Iterative methods for the numerical solution of mixed finite element approximations of the Stokes problem

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ITERATIVE METHODS FOR THE NUMERICAL SOLUTION OF MIXED FINITE ELEMENT APPROXIMATIONS OF THE STOKES PROBLEM

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SUMMARY : We describe three iterative methods for the numerical solution of mixed finite element approximations of the Stokes problem. All methods use more or less the multigrid idea. We give a convergence analysis for each method. Numerical experiments show the applicability of the methods and allow a comparison of their efficiency. Finally we give two examples of Navier-Stokes calculations using these methods as iterative Stokes Solvers.

RESUME : Nous présentons trois méthodes itératives pour la solution numérique de l'approximation par éléments finis mixtes du problème de Stokes. Les méthodes utilisent plus ou moins la méthode de multigrille Nous donnons une analyse de convergence pour chaque méthode. Quelques exemples numériques montrent l'applicabilité des méthodes. Ils permettent en outre de comparer leur efficacité. Enfin, nous résoudrons deux problèmes de Navier-Stokes stationaires utilisant ces méthodes comme solveur de Stokes.

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1. INTRODUCTION

This note gives a survey of some iterative methods for the numerical solution of mixed finite element approximations of the Stokes problem. Such algorithms are of great interest as subroutines in Navier-Stokes codes where in general many Stokes problems have to be solved.

We discuss three algorithms in detail. The first one is an improved version of the well known Uzawa algorithm. The original indefinite problem for the velocity and pressure is transformed into a positive definite problem for the pressure alone. To this problem we apply a conjugate gradient (CG) algorithm. Eache CG-step then requires the solution of several poisson equations. This is done only approximately using a multigrid (MG) algorithm. The resulting algorithm has a convergence rate which is independent of the meshsize. It is easily implemented if an efficient poisson solver is available.

The second algorithm is a preconditioned conjugate residual (CR) algorithm for the original velocity-pressure formulation. The preconditioning uses the idea of hierarchical basis functions for finite elements [17,18]. The preconditioning is very cheap, its cost corresponds to the calculation of three scalar products. The convergence rate is of the form $1-O(|\log h(h)|)$ where h is the meshsize.

The last algorithm is a direct application of the multigrid idea to the Stokes problem. Because of the indefiniteness of the problem and the poor regularity of the pressure, additional difficulties arise when compared with the existing MG - theory for elliptic problems. The MG algorithm has a convergence rate which is independent of the meshsize.

The second algorithm is most easily implemented. However it is restricted to linear finite elements and two dimensional problems. The other algorithms can be applied to a broad class of mixed finite elements in two and three dimensions (Cf. §2).

Finally, we give examples of Stokes and Navier-Stokes calculations which show the efficiency of the algorithms.

2. MIXED FINITE ELEMENT DISCRETIZATION OF THE STOKES PROBLEM

We consider the Stokes problem

$$(2.1)\qquad \begin{aligned} -\nu\Delta\underline{u} + \nabla p &= \underline{f} \quad \text{in } \Omega \\ \operatorname{div} \underline{u} &= 0 \quad \text{in } \Omega \\ \underline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here, $\underline{u} = (u_1, \ldots, u_d)$ denotes the velocity, p the pressure and $\nu > 0$ the viscosity of the fluid.

Let $H^k(\Omega)$, $k \in \mathbb{N}$, and $L^2(\Omega) = H^0(\Omega)$ be the usual Sobolev and Lebesgue spaces equipped with the norm

$$(2.2)\qquad \|v\|_k := \{ \int_\Omega \sum_{|\alpha| \leq k} |D^\alpha v|^2 \, dx \}^{1/2} .$$

Since no confusion can arise we use the same notation for the product norm of $H^k(\Omega)^d$. Finally, the scalar product of $L^2(\Omega)^d$ is denoted by $(.,.)$.

Put

$$(2.3)\qquad \begin{aligned} X &:= H^1_0(\Omega)^d = \{ \underline{u} \in H^1(\Omega)^d : \underline{u} = 0 \text{ on } \partial\Omega \} \\ M &:= L^2_0(\Omega) = \{ p \in L^2(\Omega) : \int_\Omega p dx = 0 \}. \end{aligned}$$

and introduce the bilinear forms

$$(2.4)\qquad \begin{aligned} a(\underline{u},\underline{v}) &:= \nu \int_\Omega \nabla\underline{u} \; \nabla\underline{v} \; dx \; , \\ b(\underline{u},\underline{p}) &:= - \int_\Omega p \operatorname{div} \underline{u} \; dx \; , \end{aligned}$$

$$\mathscr{L}([\underline{u},p],[\underline{v},q]) := a(\underline{u},\underline{v}) + b(\underline{v},p) + b(\underline{u},q)$$

on $X \times X$, $X \times M$ and $(X\times M) \times (X\times M)$ resp.

The weak form of (2.1) then is to find $[\underline{u},p] \in X \times M$ such that

$$(2.5)\qquad \mathscr{L}([\underline{u},p] , [\underline{v},q]) = (\underline{f},\underline{v}) \quad \forall [\underline{v},q] \in X \times M.$$

Finally, we introduce the norm

$$(2.6)\qquad |[\underline{u},p]|_1 := \{ \|\underline{u}\|_1^2 + \|p\|_0^2 \}^{1/2}$$

on $X \times M$.

