On strongly coupled elliptic systems: existence,uniqueness and representation results

O. Bennouna

INRIA
CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél. : (1) 39 63 55 11

Rapports de Recherche

N° 453

ON STRONGLY COUPLED ELLIPTIC SYSTEMS: EXISTENCE, UNIQUENESS AND REPRESENTATION RESULTS

Omar BENNOUNA

Novembre 1985

ON STRONGLY COUPLED ELLIPTIC SYSTEMS :

EXISTENCE, UNIQUENESS AND REPRESENTATION

RESULTS

Omar BENNOUNA (*)

(*) *BEL Laboratories, 38 Allée des Roses - FES - MAROC.*

PAPIER RÉCUPÉRÉ ET RECYCLÉ

RESUME

Dans ce papier on considère une classe d'équations et de systèmes elliptiques qui peuvent être réduits à la forme

$$(*) \qquad -\sum_{i,j} a_{ij} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum_{\ell} \sum_{i} b_{k\ell,i} \frac{\partial u_\ell}{\partial x_i} + \sum_{\ell} c_{k\ell}\, u_\ell = f_k \qquad \text{dans } \mathcal{O}$$

où a_{ij}, $b_{k\ell,i}$, $c_{k\ell}$, f_k *sont des données et* $\mathcal{O}$ *est un ouvert borné de* $\mathbb{R}^N$. *On donne un résultat du type principe du maximum et on démontre un résultat d'existence et d'unicité. D'autre part on donne une représentation de la solution du type Feynman-Kac.*

MOTS-CLEFS

Elliptique – Principe du maximum – Représentation du type Feynman-Kac.

ABSTRACT

In this paper we consider a class of elliptic equations and systems which can be reduced to the form :

$$(*) \qquad -\sum_{i,j} a_{ij} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum_{\ell} \sum_{i} b_{k\ell,i} \frac{\partial u_\ell}{\partial x_i} + \sum_{\ell} c_{k\ell}\, u_\ell = f_k \qquad \text{in } \mathcal{O}$$

where a_{ij}, $b_{k\ell,i}$, $c_{k\ell}$, f_k *are given data and* $\mathcal{O}$ *is an open bounded subset of* $\mathbb{R}^N$. *We give a maximum principle type result for thisclass and we prove existence and uniqueness results.*

One the other hand, we give a Feynman-Kac type representation of the solution of (*).

KEY WORDS

Elliptic – Maximum principle – Feynman-Kac representation.

ON STRONGLY COUPLED ELLIPTIC SYSTEMS : EXISTENCE, UNIQUENESS AND REPRESENTATION RESULTS

O. Bennouna

INTRODUCTION

We consider a class of elliptic equations and systems which can be reduced to the form :

$$(*)\quad \begin{cases} -\sum\limits_{i,j} a_{ij}\dfrac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum\limits_{\ell}\sum\limits_{i} b_{k\ell,i}\dfrac{\partial u_\ell}{\partial x_i} + \sum\limits_{\ell} c_{k\ell}\, u_\ell = f_k \quad \text{in } \mathcal{O} \\[2ex] u_k\big|_{\partial\mathcal{O}} = 0 \qquad 1 \le k \le M \end{cases}$$

where a_{ij}, $b_{k\ell,i}$, $c_{k\ell}$, f_k are given data and $\mathcal{O}$ is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\mathcal{O}$.

In this note we give first a maximum principle type result for this class. To obtain the existence and uniqueness result we use a perturbation technic of weakly coupled system of second order elliptic partial differential equations to which we apply accretive operators methods.

The parabolic case is studied in [2] by the method of parabolic singular integrals.

On the other hand, we give a Feynman-Kac type representation of the solution of (*) using the notion of Itô random evolution. In the parabolic case this question is solved in [1]. This type of result is made possible in the elliptic case under a stability condition of (*) which means, roughly speaking, that the dissipation term must be large enough.

Let us give an example of forth order equation which can be reduced to the form (*).

We consider

$$\begin{cases} -\Delta^2 u + b\Delta u + cu = f \quad \text{in } \mathcal{O} \\ u\big|_{\partial\mathcal{O}} = 0 \quad , \quad \Delta u\big|_{\partial\mathcal{O}} = 0 \end{cases}$$

when b, c and f are given smooth data.

