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SIMPLE COMPUTABLE BOUNDS
FOR
THE FORK-JOIN QUEUE

by

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Abstract

A simple queuing system, known as the Fork-Join queue, is considered with basic performance measure defined as the delay between the Fork and Join dates. Simple bounds are derived for some of the statistics of this quantity under standard renewal assumptions. These bounds are obtained, in both transient and steady-state regimes, by stochastically comparing the original system to two other queueing systems with a structure simpler than the original system, yet with identical stability characteristics. In steady-state, the computation reduces to standard $GI \mid GI \mid 1$ calculations and the obtained approximations thus constitute a first sizing-up of system performance. The bounding methodology is of independent interest to study various other queueing systems.

Résumé

On établit une borne supérieure et une borne inférieure pour chaque moment du temps de réponse dans une file d'attente "FORK-JOIN". Les hypothèses sont celles de processus de renouvellement et les résultats s'appliquent aux régimes transitoire et stationnaire. La méthode d'analyse consiste en une comparaison stochastique de cette file avec deux réseaux de files d'attente à forme produit qui possèdent la même condition de stabilité, si bien que le calcul de ces bornes se ramène à celui des caractéristiques de simples files GI/GI/1. Cette méthode de comparaison est d'intérêt général pour l'étude d'autres systèmes de files d'attente.

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1. INTRODUCTION:

A simple queueing system, known as the *Fork-Join* (FJ) queue, is considered in this paper. Rouhgly speaking, a K-dimensional FJ queue is a queueing system operated by K servers working in *parallel*; each server is attended by a buffer area of *infinite* capacity and individually operates according to the FIFO discipline. Customers arrive into the system in *bulks* of size K and are processed according to the following discipline:

> Upon arrival, a bulk is immediately split so that each one of the K customers composing it, is allocated to exactly one server (the so-called Fork primitive).
>
> As soon as all the K customers constituting a bulk have been serviced, the bulk is immediately and instantaneously recomposed (the so-called Join primitive) and leaves the system at once. This *synchronization* constraint is achieved by parking already serviced customers in an auxiliary buffer of infinite capacity, where they await being reunited to not yet serviced customers of the same bulk.

Such queueing models arise in many application areas, including flexible manufacturing and parallel processing (e. g. the cobegin and coend structures in concurrent languages), with a wide variety of interpretations. In the context of production systems, a bulk customer can be interpreted as a customer's order with several components, each component or suborder being attended by a separate production device. An example very similar to this one is obtained by considering the production of multipart items. In computer systems with a parallel architecture, a bulk customer can be viewed as a program composed of several subroutines, each one to be executed on a different processor.

For this type of applications, the determination of bulk response time (defined as the delay between the Fork and the Join dates) is of crucial importance in quantifying system performance. In two dimensions ($K=2$), the stationary distribution of this response time was derived by Flatto [3] under Markovian assumptions, via uniformization techniques. In more dimensions ($K \geq 2$), the problem seems to still be completely open.

In this paper, simple computable lower and upper bounds are derived for various statistics of this response time, including its moments. These bounds have both transient and steady-state versions, and are obtained by a direct stochastic comparison of the FJ queue to two other systems with K parallel servers, that exhibit stability conditions identical to the one for the FJ queue. The steady-state bounds are computable in the sense that their evaluation reduces to standard calculations for $D \mid GI \mid 1$ and $GI \mid GI \mid 1$ systems, respectively Although the results reported here are obtained under renewal type assumptions, they hold for a much wider class of systems; this is discussed in a lenghtier version of this paper [1].

The paper is organized as follows: The model for the FJ queue, the notation and working assumptions are given in Section 2. The lower and upper bounds are derived in Sections 3 and 4, respectively; the results and some of their consequences are discussed in Section 5.

2. THE MODEL:

The model discussed in this paper is introduced in this section, together with the notation and the various assumptions enforced throughout.