Let $X_h \subset X$ and $M_h \subset M$ be two families of finite element spaces satisfying the following hypothesis :

(H_1) $\inf_{\underline{v}_h \in X_h} \| \underline{v} - \underline{v}_h \|_k \leq c h^{\ell-k} \| \underline{v} \|_\ell \quad \forall\, 0 \leq k \leq 1,\ k \leq \ell \leq 2,\ \underline{v} \in H^\ell(\Omega)^d$

(H_2) $\inf_{q_h \in X_h} \| q - q_h \|_0 \leq c h^\ell \| q \|_\ell \quad \forall\, 0 \leq \ell \leq 1,\ q \in H^\ell(\Omega)$

(H_3) $\| \underline{v}_h \|_1 \leq c h^{-1} \| \underline{v}_h \|_0 \quad \forall\, \underline{v}_h \in X_h$

(H_4) There is a constant $\beta > 0$ which does not depend on h such that

$$\inf_{q_h \in M_h \{0\}} \ \sup_{\underline{v}_h \in X_h \{0\}} \ \frac{b(\underline{v}_h, q_h)}{\| \underline{v}_h \|_1 \| q_h \|_0} \geq \beta$$

(H_5) $X_h \subset X_{2h}$, $M_h \subset M_{2h}$.

Here and in the sequel, c denotes a generic constant which does not depend on h. Conditions (H_1)-(H_3) are satisfied by standard finite element spaces (cf. [9]). (H_5) is a condition on the triangulation of Ω which is easily satifsied in general. The so called Babuška-Brezzi condition (H_4) is much harder to fulfil. Before giving some examples of finite element spaces satisfying (H_4) let us note, that (H_4) together with the ellipticity of a(.,.) is equivalent to

$$(2.7) \quad \inf_{[\underline{u}_h, p_h] \in X_h \times M_h} \ \sup_{[\underline{u}_h, q_h] \in X_h \times M_h} \ \frac{\mathscr{L}([\underline{u}_h, p_h], [\underline{v}_h, q_h])}{|[\underline{u}_h, p_h]|_1 \, |[\underline{v}_h, q_h]|_1} \geq \gamma > 0$$

with a constant $\gamma > 0$ independent of h.

<u>Example 2.1</u> : (a) Let $\Omega \subset \mathbb{R}^2$ be a plane polygonal domain and J_h be a family of regular triangulations of Ω (cf. [9]). Furthermore, assume that each $T \in J_h$ is obtained by dividing a suitable $T^1 \in J_h$ into four equal triangles the vertices of which are the vertices and midpoints of T^1. Let

$$S_h^r := \{ v \in C^0(\bar{\Omega}) : v|_T \text{ is a polynomial of degree} \leq r \text{ on each } T \in T_h \}.$$

Then the couples

$$(2.8) \quad X_h = [S^1_{h/2} \cap H^1_0(\Omega)]^2, \quad M_h = S^1_h \cap L^2_0(\Omega)$$

and

$$(2.9) \quad X_h = [S^2_h \cap H^1_0(\Omega)]^2, \quad M_h = S^1_h \cap L^2_0(\Omega)$$

satisfy conditions (H_1)-(H_5) (Cf. [4,14]).

(b) Let $\Omega \subset \mathbb{R}^3$ be polyhedral and $\mathcal{T}$ be a family of regular partitions of Ω into tetrahedrons. Let S_h^r be defined as above. Then the couple

(2.10) $X_h = [S_h^2 \cap H_0^1(\Omega)]^3$, $M_h = S_h^1 \cap L_0^2(\Omega)$

satisfies conditions (H_1) to (H_5) (cf.[4,14]).

Remark 2.2 : More examples of finite element spaces satisfying the above conditions can be found in [10]. They can also be generalized to non-polyhedral domains and more general boundary conditions. □

The mixed finite element approximation of the stokes problem then is to find $[\underline{u}_h, p_h] \in X_h \times M_h$ such that

$$\mathcal{L}([\underline{u}_h,p_h],[\underline{v}_h,q_h]) = (\underline{f},\underline{v}_h) \ \forall\ [\underline{v}_h,q_h]\ X_h \times M_h \text{ or equivalently}$$

$$(2.11) \quad \begin{aligned} a(\underline{u}_h,\underline{v}_h) + b(\underline{v}_h,p_h) &= (\underline{f},\underline{v}_h) \quad \forall\ \underline{v}_h \in X_h \\ b(\underline{u}_h,q_h) &= 0 \qquad \forall\ \underline{v}_h,q_h \in X_h \times M_h \end{aligned}$$

Inequality (2.7) ensures that (2.11) has a unique solution and yields together with (H 1, 2) optimal error estimates [10, 14].

3. A COMBINED CONJUGATE GRADIENT MULTIGRID ALGORITHM

Define the operators $A : X_h \to X_h$, $B : X_h \to M_h$ and $B^* : M_h \to X_h$ by

$$(3.1) \quad \begin{aligned} (A\,\underline{u},\underline{v}) &= a(\underline{u},\underline{v}) \quad \forall\ \underline{u},\underline{v} \in X_h , \\ (B\,\underline{u},p) &= b(\underline{u},p) \quad \forall\ \underline{u} \in X_h,\ p \in M_h , \\ (B^* p,\underline{u}) &= b(\underline{u},p) \quad \forall\ \underline{u} \in X_h;\ p \in M_h , \end{aligned}$$

Then problem (2.11) can be written as

$$(3.2) \quad \begin{aligned} A\,\underline{u}_h + B^*\,p_h &= f \\ B\,\underline{u}_h &= 0. \end{aligned}$$

The following proposition, which is proved in [15], is essential for the sequel.