We denote $v = \Delta u$, then we obtain the second order elliptic system :

$$\begin{cases} -\Delta v + bv + cu = f \\ -\Delta u + v = 0 \\ u\big|_{\partial\mathcal{O}} = 0 \quad , \quad v\big|_{\partial\mathcal{O}} = 0 \, . \end{cases}$$

As we shall see below this problem is well posed, and we have a probabilistic representation of u, under some natural conditions on the data.

The paper is organized as follows : the part I contains analytic results and the part II is concerned with the Feynman-Kac type result.

I - THE ANALYTIC RESULTS

I.1. Assumptions and notations

Let $\mathcal{O}$ be an open bounded smooth domain in $\mathbb{R}^N$. We assume that :

1) Regularity conditions :

$$\phi = a_{ij}, b_{k\ell,i}, c_{k\ell}, f_k \quad ; \quad \phi \in C(\bar{\mathcal{O}}) .$$

2) Ellipticity condition :

$$\sum_{i,j} a_{ij}\, \xi_i\, \xi_j \geq \gamma\, |\xi|^2 \quad , \quad \gamma > 0 \quad , \quad \forall\, \xi \in \mathbb{R}^N .$$

3) Stability condition :

$$\sum_{k,\ell} c_{k\ell}\, v_k\, v_\ell \geq \beta\, |v|^2 \quad , \quad \beta > 0 \ \text{ large enough}$$

$$\forall\, v \in \mathbb{R}^M$$

(as we shall see in remark 2.2 it is sufficient to assume that

$$\sum_{k,\ell} c_{k\ell}\, v_k\, v_\ell \geq 0 \ ) .$$

I.2. A maximum principle for (*)

In this section we shall give a maximum principle type result.

We have

<u>Theorem 1.1</u> - Assume that conditions (1), (2), (3) hold and

(10) $$4\gamma\beta > ||b||_\infty^2$$

then we have for $u, \bar{u} \in C(\bar{\mathcal{O}}) \cap W^{2,p}(\mathcal{O})$ solutions of (*) corresponding respectively to $f, \bar{f}$

(11) $$||u-\bar{u}||_\infty \le \frac{1}{\beta - \frac{||b||_\infty^2}{4\gamma}} ||f-\bar{f}||_\infty$$

where

$$||b||_\infty^2 = \sum_i ||b_i||_\infty^2 \quad ,$$

<u>Proof</u> : It is enough to prove (11) for $u_k \in C(\bar{\mathcal{O}}) \cap C^2(\mathcal{O})$. Hence, multiplying (*) by u_k and then summing up from 1 to M yields

(12) $$-\sum_{i,j} \frac{a_{ij}}{2} \frac{\partial^2 |u|^2}{\partial x_i \partial x_j} + \sum_{i,j} \sum_\kappa a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} -$$

$$- \sum_i \sum_{\kappa,\ell} b_{k\ell,i} \frac{\partial u_\ell}{\partial x_i} u_k + \sum_{\kappa,\ell} c_{k\ell}\, u_\ell\, u_k = \sum_k f_k\, u_k \; .$$

Let x_0 be a point in $\bar{\mathcal{O}}$ such that $|u|^2 \equiv \sum_k |u_k|^2$ attains a maximum at x_0. If $|u|(x_0) = 0$ then

(13) $$||u||_\infty \le \frac{1}{\beta - \frac{||b||_\infty^2}{4\gamma}} ||f||_\infty \; .$$

If not, we have

(14) $$\sum_{i,j} \frac{a_{ij}}{2}(x_0)\, \frac{\partial^2 |u|}{\partial x_i \partial x_j}(x_0) \le 0$$

and using conditions 1), 3) we get that

(15) $$\gamma \left|\frac{\partial u}{\partial x}\right|^2(x_0) - \sum_i \sum_{k,\ell} b_{k\ell,i}(x_0)\, \frac{\partial u_\ell}{\partial x_i}(x_0)\, u_k(x_0) + \beta\, |u|^2(x_0)$$
$$\le |f|(x_0)\,|u|(x_0)\ .$$

