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existence, uniqueness and representation results**

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► **To cite this version:**

O. Bennouna. On strongly coupled elliptic systems: existence, uniqueness and representation results. RR-0453, INRIA. 1985. <inria-00076102>

HAL Id: inria-00076102

<https://hal.inria.fr/inria-00076102>

Submitted on 24 May 2006

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Rapports de Recherche

N° 453

**ON STRONGLY COUPLED
ELLIPTIC SYSTEMS:
EXISTENCE, UNIQUENESS AND
REPRESENTATION RESULTS**

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Novembre 1985

ON STRONGLY COUPLED ELLIPTIC SYSTEMS :

EXISTENCE, UNIQUENESS AND REPRESENTATION

RESULTS

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RESUME

Dans ce papier on considère une classe d'équations et de systèmes elliptiques qui peuvent être réduits à la forme

$$(*) \quad -\sum_{i,j} a_{ij} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum_{\ell} \sum_i b_{k\ell,i} \frac{\partial u_\ell}{\partial x_i} + \sum_{\ell} c_{k\ell} u_\ell = f_k \quad \text{dans } \mathcal{O}$$

où a_{ij} , $b_{k\ell,i}$, $c_{k\ell}$, f_k sont des données et \mathcal{O} est un ouvert borné de \mathbb{R}^N . On donne un résultat du type principe du maximum et on démontre un résultat d'existence et d'unicité. D'autre part on donne une représentation de la solution du type Feynman-Kac.

MOTS-CLEFS

Elliptique - Principe du maximum - Représentation du type Feynman-Kac.

ABSTRACT

In this paper we consider a class of elliptic equations and systems which can be reduced to the form :

$$(*) \quad -\sum_{i,j} a_{ij} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum_{\ell} \sum_i b_{k\ell,i} \frac{\partial u_\ell}{\partial x_i} + \sum_{\ell} c_{k\ell} u_\ell = f_k \quad \text{in } \mathcal{O}$$

where a_{ij} , $b_{k\ell,i}$, $c_{k\ell}$, f_k are given data and \mathcal{O} is an open bounded subset of \mathbb{R}^N . We give a maximum principle type result for this class and we prove existence and uniqueness results.

On the other hand, we give a Feynman-Kac type representation of the solution of (*).

KEY WORDS

Elliptic - Maximum principle - Feynman-Kac representation.

ON STRONGLY COUPLED ELLIPTIC SYSTEMS :
EXISTENCE, UNIQUENESS AND REPRESENTATION
RESULTS

O. Bennouna

INTRODUCTION

We consider a class of elliptic equations and systems which can be reduced to the form :

$$(*) \left\{ \begin{array}{l} - \sum_{i,j} a_{ij} \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \sum_{\ell} \sum_i b_{k\ell,i} \frac{\partial u_{\ell}}{\partial x_i} + \sum_{\ell} c_{k\ell} u_{\ell} = f_k \text{ in } \Omega \\ u_k|_{\partial\Omega} = 0 \quad 1 \leq k \leq M \end{array} \right.$$

where a_{ij} , $b_{k\ell,i}$, $c_{k\ell}$, f_k are given data and Ω is an open bounded subset of \mathbb{R}^N with smooth boundary $\partial\Omega$.

In this note we give first a maximum principle type result for this class. To obtain the existence and uniqueness result we use a perturbation technic of weakly coupled system of second order elliptic partial differential equations to which we apply accretive operators methods.

The parabolic case is studied in [2] by the method of parabolic singular integrals.

On the other hand, we give a Feynman-Kac type representation of the solution of (*) using the notion of Itô random evolution. In the parabolic

case this question is solved in [1]. This type of result is made possible in the elliptic case under a stability condition of (*) which means, roughly speaking, that the dissipation term must be large enough.

Let us give an example of fourth order equation which can be reduced to the form (*).

We consider

$$\left\{ \begin{array}{l} -\Delta^2 u + b\Delta u + cu = f \text{ in } 0 \\ u|_{\partial 0} = 0 \quad , \quad \Delta u|_{\partial 0} = 0 \end{array} \right.$$

when b , c and f are given smooth data.

