Dynamic behaviour of N-stage transfer lines with unreliable machines and finite buffers

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ABSTRACT

We study the dynamic behaviour of an unreliable N-machine transfer line with finite interstage buffers.

We prove the convergence towards a steady state.

RESUME

On étudie le comportement dynamique d'une ligne de transfert à N-machines séparées par zones de stockage de capacité finie.

On montre la convergence vers un régime permanent.

DYNAMIC BEHAVIOUR OF N-STAGE TRANSFER LINES WITH UNRELIABLE MACHINES AND FINITE BUFFERS

An important class of production systems are modeled by buffered transfer lines with unreliable workstations. The system states consist of the operational conditons of the workstations and the levels of the material in the buffers. The purpose of this paper is to study the DYNAMIC BEHAVIOUR of N-stage transfer lines.

The probability density functions are given by a system of partial differential equations which along the storage level variation curves becomes a system of ordinary differential equations easy to solve with the METHOD OF SUCCESSIVE APPROXIMATIONS and easy to be programmed using MACSYMA language.

We show that every dynamic state converges asymptotically towards a steady state. We estimate the speed of convergence.

0 - INTRODUCTION

Manufacturing systems are often composed of many workstations in which material must pass.

The output of the system is a random process due to the unreliabi- libility of the workstations. The workstations or machines may fail at random time while there are operating. The impaired performance of such a system can be mitigated by the use of storage buffers between processing stages. They provide material upstream of the failed machine and supply material downs- stream.

Such a system illustrated in figure 1.1 is called TRANSFER LINE.

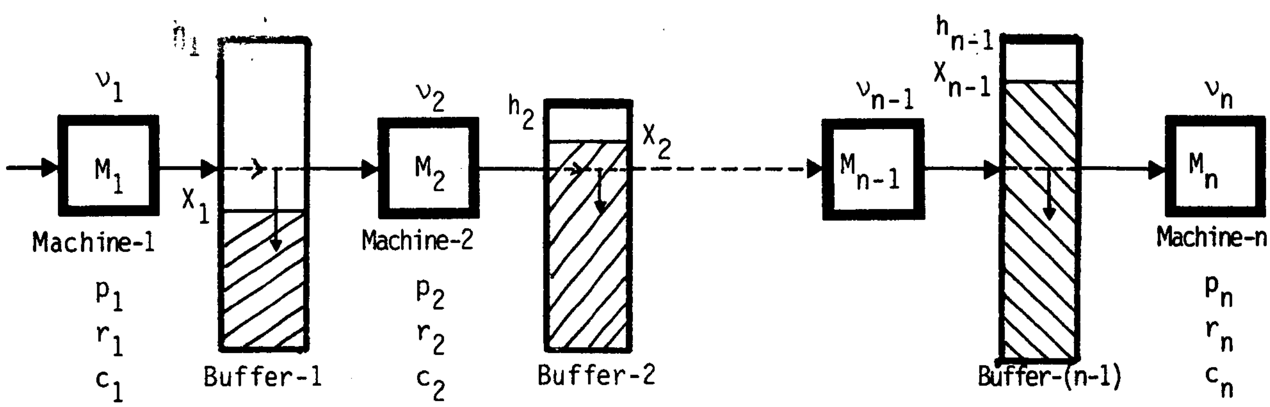

Fig. 1.1

N-STAGE TRANSFER LINE

The steady-state density functions have been studied by GERSHWIN S.B., FORESTIER J.P., PROTH J.M.,... (cf [2], [3], [4], [7],...).

The dynamic behaviour of a two-machine transfer line is given in [8].

OUTLINE

The description of the model and the assumptions are given in section I. In section II we present the storage level behaviour and the system of differential equations satisfied by the probability density functions.

In section III, we use successive approximations method to solve this system. We give a three-machine transfer line example in section IV. The asymptotic behaviour results are given in section V. The conclusion and future research directions can be found in section VI.

I - MODEL DESCRIPTION

The n-stage transfer line, illustrated in figure 1.1, consists of n workingstations or machines denoted by $M_1, M_2, \ldots, M_n$ and separated by n-1 buffers.

Material enters the system at machine 1 from outside and leaves the system after having been processed sequentially in $M_1, M_2, \ldots, M_n$.

Each machine has an unreliable workingtime, it breaks down at random time while it is able to process material and stays under repair for a random length of time. As a consequence of this feature, we associate to each machine M_i a random process $\nu_i(t)$ which is defined to be 1 if M_i is operational and 0 if M_i is under repair.

We suppose that each $\nu_i(t)$ is a Stationary MARKOV PROCESS, that means that the failure and repair rates denoted by p_i, r_i respectively do not depend on time and the failure and repair probabilities are given as follows :

$$(1.1) \qquad \begin{cases} p_i \Delta t = \text{prob}\ \{\nu_i(t+\Delta t)=0\ /\ \nu_i(t)=1\} \\ r_i \Delta t = \text{prob}\ \{\nu_i(t+\Delta t)=1\ /\ \nu_i(t)=0\} \end{cases}$$

We assume that an inexhaustible supply of material is available upstream and an unlimited storage area is present downstream.

