Dynamic behaviour of an unreliable two-machines transfer line

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TWO-MACHINES TRANSFER LINE

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ABSTRACT

We study the dynamic behaviour of an unreliable two-machines transfer line with finite interstage buffer.

We prove a convergence towards a steady-state.

RESUME

On étudie le comportement dynamique d'une ligne de transfert à deux machines séparées par une zone de stockage de capacité finie.

On montre la convergence vers un régime permanent.

1 - INTRODUCTION

In this article we consider a two-machine transfer-line (Fig.1.1). The material to be processed entersthe system at Machine-1. It flows into and out of the buffers and then to Machine-2, after which it leaves the system.

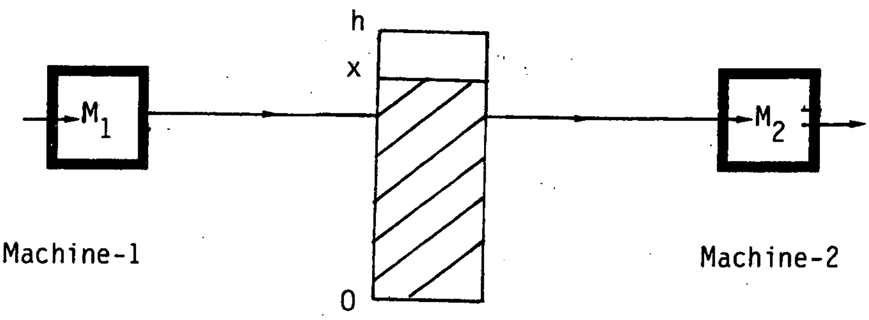

FIG. 1.1

Two machines transfer line

Each machine is unreliable, it breaks down at random time while it is operating and stays under repair for a random length of time.

As a consequence of this feature, we associate to each Machine a random process $\nu_i(t)$ $(i = 1,2)$ with values in $\{0,1\}$ (1 if M_i is operational and 0 if M_i is under repair).
Here operational means that the machine is able to process material.

ASSUMPTIONS

i) The transportation time is not taken into account.

ii) An inexhaustible supply of material is available upstream and an unlimited storage area is present downstream.

iii) We suppose that each $\nu_i(t)$ is a stationary markov process, $(i = 1,2)$.

i.e. We assume that there exists :

$$p_i,\ r_i \in R^+ \ , \qquad i = 1,2$$

such that :

$$p_i\delta = \text{prob }\{\nu_i(t+\delta) = 0 \ / \ \nu_i(t) = 1\}$$

$$r_i\delta = \text{prob }\{\nu_i(t+\delta) = 1 \ / \ \nu_i(t) = 0\}$$

iv) We consider that the production rate of M_i, when it is operating, is given by :

$$c_i \in R^+ \qquad (i = 1,2)$$

DENSITY FUNCTIONS

Let us denote by $X(t)$ the storage level at time t and by h the buffer's capacity.

If $(i,j) \in \{0,1\}$ and $x \in (0,h]$ then we give the following notation :

$$F_{ij}(x,t) = \text{prob }\{\nu_1(t) = i, \ \nu_2(t) = j, \ X(t) \leq x\}$$

The associated DENSITY FUNCTION is :

$$\boxed{f_{ij}(x,t) = \frac{\partial}{\partial x} F_{ij}(x,t)}$$

The behaviour of the transfer line can be described by the following system (cf [2], [5] ,...) :

$$(1.1) \quad \begin{cases} \dfrac{\partial f_{00}}{\partial t} = -(r_1+r_2)f_{00} + p_1f_{10} + p_2f_{01} \\[2ex] \dfrac{\partial f_{11}}{\partial t} = -(p_1+p_2)f_{11} + r_2f_{10} + r_1f_{01} + (c_2-c_1)\dfrac{\partial f_{11}}{\partial x} \\[2ex] \dfrac{\partial f_{10}}{\partial t} = r_1f_{00} + p_2f_{11} - (p_1+r_2)f_{10} - c_1\dfrac{\partial f_{10}}{\partial x} \\[2ex] \dfrac{\partial f_{01}}{\partial t} = r_2f_{00} + p_1f_{11} - (p_2+r_1)f_{01} + c_2\dfrac{\partial f_{01}}{\partial x} \end{cases}$$

This system was studied in the steady state by GERSHWIN S.B., PROTH J.M., (cf. [3], [4], [5],...)