We denote $v = \Delta u$, then we obtain the second order elliptic system :

$$\left\{ \begin{array}{l} -\Delta v + bv + cu = f \\ -\Delta u + v = 0 \\ u|_{\partial 0} = 0 \quad , \quad v|_{\partial 0} = 0 . \end{array} \right.$$

As we shall see below this problem is well posed, and we have a probabilistic representation of u , under some natural conditions on the data.

The paper is organized as follows : the part I contains analytic results and the part II is concerned with the Feynman-Kac type result.

I - THE ANALYTIC RESULTS

I.1. Assumptions and notations

Let O be an open bounded smooth domain in \mathbb{R}^N . We assume that :

1) Regularity conditions :

$$\phi = a_{ij}, b_{kl,i}, c_{kl}, f_k ; \phi \in C(\bar{O}) .$$

2) Ellipticity condition :

$$\sum_{i,j} a_{ij} \xi_i \xi_j \geq \gamma |\xi|^2 , \quad \gamma > 0 , \quad \forall \xi \in \mathbb{R}^N .$$

3) Stability condition :

$$\sum_{k,l} c_{kl} v_k v_l \geq \beta |v|^2 , \quad \beta > 0 \text{ large enough}$$

$$\forall v \in \mathbb{R}^M$$

(as we shall see in remark 2.2 it is sufficient to assume that

$$\sum_{k,l} c_{kl} v_k v_l \geq 0) .$$

I.2. A maximum principle for (*)

In this section we shall give a maximum principle type result.

We have

Theorem 1.1 - Assume that conditions (1), (2), (3) hold and

$$(10) \quad 4\gamma\beta > \|b\|_\infty^2$$

then we have for $u, \bar{u} \in C(\bar{O}) \cap W^{2,p}(O)$
solutions of (*) corresponding respectively to f, \bar{f}

$$(11) \quad \|u - \bar{u}\|_\infty \leq \frac{1}{\beta - \frac{\|b\|_\infty^2}{4\gamma}} \|f - \bar{f}\|_\infty$$

where

$$\|b\|_\infty^2 = \sum_i \|b_i\|_\infty^2,$$

Proof : It is enough to prove (11) for $u_k \in C(\bar{O}) \cap C^2(O)$. Hence, multiplying (*) by u_k and then summing up from 1 to M yields

$$(12) \quad - \sum_{i,j} \frac{a_{ij}}{2} \frac{\partial^2 |u|^2}{\partial x_i \partial x_j} + \sum_{i,j} \sum_{\kappa} a_{ij\kappa} \frac{\partial u_\kappa}{\partial x_i} \frac{\partial u_\kappa}{\partial x_j} - \\ - \sum_{i,\kappa,l} b_{\kappa l,i} \frac{\partial u_l}{\partial x_i} u_\kappa + \sum_{\kappa,l} c_{\kappa l} u_l u_\kappa = \sum_k f_k u_k.$$

Let x_0 be a point in \bar{O} such that $|u|^2 \equiv \sum_k |u_k|^2$ attains a maximum at x_0 . If $|u|(x_0) = 0$ then

$$(13) \quad \|u\|_\infty \leq \frac{1}{\beta - \frac{\|b\|_\infty^2}{4\gamma}} \|f\|_\infty.$$

If not, we have

$$(14) \quad \sum_{i,j} \frac{a_{ij}}{2}(x_0) \frac{\partial^2 |u|}{\partial x_i \partial x_j}(x_0) \leq 0$$

and using conditions 1), 3) we get that

$$(15) \quad \gamma \left| \frac{\partial u}{\partial x} \right|^2(x_0) - \sum_i \sum_{k,l} b_{kl,i}(x_0) \frac{\partial u_l}{\partial x_i}(x_0) u_k(x_0) + \beta |u|^2(x_0) \leq |f|(x_0) |u|(x_0).$$