Remarking that

(16) $$\left|\sum_i \sum_{k,\ell} b_{k\ell,i}(x)\, \frac{\partial u_\ell}{\partial x_i}\, u_k\right| \le ||b||_\infty \left|\frac{\partial u}{\partial x}\right|\, |u|$$

we obtain from (15) and (16)

(17) $$\gamma \left|\frac{\partial u}{\partial x}\right|^2(x_0) - ||b||_\infty \left|\frac{\partial u}{\partial x}\right|(x_0)\,|u|(x_0) + \beta\, |u|^2(x_0)$$
$$\le |f|(x_0)\ |u|(x_0)\ .$$

Next, noting that

$$\gamma \left|\frac{\partial u}{\partial x}\right|^2 - ||b||_\infty \left|\frac{\partial u}{\partial x}\right|\, |u| + \frac{||b||_\infty^2}{4\gamma}\, |u|^2 \ge 0$$

we get finally

(18) $$|u|(x_0) \le \frac{1}{\beta - \dfrac{||b||_\infty^2}{4\gamma}}\, ||f||_\infty$$

which implies (11).

I.3. Existence result

Let us now give the existence result. The proof is made in several steps and use accretive operators methods (see [3]).

In the following we need some additional assumptions on the data :

2') $$a_{ij} \in C^1(\bar{\mathcal{O}}) \quad , \quad \partial\mathcal{O} \in C^2$$

3') $$c_{kk} \geq \beta > 0 \quad \forall\, k \,,\ 1 \leq k \leq M \,.$$

Let us denote

$$Au \equiv -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\, \frac{\partial u}{\partial x_j}\right)$$

we have :

<u>Theorem 1.2</u> - Assume that 1), 2), 2'), 3) 3') hold. Then there exists a unique solution of (*)

(19) $$u_k \in W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}) \quad , \quad 2 \leq p < +\infty.$$

<u>Proof</u> :

. <u>Step 1</u> : Approximation of (*)

Let $\beta_\eta(t)$ be a smooth function defined by

(20) $$\beta_\eta(t) = \begin{cases} t & \text{if } |t| \leq 1/\eta - 1 \\ 1/\eta & \text{if } |t| > 1/\eta \end{cases}$$

$$0 \leq \beta_\eta \leq 1 \quad , \quad 0 < \eta < 1 \quad , \quad \eta \text{ fixed.}$$

Then we consider the problem $(*)_\eta$

$$(*)_\eta \quad \begin{cases} Au_k^\eta + \sum_\ell c_{k\ell}\, u_\ell^\eta = f_k - \beta_\eta(-\sum_\ell \sum_i \tilde{b}_{k\ell,i} \frac{\partial u_\ell^\eta}{\partial x_i}) & \text{a.e. in } \mathcal{O} \\ u_k^\eta|_{\partial\mathcal{O}} = 0 \end{cases}$$

where

$$\tilde{b}_{k\ell,i} = \begin{cases} b_{k\ell,i} & \text{for } k \neq \ell \\ b_{kk,i} - \sum_j \frac{\partial a_{ij}}{\partial x_j} & \text{for } k = \ell \end{cases}$$

We have that the operator A with domain

$$\mathcal{D}(A) = \{u \in [L^2(\mathcal{O})]^M \mid u_k \in W_0^{1,2}(\mathcal{O}) \text{ with } Au_k \in L^2(\mathcal{O})\}$$

is m-accretive in $[L^2(\mathcal{O})]^M$. Indeed, we have

(21) $$[u-\bar{u}, Au-A\bar{u}]_+ = \sum_k [u_k-\bar{u}_k, Au_k-A\bar{u}_k]_+$$

$$= \sum_k \int_{\mathcal{O}} \operatorname{sign}(u_k-\bar{u}_k)\,|u_k-\bar{u}_k|\,(-\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial(u_k-\bar{u}_k)}{\partial x_j}))dx$$

$$= \sum_k \int_{\mathcal{O}} \sum_{i,j} a_{ij} \frac{\partial(u_k-\bar{u}_k)}{\partial x_i}\frac{\partial(u_k-\bar{u}_k)}{\partial x_j} dx \geq 0$$

and $R(I+\lambda A) = L^2(\mathcal{O})$ for any $\lambda > 0$.