THE PURPOSE OF THIS WORK IS TO STUDY THE DYNAMIC BEHAVIOUR.

We use the METHOD OF SUCCESSIVE APPROXIMATIONS to solve (1.1) in section 2 and we study the asymptotic behaviour in section 3.

We can, without loss of generality, assume that :

$$p_1 = p_2 = p \quad ; \quad r_1 = r_2 = r$$

and

$$c_1 = c_2 = c$$

(We have the same results for $p_1 \neq p_2$, $r_1 \neq r_2$, $c_1 \neq c_2$).

2 - THE METHOD OF SUCCESSIVE APPROXIMATIONS TO SOLVE (1.1)

The system (1.1) can be written as follows :

$$(2.1) \qquad \begin{cases} \dfrac{\partial f_{00}}{\partial t} + 2r\, f_{00} = p\,(f_{10} + f_{01}) \\ \dfrac{\partial f_{11}}{\partial t} + 2p\, f_{11} = r\,(f_{10} + f_{01}) \\ \dfrac{\partial f_{10}}{\partial t} + c\dfrac{\partial f_{10}}{\partial x} + (p+r)\, f_{10} = r\, f_{00} + p\, f_{11} \\ \dfrac{\partial f_{01}}{\partial t} - c\, \dfrac{\partial f_{01}}{\partial x} + (p+r)\, f_{01} = r\, f_{00} + p\, f_{11} \end{cases}$$

Let us associate the following problem :

$$(P_{\alpha,\beta}) \qquad \begin{cases} \dfrac{\partial f}{\partial t} + C\dfrac{\partial f}{\partial x} + Df = Af & \text{in } \Omega \\ f(x,0) & \text{p.p. } x \in]0,h[\\ f_{10}\,(0,t) = \alpha \quad ; & f_{01}\,(h,t) = \beta \end{cases}$$

where

$$\Omega = (0,h) \times (0,+\infty)$$

$$f = (f_{00}, f_{11}, f_{10}, f_{01})$$

$$A = \begin{bmatrix} 0 & 0 & p & p \\ 0 & 0 & r & r \\ r & p & 0 & 0 \\ r & p & 0 & 0 \end{bmatrix} \quad ; \quad D = \begin{bmatrix} 2r & 0 & 0 & 0 \\ 0 & 2p & 0 & 0 \\ 0 & 0 & p+r & 0 \\ 0 & 0 & 0 & p+r \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c \end{bmatrix}$$

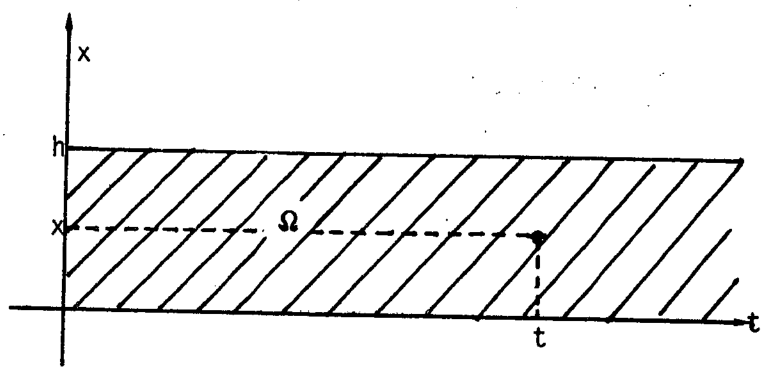

FIG. 2.1

We define the sequence of successsive approximations as follows :

$$f^{(0)}(x,t) = f(x,0) \qquad \text{p.p. } x \in (0,h)$$

and

$f^{(n+1)}(x,t)$ as a solution of the following problem :

$$(P_{n+1}) \qquad \begin{cases} \dfrac{\partial f^{(n+1)}}{\partial t} + C\dfrac{\partial f^{(n+1)}}{\partial x} + Df^{(n+1)} = Af^{(n)} & \text{in } \Omega \\ f^{(n+1)}(x,0) = f(x,0) & \text{p.p. } x \in (0,h) \\ f_{10}^{(n+1)}(0,t) = f_{10}(0,t) & \forall\, t > 0 \\ f_{01}^{(n+1)}(h,t) = f_{01}(h,t) & \forall\, t > 0 \end{cases}$$