Remarking that

$$(16) \quad \left| \sum_i \sum_{k,l} b_{kl,i}(x) \frac{\partial u_l}{\partial x_i} u_k \right| \leq \|b\|_{\infty} \left| \frac{\partial u}{\partial x} \right| |u|$$

we obtain from (15) and (16)

$$(17) \quad \gamma \left| \frac{\partial u}{\partial x} \right|^2(x_0) - \|b\|_{\infty} \left| \frac{\partial u}{\partial x} \right|(x_0) |u|(x_0) + \beta |u|^2(x_0) \leq |f|(x_0) |u|(x_0).$$

Next, noting that

$$\gamma \left| \frac{\partial u}{\partial x} \right|^2 - \|b\|_{\infty} \left| \frac{\partial u}{\partial x} \right| |u| + \frac{\|b\|_{\infty}^2}{4\gamma} |u|^2 \geq 0$$

we get finally

$$(18) \quad |u|(x_0) \leq \frac{1}{\|b\|_{\infty} - \frac{\|b\|_{\infty}^2}{4\gamma}} \|f\|_{\infty}$$

which implies (11).

1.3. Existence result

Let us now give the existence result. The proof is made in several steps and use accretive operators methods (see [3]).

In the following we need some additional assumptions on the data :

- 2') $a_{ij} \in C^1(\bar{O})$, $\partial O \in C^2$
- 3') $c_{kk} \geq \beta > 0 \quad \forall k, 1 \leq k \leq M$.

Let us denote

$$Au \equiv - \sum_{i,j} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j})$$

we have :

Theorem 1.2 - Assume that 1), 2), 2'), 3) 3') hold. Then there exists a unique solution of (*)

$$(19) \quad u_k \in W^{2,p}(O) \cap W_0^{1,p}(O) \quad , \quad 2 \leq p < +\infty.$$

Proof :

. Step 1 : Approximation of (*)

Let $\beta_\eta(t)$ be a smooth function defined by

$$(20) \quad \beta_\eta(t) = \begin{cases} t & \text{if } |t| \leq 1/\eta - 1 \\ 1/\eta & \text{if } |t| > 1/\eta \end{cases}$$

$$0 \leq \beta_\eta \leq 1 \quad , \quad 0 < \eta < 1 \quad , \quad \eta \text{ fixed.}$$

Then we consider the problem $(*)_\eta$

$$(*)_{\eta} \quad \begin{cases} Au_k^{\eta} + \sum_{\ell} c_{k\ell} u_{\ell}^{\eta} = f_k - \beta_{\eta} \left(-\sum_{\ell} \sum_i \tilde{b}_{k\ell,i} \frac{\partial u_{\ell}^{\eta}}{\partial x_i} \right) & \text{a.e. in } O \\ u_k^{\eta}|_{\partial O} = 0 \end{cases}$$

where

$$\tilde{b}_{k\ell,i} = \begin{cases} b_{k\ell,i} & \text{for } k \neq \ell \\ b_{kk,i} - \sum_j \frac{\partial a_{ij}}{\partial x_j} & \text{for } k = \ell \end{cases}$$

We have that the operator A with domain

$$\mathcal{D}(A) = \{u \in [L^2(O)]^M \mid u_k \in W_0^{1,2}(O) \text{ with } Au_k \in L^2(O)\}$$

is m -accretive in $[L^2(O)]^M$. Indeed, we have

$$\begin{aligned} (21) \quad [u - \bar{u}, Au - A\bar{u}]_+ &= \sum_k [u_k - \bar{u}_k, Au_k - A\bar{u}_k]_+ \\ &= \sum_k \int_O \text{sign}(u_k - \bar{u}_k) |u_k - \bar{u}_k| \left(-\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial (u_k - \bar{u}_k)}{\partial x_j} \right) \right) dx \\ &= \sum_k \int_O \sum_{i,j} a_{ij} \frac{\partial (u_k - \bar{u}_k)}{\partial x_i} \frac{\partial (u_k - \bar{u}_k)}{\partial x_j} dx \geq 0 \end{aligned}$$

and $R(I + \lambda A) = L^2(O)$ for any $\lambda > 0$.