Proposition 3.1 : The linear operator $L : M_h \to M_h$ defined by

(3.3) $L := B\,A^{-1}\,B^*$.

is symmetric, positive definite and continuous. Its condition number is bounded independently of h. There are two constans $0 < C_0 < C_1$ which do not depend on h such that

$$(3.4)\quad \begin{aligned} (Lp,p) &\geq C_0 \, \|p\|_0^2 \qquad &\forall\, p \in M_h \,, \\ \|Lp\|_0 &\leq C_1 \, \|p\|_0 \qquad &\forall\, p \in M_h \,. \end{aligned}$$

The couple $[u_h^*, p_h^*] \in X_h \times M_h$ is the solution of (3.2) if and only if

$$(3.5)\quad L p_h^* = g \quad , \quad u_h^* = A^{-1} \; (\underline{f} - B^* \; p_h^*)$$

where

$$(3.6)\quad g : = B \, A^{-1} \, \underline{f} \,. \qquad \square$$

Thus the indefinite problem (3.2) is reduced to the definite problem (3.5) which involves only the pressure. Since the condition number of L is bounded independently of h, a CG- algorithm can be applied efficiently to (3.5). Each evaluation of Lp then requieres the calculation of $A^{-1}\,w$ for a suitable $w \in X_h$ i.e. the solution of d discrete poisson equations with homogeneous boundary conditions. This is done only approximately using a multigrid algorithm.

Denote by $K_n : X_h \to X_h$ the linear operator which associates with $\underline{w} \in X_h$ the result of n iterations of the multigrid algorithm with starting value 0 applied to the poisson equation $A\underline{u} = \underline{w}$. Then we have for all $\underline{w} \in X_h$:

$$(3.7)\quad \| K_n \, \underline{w} - A^{-1} \underline{w} \|_1 \leq \varkappa^n \, \| A^{-1} \, \underline{w} \|_1$$

where $0 < \varkappa < 1$ is the convergence rate of the MG-algorithm. It is well known that $\varkappa$ is independent of h (Cf. [2,6,12]). The theoretical upper bounds $\varkappa \leq .205$ and $\varkappa \leq .291$ are derived in [5,13] for a special triangulation of plane, convex polygonal domains. In pratice, convergence rates $\varkappa \sim .1$ are often observed (cf. [1,7,11]).

The following proposition is proved in [15] .

<u>Proposition 3.2</u> : Assume that n is sufficiently large such that $\varkappa^n < \frac{C_0}{dC_1}$ where C_0, C_1 are the constants of (3.4). Then the operator $L_n : M_h \to M_h$ defined by

$$(3.8)\quad L_n : = B \, k_n \, B^*$$

is symmetric, positive definite and satisfies

$$\begin{aligned} &\|L_n p - L p\|_0 \leq d\, C_1\, \varkappa^n\, \|p\|_0 ,\\ (3.9)\quad &(L_n p, p) \geq (C_0 - d\, C_1\, \varkappa^n)\, \|p\|_0^2 ,\\ &\|L_n p\|_0 \leq (C_1 + d\, C_1\, \varkappa^n)\, \|p\|_0 \end{aligned}$$

for all $p \in M_h$. □

Remark 3.3 : Note that C_1/C_0 is an upper bound for the condition number of L. Estimates of C_0, C_1 and $\varkappa$ yield lower bounds for the number n of MG iterations which are necessary to satisfy $\varkappa^n < \frac{C_0}{dC}$. These estimates are far too pessimistic. It pratice $n = 2$ or $n = 3$ is sufficient. □

Proposition 3.2 shows that a CG-algorithm can be applied to the problem

$$(3.10)\quad L_n\, p = g_n\ , \text{ where } g_n := B\, K_n\, \underline{f}$$

which is an approximation of (3.5).
This gives rise to the following

Algorithm CGMGST : 0. Preprocessing phase :

Compute

$$g_n := B\, K_n\, \underline{f}\ .$$

1. Start : Given an initial guess $P^o \in M_h$ for the pressure solving (3.5).
Compute

$$q^o = L_n\, p^o$$

and put

$$r^o := q^o - g_n\ ,\ d^o := - r^o.$$

Set $i := 0$ and set ε to a small positive tolerance.

2. Iteration step : If $\|r^i\|_0 \leq \varepsilon$ goto step 3. Otherwise compute

$$q^{i+1} = L_n\, d^i$$

and put

$$\alpha^{i+1} := - \frac{(r^i, d^i)}{(d^i, q^{i+1})}\ ,$$

$$p^{i+1} := p^i + \alpha^{i+1} d^i,\quad r^{i+1} := r^i + \alpha^{i+1} q^{i+1},$$

$$\beta^{i+1} := \frac{(r^{i+1}, r^{i+1})}{(r^i, r^i)}\ ,$$

$$d^{i+1} := - r^{i+1} + \beta^{i+1} d^i.$$

Replace i by i+1 and return to the beginning of step 2.