We have the associated system :

$$(S_{n+1}) \quad \begin{cases} \dfrac{\partial f_{00}^{(n+1)}}{\partial t} + 2r\, f_{00}^{(n+1)} = p\,(f_{10}^{(n)} + f_{01}^{(n)}) \\ \dfrac{\partial f_{11}^{(n+1)}}{\partial t} + 2p\, f_{11}^{(n+1)} = r\,(f_{10}^{(n)} + f_{01}^{(n)}) \\ \dfrac{\partial f_{10}^{(n+1)}}{\partial t} + c\,\dfrac{\partial f_{10}^{(n+1)}}{\partial x} + (p+r)\, f_{10}^{(n+1)} = r\, f_{00}^{(n)} + p\, f_{11}^{(n)} \\ \dfrac{\partial f_{01}^{(n+1)}}{\partial t} - c\,\dfrac{\partial f_{01}^{(n+1)}}{\partial x} + (p+r)\, f_{01}^{(n+1)} = r\, f_{00}^{(n)} + p\, f_{11}^{(n)} \end{cases}$$

This system is hyperbolic and can be integrated along the <u>characteristic curves</u> :

$$X_i(t) \;, \quad i = 1,2,3,4$$

given by :

$$\frac{dX_1}{dt} = 0 \;, \frac{dX_2}{dt} = 0 \;, \frac{dX_3}{dt} = c \;, \frac{dX_4}{dt} = -c$$

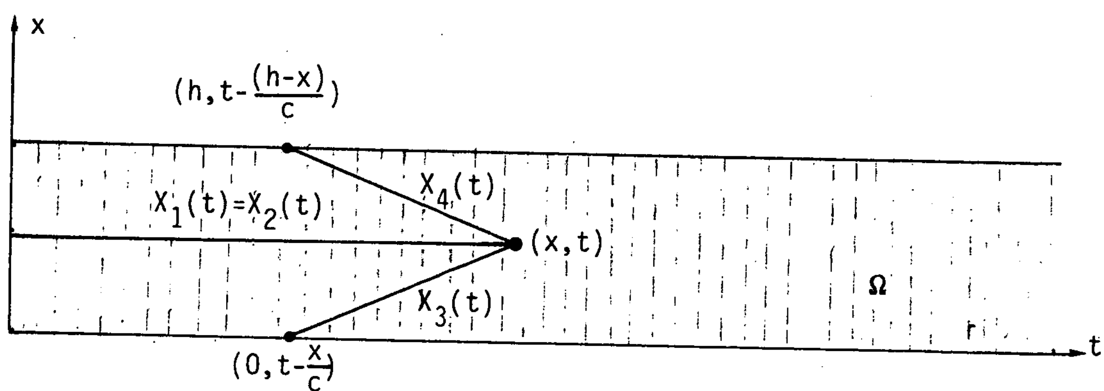

FIG.2.2

The characteristic curves

<u>REMARK 2.1</u>

> Each characteristic curve gives the evolution of the storage level.

Along the characteristic curves (S_{n+1}) becomes a system of ordinary differential equations.

Let us denote :

$$\Omega_1^+ = \{(x,t) \in \Omega : x > ct\} \quad , \quad \Omega_2^+ = \Omega \setminus \Omega_1^+$$

$$\Omega_1^- = \{(x,t) \in \Omega : x < h-ct\} , \quad \Omega_2^- = \Omega \setminus \Omega_1^-$$