On the other hand, let

$$(22) \quad Cu \equiv (C_1 u, C_2 u, \dots, C_M u)$$

$$C_k u \equiv \sum_{\ell} c_{k\ell} u_{\ell}$$

we now prove that C is an accretive operator in $[L^2(O)]^M$.

Indeed C is everywhere defined, Lipschitz continuous operator with range in $[L^2(O)]^M$ and

$$\begin{aligned}
 (23) \quad [u-\bar{u}, Cu-C\bar{u}]_+ &= \sum_{k=1}^M [u_k - \bar{u}_k, C_k u - C_k \bar{u}]_+ \\
 &= \sum_k \int_0 \text{sign}(u_k - \bar{u}_k) |u_k - \bar{u}_k| \left(\sum_l c_{kl} (u_l - \bar{u}_l) \right) dx \\
 &= \int_0 \sum_{k,l} c_{kl} (u_k - \bar{u}_k) (u_l - \bar{u}_l) dx \geq 0
 \end{aligned}$$

by condition 3).

Now from known results on the perturbation of m -accretive operator ([3]) we have that

$$(24) \quad A + C \text{ is } m\text{-accretive in } [L^2(O)]^M.$$

. Step 2 : Existence result for $(*)_\eta$

We define a mapping from $[W_0^{1,p}(O)]^M$ into itself by

$$(25) \quad S_\eta v = u^\eta$$

where u_k^η is the unique solution in $W^{2,p}(O) \cap W_0^{1,p}(O)$, $2 \leq p < +\infty$ of the problem

$$(26) \quad \begin{cases} Au_k^\eta + \sum_l c_{kl} u_l^\eta = f_k - \beta_\eta \left(-\sum_l \sum_i \tilde{b}_{kl,i} \frac{\partial v_l}{\partial x_i} \right) \\ u_k^\eta|_{\partial O} = 0. \end{cases}$$

Indeed, since $A+C$ is m -accretive in $[L^2(O)]^M$ there exists a unique solution of (26) in $W_0^{2,1}(O)$ with $Au_k \in L^2(O)$. On the other hand, we have

$$(27) \quad |\beta_\eta| \leq 1/\eta$$

multiplying (26)a) by $u_k^\eta |u_k^\eta|^{p-2}$ and integrating over O , give us

$$(28) \quad \beta \| |u_k^\eta| \|_{L^p} \leq C \| |u_k^\eta| \|_{L^p} + C_0(\eta, p) \quad (*)$$

But $u_k^\eta \in W^{2,2}(O)$ implies that $u_k^\eta \in W^{1,\bar{p}}(O)$ for some $\bar{p} > 2$. Therefore the right hand of (26)a) is in $L^{p \wedge \bar{p}}(O)$ and from regularity results of elliptic equations

$$(29) \quad u_k^\eta \in W^{2,p \wedge \bar{p}}(O) .$$

Now by a bootstrap argument we get that

$$(30) \quad u_k^\eta \in W^{2,p}(O) \quad \text{for each } p : 2 \leq p < +\infty$$

and

$$(31) \quad \| |u_k^\eta| \|_{W^{2,p}(O)} \leq C_1(\eta, p)$$

(the constants C_i do not depend on v).

Next consider the set

$$(32) \quad D_\eta = \{v \in [W_0^{1,p}(O)]^M \mid \|v\|_{W^{2,p}} \leq C_1(\eta, p), \quad 2 \leq p < +\infty\}$$

which is a convex compact subset of $[W_0^{1,p}(O)]^M$. D_η is mapped into itself under S_η and the smoothness of β_η implies that S_η is continuous. Therefore, it follows from Schauder's theorem that S_η has a fixed point.

(*) we take β large enough.

