{t} \left(\omega N_0(s) + \sum_{k=1}^{K} c_k N_k(s)\right) \mid N_1 = n\right],$$

and consider the *unconstrained problem* of minimizing $J_n^{\omega}(u)$ over the class of all policies. This unconstrained problem can be put into the framework of a *multiarmed bandit problem* [17], [18]. However, the simple structure of the present problem permits a direct solution based on dynamic progamming [1] or on a simple interchange argument [4]. Indeed, it is shown in [1] and in [4] that the *μc rule* is optimal. Alternatively stated, if the multiplier ω satisfies

$$\frac{\mu_{k-1}c_{k-1}}{\mu_0} \leq \omega \leq \frac{\mu_k c_k}{\mu_0},$$

then the policy g_{k-1} is unconstrained optimal for $k = 1, 2, \ldots, K$. In particular, if

$$\omega := \frac{\mu_k c_k}{\mu_0},$$

then the policies g_{k-1}, g_k and the randomized *μc rule* $[g_{k-1}, g_k, q]$ are all unconstrained optimal.

Remark 3.1. The unconstrained optimality of the *μc rule* can also be easily proved via the conservation law (2.7) as done in [10, pp.198-200], [12, p.175] for continuous-time queues.

We are now in position to determine the structure of the optimal policy. Observe that the means $V(g_k)$, $k = 0, 1, \ldots, K$ can easily be determined using Theorem 2.1.

Theorem 3.1. *If $V(g_K) > \alpha$, then there does not exist a policy that meets the constraint. If $V(g_K) \leq \alpha \leq V(g_0)$, then for some $q \in [0,1]$ the randomized μc rule $f_q := [g_{j-1}, g_j, q]$ is constrained optimal, where j is given by*

$$j = min\{i : V(g_i) \leq \alpha\}. \tag{3.1}$$

If $\alpha \geq V(g_0)$, then the static policy g_0 is constrained optimal.

Proof. First suppose that $V(g_K) \leq \alpha \leq V(g_0)$ so that

$$V(g_j) \leq \alpha \leq V(g_{j-1}),$$

with j given by (3.1). Since

$$V(f_0) = V(g_j), \qquad V(f_1) = V(g_{j-1}),$$

it follows by Theorem 2.2 that there is a $q \in [0,1]$ such that

$$V(f_q) = \alpha. \tag{3.2}$$

Moreover, by the *μc rule*, the policy f_q minimizes $J_n^\gamma(u)$ for all $n \in S$, where

$$\gamma := \frac{\mu_j c_j}{\mu_0}.$$

Thus, for all initial states $n \in S$ and any feasible policy u, we have

$$\begin{aligned} C(f_q) + \gamma\alpha &= J^\gamma(f_q) \\ &\leq J_n^\gamma(u) \\ &\leq C_n(u) + \gamma V_n(u). \end{aligned} \tag{3.3}$$

Equation (3.3) in turn implies

$$C(f_q) - C_n(u) \leq \gamma(V_n(u) - \alpha),$$

for all $n \in S$. Hence, $C(f_q) \leq C_n(u)$ for all $n \in S$ if the policy u meets the hard constraint (1.2). Therefore f_q is *constrained optimal*.

For the other cases, we first observe that the *μc rule* implies

$$V(g_K) \leq V_n(u), \tag{3.4}$$

$$K(g_0) \leq K_n(u), \tag{3.5}$$

for all policies u and all $n \in S$. Now if $\alpha < V(g_K)$, then from (3.4) we have $\alpha < K_n(u)$ for all $n \in S$ and all policies u, i.e., there does not exist a policy that meets the constraint (1.2). On the other hand, if $V(g_0) \leq \alpha$, then g_0 is *constrained optimal* by (3.5). ∎

Remark 3.2. Besides the randomized *μc rule* there exist other optimal constrained (nonstationary) policies (see [15, pp.126-127]).

In order to completely determine the optimal policy, it remains to calculate the bias factor q that satisfied (3.2). This is the theme of Section IV.

IV. THE OPTIMAL BIAS FACTOR

For the remainder of this paper, we will assume that one of the two following conditions holds:

i) $\alpha \leq V(g_{K-1})$ (tight constraint value),
ii) $K = 1$.

In case i (and obviously in case ii) it follows that it is only necessary to study a system with classes 0 and 1.