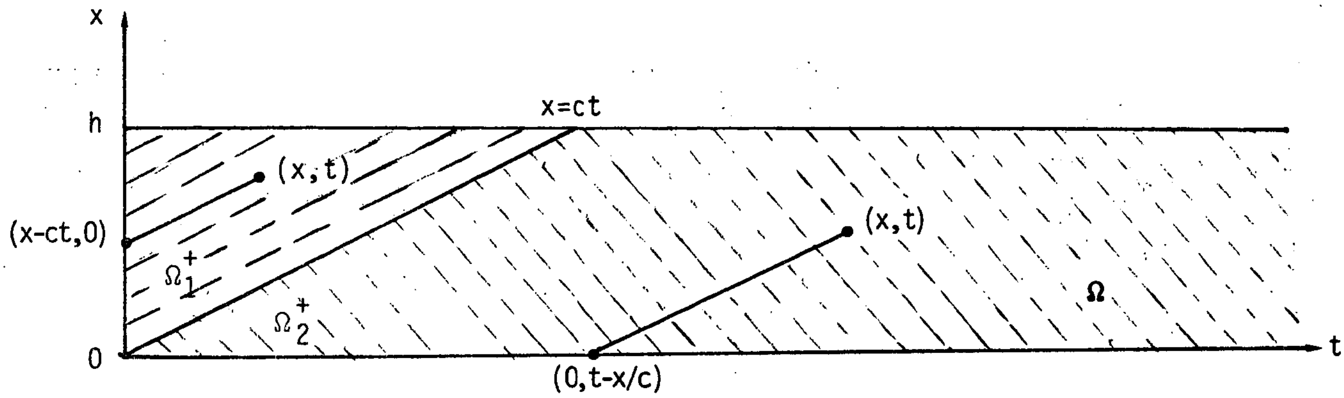

FIG. 2.3

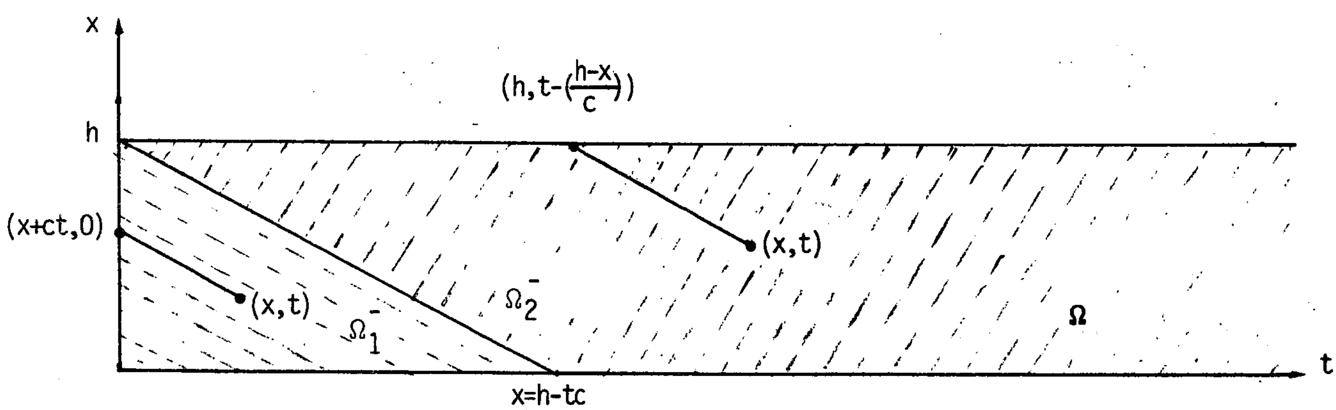

FIG. 2.4

$$(2.2)\quad\begin{cases}
\frac{d}{ds}\left[e^{-2rs}\, f_{00}^{(n+1)}(x,t-s)\right] = -\,p\, e^{-2rs}\left[f_{10}^{(n)}+f_{01}^{(n)}\right](x,t-s) \\
\qquad\qquad 0<s<t \\
\frac{d}{ds}\left[e^{-2ps}\, f_{11}^{(n+1)}(x,t-s)\right] = -r\, e^{-2ps}\left[f_{10}^{(n)}+f_{01}^{(n)}\right](x,t-s) \\
\qquad\qquad 0<s<t \\
\frac{d}{ds}\left[e^{-(p+r)s} f_{10}^{(n+1)}(x-cs,t-s)\right] = e^{-(p+r)s}\left[-rf_{00}^{(n)}-pf_{11}^{(n)}\right](x-cs,t-s) \\
\qquad\qquad \begin{cases} 0<s<t \text{ if } (x,t)\in\Omega_1^+ \\ 0<s<x/c \text{ if } (x,t)\in\Omega_2^+ \end{cases} \\
\frac{d}{ds}\left[e^{-(p+r)s} f_{01}^{(n+1)}(x+cs,t-s)\right] = e^{-(p+r)s}\left[-rf_{00}^{(n)}-pf_{11}^{(n)}\right](x+cs,t-s) \\
\qquad\qquad \begin{cases} 0<s<t \text{ if } x,t)\in\Omega_1^- \\ 0<s<\frac{h-x}{c} \text{ if } (x,t)\in\Omega_2^- \end{cases}
\end{cases}$$