. Step 3 : L^∞ -a priori estimates

Using the same technic as in the proof of theorem 1.1 and the fact that

$$(33) \quad |\beta_\eta(t)| \leq |t|$$

we get that

$$(34) \quad \|u_\eta\|_{L^\infty} \leq \frac{1}{\beta - \frac{\|b\|_\infty^2 + 4\gamma C}{4\gamma}} \|f\|_{L^\infty}.$$

. Step 4 : $W^{2,p}$ -a priori estimates

From the first equation in (*) _{η} we get that

$$(35) \quad \|Au^\eta\|_{L^p} \leq \|f\|_{L^p} + C_1(\|u^\eta\|_{W^{1,p}} + \|u^\eta\|_{L^p})$$

and using the Sobolev interpolation inequality

$$(36) \quad \|u^\eta\|_{W^{1,p}} \leq C_2(\delta \|u^\eta\|_{W^{2,p}} + \frac{1}{\delta} \|u^\eta\|_{L^p}), \quad \delta > 0$$

we obtain

$$(37) \quad \|Au^\eta\|_{L^p} \leq \|f\|_{L^p} + C_3\delta \|u^\eta\|_{W^{2,p}} + \frac{C_3}{\delta} \|u^\eta\|_{L^p}.$$

Now, using the estimate (34) we get

$$(38) \quad \|Au^\eta\|_{L^p} \leq \frac{C_4}{\delta} \|f\|_{L^\infty} + C_3\delta \|u^\eta\|_{W^{2,p}}.$$

On the other hand, from the standard elliptic theory we have

$$(39) \quad \|u^\eta\|_{W^{2,p}} \leq C_5(\|Au^\eta\|_{L^p} + \|u^\eta\|_{L^p})$$

Therefore, combining (38) and (39) we get

$$(40) \quad \| |u^\eta| \|_{W^{2,p}} \leq \frac{C_6}{\delta} \| |f| \|_{L^\infty} + C_6 \delta \| |u^\eta| \|_{W^{2,p}}$$

and for δ well chosen we get

$$(41) \quad \| |u^\eta| \|_{W^{2,p}} \leq C \| |f| \|_{L^\infty}$$

where C does not depend on η .

. Step 5 : Passage to limit

Now, from (41) there exists a subsequence u^{η_n} and a function u such that

$$u_k^{\eta_n} \rightarrow u_k \text{ in } W^{2,p}\text{-weakly } (2 \leq p < +\infty) \text{ and in } C(\bar{O})$$

(42)

$$\frac{\partial u_k^{\eta_n}}{\partial x} \rightarrow \frac{\partial u_k}{\partial x} \text{ in } C(\bar{O}) .$$

Next, set $\eta = \eta_n$ and let $\eta_n \rightarrow 0$ in $(*)_{\eta}$ yields the existence result for $(*)$.

. Step 6 : Uniqueness

Let $\tilde{u} \in W^{2,p}(O) \cap W_0^{1,p}(O)$ be another solution of $(*)$. Then we get that

$$(43) \quad - \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (u_k - \tilde{u}_k) - \sum_{\ell,i} b_{k\ell,i} \frac{\partial}{\partial x_i} (u_\ell - \tilde{u}_\ell) + \sum_{\ell} c_{k\ell} (u_\ell - \tilde{u}_\ell) = 0 \quad \text{a.e. in } O$$

and by the maximum principle result we deduce that

$$(44) \quad u - \tilde{u} \equiv 0 .$$

which completes the proof.

Remark 1.1 : From this proof we see that when $p = 2$ it is enough to assume condition 3). But as we shall see below condition 3') is not a restriction at least when θ is bounded.

II - THE PROBABILISTIC RESULTS

II.1. The Itô Random evolution

Let $\Omega = C([0, \infty); \mathbb{R}^N)$ be the space of continuous functions on $[0, \infty)$ into \mathbb{R}^N .

Given $t \geq 0$, $\omega \in \Omega$, let $y(t, \omega)$ denotes the position of ω at time t .