The transportation time is not taken into account. We suppose that each machine M_i has a time independant production rate denoted c_i.

II - STORAGE BEHAVIOUR AND DENSITY FUNCTIONS

The n-stage transfer line storage level is given by

$$X(t) = (X_1(t), X_2(t), \ldots, X_{n-1}(t))$$

where $X_i(t)$ can rise or fall depending on the adjacent machines states $\nu_i(t)$ and $\nu_{i+1}(t)$ (figure 2.1).

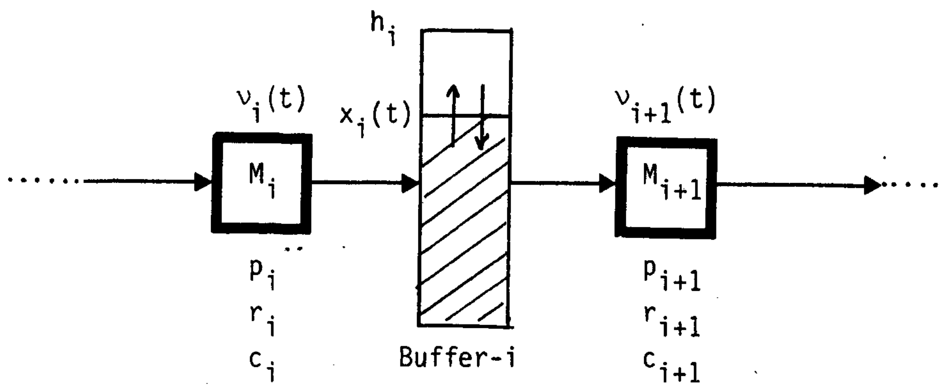

Fig. 2.1

If M_i is operational (resp. under repair) and M_{i+1} under repair (resp. operational) then :

$$\frac{d}{dt} X_i(t) = c_i \quad (\text{resp.} \frac{d}{dt} X_i(t) = -c_{i+1})$$

Clearly

$$\frac{d}{dt} X_i(t) = c_i - c_{i+1}$$

when both adjacent machines are operational and

$$\frac{d}{dt} X_i(t) = 0$$

when they are under repair.

Thus for a given machines-state $\nu = (\nu_1, \ldots, \nu_n)$, the storage level behaviour is described as follows :

(2.1) $$\frac{d}{dt} X(t) = D^\nu \; , \; (\frac{d}{dt} X_i(t) = D_i^\nu) \text{ where } D^\nu = (D_1^\nu, \ldots, D_{n-1}^\nu)$$

and

$$D_i^\nu = \begin{cases} c_i - c_{i+1} & \text{when } \nu_i = \nu_{i+1} = 1 \\ 0 & \text{when } \nu_i = \nu_{i+1} = 0 \\ c_i & \text{when } \nu_i = 1 \; , \; \nu_{i+1} = 0 \\ -c_{i+1} & \text{when } \nu_i = 0 \; , \; \nu_{i+1} = 1 \end{cases}$$

Therfore, $X(s) = X(t) + D^{\nu}.\Delta t$ for s in $[t,t+\Delta t]$ and the necessary storage level at time t form which X can be reached at time $t+\Delta t$ is given by :

(2.2) $$X^{\nu} = X - D^{\nu}.\Delta t$$

For a first order approach, the probability of finding the storage level between X and $X + \Delta X$, and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ as machines-state can be given as follows :

(2.3) $$\text{Prob}\ \{\nu = (\nu_1,\nu_2,\ldots,\nu_n) \text{ and } X \leq X(t) < X+\Delta X\} = f_{\nu}(X,t).|\Delta X|$$

where $f_{\nu}(X,t)$ is the corresponding <u>density function</u>, $\Delta X = (\Delta X_1, \Delta X_2, \ldots, \Delta X_{n-1})$ and $|\Delta X| = \Delta X_1.\Delta X_2.\ldots.\Delta X_{n-1}$

The transition probability matrix $T = (a_{\alpha}^{\nu})_{\alpha,\nu}$, for a first order approach, is given in figure 2.2 We say that two machines states α and ν are <u>adjacent</u> if $a_{\alpha}^{\nu} \neq 0$.