On the other hand, let

(22) $$Cu \equiv (C_1u, C_2u, \ldots, C_Mu)$$

$$C_ku \equiv \sum_\ell c_{k\ell}\, u_\ell$$

we now prove that C is an accretive operator in $[L^2(\mathcal{O})]^M$.

Indeed C is everywhere defined, Lipschitz continuous operator with range in $[L^2(\mathcal{O})]^M$ and

$$(23)\qquad [u-\bar{u},Cu-C\bar{u}]_+ = \sum_{k=1}^{M} [u_k-\bar{u}_k,C_k u-C_k\bar{u}]_+$$
$$= \sum_k \int_{\mathcal{O}} \operatorname{sign}(u_k-\bar{u}_k)|u_k-\bar{u}_k|(\sum_\ell c_{k\ell}(u_\ell-\bar{u}_\ell))dx$$
$$= \int_{\mathcal{O}} \sum_{k,\ell} c_{k\ell}(u_k-\bar{u}_k)(u_\ell-\bar{u}_\ell)dx \geq 0$$

by condition 3).

Now from known results on the perturbation of m-accretive operator ([3]) we have that

$$(24)\qquad A + C \text{ is m-accretive in } [L^2(\mathcal{O})]^M .$$

. <u>Step 2</u> : Existence result for $(*)_\eta$

We define a mapping from $[W_0^{1,p}(\mathcal{O})]^M$ into itself by

$$(25)\qquad S_\eta v = u^\eta$$

where u_k^η is the unique solution in $W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O})$, $2 \leq p < +\infty$ of the problem

$$(26)\qquad \begin{cases} Au_k^\eta + \sum_\ell c_{k\ell}\, u_\ell^\eta = f_k - \beta_\eta(-\sum_\ell \sum_i \tilde{b}_{k\ell,i} \dfrac{\partial v_\ell}{\partial x_i}) \\ u_k^\eta|_{\partial\mathcal{O}} = 0 . \end{cases}$$

Indeed, since $A+C$ is m-accretive in $[L^2(\mathcal{O})]^M$ there exists a unique solution of (26) in $W_0^{2,1}(\mathcal{O})$ with $Au_k \in L^2(\mathcal{O})$. On the other hand, we have

(27) $$|\beta_\eta| \le 1/\eta$$

multiplying (26)a) by $u_k^\eta\,|u_k^\eta|^{p-2}$ and integrating over $\mathcal{O}$, give us

(28) $$\beta\,||u_k^\eta||_{L^p} \le C\,||u_k^\eta||_{L^p} + C_0(\eta,p)\ ^{(*)}.$$

But $u_k^\eta \in W^{2,2}(\mathcal{O})$ implies that $u_k^\eta \in W^{1,\bar{p}}(\mathcal{O})$ for some $\bar{p} > 2$. Therefore the right hand of (26)a) is in $L^{p\wedge\bar{p}}(\mathcal{O})$ and from regularity results of elliptic equations

(29) $$u_k^\eta \in W^{2,p\wedge\bar{p}}(\mathcal{O}) .$$

Now by a bootstrap argument we get that

(30) $$u_k^\eta \in W^{2,p}(\mathcal{O}) \quad \text{for each} \quad p : 2 \le p < +\infty$$

and

(31) $$||u_k^\eta||_{W^{2,p}(\mathcal{O})} \le C_1(\eta,p)$$

(the constants C_i do not depend on v).

Next consider the set

(32) $$D_\eta = \{v \in [W_0^{1,p}(\mathcal{O})]^M \mid \; ||v||_{W^{2,p}} \le C_1(\eta,p) \quad , \quad 2 \le p < +\infty\}$$

which is a convex compact subset of $[W_0^{1,p}(\mathcal{O})]^M$. D_η is mapped into itself under S_η and the smoothness of β_η implies that S_η is continuous. Therefore, it follows from Schauder's theorem that S_η has a fixed point.

(*) we take β large enough.