Because the function $q \longrightarrow V(f_q)$ is clearly strictly increasing and continuous in $[0,1]$ by Theorem 2.2 (here $f_q := [\,g_0,\ g_1,\ q\,]$ as in Theorem 3.1), the bias factor that satisfies (3.2) can easily be obtained numerically once $V(f_q)$ is known for all $q \in (0,1)$. For the remainder of this paper, we focus our attention on the determination of $V(f_q)$ for fixed $q \in (0,1)$.

The sample path equations (2.3) for the randomized policy f_q become

$$\begin{aligned} N_0(t+1) &= N_0(t) + A_0(t) - B_0(t)\mathbf{1}(N_0(t) > 0)\,[\mathbf{1}(N_1(t) = 0) + (1 - U(t))\mathbf{1}(N_1(t) > 0)], \\ N_1(t+1) &= N_1(t) + A_1(t) - B_1(t)\mathbf{1}(N_1(t) > 0)\,[\mathbf{1}(N_0(t) = 0) + U(t)\mathbf{1}(N_0(t) > 0)], \end{aligned} \tag{4.1}$$

where $\{\,U(t),\ t \geq 1\,\}$ is a sequence of i.i.d. $\{0,1\}$-valued random variables such that $P(\,U(t) = 1\,) = q$.

At this point, it is natural to try to calculate $V(f_q)$ directly from the equations (4.1), as done in the proof of Theorem 2.1. The reader can verify, however, that this approach introduces more unknowns than independent equations when $q \in (0,1)$, even if the conservation law (2.7) is employed.

We are therefore motivated to take the more indirect, but classical approach in queueing theory, of using *generating functions* to derive $V(f_q)$. This method also enables the determination of higher moments.

For $|x| \leq 1$, $|y| \leq 1$, we define

$$F_q(x,y) = E_q\left[\,x^{N_1} y^{N_0}\,\right],$$

the generating function under f_q for the equilibrium random variables N_1, N_0 (see Section II), which is well-defined when $\rho < 1$.

Theorem 4.1. *For all $|x| \leq 1$, $|y| \leq 1$, the generating function $F_q(x,y)$ satisfies the functional equation*

$$K(x,y)F_q(x,y) = A(x,y)\,[qF_q(0,y) - (1-q)F_q(x,0)] + B(x,y)(1-\rho), \tag{4.2}$$

where

$$\begin{aligned} K(x,y) &:= \frac{1 - G(x,y)\,[1 - q\mu_1(1 - 1/x) - (1-q)\mu_0(1 - 1/y)]}{G(x,y)}, \\ G(x,y) &:= E\left[x^{A_1(t)} y^{A_0(t)}\right], \\ A(x,y) &:= \mu_1(1 - \frac{1}{x}) - \mu_0(1 - \frac{1}{y}), \\ B(x,y) &:= (1-q)\mu_1(1 - \frac{1}{x}) + q\mu_0(1 - \frac{1}{y}). \end{aligned}$$

Proof. Employing (4.1) and the statistical properties of $B_i(t)$, $A_i(t)$ and $U(t)$ for $i = 0, 1$ we obtain

$$\begin{aligned} E_q\left[x^{N_1(t+1)} y^{N_0(t+1)}\right] = G(x,y)\{&P_q\,(N_1(t) = N_0(t) = 0) \\ &+ (1 - \mu_1(1 - \frac{1}{x}))E_q\left[x^{N_1(t)}\mathbf{1}(N_1(t) > 0, N_0(t) = 0)\right] \\ &+ (1 - \mu_0(1 - \frac{1}{y}))E_q\left[y^{N_0(t)}\mathbf{1}(N_1(t) = 0, N_0(t) > 0)\right] \\ &+ (1 - q\mu_1(1 - \frac{1}{x}) - (1-q)\mu_0(1 - \frac{1}{y})) \\ &\times E_q\left[x^{N_1(t)} y^{N_0(t)}\mathbf{1}(N_1(t) > 0, N_0(t) > 0)\right]\}. \end{aligned}$$

Letting t approach infinity, we obtain

$$K(x,y)F_q(x,y) = \left[\mu_1(1-\frac{1}{x}) - \mu_0(1-\frac{1}{y})\right][qF_q(0,y) - (1-q)F_q(x,0)] \\ + \left[(1-q)\mu_1(1-\frac{1}{x}) + q\mu_0(1-\frac{1}{y})\right] F_q(0,0).$$

It remains to show that $F_q(0,0) = 1-\rho$. To this end, let $y=1$ in the above equation, divide by $x-1$ and let x go to 1, and vice versa, to obtain

$$\frac{\lambda_1}{\mu_1} = q(1-F_q(0,1)) + (1-q)\left(F_q(1,0) - F_q(0,0)\right),$$
$$\frac{\lambda_0}{\mu_0} = (1-q)(1-F_q(1,0)) + q\left(F_q(0,1) - F_q(0,0)\right).$$