Using (2.3) and the transition probability matrix $T = (a_{\alpha}^{\nu})$, with $a_{\nu}^{\nu} = 1-b_{\nu}^{\nu}.\Delta t$ as diagonal (see fig. 2.2 and 4.2), $f_{\nu}(X,t+\Delta t)$ can be given in connection with $f_{\alpha}(X^{\alpha},t)$, for every adjacent machine state α, as follows :

$$f_{\nu}(X,t+\Delta t).|\Delta X| = (1-b^{\nu}\Delta t)\ f_{\nu}(X^{\nu},t)|\Delta X| + \sum_{\alpha\neq\nu} a_{\alpha}^{\nu}\Delta t\ f_{\alpha}(X^{\alpha},t)|\Delta X|$$

Hence

(2.4) $$\frac{f_{\nu}(X,t+\Delta t)-f_{\nu}(X,t)}{\Delta t} = \frac{f_{\nu}(X^{\nu},t) - f_{\nu}(X,t)}{\Delta t} - b^{\nu}f_{\nu}(X^{\nu},t) + \sum_{\alpha\neq\nu} a_{\alpha}^{\nu}f_{\alpha}(X^{\alpha},t)$$

since

$$X^{\nu} = X-D^{\nu}.\Delta t = (X_1-D_1^{\nu}\Delta t, X_2-D_2^{\nu}\Delta t, \ldots, X_{n-1}-D_{n-1}^{\nu}\Delta t)$$

we have

$$f_{\nu}(X^{\nu},t) = f_{\nu}(X,t) - \sum_{i=1}^{n-1} D_i^{\nu}\Delta t \frac{\partial f_{\nu}}{\partial X_i}(X,t) + o(\Delta t)$$

with

$$\lim_{\Delta t\to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

$\nu(t)$ \ $\nu(t+\Delta t)$	(11...111)	(11...110)	(11...101)	(11...011)	.....
(11...111)	$1-(p_1+p_2...,+p_n)\Delta t$	$p_n\Delta t$	$p_{n-1}\cdot\Delta t$	$p_{n-2}\Delta t$	.....
(11...110)	$r_n\cdot\Delta t$	$1-(p_1+p_2+...+p_{n-1}+r_n)\Delta t$	0	$r_n\cdot\Delta t$	.....
(11...101)	$r_{n-1}\Delta t$	0	$1-(p_1+...p_{n-2}+r_{n-1}+p_n)\Delta t$	$r_{n-1}\Delta t$	......
(11...011)	$r_{n-2}\cdot\Delta t$	0	0	$1-(p_1+...+r_{n-2}+p_{n-1}+p_n)\Delta t$	......
....					.....
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$b^{11...1}=p_1+p_2+...+p_n$; $b^{11...10}=p_1+...+p_{n-1}+r_n$; .....; $b^{101...1}=p_1+r_2+p_3+...+p_n$

Fig. 2.2

N-MACHINE TRANSITION PROBABILITY MATRIX

Therefore

$$(2.5)\qquad \frac{f_\nu(X,t+\Delta t)-f_\nu(X,t)}{\Delta t} = -\sum_{i=1}^{n-1} D_i^\nu \frac{\partial f_\nu}{\partial X_i}(X,t) - b^\nu f_\nu(X^\nu,t) + \sum_{\alpha\neq\nu} a_\alpha^\nu f_\alpha(X^\alpha,t)$$

Now, letting Δt tend to 0 in (2.5) one obtains :

$$(2.6)_\nu\qquad \frac{\partial f_\nu}{\partial t}(X,t) + \sum_{n=1}^{n-1} D_i^\nu \frac{\partial f_\nu}{\partial X_i}(X,t) + b^\nu f_\nu(X,t) = \sum_{\alpha\neq\nu} a_\alpha^\nu f_\alpha(X,t)$$

This equation is given in the following domain Ω.

$\Omega = \Sigma \times (]0,+\infty[)$ where $\Sigma = \{X=(X_1,X_2,\ldots,X_{n-1}) \in R^{n-1} : 0 < X_i < h_i \;\forall X_i\}$

is the storage level space, $\partial\Sigma$ its frontier and $h = (h_1,h_2,\ldots,h_{n-1})$ the buffers-capacity (see figure 1.1.).

The vector equation associated to $(2.6)_\nu$ is given as follows :

$$(2.7)\qquad \frac{\partial f}{\partial t} + \sum_{i=1}^{n-1} C_i \frac{\partial f}{\partial X_i} + Bf = Af \qquad \text{in } \Omega$$

where $f(X,t) = (f_\nu(X,t))_\nu$, C_i and B are diagonal matrices defined by $(C_i)_{\nu\nu} = D_i^\nu$ and $B_{\nu\nu} = b^\nu$.

Matrix A coincides with transition matrix T except for its diagonal which is null.

In order to have uniqueness of the density functions satisfying $(2.6)_\nu$ one is led to find a solution of (2.7) for given $f(X,0)$ in Σ and given $f(X,t)$ in a part Γ of $\partial\Sigma \times (]0,+\infty[)$.

Therefore, the density functions are defined by the following problem :

$$P\begin{cases} \text{Find } f(X,t) \text{ such that} \\ \dfrac{\partial f}{\partial t} + \displaystyle\sum_{i=1}^{n-1} C_i \frac{\partial f}{\partial X_i} + Bf = Af \qquad \text{in } \Omega \\ \text{whith } f(X,0) \text{ given in } \Sigma \text{ and } f(X,t) \text{ given in } \Gamma \end{cases}$$

The system of partial differential equations given by $(2.16)_\nu$ becomes a system of ordinary differential equations along the storage level variation curves.