The model emphasizes sample path representation for the quantities of interest and as further developments will demonstrate, this approach is quite fruitful in establishing bounds. To that end, an underlying probability triple (Ω, F, P) is given on which all the random variables (RV) mentionned in this paper are defined. A positive integer K is given and held fixed hereafter. As a convention, the k-th component RV of any R^K-valued RV is denoted by the same symbol as this RV but superscripted by k; a similar convention is adopted to denote the components of any vector in R^K. This probability triple (Ω, F, P) is assumed to simultaneously carry an R_+^K-valued RV W, together with the sequences of RV's $\{\tau_{n+1}\}_0^\infty$ and $\{\sigma_n\}_0^\infty$ which are R_+- and R_+^K-valued, respectively.

In the context of the FJ queue, these RV's are given the following interpretation: Each one of the K *parallel* severs composing the system is attended by a buffer area of *infinite* capacity and operates according to the FIFO discipline. The customers to the system come in bulk of size K. The sequence $\{\tau_{n+1}\}_0^\infty$ represents the bulk inter-arrival times, i. e., the customer bulks arrive into the system at times $\{A_n\}_0^\infty$, defined by

$$A_n := \sum_{m=0}^{n-1} \tau_{m+1} \qquad n=1,2,...(2.1)$$

with $A_0 := 0$. Upon its arrival in the system, the n-th bulk is split into its K constituting customers; the type k customer of the bulk requires an amount of processing time σ_n^k from the k-th server and is thus routed directly to the buffer area of this k-th server, $1 \leq k \leq K$. At time $t=0$, an initial load is already awaiting service in the various buffer areas; the RV W^k then represents the amount of time required by the k-th server to clear this initial load from the k-th queueing area.

In order to define a reasonable performance measure for the FJ queue, consider the sequence of R_+^K-valued RV's $\{W_n\}_0^\infty$ generated by the recursions

$$W_{n+1}^k = [W_n^k + \sigma_n^k - \tau_{n+1}]^+ , 1 \leq k \leq K, \qquad n=0,1,...(2.2)$$

with $W_0^k = W^k$, $1 \leq k \leq K$. The RV W_n^k is the *waiting time* of the k-th customer from the n-th bulk and its *response time*, denoted by R_n^k, is defined by

$$R_n^k = W_n^k + \sigma_n^k , 1 \leq k \leq K. \qquad n=0,1,...(2.3)$$

Global FIFO is enforced in this system in the sense that the n-th customer bulk is declared serviced *if and only if* each one of its K constituting customers has been serviced. It is then natural to define the *system response time* for the n-th customer bulk as the RV T_n given by

$$T_n = \max_{1 \leq k \leq K} R_n^k . \qquad n=0,1,...(2.4)$$

In this paper, the task of evaluating various statistics of system response time is taken on, be it in *transient* or in *steady state*. The discussion is conducted under the following basic assumptions.

(A1): The RV W and the sequences of RV's $\{\tau_{n+1}\}_0^\infty$ and $\{\sigma_n\}_0^\infty$ are mutually independent.

(A2): The RV's $\{\tau_{n+1}\}_0^\infty$ form a finite mean renewal sequence with common probability distribution function $A(\bullet)$.

(A3): The sequences $\{\sigma_n^k\}_0^\infty$, $1 \leq k \leq K$, are mutually independent sequences.

(A4): The sequence $\{\sigma_n^k\}_0^\infty$ is a finite mean renewal sequence, with common probability distribution $B_k(\bullet)$, $1 \leq k \leq K$.

Most of the results reported here can be obtained under a much weaker set of assumptions, as shown in [1]. To fix the notation, the arrival rate λ and the service rates μ_k, $1 \leq k \leq K$, are defined as usual by

$$\frac{1}{\lambda} := E[\tau_{n+1}] = \int_0^\infty t dA(t), \tag{2.5}$$

and

$$\frac{1}{\mu_k} := E[\sigma_n^k] = \int_0^\infty t dB_k(t), \ 1 \leq k \leq K. \tag{2.6}$$

Under the assumptions (A1)-(A4), the queueing system associated with a processor operates like a standard $GI \mid GI \mid 1$ queueing system. However, these K $GI \mid GI \mid 1$ systems are *not* independent in general since they have *identical* inputs. It is this very lack of independence that makes the computation of the statistics of the RV's $\{T_n\}_0^\infty$ extremely hard, not to say impossible.