. <u>Step 3</u> : L^∞-a priori estimates

Using the same technic as in the proof of theorem 1.1 and the fact that

$$|\beta_\eta(t)| \le |t| \tag{33}$$

we get that

$$||u_\eta||_{L^\infty} \le \frac{1}{\beta - \frac{||b||_\infty^2 + 4\gamma C}{4\gamma}} ||f||_{L^\infty} . \tag{34}$$

. <u>Step 4</u> : $W^{2,p}$- apriori estimates

From the first equation in $(*)_\eta$ we get that

$$||Au^\eta||_{L^p} \le ||f||_{L^p} + C_1(||u^\eta||_{W^{1,p}} + ||u^\eta||_{L^p}) \tag{35}$$

and using the Sobolev interpolation inequality

$$||u^\eta||_{W^{1,p}} \le C_2(\delta||u^\eta||_{W^{2,p}} + \frac{1}{\delta} ||u^\eta||_{L^p}) , \delta > 0 \tag{36}$$

we obtain

$$||Au^\eta||_{L^p} \le ||f||_{L^p} + C_3\delta||u^\eta||_{W^{2,p}} + \frac{C_3}{\delta} ||u^\eta||_{L^p} . \tag{37}$$

Now, using the estimate (34) we get

$$||Au^\eta||_{L^p} \le \frac{C_4}{\delta} ||f||_{L^\infty} + C_3\delta||u^\eta||_{W^{2,p}} . \tag{38}$$

On the other hand, from the standard elliptic theory we have

$$||u^\eta||_{W^{2,p}} \le C_5(||Au^\eta||_{L^p} + ||u^\eta||_{L^p}) \tag{39}$$

Therefore, combining (38) and (39) we get

(40) $$||u^{\eta}||_{W^{2,p}} \le \frac{C_6}{\delta}\,||f||_{L^\infty} + C_6\delta||u^{\eta}||_{W^{2,p}}$$

and for δ well chosen we get

(41) $$||u^{\eta}||_{W^{2,p}} \le C\,||f||_{L^\infty}$$

where C does not depend on η.

. Step 5 : Passage to limit

Now, from (41) there exists a subsequence u^{η_n} and a function u such that

(42) $$\begin{gathered} u_k^{\eta_n} \to u_k \quad \text{in } W^{2,p}\text{-weakly} \quad (2 \le p < +\infty) \quad \text{and in } C(\overline{\mathcal{O}}) \\ \frac{\partial u_k^{\eta_n}}{\partial x} \to \frac{\partial u_k}{\partial x} \quad \text{in } C(\overline{\mathcal{O}}) . \end{gathered}$$

Next, set $\eta = \eta_n$ and let $\eta_n \to 0$ in $(*)_\eta$ yields the existence result for $(*)$.

. Step 6 : Uniqueness

Let $\tilde{u} \in W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O})$ be another solution of $(*)$. Then we get that

(43) $$\begin{aligned} -\sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}(u_k - \tilde{u}_k) - \sum_{\ell}\sum_{i} b_{k\ell,i} \frac{\partial}{\partial x_i}(u_\ell - \tilde{u}_\ell) + \\ + \sum_{\ell} c_{k\ell}(u_\ell - \tilde{u}_\ell) = 0 \quad \text{a.e. in } \mathcal{O} \end{aligned}$$

and by the maximum principle result we deduce that

(44) $$u - \tilde{u} \equiv 0 .$$

which completes the proof.

Remark 1.1 : From this proof we see that when $p = 2$ it is enough to assume condition 3). But as we shall see below condition 3') is not a restriction at least when $\mathcal{O}$ is bounded.

II - THE PROBABILISTIC RESULTS

II.1. The Itô Random evolution

Let $\Omega = C([0,\infty);\mathbb{R}^N)$ be the space of continuous functions on $[0,\infty)$ into $\mathbb{R}^N$.

Given $t \geq 0$, $\omega \in \Omega$, let $y(t,\omega)$ denotes the position of ω at time t.