For $t \geq s \geq 0$, we consider the σ -fields

$$\mu_t^s = \mathcal{B}(y(\lambda); s \leq \lambda \leq t)$$

$$\mu^s = \mu_\infty^s.$$

We assume that 1) holds and

$$4) \quad b_{k\ell, i}, c_{k\ell} \in L^\infty(\mathbb{R}^N); a_{ij} \in W^{1, \infty}(\mathbb{R}^N)$$

$$5) \quad a_{ij} = a_{ji} \quad (\text{which is not a restriction since } a_{ij} \in W^{1, \infty}(\mathbb{R}^N)).$$

Then in [4] it was shown that, for x, s fixed, there exists a unique probability measure $Q_{x, s}$ on (Ω, μ^s) such that

- i) $Q_{x,s}(y(s) = x) = 1$
- ii) $\phi(y(t)) + \int_s^t A\phi(y(\lambda))d\lambda$ is a $Q_{x,s}$ -martingale
 $\forall \phi \in \mathcal{D}(\mathbb{R}^N)$,

where

$$(45) \quad A\phi(x) = - \sum_{i,j} a_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x).$$

Moreover, there exists a standard Wiener process $w(t)$ such that

$$y(t) = x + \int_s^t \sigma(y(\lambda))dw(\lambda) \quad \text{a.s. } - Q_{x,s}, \quad t \geq s$$

where $\sigma(x)$ is such that $a = \frac{1}{2} \sigma^2$.

Let us consider the system $F = (\Omega, \mu^s, Q_{x,s}, y(t))$, then we have the definition, for $z \in \mathbb{R}^M$ fixed :

Definition 2.1 - A stochastic process $\xi_z(t)$ is called the Itô random evolution associated to F with respect to the coefficients $b_i, c, 1 \leq i \leq N$, if

1) $\xi_z(t)$ is μ_t^s -adapted and continuous in t with probability one

2) $Q_{x,s}(\xi_z(s)=z) = 1$

3) $d\xi_z(t) = - \xi_z(t)c(y(t))dt + \frac{1}{2} \sum_{i,j=1}^N \xi_z(t)(a^{-1})_{ij}(y(t)) b_j(y(t))dy_i(t)$

a.s. $Q_{x,s}, \quad t \geq s.$

we have the result

Theorem 2.1 - Under conditions 1), 4), 5) there exists a unique, in the a.s.- $Q_{s,x}$ sense, Itô random evolution.

The proof of this result is similar to the one given in [1].

Under some additional condition we shall prove that $\xi_z(t)$ decreases exponentially as $t \rightarrow \infty$. More precisely, let us assume that

$$3) \quad \sum_{k,l} c_{k,l} z_k z_l \geq \beta |z|^2, \quad \forall z \in \mathbb{R}^M$$

then we have

Theorem 2.2 - Assume that 1), 2), 3), 4), 5) hold. Then we have the estimate

$$(46) \quad E_{x,s} |\xi_z(t)|^2 \leq |z|^2 \exp[-(\beta - \frac{\|b\|_\infty^2}{4\gamma})(t-s)]$$

$$\forall t \geq s \text{ and } z \in \mathbb{R}^M.$$

Proof : Indeed, we have

$$(47) \quad d\xi_z(t)\xi_z^*(t) = \frac{1}{2} \xi_z(t) \left[\sum_{i,j} (a^{-1})_{ij}(y(t)) b_j^*(y(t)) \xi_z^*(t) \right] dy_i(t) +$$

$$+ \frac{1}{2} \left[\sum_{i,j} (a^{-1})_{ij}(y(t)) \xi_z(t) b_j(y(t)) \right] \xi_z^*(t) dy_i(t) -$$

$$- \xi_z(t) c^*(y(t)) \xi_z^*(t) dt - \xi_z(t) c(y(t)) \xi_z^*(t) dt +$$

$$+ \frac{1}{4} \sum_{i,j} (a^{-1})_{ij} \xi_z(t) b_j(y(t)) b_i^*(y(t)) \xi_z^*(t) dt.$$

Hence, as $a^{-1} \leq \frac{1}{\gamma}$ we get that

$$(48) \quad E_{x,s} |\xi_z(t)|^2 \leq |z|^2 - E_{x,s} \int_s^t \xi_z(\lambda) [c(y(\lambda)) + c^*(y(\lambda))] \xi_z^*(\lambda) d\lambda + \\ + \frac{1}{4\gamma} \|b\|_\infty^2 E_{x,s} \int_s^t |\xi_z(\lambda)|^2 d\lambda .$$

Using condition (3) we get

$$(49) \quad E_{x,s} |\xi_z(t)|^2 \leq |z|^2 + \left(\frac{1}{4\gamma} \|b\|_\infty^2 - \beta\right) \int_s^t E_{x,s} |\xi_z(\lambda)|^2 d\lambda .$$

By Gronwall's lemma we obtain (46).