In view of these difficulties, it seems relevant to seek ways of generating *bounds* and *approximations* to the various statistics $\{T_n\}_0^\infty$. The present paper is devoted to the derivation of *simple computable* bounds through direct bounding arguments that explicitly, and in a crucial way, exploit the sample path nature of the recursions (2.2).

At this point, it is appropriate to observe that the stability condition for the FJ queue can be easily obtained from standard results on $GI \mid GI \mid 1$ queues. Here, stability is understood as the convergence *in distribution* of the sequence of RV's $\{W_n\}_0^\infty$ to a *proper* R_+^K-valued RV, *independently of the initial condition* W, and is equivalent to the existence of a R_+^K-valued RV W_∞ defined on the sample space Ω with the property that

$$\lim_{n\to\infty} P[W_n^k \leq x^k, 1 \leq k \leq K \mid W] = P[W_\infty^k \leq x^k, 1 \leq k \leq K] \tag{2.7}$$

for every x in R_+^K point of continuity for the probability distribution of W_∞. Of course, if (2.7) holds, then necessarily

$$\lim_{n\to\infty} P[R_n^k \leq x^k, 1 \leq k \leq K \mid W] = P[R_\infty^k \leq x^k, 1 \leq k \leq K] \tag{2.8}$$

for every x in a dense subset of R_+^K, with $R_\infty^k := W_\infty^k + \sigma_1^k$, $1 \leq k \leq K$. By virtue of (2.4), this last remark readily yields that for all x in a dense subset of R_+,

$$\lim_{n\to\infty} P[T_n \leq x \mid W] = P[T_\infty \leq x] \tag{2.9}$$

with $T_\infty := \max_{1 \leq k \leq K} R_\infty^k$. Moreover, under additional finiteness conditions on second moments, it can be shown [1] that

$$\lim_{n \to \infty} E[T_n \mid W] = E[T_\infty]. \tag{2.10}$$

Proposition 2.1. *Under the assumptions (A1)-(A4), the FJ queue is stable if and only if*

$$\rho := \max_{1 \le k \le K} \rho_k < 1, \tag{2.11}$$

where

$$\rho_k := \frac{\lambda}{\mu_k}, \quad 1 \le k \le K. \tag{2.12}$$

The intuition behind the result is that the FJ queue is stable *iff* each queue in isolation is stable. A formal proof is available in [1] where it is shown that the k-th component of W_∞ can be obtained as the a.s. limit of the Loynes sequence [6] associated with the k-th $GI \mid GI \mid 1$ queue generated by $W^k, \{\tau_{n+1}\}_0^\infty$ and $\{\sigma_n^k\}_0^\infty$.

The next two sections are devoted to the derivation of lower and upper bounds on system response time statistics. The key idea is to construct two queueing systems that in the sense of some *stochastic ordering*, bound the original system, from above and from below, respectively and that are more *tractable analytically* than the original system. The approach is motivated by the idea that increased variablity in some of the stochastic components of a queueing system should results in a greater variability of the waiting times. Many results in that vein have appeared in the litterature and the reader is refered to the work of Stoyan [8] and Whitt [9] for an introduction to this topic.

3. A LOWER BOUND:

The discussion given in this section finds its origin in a folk theorem of Queueing Theory stating that *determinism minimizes waiting times* in many queueing systems. For $G \mid G \mid 1$ queues, such results have been established under a variety of assumptions by a number of authors, including Hajek [4], Humblet [5] and Rogozin [7], to name a few.

Here, following this idea amounts to constructing a new FJ queueing system from the original one that is of the same type but has a *deterministic* input stream. More precisely, on the *same* probability triple (Ω, F, P), a second FJ queueing system is generated that has same initial system load W and same service time sequences $\{\sigma_n^k\}_0^\infty$, $1 \leq k \leq K$, as the original FJ queueing system, but with *deterministic* inter-arrival time sequence $\{\underline{\tau}_{n+1}\}_0^\infty$. For this last sequence to be a renewal sequence, as specified by (A2), it is necessary to have

$$\underline{\tau}_{n+1} = \frac{1}{\lambda} \qquad n=0,1,...(3.1)$$

The waiting times for this new FJ queueing system can be grouped into K sequences $\{\underline{W}_n^k\}_0^\infty$ of R_+-valued RV's which are obtained through the recursions