Indeed,

$$(2.8)\qquad \frac{d}{ds}\{e^{-b^\nu s} f_\nu(X-D^\nu s,t-s)\} = [-b^\nu f_\nu - \sum_{i=1}^{n-1} D_i^\nu \frac{\partial f_\nu}{\partial X} - \frac{\partial f_\nu}{\partial t}](X-D^\nu s,t-s)$$

$$0 < s < T_\nu$$

where $X-D^\nu T_\nu \in \Gamma$ and $T_\nu < t$ when $D^\nu \neq 0$ (non constant storage level) and $T_\nu=t$ when $D^\nu = 0$ (constant storage level).

Using $(2.6)_\nu$, (2.8) yields :

$$(2.9)_\nu \qquad \frac{d}{ds}\{e^{-b^\nu s} f_\nu(X-D^\nu s,t-s)\} = -e^{-b^\nu s} \sum_{\alpha\neq\nu} a_\alpha^\nu f_\alpha(X-D^\nu s,t-s) \ , \ 0 < s < T_\nu$$

REMARK 2.1

The system $(2.6)_\nu$ is hyperbolic and $(X-D^\nu s,t-s)$ for $0 < s < T$ are the corresponding characterisitic curves.

III - THE METHOD OF SUCCESSIVE APPROXIMATIONS

Problem P can be solved using successive approximations method.

The successive approximations are the functions $f^0(X,t), f^1(X,t),\dots,f^{K+1}(X,t)\dots$ défined recursively as follows :

$f^0(X,t) = f(X,0)$ and $f^{K+1}(X,t)$ given as the solution of the next problem :

$$P_{n+1}\begin{cases} \dfrac{\partial f^{k+1}}{\partial t} + \displaystyle\sum_{i=1}^{n-1} C_i \frac{\partial f^{k+1}}{\partial X_i} + Bf^{K+1} = Af^K \text{ in } \Omega \\ \\ f^{K+1}(X,0) = f(X,0) \text{ in } \Sigma \text{ and } f^{K+1}(X,t) = f(X,t) \text{ in } \Gamma \end{cases}$$

which, along the storage level variation curves becomes

$$(3.1)\qquad \frac{d}{ds}\{e^{-b^\nu s} f_\nu^{K+1}(X-D^\nu s,t-s)\} = -e^{-b^\nu s} \sum_{\alpha\neq\nu} a_\alpha^\nu f_\alpha^K(X-D^\nu s,t-s) \ , \ 0 < s < T_\nu$$

where $T_\nu < t$ and $X-D^\nu T_\nu \in \Gamma$ when $D^\nu \neq 0$

and $T_\nu = t$ when $D^\nu = 0$

Integrating (3.1) from 0 to T_ν, one obtains :

$$(3.2)\qquad f_\nu^{K+1}(X,t) = e^{-b^\nu T_\nu} f_\nu(X-D^\nu T_\nu,t-T_\nu) + \int_0^{T_\nu} e^{-b^\nu s} \sum_{\alpha\neq\nu} a_\alpha^\nu f_\alpha^K (X-D^\nu s,t-s)ds$$

It is well known that this sequence converges towards the solution of P.

REMARK 3.1

Equation (3.2) is easy to be programmed using MACSYMA LANGUAGE.

REMARK 3.2

If the given functions, $f(X,0)$ in Σ and $f(X,t)$ in Γ, are not negative the corresponding solutions $f_\nu(X,t)$ of $(2.6)_\nu$ are also not negative and can be used as density funtion.

IV - THREE-MACHINE TRANSFER LINE

Let us represent this example in figure 4.1.

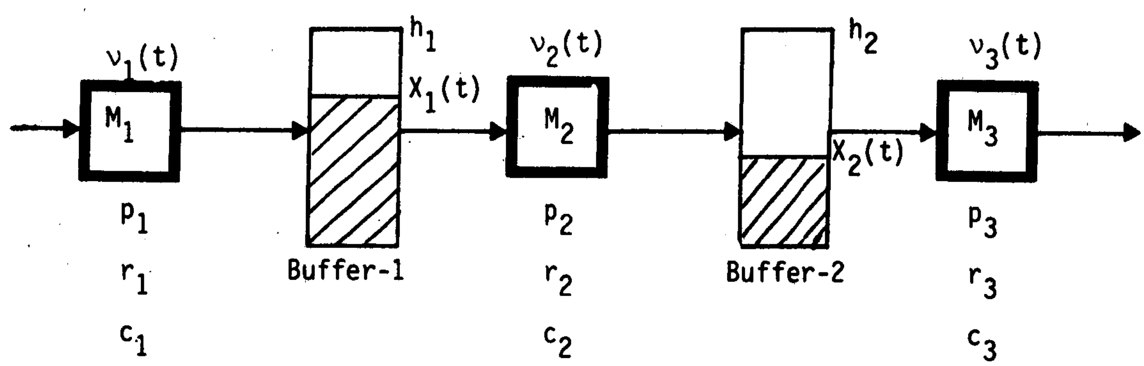

Fig. 4.1

The transition probability matrix $T = (a_\alpha^\nu)_{\alpha,\nu}$, for a first order approach is given in figure 4.2.