<u>Therefore</u> :(*)

$$(2.3)\quad\begin{cases}
f_{00}^{(n+1)}(x,t) = e^{-2rt} f_{00}(x,0) + p\int_0^t e^{-2rs}\left[f_{10}^{(n)}+f_{01}^{(n)}\right](x,t-s)ds \\
f_{11}^{(n+1)}(x,t) = e^{-2pt} f_{11}(x,0) + r\int_0^t e^{-2ps}\left[f_{10}^{(n)}+f_{01}^{(n)}\right](x,t-s)ds \\
f_{10}^{(n+1)}(x,t) = \begin{cases} e^{-(r+p)t} f_{10}(x-ct,0)+\int_0^t e^{-(p+r)s}\left[rf_{00}^{(n)}+pf_{11}^{(n)}\right](x-cs,t-s)ds \\ \qquad \text{if } (x,t)\in\Omega_1^+ \\ e^{-(r+p)x/c} f_{10}(0,t-x/c)+\int_0^{x/c} e^{-(p+r)s}\left[rf_{00}^{(n)}+pf_{11}^{(n)}\right](x-cs,t-s)ds \\ \qquad \text{if } (x,t)\in\Omega_2^+ \end{cases} \\
f_{01}^{(n+1)}(x,t) = \begin{cases} e^{-(r+p)t} f_{01}(x+ct,0)+\int_0^t e^{-(p+r)s}\left[rf_{00}^{(n)}+pf_{11}^{(n)}\right](x+cs,t-s)ds \\ \qquad \text{if } (x,t)\in\Omega_1^- \\ e^{-(r+p)(\frac{h-x}{c})} f_{01}(h,t-(\frac{h-x}{c}))+\int_0^{\frac{h-x}{c}} e^{-(p+r)s}\left[rf_{00}^{(n)}+pf_{11}^{(n)}\right](x+cs,ts)ds \\ \qquad \text{if } (x,t)\in\Omega_2^- \end{cases}
\end{cases}$$

(*) We use MACSYMA language to solve (2.3).

Since :

$$f^{(n)} = f^{(0)} + \sum_{k=1}^{n} [f^{(k)} - f^{(k-1)}]$$

We have :

$$\lim_{n \to \infty} f^{(n)} = f^{(0)} + \sum_{k=1}^{\infty} [f^{(k)} - f^{(k-1)}] \qquad (2.4)$$

For the proof of the convergence of the infinite series (2.4) we can show that :

$$\sum_{k=0}^{\infty} \frac{(Mt)^k}{k!} \quad \|f(0)\|_{L^\infty\,(0,h)}$$

is a majorant.

Indeed, if we consider that

$$\frac{x}{c} < t \quad \text{when} \quad (x,t) \in \Omega_2^+$$

and

$$\frac{h-x}{c} < t \text{ when} \quad (x,t) \in \Omega_2^-$$

Then

$$\|f^{(1)}(t) - f^{(0)}(t)\|_{L^\infty\,(0,h)} \leq (Mt)\ \|f(0)\|_{L^\infty\,(0,h)}$$

$$\|f^{(2)}(t) - f^{(1)}(t)\|_{L^\infty\,(0,h)} \leq \frac{(Mt)^2}{2}\ f(0)_{L^\infty\,(0,h)}$$

$$\|f^{(k)}(t) - f^{(k-1)}(t)\|_{L^\infty\,(0,h)} \leq \frac{(Mt)^k}{k!}\ \|f(0)\|_{L^\infty\,(0,h)}$$