$$\underline{W}_{n+1}^k = [\underline{W}_n^k + \sigma_n^k - \frac{1}{\lambda}]^+ , 1 \leq k \leq K, \qquad n=0,1,...(3.2)$$

with $\underline{W}_0^k = W^k$, $1 \leq k \leq K$. In analogy with (2.3)-(2.4), the corresponding response times $\{\underline{R}_n^k\}_0^\infty$, $1 \leq k \leq K$, and $\{\underline{T}_n\}_0^\infty$ are then defined simply by

$$\underline{R}_n^k = \underline{W}_n^k + \sigma_n^k , 1 \leq k \leq K \qquad n=0,1,...(3.3)$$

and

$$\underline{T}_n = \max_{1 \leq k \leq K} \underline{R}_n^k . \qquad n=0,1,...(3.4)$$

It is noteworthy that the FJ queue with deterministic inter-arrival stream (3.1) is stable iff the original FJ queue is stable; indeed, according to Proposition 2.1, stability is completely characterized by the coefficient ρ which coincides for such systems owing to (3.1). The next proposition contains the key comparison result for generating lower bounds to the system response time statistics.

Theorem 3.1. *Let $\underline{S}$ be the σ-field of events generated on the sample space Ω by the RV's W and $\{\sigma_n\}_0^\infty$. Under the enforced assumptions (A1)-(A4), the inequalities*

$$\underline{W}_n^k \leq E[W_n^k \mid \underline{S}] , 1 \leq k \leq K, \qquad n=0,1,...(3.5)$$

and

$$\underline{R}_n^k \leq E[R_n^k \mid \underline{S}] , 1 \leq k \leq K, \qquad n=0,1,...(3.6)$$

hold, whence

$$\underline{T}_n \leq E[T_n \mid \underline{S}] . \qquad n=0,1,...(3.7)$$

The arguments giving these results are simple and use in a direct way the sample path recursions (2.2)-(2.3) and (3.2)-(3.3), together with Jensen's inequality applied to the *convex monotone non-decreasing* function $R \to R : x \to x^+$.

Proof: The proof proceeds by induction. First fix k, $1 \leq k \leq K$. Since the RV W^k is $\underline{S}$-measurable, (3.5) is obviously true owing to the fact that in both FJ systems, the initial loads at the k-th server, and thus the initial clearing times, are identical and equal to W^k.

Take as induction hypothesis that *(3.5) holds true for some* $n = m \geq 0$. Now, for such m, Jensen's inequality gives

$$E[W^k_{m+1} \mid \underline{S}] \geq [E[W^k_m \mid \underline{S}] + \sigma^k_m - E[\tau_{m+1} \mid \underline{S}]]^+ \qquad (3.8)$$

since the RV σ^k_m is $\underline{S}$-measurable. Moreover, assumption (A1) immediately implies that

$$E[\tau_{n+1} \mid \underline{S}] := E[\tau_{n+1}] = \frac{1}{\lambda} \qquad n=0,1,...(3.9)$$

Substitution of (3.9) into (3.8) and use of the induction hypothesis easily yield that

$$E[W^k_{m+1} \mid \underline{S}] \geq [\underline{W}^k_m + \sigma^k_m - \frac{1}{\lambda}]^+ = \underline{W}^k_{m+1}. \qquad (3.10)$$

where the last equality follows from (3.2). This shows that (3.5) holds for $n = m+1$ and since it holds for $n = 0$, it holds by induction for all $n \geq 0$.

The inequalities (3.6) are now easy consequences of (3.5), and of the defining relations (2.3) and (3.3), together with the $\underline{S}$-measurability of the service times. To get (3.7), observe that

$$E[T_n \mid \underline{S}] \geq \max_{1 \leq k \leq K} E[R^k_n \mid \underline{S}]. \qquad n=0,1,...(3.11)$$

by standard properties of conditional expectations. The first part of the proof and the defining relation (3.4) now imply that

$$E[T_n \mid \underline{S}] \geq \max_{1 \leq k \leq K} \underline{R}^k_n = \underline{T}_n \qquad n=0,1,...(3.12)$$

and (3.7) thus readily follows. •

The following corollary is an easy consequence of Theorem 3.1. and of Jensen's inequality.