$\nu(t)$ \ $\nu(t+\Delta t)$	(111)	(110)	(101)	(011)	(100)	(010)	(001)	(000)
(111)	$1-b^{111}.\Delta t$	$p_3.\Delta t$	$p_2.\Delta t$	$p_1.\Delta t$	0	0	0	0
(110)	$r_3.\Delta t$	$1-b^{110}.\Delta t$	0	0	$p_2.\Delta t$	$p_1.\Delta t$	0	0
(101)	$r_2.\Delta t$	0	$1-b^{101}.\Delta t$	0	$p_3.\Delta t$	0	$p_1.\Delta t$	0
(011)	$r_1.\Delta t$	0	0	$1-b^{011}.\Delta t$	0	$p_3.\Delta t$	$p_2.\Delta t$	0
(100)	0	$r_2.\Delta t$	$r_3.\Delta t$	0	$1-b^{100}.\Delta t$	0	0	$p_1\Delta t$
(010)	0	$r_1.\Delta t$	0	$r_3.\Delta t$	0	$1-b^{010}.\Delta t$	0	$p_2.\Delta t$
(001)	0	0	$r_1.\Delta t$	$r_2.\Delta t$	0	0	$1-b^{001}.\Delta t$	$p_3.\Delta t$
(000)	0	0	0	0	$r_1.\Delta t$	$r_2.\Delta t$	$r_3.\Delta t$	$1-b^{000}.\Delta t$

$b^{111} = p_1+p_2+p_3$, $b^{110} = p_1+p_2+r_3$, $b^{101} = p_1+r_2+p_3$, $b^{011} = r_1+p_2+p_3$,

$b^{100} = p_1+r_2+r_3$, $b^{010} = r_1+p_2+r_3$, $b^{001} = r_1+r_2+p_3$, $b^{000} = r_1+r_2+r_3$

Fig. 4.2

The density functions $f_\nu(X,t)$ are given in the following system :

$$
(4.1)\quad \begin{cases}
\dfrac{\partial f_{111}}{\partial t}+(c_1-c_2)\dfrac{\partial f_{111}}{\partial X_1}+(c_2-c_3)\dfrac{\partial f_{111}}{\partial X_2}+b^{111}.f_{111}=r_3f_{110}+r_2f_{101}+r_1f_{011} \\[2ex]
\dfrac{\partial f_{110}}{\partial t}+(c_1-c_2)\dfrac{\partial f_{110}}{\partial X_1}+c_2\dfrac{\partial f_{110}}{\partial X_2}+b^{110}.f_{110}= p_3f_{111}+r_2f_{100}+r_1f_{010} \\[2ex]
\dfrac{\partial f_{101}}{\partial t}+ c_1\dfrac{\partial f_{101}}{\partial X_1} - c_3\dfrac{\partial f_{101}}{\partial X_2} + b^{101}.\ f_{101} = p_2f_{111} + r_3f_{100} + r_1f_{001} \\[2ex]
\dfrac{\partial f_{011}}{\partial t} - c_2\dfrac{\partial f_{011}}{\partial X_1} +(c_2-c_3)\dfrac{\partial f_{011}}{\partial X_2}+b^{011}.f_{011}=p_1f_{111} + r_2f_{001} + r_3f_{010} \\[2ex]
\dfrac{\partial f_{100}}{\partial t} + c_1\dfrac{\partial f_{100}}{\partial X_1} + 0 + b^{100}.f_{100} = p_2f_{110} + p_3f_{101} + r_1f_{000} \\[2ex]
\dfrac{\partial f_{010}}{\partial t} - c_2\dfrac{\partial f_{010}}{\partial X_1}+ c_2\dfrac{\partial f_{010}}{\partial X_2} + b^{010}.f_{010} = p_1f_{110} + p_3f_{011} + r_2f_{000}
\end{cases}
$$

$$\left\{\begin{array}{l}\dfrac{\partial f_{001}}{\partial t} + 0 - c_3 \dfrac{\partial f_{001}}{\partial X_2} + b^{001} f_{001} = p_1 f_{101} + p_2 f_{011} + r_3 f_{000} \\ \dfrac{\partial f_{000}}{\partial t} + 0 + 0 + b^{000} \cdot f_{000} = p_1 f_{100} + p_2 f_{010} + p_3 f_{001}\end{array}\right.$$

$$(X,t) \in \Sigma \times (]0,+\infty[) = \Omega$$

where $\Sigma = \{(X_1,X_2) \in R^2 : 0 < X_1 < h_1 \ , \ 0 < X_2 < h_2\}$ is the storage level space.