We have then proved the following

THEOREM 2.1

If $f(x,0) \in L^\infty\,(0,h)$ and $M = \max_i \left(\sum_j |A_{ij}|\right)$ where $(A_{ij})_{1 \leq i,j \leq 4} = A$. Then, there exists a UNIQUE SOLUTION of $(P_{\alpha,\beta})$ such that :

$$\|f(t)\|_{L^\infty\,(0,h)} \leq \|f(0)\|_{L^\infty\,(0,h)} \exp(Mt)$$

LEMMA 2.1

If $f(x,t)$ is a solution of $P_{\alpha,\beta}$ such that $\alpha > 0$, $\beta > 0$, $f(x,0) \geqq 0$ p.p, $x \in (0,h)$ then :

$$f(x,t) \geqq 0 \quad \forall\, t > 0, \quad \text{p.p. } x \in (0,h)$$

3. ASYMPTOTIC BEHAVIOUR

Let us write the associated steady-state equation :

(G) $$C \frac{dg}{dx} + Dg = Ag \qquad \text{in } (0,h)$$

where

$$g = (g_{00}, g_{11}, g_{10}, g_{01})$$

Let us state the following

THEOREM 3.1

(3.1) If $f(x,t)$ is a solution of $(P_{0,0})$ then

$$\|f(t)\|_{L^\infty(0,h)} \leqq e^{-Mt} \|f(0)\|_{L^\infty(0,h)} \qquad \forall\, t > 0$$

COROLLARY

Every solution $f(x,t)$ of the dynamic problem $(P_{\alpha,\beta})$ converges towards the solution $g(x)$ of the steady-state equation (G) such that $g_{10}(0) = \alpha$ and $g_{01}(h) = \beta$, when $t \longrightarrow \infty$.

(3.2) Moreover :

$$\|f(t)-g\|_{L^\infty(0,h)} \leqq e^{-Mt} \|f(0)-g\|_{L^\infty(0,h)} \quad \forall\, t > 0$$

(This inequality estimates the speed of convergence).

REMARK 3.1

If $\gamma \in R^4$ such that $D\gamma = A\gamma$ then $\gamma = (\lambda p/r, \lambda r/p, \lambda, \lambda)$ for any $\lambda \in R$

From Lemma 2.1 we deduce the following

LEMMA 3.1

If

(3.3) $$\begin{cases} |f_{00}(x,0)| \leq \lambda\ p/r \ ; & |f_{11}(x,0)| \leq \lambda\ r/p \qquad \text{p.p. } x \in (0,h) \\ |f_{10}(x,0)| \leq \lambda \ ; & |f_{01}(x,0)| \leq \lambda \qquad \text{p.p. } x \in (0,h) \end{cases}$$

Then

(3.4) $$\begin{cases} |f_{00}(x,t)| \leq \lambda\ p/r \ , & |f_{11}(x,t)| \leq \lambda\ r/p \\ |f_{10}(x,t)| \leq \lambda \ , & |f_{01}(x,t)| \leq \lambda \end{cases}$$

$\forall\, t > 0$, p.p. $x \in (0,h)$

PROOF OF THEOREM 3.1

i) Estimation of $f_{10}(x,t)$ for $t \geqslant (h/c)$

(3.5) $$\frac{\partial f_{10}}{\partial t} + c\,\frac{\partial f_{10}}{\partial x} + (p+r)\ f_{10} = r\ f_{00} + p\ f_{11}$$

Along the characteristic curves, (3.5) becomes :

$$\frac{d}{ds}\left[e^{-(p+r)s} f_{10}(x-cs,t-s)\right] = e^{-(p+r)s}\left[-rf_{00}-pf_{11}\right](x-cs,t-s)$$

where

$$o < s < t \quad \text{if} \quad (x,t) \in \Omega_1^+$$

$$0 < s < x/c \quad \text{if} \quad (x,t) \in \Omega_2^+$$

Using (3.4) we deduce :

$$|f_{10}(x,t)| \leq \lambda \int_0^{x/c} (p+r)\ e^{-(p+r)s}\, ds \quad \text{for } (x,t) \in \Omega_2^+$$

Since for $t > h/c$ we have $(x,t) \in \Omega_2^+$

$$|f_{10}(x,t)| \leq \lambda\ (1- e^{-(p+r)x/c}) \leq \lambda\ (1- e^{-(p+r)h/c})$$

Therefore

(3.6) $$\forall\, t > h/c\ , \ |f_{10}(x,t)| \leq \Theta\,\lambda \ ; \quad 0 < \Theta < 1$$