3. Postprocessing phase : Compute

$$\underline{u}^i = K_n (\underline{f} - B^* p^i)$$

and take $[\underline{u}^i, p^i] \in X_h \times M_h$ as final guess for the solution of (3.5), (3.6). □

In [15] we proved the error estimate

$$(3.11) \quad \| \underline{u}^i - \underline{u}_h^* \|_1 + \| p^i - p_h^* \|_0 \leq O(\varepsilon + \varkappa^n)$$

for the last iterate of cgmgst. Here $[\underline{u}_h^*, p_h^*] \in X_h \times M_h$ is the exact solution of (2.11). The term $\varkappa^n$ is the relative accuracy with which g_n, the last residue and $\underline{u}^i$ are calculated. Hence steps 1 and 2 need only be performed with a moderate accuracy for the poisson problems, i.e. a small value of n. Once $\| r^i \|_0 \leq \varepsilon$ is obtained, one switches to a higher accuracy, i.e. greater value of n, in the solution of the poisson equations. This strategy improves the efficiency of cgmgst considerably. Moreover, inequ. (3.11) suggests that it is useless to choose ε smaller than the accuracy with which g_n is calculated.

4. A PRECONDITIONED CONJUGATE RESIDUAL ALGORITHM

In this section we consider a special discretization of the stokes problem. We assume that Ω is a plane polygonal domain and that we are given a sequence of triangulations J_{h_j}, j = 0, 1, ..., R with $h_j = \frac{1}{2} h_{j-1}$.

Here, $J_{h/2}$ is obtained from J_h by dividing each $T \in J_h$ into four equal triangles the vertices of which are the midpoints of sides and vertices of T.

The mixed finite element approximation of the stokes problem on level h_k, k = 1, ..., R, is given by the spaces of (2.8), i.e.

$$X_{h_k} = [S^1_{h_k} \cap H^1_0(\Omega)]^2 \quad , \quad M_{h_k} := S^1_{h_{k-1}} \cap L^2_0(\Omega).$$

Note that the velocity is approximated on a finer triangulation than the pressure. If no confusion can arise, we replace subscripts h_j by j. Actually, we want to solve the discrete stokes problem on level h_R. The coarser triangulations are only auxiliary ones.

Denote by Ω_j, j = 0, ..., R, the set of vertices corresponding to the triangulation J_j and by $I_j : C(\bar{\Omega}) \to S_j$ the standard pointwise interpolation operator. Define the mesh dependent scalar product

$$(4.1) \quad ((\varphi,\psi))_{1,R} := (\nabla I_0\varphi, \nabla I_0\psi) + \sum_{j=1}^{R} \sum_{j \in \Omega_j \setminus \Omega_{j-1}} [I_j\varphi - I_{j-1}\varphi][I_j\psi - I_{j-1}\psi](x)$$

On $S_R^1 \cap H_0^1(\Omega)$. The corresponding norm is denoted by $|||\cdot|||_{1,R}$. We use the same notations for the corresponding scalar product and norm or X_R. The following Lemma which is proved in [17, 18] shows that $|||\cdot|||_{1,R}$ is a good approximation for the H^1-norm.

<u>Lemma 4.1</u> : There are two constants $0 < C_0 < C_1$ which do not depend on h_0 and h_R such that

(4.2) $\quad C_0(R+1)^{-2} \;\|\nabla\varphi\|_0 \leq |||\varphi|||_{1,R} \leq C_1 \;|||\nabla\varphi|||_0$

for all $\varphi \in S_R^1 \cap H_0^1(\Omega)$. □

The continuity of $\mathcal{L}$ and equations (2.7), (4.2) immediately imply

<u>Corollary 4.2</u> : There are two constants $0 < \gamma < C_L$ which do not depend on h_o, h_R such that

(4.3)
$$|\mathcal{L}([\underline{u},p],[\underline{v},q])| \leq C_L \;\{|||\underline{u}|||_{1,R}^2 + \|p\|_0^2\}^{1/2}\{|||\underline{v}|||_{1,R}^2 + \|q\|_0^2\}^{1/2} \quad \forall[\underline{u},p],[\underline{v},q]\in X_R \times M_R$$

and

(4.4)
$$\operatorname*{Inf}_{[\underline{u},p]\, X_R\times M_R} \;\operatorname*{Sup}_{[\underline{v},q]\, X_R\times M_R} \frac{\mathcal{L}([\underline{u},p],[\underline{v},q])}{\{|||\underline{u}|||_{1,R}^2+\|p\|_0^2\}^{1/2}\;\{|||\underline{v}|||_{1,R}^2+\|q\|_0^2\}^{1/2}} \geq \gamma\;(R+1)^{-2}.$$
□

The crucial point is to interprete Corollary 4.2 as a preconditioning result for the discrete stokes problem. To this end we have to introduce the notion of hierarchical basis functions (cf.[17, 18]). Denote by ψ_i^j, $0 \leq j \leq R$, $1 \leq i \leq N_j := \dim\,[S_j^1 \cap H_0^1(\Omega)]$ the hierarchical basis functions.

The ψ_i^j are defined recursively : If $j = 0$, they are the usual nodal basis of $S_0^1 \cap H_0^1(\Omega)$, if $j > 0$, they consist of the ψ_i^{j-1} plus the nodal basis functions corresponding to the interior nodes of $\Omega_j | \Omega_{j-1}$. Figure 1 shows the hierarchical basis for $j = 0, 1, 2$ in one space dimension.