In a vectorial notation, (4.1) becomes

$$\frac{\partial f}{\partial t} + C_1 \frac{\partial f}{\partial X_1} + C_2 \frac{\partial f}{\partial X_2} + Bf = Af \quad \text{in } \Omega$$

where $f(X,t) = (f_{111}, f_{110}, f_{101}, f_{011}, f_{100}, f_{010}, f_{001}, f_{000}) \quad (X,t)$

$$C_1 = \begin{bmatrix} c_1-c_2 &&&&&&& \\ & c_1-c_2 &&&&&& \\ && c_1 &&&&& \\ &&& -c_2 &&&& \\ &&&& c_1 &&& \\ &&&&& -c_2 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{bmatrix} \; ; \quad C_2 = \begin{bmatrix} c_2-c_3 &&&&&&& \\ & c_2 &&&&&& \\ && -c_3 &&&&& \\ &&& c_2-c_3 &&&& \\ &&&& 0 &&& \\ &&&&& c_2 && \\ &&&&&& -c_3 & \\ &&&&&&& 0 \end{bmatrix}$$

$$B = \begin{bmatrix} (p_1+p_2+p_3) &&&&&&& \\ & (p_1+p_2+r_3) &&&&&& \\ && (p_1+r_2+p_3) &&&&& \\ &&& (r_1+p_2+p_3) &&&& \\ &&&& (p_1+r_2+r_3) &&& \\ &&&&& (r_1+p_2+r_3) && \\ &&&&&& (r_1+r_2+p_3) & \\ &&&&&&& (r_1+r_2+r_3) \end{bmatrix} \quad A = \begin{bmatrix} 0 & r_3 & r_2 & r_1 & 0 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 & r_2 & r_1 & 0 & 0 \\ p_2 & 0 & 0 & 0 & r_3 & 0 & r_1 & 0 \\ p_1 & 0 & 0 & 0 & 0 & r_2 & r_3 & 0 \\ 0 & p_2 & p_3 & 0 & 0 & 0 & 0 & r_1 \\ 0 & p_1 & 0 & p_3 & 0 & 0 & 0 & r_2 \\ 0 & 0 & p_1 & p_2 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 0 & p_1 & p_2 & p_3 & 0 \end{bmatrix}$$

System (4.1) along the storage level variation curves becomes :

$$\frac{d}{ds}\{e^{-b^{111}s}.f_{111}(X_1-(c_1-c_2)s,X_2-(c_2-c_3)s,t-s)\} =$$
$$e^{-b^{111}s}[-r_3f_{110}-r_2f_{101}-r_1f_{011}]\ (X_1-(c_1-c_2)s,X_2-(c_2-c_3)s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{110}s}.f_{110}(X_1-(c_1-c_2)s,X_2-c_2s,\ t-s)\} =$$
$$e^{-b^{110}s}[-p_3f_{111}-r_2f_{100}-r_1f_{010}]\ (X_1-(c_1-c_2)s,X_2-c_2s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{101}s}.f_{101}(X_1-c_1s,X_2+c_3s,t-s)\} =$$
$$e^{-b^{101}s}[-p_2f_{111}-r_3f_{100}-r_1f_{001}]\ (X_1-c_1s,X_2+c_3s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{011}s}.f_{011}(X_1+c_2s,X_2-(c_2-c_3)s,t-s)\} =$$
$$e^{-b^{011}s}[-p_1f_{111}-r_2f_{001}-r_3f_{010}]\ (X_1+c_2s,X_2-(c_2-c_3)s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{100}s}.f_{100}(X_1-c_1s,X_2,t-s)\} =$$
$$e^{-b^{100}s}[-p_2f_{110}-p_3f_{101}-r_1f_{000}](X_1-c_1s,X_2,t-s)$$

$$\frac{d}{ds}\{e^{-b^{010}s}.f_{010}(X_1+c_2s,X_2-c_2s,t-s)\} =$$
$$e^{-b^{010}s}[-p_1f_{110}-p_3f_{011}-r_2f_{000}]\ (X_1+c_2s,X_2-c_2s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{001}s}.f_{001}(X_1,X_2+c_3s,t-s)\} =$$
$$e^{-b^{001}s}[-p_1f_{101}-p_2f_{011}-r_3f_{000}]\ (X_1,X_2+c_3s,t-s)$$

$$\frac{d}{ds}\{e^{-b^{000}s}.f_{000}(X_1,X_2,t-s)\} =$$
$$e^{-b^{000}s}[-p_1f_{100}-p_2f_{010}-p_3f_{001}]\ (X_1,X_2,t-s)$$

V- ASYMPTOTIC BEHAVIOUR

Every dynamic state converges asymptotically towards a steady state. Indeed, we prove that the density functions, null over Γ, decrease exponentially.