Similarly

$$\forall\, t > h/c \ , \quad |f_{01}(x,t)| \leq \Theta\,\lambda \qquad ; \qquad 0 < \Theta < 1$$

ii) Estimation of $f_{00}(x,t)$

(3.8) $$\begin{cases} \dfrac{\partial f_{00}}{\partial t} + 2\,r\,f_{00} = p(f_{10}+f_{01}) & \text{in } \Omega_1 = \{(x,t) \in \Omega : t > h/c\} \\ |f_{00}(x,h/c)| \leq \lambda p/r \end{cases}$$

Along the characteristic curves, (3.8) becomes :

$$\frac{d}{ds}\left[e^{-2rs} f_{00}(x,t-s)\right] = -e^{-2rs} p(f_{10}+f_{01})(x,t-s) \quad , \quad 0 < s < t$$

Hence :

$$f_{00}(x,t) = e^{-2rt} f_{00}(x,0) + p \int_0^t e^{-2rs} [f_{10}+f_{01}](x,t-s)ds$$

We deduce from (3.6), (3.7) and (3.8) :

$$|f_{00}(x,t)| \leq e^{-2rt} \lambda p/r + 2p\Theta\lambda \int_0^t e^{-2rs} ds \qquad \forall\, t \geq 2\,h/c$$

Hence:

$$|f_{00}(x,t)| \leq e^{-2rt} \lambda p/r + \Theta\,\lambda p/r \left[1-e^{-2rt}\right]$$

Thus :

$$|f_{00}(x,t)| \leq \lambda p/r \left[\Theta + (1-\Theta)\, e^{-2rt}\right] \leq \lambda p/r\left[\Theta + (1-\Theta)\, e^{-2rh/c}\right]$$

Then, for :

$t \geq 2h/c$,

(3.9) $$|f_{00}(x,t)| \leq \Theta' \lambda p/r$$

where

$$\Theta' = \Theta + (1-\Theta)\, e^{-2rh/c} \quad , \qquad 0 < \Theta' < 1$$

Similarly, for $t \geq 2h/c$

(3.10) $$|f_{11}(x,t)| \leq \Theta'' \lambda p/r \quad , \qquad 0 < \Theta'' < 1$$

Making similar estimations for

$$\Omega_1 = (0,h) \times (2h/c, +\infty)$$

for $x \in (0,h)$ and $t \geq 4\,h/c$

$$\begin{cases} |f_{10}(x,t)| \leq \Theta^2 \lambda \\ |f_{01}(x,t)| \leq \Theta^2 \lambda \\ |f_{00}(x,t)| \leq \Theta^2 \lambda p/r \\ |f_{11}(x,t)| \leq \Theta^2 \lambda r/p \end{cases}$$

Then we have $|f(t)|_{L^\infty(0,h)} \leq k^n \|f(0)\|_{L^\infty}$

for

$$t \geq 2nh/c \quad , \qquad 0 < k < 1$$

Finally, if

$$k = e^{-t_0} \qquad \text{and} \qquad M = \frac{ct_0}{h} \quad ,$$

$$\|f(t)\|_{L^\infty(0,h)} \leq e^{-Mt} \|f(0)\|_{L^\infty(0,h)} \; , \; \forall \; t > 0$$

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BIBLIOGRAPHIE

[1] BUZACOTT J.A., Automatic Transfer Lines with Buffer Stocks, The International Journal of production Research (1967), vol.5, n°3.

[2] FORESTIER J.P., Modélisation stochastique et comportement asymptotique d'un système automatisé de production. R.A.I.R.O. Automatique, vol.14, n°2, 1980, p.127 à 143.

[3] GERSHWIN S.B. and SCHICK I.C. (1980), Modeling and analysis of two -and three- stage transfer lines with unreliable machines and finite buffers. MIT laboratory for Information and Decision Systems Report LIDS-R-1039.

[4] GERSHWIN S.B. and ODED BERMAN, Analysis of transfer lines consisting of the unreliable machines with random processing times and finite storage buffer AIIE Transaction, vol.13, n°1.

[5] PROTH J.M., Study of two unreliable machines with random processing times and finite storage buffers : a markovian approach, Working Paper 81-37, European Institute of Advanced Studies in Management, Brussels, Belgium (1981).

[6] COILLARD P., et PROTH J.M., Effet des stocks tampons dans une fabrication en ligne. Revue Belge de statistique, d'information et de recherche opérationnelle, vol.24, n°2.

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