Corollary 3.2. *Under the enforced assumptions (A1)-(A4), the inequalities*

$$E[\underline{T}_n] \leq E[T_n]. \qquad n=0,1,...(3.13)$$

hold; more generally, for any convex monotone non-decreasing mapping $\phi : R \to R : x \to \phi(x)$,

$$E[\phi(\underline{T}_n)] \leq E[\phi(T_n)] \qquad n=0,1,...(3.14)$$

whenever $E[T_n] < \infty$ *and* $E[\mid \phi(T_n) \mid] < \infty$.

The bounds (3.13) and (3.14) are essentially *transient* in nature; as they stand, these bounds would be of limited interest if it were not for the fact that they easily carry over to steady-state and that in that form they are *computable.* To see this, observe that in the FJ queue with *deterministic* arrival stream (and in that case only !), the K processors operate like K *independent* $D \mid GI \mid 1$ queueing systems. As a result, the K sequences of waiting times $\{\underline{W}_n^k\}_0^\infty$, $1 \leq k \leq K$, are *mutually independent* (given W), and so are the sequences of response times $\{\underline{R}_n^k\}_0^\infty$, $1 \leq k \leq K$. Hence, for all x in a dense subset of R_+, the defining relation (3.4) readily implies that

$$P[\underline{T}_n \leq x \mid W] = \prod_{k=1}^{K} P[\underline{R}_n^k \leq x \mid W] \qquad n=0,1,... (3.15)$$

owing to the enforced independence assumptions and therefore

$$E[\underline{T}_n] = \int_0^\infty \left[1 - E[\prod_{k=1}^{K} P[\underline{R}_n^k \leq x \mid W]]\right] dx. \qquad n=0,1,... (3.16)$$

When $\rho < 1$, the FJ queue with deterministic arrivals is also stable and the convergences (2.7)-(2.9) apply. It thus follows from (3.15) that for all x in a dense subset of R_+,

$$P[\underline{T}_\infty \leq x] = \lim_{n \to \infty} P[\underline{T}_n \leq x \mid W] = \prod_{k=1}^{K} P[\underline{R}_\infty^k \leq x] \qquad (3.17)$$

(with the notation introduced at the end of the previous section) and consequently

$$P[\underline{R}_\infty^k \leq x] = \lim_{n \to \infty} P[\underline{R}_n^k \leq x \mid W] = \int_0^\infty P[\underline{W}_\infty^k \leq x - \sigma] dB_k(\sigma) \qquad (3.18)$$

for all $1 \leq k \leq K$. Under additional finiteness conditions on second moments, the convergence (2.10) takes the form

$$E[\underline{T}_\infty] = \lim_{n \to \infty} E[\underline{T}_n] = \int_0^\infty \left[1 - \prod_{k=1}^{K} P[\underline{R}_\infty^k \leq x]\right] dx. \qquad (3.19)$$

4. AN UPPER BOUND:

In this section, upper bounds to the response time statistics are established under an additional assumption on the arrival stream $\{\tau_{n+1}\}_0^\infty$, namely,

A(5): The RV's $\{\tau_{n+1}\}_0^\infty$ are divisible in the sense that the representation

$$\tau_{n+1} = \frac{1}{K}\sum_{k=1}^{K} \overline{\tau}_{n+1}^k \qquad n=0,1,...(4.1)$$

holds, where the sequences of RV's $\{\overline{\tau}_{n+1}^k\}_0^\infty$, $1 \le k \le K$, form mutually independent renewal sequences with common probability distribution $\overline{A}(\bullet)$.