To be more precise, let us give the following

THEOREM 5.1

If $f(X,t)$ is a solution of P, null over Γ, then

(5.1) $$\|f(t)\|_{L^\infty(0,h)} \leq e^{-Mt} \|f(0)\|_{L^\infty(0,h)} \quad \forall \; t > 0$$

The steady state density functions $g_\nu(X)$ are given by the following equation :

(5.2) $$\sum_{i=1}^{n-1} C_i \frac{\partial g}{\partial X_i} + Bg = Ag \quad \text{in } \Sigma$$

where $g(X) = (g_\nu(X))_\nu$ and C_i, B, A are given in section 2.

Theorem 5.1 yields :

COROLLARY 5.1

Every solution $g(X)$ of (5.2) is the limit of a function $f(X,t)$ satisfying P with $f(X,t) - g(X)$ null over Γ. The following inequality gives the speed of convergence

(5.3) $$\|f(t) - g\|_{L^\infty(0,h)} \leq e^{-Mt} \|f(0) - g\|_{L^\infty(0,h)} \quad \forall \; t > 0$$

PROOF OF THEOREM 5.1

Since $\det(A-B) = 0$, we can take $\gamma = (\gamma_\nu)_\nu \neq 0$ such that

(5.4) $$A\gamma = B\gamma \quad \text{and} \quad |f_\nu(X,0)| \leq \gamma_\nu \quad \forall \; \nu$$

Then

(5.5) $$|f_\nu(X,t)| \leq \gamma_\nu \qquad \forall \; \nu \qquad \forall \; t > 0 \quad \text{(see remark 3.2)}$$

Let us first consider the machines-state which yields a storage level variation ($D^\nu \neq 0$)

If $T = \max \{\frac{h_i}{|c_i - c_{i+1}|}, \frac{h_i}{c_i}, \frac{h_i}{c_{i+1}}, \ i = 1,2,\ldots,n-1\}$ and $t > T$

(5.6) $$f_\nu(X,t) = e^{-b^\nu T_\nu} f_\nu(X - D^\nu T_\nu, t - T_\nu) + \int_0^{T_\nu} e^{-b^\nu s} \sum_{\alpha \neq \nu} a_\alpha^\nu f_\alpha(X - D^\nu s, t-s)\,ds$$

since f is null over Γ, we have

(5.7) $$f_\nu(X,t) = \int_0^{T_\nu} e^{-b^\nu s} \sum_{\alpha \neq \nu} a_\alpha^\nu f_\alpha(X - D^\nu s, t-s)\,ds$$

Thus using (5.4) and (5.5)

(5.8) $$|f_\nu(X,t)| \leq b^\nu \gamma_\nu \int_0^{T_\nu} e^{-b^\nu s}\,ds = \gamma_\nu (1 - e^{-b^\nu T_\nu}) \leq \Theta_\nu \gamma_\nu \quad \forall\, t > T$$

Let us consider know the following problem for the machines-state $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ which do not yield a storage level variation $(D^\nu = 0)$.

(5.9) $$\begin{cases} \dfrac{\partial f_\nu}{\partial t} + \displaystyle\sum_{i=1}^{n-1} C_i \dfrac{\partial f_\nu}{\partial X_i} + B f_\nu = A f_\nu & \text{in } \Sigma \times (]T, +\infty[) \\ |f_\nu(X,T)| < \gamma_\nu \end{cases}$$

Along the characteristic curves (5.9) becomes :

(5.10) $$\frac{d}{ds}[e^{-b^\nu s} f_\nu(X, t-s)] = e^{-b^\nu s} \sum_{\alpha \neq \nu} a_\alpha^\nu f_\alpha(X, t-s) \qquad 0 < s < t-T$$

Integrating (5.10) from 0 to $t-T$,

$$f_\nu(X,t) = e^{-b^\nu (t-T)} f_\nu(X,T) + \int_0^{t-T} e^{-b^\nu s} \sum_{\alpha \neq \nu} a_\alpha^\nu f_\alpha(X, t-s)\,ds$$

Thus, using (5.4) and (5.8)

$$|f_\nu(X,t)| \leq \gamma_\nu [\hat\Theta + (1-\hat\Theta) e^{-b^\nu T}] = \gamma_\nu \Theta, \ \forall\, t \geq 2T$$

with $0 < \Theta < 1$ and $\hat\Theta = \max \{\Theta_\nu : D^\nu \neq 0\}$

We have proved that it exists K such that $0 < K < 1$ and

$$|f_\nu(X,t)| \leq K \gamma_\nu \qquad \forall\, t \geq 2T, \ \forall \nu$$

The same reasoning shows that it exists K satisfying $0 < K < 1$ and $|f_\nu(X,T)| \leq K^2\gamma_\nu \quad \forall\, t \geq 4T, \quad \forall\, \nu$