II.2. A representation result

Our objective here is to give a probabilistic interpretation of the functions u_k .

To this end we will rely on the results of the analytic part.

We will be interested only in the case of $s = 0$ and denote

$$Q_x = Q_{x,0}.$$

Let τ be the first exit time of the process $y(t)$ from the domain O ,

$$(50) \quad \tau = \inf\{t \geq 0 \mid y(t) \notin O\}$$

which is an μ_t^0 -stopping time.

Our aim is to prove that

$$(51) \quad u_k(x) = E_x \int_0^\tau \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda$$

where e_k is the unit vector in the direction coordinate x_k .

We have :

Theorem 2.3 - Assume that 1), 2), 2'), 3), 3'), 4), 5) hold. Then we have

$$(52) \quad u_k(x) = E_x \int_0^T \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda, \quad 1 \leq k \leq M.$$

Proof : Let u_k^n be functions belonging in $\mathcal{D}(O)$ such that

$$(53) \quad u_k^n \rightarrow u_k \text{ in } W^{2,p}(O).$$

Since $p > N$, we have

$$(54) \quad Du_k^n \rightarrow Du_k \text{ in } C(\bar{O})$$

$$u_k^n \rightarrow u_k \text{ in } C(\bar{O}).$$

By defining u_k^n outside O in a convenient way, we can assume that $u_k^n \in \mathcal{D}(\mathbb{R}^N)$.

We have the vector version of Itô's formula which is derived from component-wise considerations,

$$(55) \quad du^n(y(t)) = \sum_i \frac{\partial u^n}{\partial x_i}(y(t)) dy_i(t) + \sum_{i,j} a_{ij}(y(t)) \frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(t)) dt$$

a.s. Q_x

On the other hand,

$$(56) \quad d\xi_{e_k}(t) = -\xi_{e_k}(t) c(y(t)) dt + \frac{1}{2} \sum_{i,j} \xi_{e_k}(t) (a^{-1})_{ij}(y(t)) b_j(y(t)) dy_i(t)$$

a.s. Q_x .

Therefore, we have

$$\begin{aligned}
 (57) \quad d\xi_{e_k}(t)u^n(y(t)) &= \sum_{i,j} \xi_{e_k}(t) ((a^{-1})_{ij}(y(t))b_j(y(t)) + \\
 &+ \frac{\partial u^n}{\partial x_i}(y(t)))dy_i(t) + \\
 &+ \xi_{e_k}(t) \left[\sum_{i,j} a_{ij}(y(t)) \frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(t)) + \right. \\
 &\left. + \sum_i b_i(y(t)) \frac{\partial u^n}{\partial x_i}(y(t)) - c(y(t))u^n(y(t)) \right] dt .
 \end{aligned}$$

Hence, we get that for $T > 0$ fixed

$$\begin{aligned}
 (58) \quad E_x \xi_{e_k}(T\wedge\tau)u^n(y(T\wedge\tau)) &= u_k^n(x) + E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) \\
 &\left[\sum_{i,j} a_{ij}(y(\lambda)) \frac{\partial^2 u^n}{\partial x_i \partial x_j}(y(\lambda)) + \sum_i b_i(y(\lambda)) \frac{\partial u^n}{\partial x_i}(y(\lambda)) - \right. \\
 &\left. - c(y(\lambda))u^n(y(\lambda)) \right] d\lambda .
 \end{aligned}$$