Adding these two equations, we get the desired result. ∎

The functional equation (4.2) operates on a function of two complex variables, $F_q(x,y)$, and contains two unknown functions of one complex variable, $F_q(x,0)$ and $F_q(0,y)$. Such functional equations frequently appear in studies of random walks on $\mathbf{N}^2$; for instance see [5], [6], [7], [14]. In each case, it is desired to find the unique solution that is analytic for $|x| < 1$, $|y| < 1$, continuous for $|x| \le 1$, $|y| \le 1$ and satisfies the normalization condition $F_q(1,1) = 1$. The uniqueness of such a solution follows both from analytical and probabilistic arguments (see Appendix).

Recent developments in the study of such functional equation [5] do not however give a solution of (4.2) for *all* parameters λ_1, λ_0, μ_1, μ_0 satisfying the condition $\rho < 1$ *and* for all $q \in (0,1)$ if the generating function $G(x,y)$ is arbitrary (see Remark 4.1). It is possible, however, to solve (4.2) for particular arrival processes.

For illustration, we henceforth consider the natural case when $A_1(t)$ and $A_0(t)$ are *independent* and *geometrically distributed*.

Then

$$G(x,y) = [1+\lambda_1(1-x)]^{-1}[1+\lambda_0(1-y)]^{-1}$$

and

$$K(x,y) = \lambda_1(1-x) + \lambda_0(1-y) + \lambda_1\lambda_0(1-x)(1-y) + q\mu_1(1-\frac{1}{x}) + (1-q)\mu_0(1-\frac{1}{y}),$$

for all $|x| \le 1$, $|y| \le 1$. In this case, (4.2) is very similar to an equation solved by Nain [14], and the results there can be adapted to the present case, giving, for example, $V(f_q)$ in closed form for all $q \in [0,1]$. This is done in the Appendix, where the method of [14] is briefly described.

A numerical procedure can thus be written to calculate the $q \in [0,1]$ satisfying (3.2). Some numerical results for varying values of $\lambda_1, \lambda_0, \mu_1, \mu_0, \alpha$ are presented in Figure 1 and Table 1.

The calculations were performed on a Honeywell Bull DPS68 computer.

In Figure 1 we have plotted $V(f_q)$ versus the bias factor q. Each point was computed with a precision of 10^{-5} and the CPU utilisation time per point was about 70 seconds. The shape of the curves displayed in Figure 1 appears to be typical, since in each case the saddle point is exactly located at point q such that $\lambda_0 = (1-q)\mu_0$, which actually turns out to be an interesting point both from an analytical (see Appendix) and from a queueing point of view.

Table 1 gives the optimal bias factor for varying values of the constraint α. The numbers in parentheses indicate the corresponding CPU utilization times, where the required precision for the computation of q_{opt} was such that $|V(f_{q_{opt}}) - \alpha| < 10^{-3}$. The differences appearing between the various CPU utilization times come from the fact that the computation of $V(f_q)$, for given q, needs the resolution of an integral equation (for computing $\gamma(1)$, see Appendix as well as [5, pp.349-352], [14]) which almost behaves as a singular integral equation when λ_0 tends to $(1-q)\mu_0$ from below. In that case, the computational effort must be greater in order to determine $V(f_q)$ with the desired precision.

Remark 4.1. (4.2) can be reduced to the equation studied in [5, p.81, eq.119], i.e., the *kernel* $K(x,y)$ of equation (4.2) can be put into the form

$$K(x,y) = \frac{xy - \Psi(x,y)}{G(x,y)},$$

where $\Psi(x,y)$ is a generating function of two nonnegative integer-valued random variables, X and Y, with $E[X] = 1 + \lambda_1 - q\mu_1$ and $E[Y] = 1 + \lambda_0 - (1-q)\mu_0$. The method proposed in [5] requires however the hypotheses $E[X] < 1$, $E[Y] < 1$, which in our case translates to $\lambda_1 < q\mu_1$, $\lambda_0 < (1-q)\mu_0$. These conditions are thus too restrictive to calculate $V(f_q)$ for *all* $q \in [0,1]$, as needed.