For $t \geq s \geq 0$, we consider the σ-fields

$$\mu_t^s = \mathcal{B}(y(\lambda); s \leq \lambda \leq t)$$

$$\mu^s = \mu_\infty^s \quad .$$

We assume that 1) holds and

4) $$b_{k\ell,i} \, , \, c_{k\ell} \in L^\infty(\mathbb{R}^N) \; ; \; a_{ij} \in W^{1,\infty}(\mathbb{R}^N)$$

5) $$a_{ij} = a_{ji} \quad \text{(which is not a restriction since } a_{ij} \in W^{1,\infty}(\mathbb{R}^N)\text{)}.$$

Then in [4] it was shown that, for x,s fixed, there exists a unique probability measure $Q_{x,s}$ on (Ω,μ^s) such that

i) $\quad Q_{x,s}(y(s) = x) = 1$

ii) $\quad \phi(y(t)) + \int_s^t A\phi(y(\lambda))d\lambda$ is a $Q_{x,s}$-martingale

$\forall \phi \in \mathcal{D}(\mathbb{R}^N)$,

where

(45) $$A\phi(x) = -\sum_{i,j} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) .$$

Moreover, there exists a standard Wiener process $w(t)$ such that

$$y(t) = x + \int_s^t \sigma(y(\lambda))dw(\lambda) \quad \text{a.s.} - Q_{x,s} \quad , \quad t \geq s$$

where $\sigma(x)$ is such that $a = \frac{1}{2}\sigma^2$.

Let us consider the system $F = (\Omega, \mu^s, Q_{x,s}, y(t))$, then we have the definition, for $z \in \mathbb{R}^M$ fixed :

<u>Definition 2.1</u> - A stochastic process $\xi_z(t)$ is called the Itô random evolution associated to F with respect to the coefficients b_i, c, $1 \leq i \leq N$, if

1) $\xi_z(t)$ is μ_t^s-adapted and continuous in t with probability one

2) $Q_{x,s}(\xi_z(s)=z) = 1$

3) $$d\xi_z(t) = -\xi_z(t)c(y(t))dt + \frac{1}{2}\sum_{i,j=1}^{N} \xi_z(t)(a^{-1})_{ij}(y(t)) b_j(y(t))dy_i(t)$$

a.s. $Q_{x,s}$, $t \geq s$.

We have the result

<u>Theorem 2.1</u> - Under conditions 1), 4), 5) there exists a unique, in the a.s.-$Q_{s,x}$ sense, Itô random evolution.

The proof of this result is similar to the one given in [1].

Under some additional condition we shall prove that $\xi_z(t)$ decreases exponentially as $t \to \infty$. More precisely, let us assume that

3) $$\sum_{k,\ell} c_{k,\ell} z_k z_\ell \geq \beta |z|^2 \quad , \quad \forall z \in \mathbb{R}^M$$

then we have

<u>Theorem 2.2</u> - Assume that 1), 2), 3), 4), 5) hold. Then we have the estimate

(46) $$E_{x,s} |\xi_z(t)|^2 \leq |z|^2 \exp[-(\beta - \frac{||b||_\infty^2}{4\gamma})(t-s)]$$

$$\forall t \geq s \text{ and } z \in \mathbb{R}^M .$$

<u>Proof</u> : Indeed, we have

(47) $$d\xi_z(t)\xi_z^*(t) = \frac{1}{2} \xi_z(t)[\sum_{i,j} (a^{-1})_{ij}(y(t))b_j^*(y(t))\xi_z^*(t)]dy_i(t) +$$

$$+ \frac{1}{2} [\sum_{i,j} (a^{-1})_{ij}(y(t))\xi_z(t)b_j(y(t))]\xi_z^*(t)dy_i(t) -$$

$$- \xi_z(t)c^*(y(t))\xi_z^*(t)dt - \xi_z(t)c(y(t))\xi_z^*(t)dt +$$

$$+ \frac{1}{4} \sum_{i,j} (a^{-1})_{ij} \xi_z(t)b_j(y(t))b_i^*(y(t))\xi_z^*(t)dt .$$

Hence, as $a^{-1} \le \frac{I}{\gamma}$ we get that

(48) $$E_{x,s} |\xi_z(t)|^2 \le |z|^2 - E_{x,s} \int_s^t \xi_z(\lambda)[c(y(\lambda)+c^*(y(\lambda))]\xi_z^*(\lambda)d\lambda + $$

$$+ \frac{1}{4\gamma} ||b||_\infty^2 \; E_{x,s} \int_s^t |\xi_z(\lambda)|^2 d\lambda \quad .$$

Using condition (3) we get

(49) $$E_{x,s} |\xi_z(t)|^2 \le |z|^2 + (\frac{1}{4\gamma} ||b||_\infty^2 - \beta) \int_s^t E_{x,s} |\xi_z(\lambda)|^2 d\lambda \; .$$

By Gronwall's lemma we obtain (46).

II.2. A representation result

Our objective here is to give a probabilistic interpretation of the functions u_k.

To this end we will rely on the results of the analytic part.

We will be interested only in the case of $s = 0$ and denote $Q_x = Q_{x,0}$.

Let τ be the first exit time of the process $y(t)$ from the domain $\mathcal{O}$,

(50) $$\tau = \inf\{t \ge 0 \; y(t) \notin \mathcal{O}\}$$

which is an μ_t^0-stopping time.

Our aim is to prove that

(51) $$u_k(x) = E_x \int_0^\tau \xi_{e_k}(\lambda) \; f(y(\lambda))d\lambda$$

where e_k is the unit vector in the direction coordinate x_k.

We have :

<u>Theorem 2.3</u> - Assume that 1), 2), 2'), 3), 3'), 4), 5) hold. Then we have

(52) $$u_k(x) = E_x \int_0^\tau \xi_{e_k}(\lambda) f(y(\lambda))d\lambda \quad , \quad 1 \le k \le M .$$

<u>Proof</u> : Let u_k^n be functions belonging in $\mathcal{D}(\mathcal{O})$ such that

(53) $$u_k^n \to u_k \quad \text{in} \quad W^{2,p}(\mathcal{O}) .$$

Since $p > N$, we have

(54) $$Du_k^n \to Du_k \quad \text{in} \quad C(\bar{\mathcal{O}})$$

$$u_k^n \to u_k \quad \text{in} \quad C(\bar{\mathcal{O}}) .$$

By defining u_k^n outside $\mathcal{O}$ in a convenient way, we can assume that $u_k^n \in \mathcal{D}(\mathbb{R}^N)$.

We have the vector version of Itô's formula which is derived from component-wise considerations,

(55) $$du^n(y(t)) = \sum_i \frac{\partial u^n}{\partial x_i}(y(t))dy_i(t) + \sum_{i,j} a_{ij}(y(t))\frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(t))dt$$

$$\text{a.s.} \quad Q_x$$

On the other hand,

(56) $$d\xi_{e_k}(t) = - \xi_{e_k}(t)c(y(t))dt + \frac{1}{2} \sum_{i,j} \xi_z(t)(a^{-1})_{ij}(y(t))b_j(y(t))$$

$$dy_i(t) \quad \text{a.s.} \quad Q_x .$$

Therefore, we have

$$d\xi_{e_k}(t)u^n(y(t)) = \sum_{i,j} \xi_{e_k}(t)((a^{-1})_{ij}(y(t))b_j(y(t)) + \tag{57}$$

$$+ \frac{\partial u^n}{\partial x_i}(y(t)))dy_i(t) +$$

$$+ \xi_{e_k}(t)\,[\sum_{i,j} a_{ij}(y(t))\frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(t)) +$$

$$+ \sum_i b_i(y(t))\frac{\partial u^n}{\partial x_i}(y(t)) - c(y,t))u^n(y(t))\,]dt \quad .$$

Hence, we get that for $T > 0$ fixed

$$E_x\, \xi_{e_k}(T\wedge\tau)u^n(y(T\wedge\tau)) = u_k^n(x) + E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) \tag{58}$$

$$[\sum_{i,j} a_{ij}(y(\lambda))\frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(\lambda)) + \sum_i b_i(y(\lambda))\frac{\partial u^n}{\partial x_i}(y(\lambda)) -$$

$$- c(y(\lambda))u^n(y(\lambda))\,]d\lambda \;.$$

Remarking that

$$E_x \int_0^{T\wedge\tau} |\xi_{e_k}(\lambda)|\;|Lu^n(y(\lambda)) - Lu(y(\lambda))|d\lambda \le \tag{59}$$

$$\le (E_x\,[\int_0^T |\xi_{e_k}(\lambda)|^q d\lambda])^{1/q}(E_x\,[\int_0^T |L(u^n-u)(y(\lambda))|^p d\lambda])^{1/p}$$