Remarking that

$$\begin{aligned}
 (59) \quad E_x \int_0^{T\wedge\tau} |\xi_{e_k}(\lambda)| |Lu^n(y(\lambda)) - Lu(y(\lambda))| d\lambda &\leq \\
 &\leq (E_x \left[\int_0^T |\xi_{e_k}(\lambda)|^q d\lambda \right])^{1/q} (E_x \left[\int_0^T |L(u^n - u)(y(\lambda))|^p d\lambda \right])^{1/p}
 \end{aligned}$$

with $1/q + 1/p = 1$. Using the L_p -estimate we get that

$$\begin{aligned}
 (60) \quad E_x \left[\int_0^T |L(u^n - u)(y(\lambda))|^p d\lambda \right] &\leq C_{T,p} \|L(u^n - u)\|_{L^p} \rightarrow 0 \\
 &\text{as } n \rightarrow \infty
 \end{aligned}$$

and as we have by elementary calculation

$$(61) \quad E_x \left[\int_0^T |\xi_{e_k}(\lambda)|^q d\lambda \right] \leq T \exp\left(\frac{T}{2} q(q-1) [\|\sigma b\|_\infty^2 + \|c\|_\infty] \right).$$

we can let $n \rightarrow \infty$ in (58) and we get that

$$(62) \quad E_x \xi_{e_k} (T\wedge\tau) u(y(T\wedge\tau)) = u_k(x) + E_x \left(\int_0^{T\wedge\tau} \xi_{e_k}(\lambda) \left[\sum_{i,j} a_{ij}(y(\lambda)) \frac{\partial^2 u}{\partial x_i \partial x_j}(y(\lambda)) + \sum_i b_i(y(\lambda)) \frac{\partial u}{\partial x_i}(y(\lambda)) - c(y(\lambda)) u(y(\lambda)) \right] d\lambda \right).$$

Using equation (*), we obtain

$$(63) \quad E_x \xi_{e_k} (T\wedge\tau) u(y(T\wedge\tau)) = u_k(x) - E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda.$$

Furthermore as u, f are bounded by deterministic constant, we can let $T \rightarrow \infty$ in (63), thus:

$$(64) \quad E_x \int_0^{T\wedge\tau} \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda \rightarrow E_x \int_0^T \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda \text{ as } T \rightarrow \infty$$

and using the estimate (46) we obtain that

$$(66) \quad E_x \xi_{e_k} (T\wedge\tau) u(y(T\wedge\tau)) = E_x \xi_{e_k} (T) u(y(T)) \chi_{T < \tau} \rightarrow 0 \text{ as } T \rightarrow \infty.$$

Therefore it follows that

$$(66) \quad u_k(x) = E_x \int_0^T \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda$$

which completes the proof.

Remark 2.1 ; When $0 \equiv \mathbb{R}^N$ we have the representation of the solution of (*) given by

$$(67) \quad u_k(x) = E_x \int_0^\infty \xi_{e_k}(\lambda) f(y(\lambda)) d\lambda$$

which makes sense using the estimate (46).

Remark 2.2 : When \bar{O} is bounded, condition (3) is not a restriction. Indeed, making the following change of unknown functions

$$(68) \quad u_k = \omega v_k$$

with

$$(69) \quad \omega(x) = 2 - \exp[-\alpha(|x-x_0|^2 + 1)^{1/2}]$$

where $x_0 \notin \bar{O}$ and α is a constant to be chosen sufficiently large.

Hence we obtain the following system

$$(70) \quad \begin{aligned} & - \sum_{i,j} a_{ij} \frac{\partial^2 v_k}{\partial x_i \partial x_j} - \sum_{i,l} b_{kl,i} \frac{\partial v_l}{\partial x_i} - \sum_{i,j} \frac{2}{\omega} \frac{\partial \omega}{\partial x_j} a_{ij} \frac{\partial v_k}{\partial x_i} + \\ & + \sum_l (c_{kl} v_l - \frac{1}{\omega} \sum_{i,j} a_{ij} \frac{\partial^2 \omega}{\partial x_i \partial x_j} v_k - \frac{1}{\omega} \sum_{i,j} b_{kl,i} \frac{\partial \omega}{\partial x_i} v_l) \\ & = \frac{1}{\omega} f_k \end{aligned}$$

which verifies condition (3) for adequate α .

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Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique