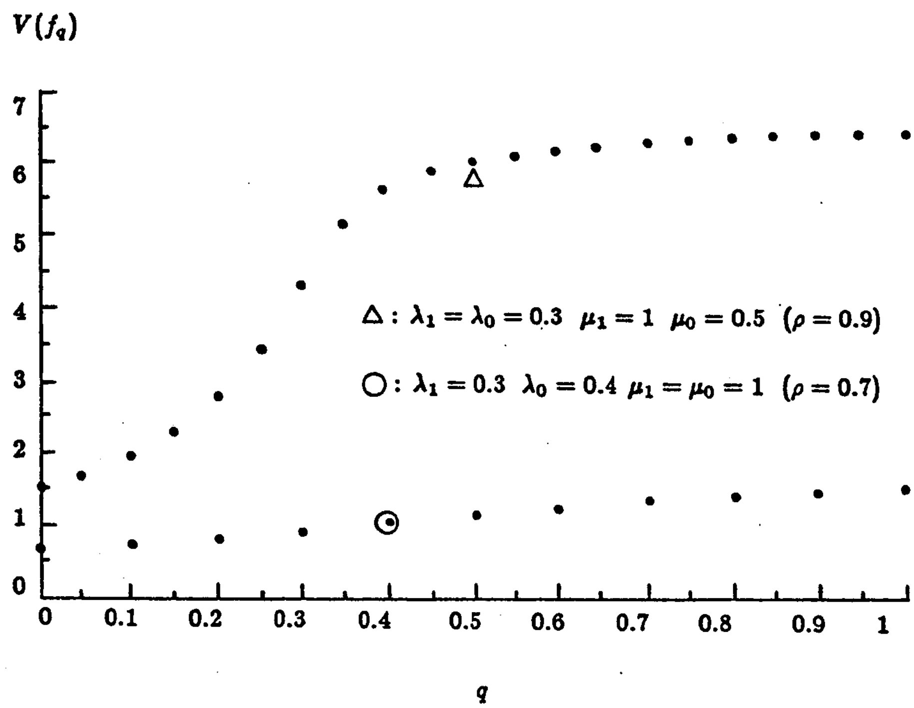

Figure 1

$V(f_q)$ versus q for $K = 1$ and for
geometric and independent arrival processes

Case ○ : $\lambda_1 = 0.3$	$\lambda_0 = 0.4$	$\mu_1 = 1$	$\mu_0 = 1$	$(\rho = 0.7)$	
	$V_{f_0} = 0.666$	$V_{f_1} = 1.504$			
$p = 0.4$	$(\alpha = 1.169)$	$\Longrightarrow$	$q_{opt} = 0.506$	(744 s)	
$p = 0.6$	$(\alpha = 1.002)$	$\Longrightarrow$	$q_{opt} = 0.357$	(679 s)	
$p = 0.8$	$(\alpha = 0.834)$	$\Longrightarrow$	$q_{opt} = 0.203$	(435 s)	

Case △ : $\lambda_1 = 0.3$	$\lambda_0 = 0.3$	$\mu_1 = 1$	$\mu_0 = 0.5$	$(\rho = 0.9)$	
	$V_{f_0} = 1.500$	$V_{f_1} = 6.386$			
$p = 0.4$	$(\alpha = 4.431)$	$\Longrightarrow$	$q_{opt} = 0.305$	(1385 s)	
$p = 0.6$	$(\alpha = 3.454)$	$\Longrightarrow$	$q_{opt} = 0.251$	(866 s)	
$p = 0.8$	$(\alpha = 2.477)$	$\Longrightarrow$	$q_{opt} = 0.172$	(565 s)	

$$(\alpha = (1-p)V(f_1) + pV(f_0)\ ,\ p \in [0,1])$$

Table 1

Computation of the optimal bias factor q_{opt} for varying values of the constraint α

APPENDIX

The method employed in [14] to solve an equation analogous to (4.2), was first proposed in [6] (see also [7]). The basic idea is to convert the problem to a *boundary value problem* (Dirichlet or Riemann-Hilbert problems) on a closed curve. More precisely, beginning with the relation

$$A(x,y)\left[qF_q(0,y)-(1-q)F_q(x,0)\right]+B(x,y)(1-\rho)=0,$$

which must hold for all couples (x,y) such that $|x|\le 1$, $|y|\le 1$ and $K(x,y)=0$ (see (4.2)), one can show that for all $q\in[0,1]$, the function $F_q(0,y)$ (or $F_q(x,0)$) can be analytically continued to a closed curve **L**, containing the point 0 (*as well the point* 1 *if* $\rho<1$), and that on **L**, $F_q(0,y)$ satisfies the following boundary condition

$$Re\left[iF_q(0,y)\right]=\Delta(y),$$

where $\Delta(y)$ is a known real function, continuous on **L**.