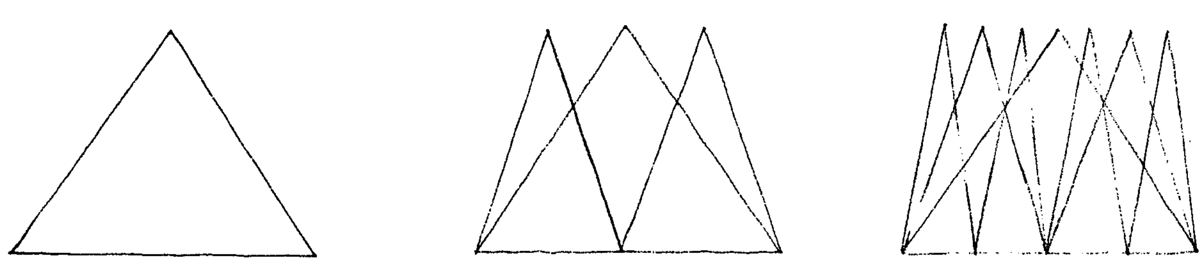


Figure 1 : Hierarchical basis for j = 0, 1, 2 in 1 D.

In the sequel we use the convention that for any basis of $X_R \times M_R$ we first number the basis functions for the x-component of the velocity, than those for its y-component and finally those for the pressure. Denote by $\tilde{S}$ the transformation matrix from the hierarchical to the nodal basis of $S_R^I \cap H_0^1(\Omega)$ and put

$$S: = \begin{pmatrix} \tilde{S} & 0 & 0 \\ 0 & \tilde{S} & 0 \\ 0 & 0 & I \end{pmatrix}$$

Let B be the stifness matrix of the scalar product $((.,.))_{1,R} + (.,.)$ on $X_R \times M_R$ corresponding to the hierarchical/nodal basis on X_R and M_R resp. Note that

$$B = \begin{pmatrix} \tilde{B} & 0 & 0 \\ 0 & \tilde{B} & 0 \\ 0 & 0 & I \end{pmatrix}$$

Where $\tilde{B}$ is a diagonal matrix except a small diagonal block corresponding to $(\nabla I_0\varphi, \nabla I_0\psi)$. Finally, denote by

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & 0 \end{pmatrix}$$

The stifness matrix of $\mathscr{L}$ with respect to the nodal basis of $X_R \times M_R$. Since by lemma 4.1 B is symmetric and positive definite, it can be decomposed as $B = LL^T$ with a lower triangular matrix L. Put

(4.5) $Q : = S^{-T}L.$

Then equations (4.3), (4.4) are equivalent to

$$\gamma(R+1)^{-2} \leq \inf_{x\in\mathbb{R}^k} \sup_{y\in\mathbb{R}^k} \frac{x^T S^T A S y}{\{x^T LL^T x\}^{1/2} \{y^T LL^T y\}^{1/2}}$$

$$\leq \sup_{x\in\mathbb{R}^k} \sup_{y\in\mathbb{R}^k} \frac{x^T S^T A S y}{\{x^T LL^T x\}^{1/2} \{y^T LL^T y\}^{1/2}}$$

$$\leq c_L$$

where $K : = \dim [X_R \times M_R]$. This proves

Proposition 4.3 : The condition number of $Q^{-1} A Q^{-T}$ is bounded by $\frac{C_L}{\gamma}(R+1)^2 \sim |\log h|^2$. A conjugate residual algorithm applied to the discrete Stokes problem with $Q Q^T$ as preconditionning matrix has the quasi optimal convergence rate $1 - O(|\log h|)$. □

Remark 4.4 : The solution of $Q Q^T Y = Z$ is done in three steps :
(1) transform the velocity components of z into their hierarchical basis representation, giving the vector $y^{(1)}$. Denote the velocity components belonging to the coarsest level by w.
(2) Solve the discrete Poisson problem $\Delta X = W$ on the coarsest level. Replace the components of $y^{(1)}$ corresponding to W by X, giving $y^{(2)}$.
(3) Transform the velocity components of $y^{(2)}$ into their nodal basis representation. This gives the solution vector y. □

Note that the pressure components remain unaffected.

Recalling that the discrete Stokes problem on $X_R \times M_R$ has the form

(4.6) $A\, x = b$,

where x contains the velocity and pressure components, the preconditioned conjugate residual algorithm is as follows :

Algorithm pcrst : 1. Start : Given an initial guess X^0 for the solution of (4.6) compute

$$r^0 := b - A\, x^0 \quad , \quad z^0 := Q^{-T} Q^{-1} r^0 ,$$
$$q^0 := A\, z^0 \quad , \quad W^0 := Q^{-T} Q^{-1} q^0 ,$$

Set $i := 0$, and set ε to a small positive tolerance.

2. Iteration step : If $\|r^i\|_0 \leq \varepsilon$, take x^i as final approximation for the solution of (4.6) and terminate the algorithm. Otherwise compute

$$\alpha^i = \frac{(r^i, q^i)}{(q^i, w^i)}$$

and put

$$X^{i+1} := X^i + \alpha^i z^i \quad , \quad r^{i+1} := r^i - \alpha^i q^i ,$$

$$\gamma^i := \frac{(w^i, A\, w^i)}{(q^i, w^i)} ,$$

$$\delta^i := \begin{cases} 0 & , \text{ if } i = 0, \\ \dfrac{(q^i, w^i)}{(q^{i-1}, w^{i-1})} & , \text{ if } i > 0, \end{cases}$$

$$z^{i+1} := w^i - \gamma^i z^i - \delta^i z^{i-1},$$
$$q^{i+1} := A z^{i+1}, \quad w^{i+1} = Q^{-T} Q^{-1} q^{i+1}.$$