In the language of Laplace transforms, this concept of divisibility is equivalent to the existence of a probability distribution $\overline{A}(\bullet)$ on R_+ such that

$$A^*(s) = [\overline{A}^*(\frac{s}{K})]^K \qquad (4.2)$$

with the superscript $*$ denoting the corresponding Laplace transform. A typical example of such divisible distributions is given by the class of K-stage Erlangian distributions. Observe that the renewal assumption imposed on the sequences $\{\overline{\tau}_{n+1}^k\}_0^\infty$, $1 \le k \le K$, necessarily implies that

$$E[\tau_{n+1}] = E[\overline{\tau}_{n+1}^k] = \frac{1}{\lambda},\ 1 \le k \le K. \qquad n=0,1,...(4.3)$$

The basic idea of this section is to used the representation (4.1) to construct on the *same* probability triple (Ω, F, P) a new queueing system composed of K parallel $GI \mid GI \mid 1$ queues. In this new system, the k-th queue has an initial system load W^k and service times sequences $\{\sigma_n^k\}_0^\infty$, identical to the ones it had in the original FJ queue, but with inter-arrival time sequences $\{\overline{\tau}_{n+1}^k\}_0^\infty$, $1 \le k \le K$. The key feature of this new system is that arrivals are *not* synchronized anymore; in particular, arrivals into the k-th queue take place at times $\{\overline{A}_n^k\}_0^\infty$ defined by

$$\overline{A}_n^k = \sum_{m=0}^{n-1} \overline{\tau}_{m+1}^k, \qquad n=1,2,...(4.4)$$

with $\overline{A}_0^k = 0$, $1 \le k \le K$.

The waiting times for the k-th queue in this new system form a sequence $\{\overline{W}_n^k\}_0^\infty$ of R_+-valued RV's which are given by the recursions

$$\overline{W}_{n+1}^k = [\overline{W}_n^k + \sigma_n^k - \overline{\tau}_{n+1}^k]^+ , \qquad n=0,1,...(4.5)$$

with $\overline{W}_0^k = W^k$, $1 \le k \le K$. As usual, the corresponding response times $\{\overline{R}_n^k\}_0^\infty$, $1 \le k \le K$, are then defined simply by

$$\overline{R}_n^k = \overline{W}_n^k + \sigma_n^k ,\ 1 \le k \le K. \qquad n=0,1,...(4.6)$$

Now, in analogy with (2.4), define the global system response time sequence $\{\overline{T}_n\}_0^\infty$ by

$$\overline{T}_n = \max_{1 \le k \le K} \overline{R}_n^k. \qquad n=0,1,...(4.7)$$

For this parallel system to be stable, it is necessary and sufficient that all components be stable; it then follows from standard $GI \mid GI \mid 1$ theory (and from (4.2)) that this happens if and only if $\rho < 1$, i. e., the same stability condition as for the FJ queue! The next proposition contains the key comparison result for generating lower bounds to the system response time statistics.

Theorem 4.1. *Let $\bar{S}$ be the σ-field of events generated on the sample space Ω by the RV's W and $\{(\sigma_n, \tau_{n+1})\}_0^\infty$. Under the enforced assumptions (A1)-(A5), the inequalities*

$$W_n^k \leq E[\overline{W}_n^k \mid \bar{S}], \ 1 \leq k \leq K, \qquad n=0,1,... (4.8)$$

and

$$R_n^k \leq E[\overline{R}_n^k \mid \bar{S}], \ 1 \leq k \leq K, \qquad n=0,1,... (4.9)$$

hold, whence

$$T_n \leq E[\overline{T}_n \mid \bar{S}]. \qquad n=0,1,... (4.10)$$

These results are established by arguments very similar to the ones giving Theorem 3.1 when combined to the following fact.

Theorem 4.2. *Under the enforced assumptions (A1)-(A5), the following identity*

$$E[\bar{\tau}_{n+1}^k \mid \bar{S}] = \tau_{n+1}, \ 1 \leq k \leq K \qquad n=0,1,... (4.11)$$

holds.