The solution of this Dirichlet problem is therefore furnished by *Schwarz's formula* [9, p.221], which gives $F_q(0,y)$ in the interior of **L**, and, in particular, the derivative of $F_q(0,y)$ at $y=1$ (see below). This solution is unique up to an additive constant, which is fixed by the condition $F_q(0,0)=1-\rho$ (see Section IV).

$V(f_q)$ can now be obtained as follows: noting that $V(f_q)=\frac{\partial}{\partial y}F_q(x,y)\Big|_{(x,y)=(1,1)}$, we get using (4.2) after routine calculations

$$V(f_q)=\begin{cases}\dfrac{(\sigma_{00}+\lambda_0(1-\lambda_0))/2-\mu_0 qA}{(1-q)\mu_0-\lambda_0}, & \text{if } \lambda_0\ne(1-q)\mu_0,\\[2ex] \dfrac{(\sigma_{00}+\lambda_0(1-\lambda_0))/2+(\mu_1-\mu_0)qA+\sigma_{01}-\lambda_0(\lambda_1-B)}{q\mu_1-\lambda_1}, & \text{otherwise,}\end{cases}$$

where $A:=\frac{\partial}{\partial y}F_q(0,y)\Big|_{y=1}$ and $B:=\frac{\partial}{\partial x}F_q(x,0)\Big|_{x=1}$.

Note that $\sigma_{jj}=\lambda_j(1+\lambda_j)$, $j=0,1$ and $\sigma_{01}=0$ when the arrival processes are independent and geometrically distributed. Following now the derivation in [14], we obtain for A and B :

- for $\lambda_0<(1-q)\mu_0$

$$A=\frac{2(1-\rho)\gamma\prime(1)(\gamma^2(1)-1)}{\pi}\int_0^\pi\frac{f(\phi)\sin\phi\,d\phi}{\left[1-2\gamma(1)\cos\phi+\gamma^2(1)\right]^2},\tag{a.1}$$

- for $\lambda_0)\ge(1-q)\mu_0$ (and therefore $\lambda_1<q\mu_1$ since $\rho<1$, see Section II)

$$A=\frac{(\lambda_0-(1-q)\mu_0)(1-q)B}{q(q\mu_1-\lambda_1)}+\frac{\lambda_0(q\mu_1-\lambda_1)+\lambda_1((1-q)\mu_0-\lambda_0)(\lambda_0+\dfrac{(1-q)\mu_0-\lambda_0}{q\mu_1-\lambda_1})}{q\mu_0\mu_1(1-\rho)},$$

where

$$i)\ f(\phi):=\frac{\mu_0\mu_1(1-\dfrac{1}{h(\theta(\phi))})\sin(\theta(\phi))}{q\rho(\theta(\phi))\left(\left[\mu_1(1-\dfrac{1}{h(\theta(\phi))})-\mu_0(1-\dfrac{\cos(\theta(\phi))}{\rho(\theta(\phi))})\right]^2+\left[\dfrac{\mu_0\sin(\theta(\phi))}{\rho(\theta(\phi))}\right]^2\right)},$$

ii) $x = h(\phi)$ is the unique solution in (0,1) to

$$0 = \lambda_1(1-x)(1+\lambda_0) + q\mu_1(1-\frac{1}{x}) + \lambda_0 + (1-q)\mu_0 - 2\cos(\phi)\sqrt{\lambda_0\mu_0(1-q)(1+\lambda_1(1-x))},$$

iii) $\rho(\phi) := \sqrt{\dfrac{(1-q)\mu_0}{\lambda_0(1+\lambda_1(1-h(\phi)))}}$,

iv) $\gamma(.)$ is the inverse mapping of the conformal mapping $\gamma_0(.)$ which maps the unit disk $|z| < 1$ onto the interior of the closed curve $\mathbf{L} := \{\, x : x = \rho(\phi)e^{i\phi},\ \phi \in [0, 2\pi] \,\}$. $\theta(.)$ is the angular deformation of this transformation.

The existence of $\gamma_0(.)$ is guaranteed by *Riemann's mapping theorem* [5, p.66]; $\gamma_0(.)$ and $\theta(.)$ can be determined via *Theodorsen's procedure* [5, p.70, pp.349-350], [14].

B is given by (a.1) by interchanging indicies 0 and 1 and replacing q by $1-q$ (including in *i*)-*iv*)).

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