Replace i by i+1 and return to the beginning of step 2. □

5. A MULTIGRID ALGORITHM

In this section we consider any mixed finite element approximation of the Stokes problem satisfying conditions (H1) - (H5) of §2. However, we assume that we have a sequence of finite dimensional spaces X_{h_j}, M_{h_j} $j = 0,1,\dots,R$, with $h_j = \frac{1}{2} h_{j-1}$. As in the last section, we replace subscripts h_j by j. On level j we have to solve the problem

$$(5.1) \quad \mathcal{L}([\underline{u}_j, p_j], [\underline{v}_j, q_j]) = G_j([\underline{v}_j, q_j]) \quad \forall [\underline{v}_j, q_j] \in X_j \times M_j$$

with linear forms G_j which will be defined recursively, especially, we have on the first level

$$(5.2) \quad G_R([\underline{v}, q]) := (\underline{f}, \underline{v}) \ \forall [\underline{v}, q] \in X_R \times M_R$$

The multigrid algorithm then is as follows :

Algorithm mgst : (one iteration loop on level j with ν smoothing steps)

1. Smoothing : Given an initial approximation $[\underline{u}^0, p^0] \in X_j \times M_j$ for the solution of problem (5.1) on level j. For $i = 0, \dots, \nu - 1$ calculate $[\underline{u}^{i+1/2}, p^{i+1/2}]$ and $[\underline{u}^{i+1}, p^{i+1}] \in X_j \times M_j$ solution of

$$(\underline{u}^{i+1/2}, \underline{v})_0 + h^2 (p^{i+1/2}, q)_0$$
$$= w^2 \{G_j([\underline{v}, q]) - \mathcal{L}([\underline{u}^i, p^i], [\underline{v}, q])\} \quad \forall [\underline{v}, q] \in X_j \times M_j$$

and

$$(\underline{u}^{i+1} - \underline{u}^i, \underline{v})_0 + h^2 (p^{i+1} - p^i, q)_0$$
$$= \mathcal{L}([\underline{u}^{i+1/2}, p^{i+1/2}], [\underline{v}, q]) \quad \forall [\underline{v}, q] \in X_j \times M_j.$$

2. Correction : Put

$$(5.3) \quad G_{j-1}([\underline{v}, q]) := G_j([\underline{v}, q]) - \mathcal{L}[\underline{u}^\nu, p^\nu], [\underline{v}, q]) \quad \forall [\underline{v}, q] \in X_{j-1} \times M_{j-1}$$

If $j = 1$, compute the exact solution $[\underline{u}^*_{j-1}, p^*_{j-1}] \in X_{j-1} \times M_{j-1}$ of problem (5.1) on level $j-1$ and put $\tilde{\underline{u}}_{j-1} = \underline{u}^*_{j-1}$, $\tilde{p}_{j-1} = p^*_{j-1}$.

Otherwise calculate an approximation $[\tilde{u}_{j-1},\tilde{p}_{j-1}] \in X_{j-1} \times M_{j-1}$ to the exact solution of problem (5.1) on level j-1 by applying $\mu \geq 2$ iterations of the algorithm on level j-1 with starting value 0. Put

$$\underline{u}^{\nu+1} := \underline{u}^{\nu} + \underline{\tilde{u}}_{j-1}, \quad p^{\nu+1} := p^{\nu} + \tilde{p}_{j-1}. \qquad \square$$

Remark 5.1 : (1) writing the problem on level j in matrix vector notation as $A\,x = b$, the smoothing corresponds to ν Jacobi iterations applied to the squared problem $A^2\,x = Ab$. This takes account of the indefiniteness of the problem.

(2) Note the different scaling of the velocity and pressure components. This is due to the different regularity of these components. In practice, one replaces the L^2-scalar product by the equivalent mesh dependent scalar product

$$((\varphi,\psi))_{o,j} := h_j^2 \sum_{x \in \Omega_j} \varphi(x) \quad \psi(x)$$

which gives a diagonal mass matrix. $\square$

A detailed convergence analysis of the multigrid algorithm is given in [16]. The main difficulties are the indefiniteness of the Stokes problem and the different regularity of the velocity and pressure. The analysis consists in establishing a smoothing and an approximation property which are measured in scales of mesh dependent norms. The smoothing property means that high frequency error components are rapidly damped out by the smoothing part of mgst. The approximation property says that the slowly convergent low frequency error components are well reduced by the coarse grid correction. Combining these two properties one obtains the convergence rate $\frac{C}{\sqrt{\nu}}$ with a constant C independent of h. (cf. [16]). Using a conjugate residual algorithm as smoother the convergence rate is improved to $\frac{C}{\nu}$.

Remark 5.2 : Combining the analysis of [15] and [16] it is easy to see that the algorithm cgmgst described in §3 can also be used as smoothing operator in the multigrid algorithm. This will be analysed in more detail in a following paper. $\square$

6. NUMERICAL RESULTS

In this section we present some numerical results obtained for stokes and Navier-Stokes problem using the algorithms of §§ 3-5 as Stokes solvers. We consider the three regions described in Figure 1 : the unit square Ω_C, an L shaped domain Ω_L and a unit square with a slit Ω_S. This allows us to test the influence of singularities caused by reetrant corners.