Proof of Theorem 4.2: First, observe that

$$E[\bar{\tau}_{n+1}^k \mid \bar{S}] = E[\bar{\tau}_{n+1}^k \mid \tau_{n+1}], \ 1 \leq k \leq K \qquad n=0,1,... (4.12)$$

owing to the independence assumptions contained in (A.1) and (A.5). Moreover, for each $n=0,1,...$, the RV's $\{\bar{\tau}_{n+1}^k\}_1^K$ are i.i.d. and the joint probability distribution of the pairs of RV's $\bar{\tau}_{n+1}^k$ and τ_{n+1}, $1 \leq k \leq K$ are thus *independent* of k. Consequently,

$$E[\bar{\tau}_{n+1}^1 \mid \tau_{n+1}] = \cdots = E[\bar{\tau}_{n+1}^K \mid \tau_{n+1}] \qquad n=0,1,... (4.13)$$

and therefore

$$\tau_{n+1} = E[\tau_{n+1} \mid \tau_{n+1}] \qquad (4.14)$$

$$= E[\frac{1}{K}\sum_{l=1}^{K} \bar{\tau}_{n+1}^l \mid \tau_{n+1}] \qquad (4.15)$$

$$= E[\bar{\tau}_{n+1}^k \mid \tau_{n+1}], \ 1 \leq k \leq K, \qquad (4.16)$$

by elementary arguments. This completes the proof of (4.11).

Proof of Theorem 4.1: The proof proceeds by induction. First fix k, $1 \leq k \leq K$. Since the RV W^k is $\bar{S}$-measurable, (4.8) is obviously true owing to the fact that the initial loads at the k-th processor, and thus the initial clearing times, are identical and equal to W^k in both systems.

Take as induction hypothesis that (4.8) *holds true for some* $n = m \geq 0$. Now, for such m, Jensen's inequality gives

$$E[\overline{W}^k_{m+1} \mid \overline{S}] \geq [E[\overline{W}^k_m \mid \overline{S}] + \sigma^k_m - E[\overline{\tau}^k_{m+1} \mid \overline{S}]]^+ \tag{4.17}$$

since the RV σ^k_m is $\overline{S}$-measurable. Owing to Theorem 4.2., this inequality takes the form

$$E[\overline{W}^k_{m+1} \mid S] \geq [E[\overline{W}^k_m \mid \overline{S}] + \sigma^k_m - \tau_{m+1}]^+ \tag{4.18}$$

and the induction hypothesis now easily yields

$$E[\overline{W}^k_{m+1} \mid \overline{S}] \geq [W^k_m + \sigma^k_m - \tau_{m+1}]^+ = W^k_{m+1} \tag{4.19}$$

where the last equality follows from (4.5). This shows that (4.8) holds for $n = m+1$ and since it holds for $n = 0$, it holds by induction for all $n \geq 0$.

The second part of the proof is now identical to the second part of the proof of Theorem 3.1; details are omitted for sake of brevity.

The following corollary is now easily obtained from Theorem 4.1.

Corollary 4.3. *Under the enforced assumptions (A1)-(A5), the inequalities*

$$E[T_n] \leq E[\overline{T}_n]. \qquad n=0,1,... \tag{4.20}$$

hold; more generally, for any convex monotone non-decreasing mapping $\phi : R \to R : x \to \phi(x)$,

$$E[\phi(T_n)] \leq E[\phi(\overline{T}_n)] \qquad n=0,1,... \tag{4.21}$$

whenever $E[\overline{T}_n] < \infty$ *and* $E[\,|\phi(\overline{T}_n)|\,] < \infty$.

Again, as in the previous section, these bounds obtained on various statistics of the system response times are essentially *transient* in nature but readily carry over to steady-state by the usual limiting arguments. This would be of limited interest if it were not for the fact that their steady-state versions are *more easily computable* than the original quantities. Indeed, the K parallel $GI \mid GI \mid 1$ queues with asynchronous arrivals that compose this system, operate *independently* of each other. Therefore, the K sequences of waiting times $\{\overline{W}^k_n\}_0^\infty$, $1 \leq k \leq K$, are *mutually independent* (given W), and so are the sequences of response times $\{\overline{R}^k_n\}_0^\infty$, $1 \leq k \leq K$. Hence, by the same reasoning as the one given at the end of the previous section, it follows from (4.7) that for all x in R_+,

$$P[\overline{T}_n \leq x \mid W] = \prod_{k=1}^{K} P[\overline{R}^k_n \leq x \mid W], \qquad n=0,1,... \tag{4.22}$$

and

$$E[\overline{T}_n] = \int_0^\infty \left[1 - E[\prod_{k=1}^{K} P[\overline{R}^k_n \leq x \mid W]]\right] dx. \qquad n=0,1,... \tag{4.23}$$

When $\rho < 1$, it is easy to see from (4.22) (and standard results on $GI \mid GI \mid 1$ [2]) that for all x in a dense subset of R_+,

$$P[\overline{T}_\infty \leq x] = \lim_{n \to \infty} P[\overline{T}_n \leq x \mid W] = \prod_{k=1}^{K} P[\overline{R}^k_\infty \leq x], \tag{4.24}$$

where

$$P[\overline{R}^k_\infty \leq x] = \lim_{n \to \infty} P[\overline{R}^k_n \leq x \mid W] = \int_0^\infty P[\overline{W}^k_\infty \leq x - \sigma] dB_k(\sigma) \tag{4.25}$$

for all $1 \leq k \leq K$. Here the RV's $\overline{W}^k_\infty$, $1 \leq k \leq K$ are defined via the appropriate version (2.7) and $\overline{R}^k_\infty = \overline{W}^k_\infty + \sigma^1_k$, $1 \leq k \leq K$. Again, under additional finiteness condiditions on second moments, the relations

$$E[\overline{T}_\infty] = \lim_{n \to \infty} E[\overline{T}_n] = \int_0^\infty \left[1 - \prod_{k=1}^{K} P[\overline{R}^k_\infty \leq x] \right] dx \tag{4.26}$$

are obtained.

5. DISCUSSION AND CONCLUSIONS:

Both bounding systems introduced in the previous sections have a stability behavior *identical* to the original FJ queue, a highly desirable property when generating approximations. In all three systems, stability is obtained if and only if $\rho < 1$, as explained in previous sections.

Under the assumptions (A1)-(A5), the transient bounds

$$E[\underline{T}_n] \leq E[T_n] \leq E[\overline{T}_n] \qquad n=0,1,... (5.1)$$

are readily obtained by combining (3.13) and (4.20). Now. if the original FJ queue is stable and additional finitness assumptions are made on second moments, then the convergence

$$\lim_{n\to\infty} E[\underline{T}_n] \leq \lim_{n\to\infty} E[T_n] \leq \lim_{n\to\infty} E[\overline{T}_n] \qquad (5.2)$$

takes place owing to (2.10), (3.19) and (4.26), with

$$E[\underline{T}_\infty] \leq E[T_\infty] \leq E[\overline{T}_\infty] \qquad n=0,1,... (5.3)$$

in the limit. As pointed out in Sections 3 and 4, the quantities $E[\underline{T}_\infty]$ and $E[\overline{T}_\infty]$ can in principle be evaluated through calculations on independent $GI \mid GI \mid 1$ systems. The tightness of the bounds (5.3) will be reported in [1].

More interesting is the following observation: The *rate* at which $\{E[\underline{T}_n]\}_0^\infty$ and $\{E[\overline{T}_n]\}_0^\infty$ converge to $E[\underline{T}_\infty]$ and $E[\overline{T}_\infty]$, respectively, can be obtained from corresponding rate of convergence results for $GI \mid GI \mid 1$ systems. This fact and the chain of inequalities (5.3) provide an easy way to obtain information on the rate of convergence of $\{E[T_n]\}_0^\infty$ to $E[T_\infty]$. This question is addressed in some details in [1].

Finally, it is noteworthy that the corollaries 3.2 and 4.3 are really statements on the *stochastic ordering* between response times, as understood by Stoyan [8], Whitt [9] and many other authors. More precisely, let X and Y be any two R_+-valued RV's defined on Ω. The (distribution of the) RV X is said to be greater than the (distribution of the) RV Y in the *stochastic convex increasing order* iff

$$E[\phi(Y)] \leq E[\phi(X)] \qquad (5.4)$$

for any convex monotone non-increasing mapping $\phi: R_+ \to R$ for which (5.4) makes sense; this is denoted in short by $X \leq_{ci} Y$. With this notation, corollaries 3.2 and 4.3 can be restated simply as saying that

$$\underline{T}_n \leq_{ci} T_n \leq_{ci} \overline{T}_n . \qquad n=0,1,... (5.5)$$

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