If we repeat this procedure n-times, we obtain :

$$|f_\nu(X,t)| \leq k^n\gamma_\nu \quad \forall\, t \geq 2nT\ , \quad 0 < K < 1.$$

Thus

$$\|f(t)\|_{L^\infty(0,h)} \leq K^n \|f(0)\|_{L^\infty(0,h)} \quad \forall\, t \geq 2nT, \quad 0 < K < 1$$

Finally, if $K = e^{-t_0}$ and $M=\frac{t_0}{4T}$ then (see figure 5.1)

$$\|f(t)\|_{L^\infty(0,h)} \leq e^{-Mt}\|f(0)\|_{L^\infty(0,h)} \quad \forall\, t > 0$$

■

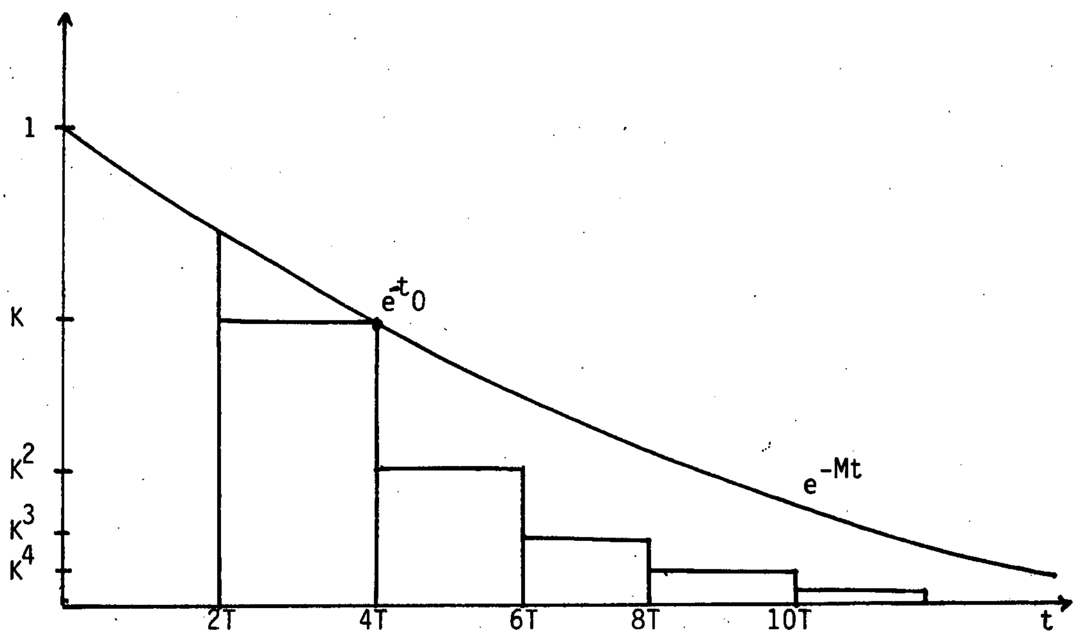

Fig. 5.1 SPEED OF CONVERGENCE

VI - CONCLUSION

The N-machine transfer lines are described in connection with storage level variations. That was not possible in earlier steady state works.

The N-machine transfer line density functions can be always approached using successive approximations method, but the number of different density fuctions grows exponentially with N which inhibits performing applications.
It is suitable to look for other models with less detailed description or to apply this model in a shrewder way.

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BIBLIOGRAPHIE

[1] BUZACOTT J.A., Automatic Transfer Lines with Buffer Stocks, The International Journal of production Research (1967), vol.5, n°3.

[2] FORESTIER J.P., Modélisation stochastique et comportement asymptotique d'un système automatisé de production. R.A.I.R.O. Automatique, vol.14, n°2, 1980, p.127 à 143.

[3] DUBOIS D. et FORESTIER J.P., Productivité et en-cours Moyens d'un ensemble de deux machines séparées par une zone de stockage. RAIRO Automatique, Vol 16, n° 2, 1982, p. 105 à 132.

[4] GERSHWIN S.B. and SCHICK I.C. (1980), Modeling and analysis of two -and three- stage transfer lines with unreliable machines and finite buffers. MIT laboratory for Information and Decision Systems Report LIDS-R-1039.

[5] GERSHWIN S.B. and ODED BERMAN, Analysis of transfer lines consisting of the unreliable machines with random processing times and finite storage buffer AIIE Transaction, vol.13, n°1.

[6] PROTH J.M., Study of two unreliable machines with random processing times and finite storage buffers : a markovian approach, Working Paper 81-37, European Institute of Advanced Studies in Management, Brussels, Belgium (1981).

[7] COILLARD P., et PROTH J.M., Effet des stocks tampons dans une fabrication en ligne. Revue Belge de statistique, d'information et de recherche opérationnelle, vol.24, n°2.

[8] WESFREID Eva, Dynamic behaviour of an unreliable two-machines transfer line. Rapport de Recherche INRIA n° 421, juillet 1985.

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