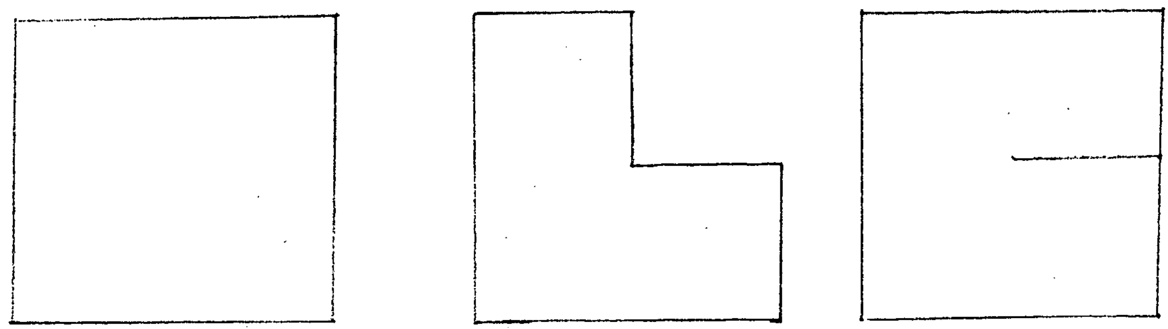


Figure 1.

We use rightangled isosceles triangles with short sides of length h for the triangulation and continuous piecewise linear finite elements on J_h and J_{2h}, resp, to approximate the velocity and the pressure, resp. The calculations are done with the meshsizes h = 1/4, 1/8, 1/16, 1/32 for the velocity. The mesh corresponding to h = 1/4 is used as coarsest grid for the preconditioning procedure of algorithm PCRST. W. Hackbusch Kindly provided us his program HELMH [11] for the multigrid part of algorithm CGMGST. For more general geometries and triangulations one could also use a simplified version of PLTMG [1]. The multigrid algorithm MGST uses five (ν=5) Jacobi iterations for the squared system per w.cycle (μ=2). The numerical tests for this algorithm are performed by KH. Untiet at the university of Bochum on a CDC 175. The other tests were done on the Bull CII-HB-DPS 68/ Multics of INRIA.

We consider five examples. In the first two examples the exact solution is given by

$$\text{Ex 1 :}\quad \underline{u}(x,y) := \begin{pmatrix} 200\, x^2(1-x)^2\, y\,(1-2y)(1-y) \\ -200\, x\,(1-x)(1-x)\, y^2\,(1-y)^2 \end{pmatrix}$$

$$p(x,y) := 100\, x\,(1-x)\, y\,(1-y) - \frac{25}{9}$$

$$\text{Ex 2 :}\quad \underline{u}(x,y) := \begin{pmatrix} 2\pi \operatorname{Sin}^2(2\pi x)\operatorname{Sin}(4\pi Y) \\ -2\pi \operatorname{Sin}(4\pi x)\operatorname{Sin}^2(2\pi Y) \end{pmatrix}$$

$$p(x,y) := 4\pi^2 \operatorname{Sin}(4\pi x)\operatorname{Sin}(4\pi Y).$$

For the other three examples the exact solution is unknown. The right hand side is given by

$$\text{Ex 3 :}\quad \underline{f}(x,y) := \underline{e} := \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{Ex 4 :}\quad \underline{f}(x,y) := 100\, x\,(1-x)\, y\,(1-y)\, \underline{e}$$

$$\text{Ex 5 :}\quad \underline{f}(x,y) := 100 \exp[-100(x^2+y^2)]\underline{e}.$$

Due to the boundary conditions, example 1 is tested only one the unit square Ω_C.

In example (1) and (2) the iterations terminate if the L^2-norm of the difference between calculated and exact pressure is lees then 0.1 h. The factor h takes account of the error estimates for the finite element approximation (cf.[14]). In example (3)-(5) the iterations terminate if the L^2-norm of the residual is lees than 10^{-4}. The cg-algorithms are restarted every 10 iterations. We always perform 3 multigrid iterations per Poisson equation in algorithm CGMGST. The result of the multigrid iteration is stored and taken as initial value for the next call of the multigrid algorithm. In all algorithms, zero is used as starting value on the coarsest grid. On finer grids, we take the interpolate of the result of the calculation on the next coarser grid as starting value.

Tables 1-5 show the convergence rates for the three algorithms and the five examples. For each example the numbers indicate from left to right the convergence rate of cgmgst, pcrst and mgst and from top to bottom the rates on the meshes corresponding to h = 1/8, 1/16, 1/32. A hyphen indicates that the starting value satisfies the stopping criterion

h^{-1}	Ω_C		
8	.674	.935	-
16	.509	.834	.32
32	.731	.872	.38

Table 1 : exact solution polynomial

h^{-1}	Ω_C			Ω_L		Ω_S	
8	-	-	-	-	-	-	-
16	.818	.935	.47	.766	.944	.760	.934
32	.905	.946	.50	.850	.942	.879	.942

Table 2 : exact solution sin function

h^{-1}	Ω_C		Ω_L		Ω_S	
8	.507	.972	.635	.973	.583	.961
16	-	.948	.781	.942	.716	.940
32	.709	.941	.816	.941	.840	.934

Table 3 : right hand side constant

h^{-1}	Ω_C		Ω_L		Ω_S	
8	.800	.965	.801	.972	.636	.966
16	.838	.944	.827	.965	.802	.952
32	.842	.946	.890	.948	.800	.949

Table 4 : Right hand side polynomial

h^{-1}	Ω_C		Ω_L		Ω_S	
8	.841	.961	.763	.968	.815	.967
16	.921	.960	.886	.965	.892	.966
32	.920	.966	.833	.963	.859	.961

Table 5 : right hand side exponential

The results show that the convergence rates of cgmgst and pcrst are rather independent of the geometry. This is consistent with the convergence analysis for these algorithms which do not need any regularity results. In contrast, the convergence analysis for the multigrid algorithm requires H^2-regularity of the Stokes problem which only holds for convex polygonal domains. The above results show that we can expect convergence rates of about .8 - .9 for cgmgst and .94 - .97 for pcrst. In order to compare them we have to take into account the different complexity of the algorithms. In table 6 we give the asymptotic number of additions and multiplications per velocity grid point and per iteration

	cgmgst	pcrst	mgst
cost per iteration	412 A 138M	67 A 39M	840 A 252M
overhead	6.5 Iter	1.5 Iter	

Table 6 : Operation counts

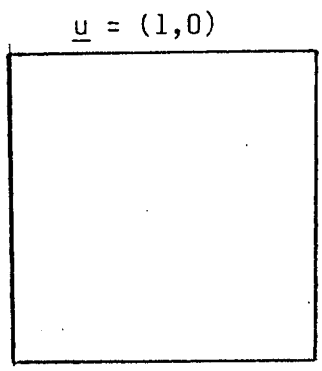
Hence, we have

1 Iteration mgst ≃ 2 iterations cgmgst
≃ 10 iterations pcrst

The overhead of cgmgst is due to the pre-and postprocessing phases which have to be done with a higher accuracy. Comparing operation counts and convergence rates we see that cgmgst and pcrst roughly have the same efficiency and that for some examples pcrst is 20% more efficient than cgmgst. Concerning the work to implement the algorithms we have pcrst > cgmgst > mgst.

However, pcrst is restricted to a special discretization and plane problems, whereas cgmgst and mgst are applicable to other discretization and three dimensional problems.

Finally, we want to present two examples for the application of our algorithms to the stationary Navier-Stokes equations. We consider the well known problems of driven cavity flow and flow across a step. For the second example the total inflow is normalized to one.



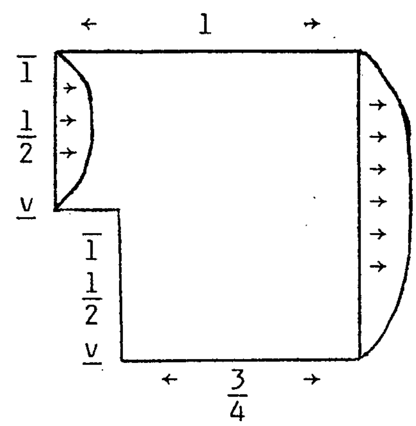


Figure 2 : driven cavity flow around a step

The nonlinearity introduces the additional term

$$\frac{Re}{2} \int_R \{[(\underline{u}.\nabla)\underline{u}]\ \underline{v} - [(\underline{u}.\nabla)v]\ \underline{u}\}\ dx$$

on the left hand side of (2.11), where Re denotes the Reynolds number. We apply the standard fixed point iteration to the nonlinear problem. The iteration terminates if the L^2-norm of the residual (of the nonlinear problem) is less than 10^{-2}. The Stokes problems occuring in each nonlinear iteration are solved approximately using algorithms cgmgst or pcrst. The characteristics of these algorithms are as described above. For the two algorithms we list in table 7 the number of nonlinear iterations (=Stokes problems to solve), number of linear iterations and mean convergence rate per stokes problem.

problem	1/h	cgmgst it nonl	cgmgst it lin	cgmgst $\varkappa$	pcrst it nonl	pcrst it lin	pcrst $\varkappa$
driven	8	2	20	.925	2	70	.943
cavity	16	1	10	.867	1	40	.935
Re=10	32	1	10	.900	1	20	.917
driven	8	7	55	.790	6	260	.940
cavity	16	3	30	.860	2	70	.932
Re=50	32	1	10	.881	1	30	.928
step	8	3	30	.760	3	130	.970
Re=10	16	2	20	.830	2	60	.927
	32	1	10	.842	1	40	.945

Table 7 :

Due to our extremely simple nonlinear iteration we could not achieve convergence for the flow accross the step at Re=50. The results show that pcrst needs roughly four times as many linear iterations as cgmgst to solve one example on h = 1/32 starting from h = 1/8. Recalling the number of operation counts this gives an advantage of roughly 25% for pcrst. Concerning the computing time pcrst was nearly twice as fast as cgmgst. This is due to the rather big overhead of cgmgst. Algorithm pcrst took between 1 and 3 minutes on the Bull C II - HB - DPS 68 Multics to solve one nonlinear problem on h = 1/32 starting from h = 1/8.

7. CONCLUSIONS

We presented three algorithms for the solution of mixed finite element approximations of the Stokes problem. All algorithms use more or less the multigrid idea.

The most easy one to implement is pcrst. It is a preconditioned conjugate residual algorithm. This algorithm is restricted to the discretization by linear finite elements and two dimensional problems. The convergence rates are worse than for the other two algorithms. But due to the low cost it is competitive with the other algorithms.

The algorithm cgmgst combines multigrid and conjugate gradient ideas and is easy to implement if a multigrid code for the poisson equation is available. It is applicable to general mixed finite element discretizations and to three dimensional problems too. The convergence rate is independent of the meshsize and turns out to be about .8 - .9.

The most difficult algorithm to implement is mgst which applies the multigrid idea directly to the Stokes problem. Practical experiences for this algorithm are actually limited. However, the first results are promising and it seems that it will be by far the fastest algorithm. This is also supported by results already obtained for multigrid methods applied to finite difference discretizations of the Stokes problem [8] .

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