with $1/q + 1/p = 1$. Using the L_p-estimate we get that

$$E_x\,[\int_0^T |L(u^n-u)(y(\lambda))|^p d\lambda] \le C_{T,p}\;||L(u^n-u)||_{L^p} \to 0 \tag{60}$$

$$\text{as} \quad n \to \infty$$

and as we have by elementary calculation

(61) $$E_x \left[\int_0^T |\xi_{e_k}(\lambda)|^q d\lambda\right] \le T \exp\left(\frac{T}{2} q(q-1)[||\sigma b||_\infty^2 + ||c||_\infty]\right) .$$

we can let $n \to \infty$ in (58) and we get that

(62) $$E_x\, \xi_{e_k}(T\wedge\tau) u(y(T\wedge\tau)) = u_k(x) + E_x\Big(\int_0^{T\wedge\tau} \xi_{e_k}(\lambda)$$
$$\Big[\sum_{i,j} a_{ij}(y(\lambda)) \frac{\partial^2 u}{\partial x_i \partial x_j}(y(\lambda)) + \sum_i b_i(y(\lambda)) \frac{\partial u}{\partial x_i}(y(\lambda)) -$$
$$-c(y(\lambda)) u(y(\lambda)) d\lambda\Big].$$

Using equation (*), we obtain

(63) $$E_x\, \xi_{e_k}(T\wedge\tau) u(y(T\wedge\tau)) = u_k(x) - E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda .$$

Furthermore as u,f are bounded by deterministic constant, we can let $T \to \infty$ in (63), thus.

(64) $$E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda \to E_x \int_0^{\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda \quad \text{as} \quad T \to \infty$$

and using the estimate (46) we obtain that

(66) $$E_x\, \xi_{e_k}(T\wedge\tau) u(y(T\wedge\tau)) = E_x\, \xi_{e_k}(T) u(y(T)) \chi_{T<\tau} \to 0 \quad \text{as} \quad T \to \infty .$$

Therefore it follows that

(66) $$u_k(x) = E_x \int_0^{\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda$$

which completes the proof.

Remark 2.1 ; When $\mathcal{O} \equiv \mathbb{R}^N$ we have the representation of the solution of (*) given by

(67) $$u_k(x) = E_x \int_0^{\infty} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda$$

which makes sense using the estimate (46).

Remark 2.2 : When $\mathcal{O}$ is bounded, condition (3) is not a restriction. Indeed, making the following change of unknown functions

$$u_k = \omega v_k \tag{68}$$

with

$$\omega(x) = 2 - \exp[-\alpha(|x-x_0|^2 + 1)^{1/2}] \tag{69}$$

where $x_0 \notin \bar{\mathcal{O}}$ and α is a constant to be chosen sufficiently large.

Hence we obtain the following system

$$\begin{aligned} -\sum_{i,j} a_{ij} \frac{\partial^2 v_k}{\partial x_i \partial x_j} - \sum_{i,\ell} b_{k\ell,i} \frac{\partial v_\ell}{\partial x_i} - \sum_{i,j} \frac{2}{\omega} \frac{\partial \omega}{\partial x_j} a_{ij} \frac{\partial v_k}{\partial x_i} + \\ + \sum_{\ell} (c_{k\ell} v_\ell - \frac{1}{\omega} \sum_{i,j} a_{ij} \frac{\partial^2 \omega}{\partial x_i \partial x_j} v_k - \frac{1}{\omega} \sum_{i,j} b_{k\ell,i} \frac{\partial \omega}{\partial x_i} v_\ell) \\ = \frac{1}{\omega} f_k \end{aligned} \tag{70}$$

which verifies condition (3) for adequate α.

REFERENCES

[1] D.W. STROOCK, On certain systems of parabolic equations, Comm. on Pure and Applied Math., Vol. 23 (1970).

[2] E. FABES, N. RIVIERE, Systems of parabolic equations with uniformly continuous coefficients, J. Anal. Math., Vol. 17 (1966).

[3] V. BARBU, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Int. Publishing, 1976.

[4] D.W. STROOCK, S.R.S. VARADHAN, Diffusion processes with continuous coefficients I,II, Comm. Pure and Appl. Math., Vol. 22 (1969).

Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique