Optimal stopping times for detecting changes in distributions

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Optimal Stopping Times for Detecting Changes in Distributions.

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ABSTRACT

It is shown that Page's stopping time is optimum, for the detection of changes in distributions, in a well defined sense. This work is a generalization of an existing result where it was shown that Page's stopping time is optimum asymptotically.

RESUME

On montre que le temps d' arret de Page est optimum, dans un sens bien defini, pour la detection de changements dans les distributions. Ce travail generalise un resultat existant montrant l' optimalite asymptotique du temps d'arret de Page.

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1. Introduction. Let us assume that $X_1, X_2, \ldots$ are independent and identically distributed random variables that are observed sequentially. Let also $X_1, \ldots, X_{m-1}$ have distribution function F_0 and $X_m, X_{m+1}, \ldots$ distribution function $F_1 \neq F_0$. The two distributions are known but the time of change m is assumed unknown. Let P_m denote the true distribution of $X_1, X_2, \ldots$ when the change occurs at m and E_m the expectation under this density. With $\mathcal{F}^x = \{\mathcal{F}_n^x, n \geq 1\}$ we denote the natural filtration defined by the sequence $X_1, X_2, \ldots$ and with $\mathcal{G} = \{\mathcal{G}_n, n \geq 0\}$ another filtration such that $\mathcal{F}_n^x \subset \mathcal{G}_n$. The reason for introducing this new filtration is to allow randomized stopping times. We extend the measures P_m defined on $\mathcal{F}^x$ to P_m' defined on $\mathcal{G}$ as follows: if $A \in \mathcal{F}^x$ then $P_m'\{A / \mathcal{G}_n\} = P_m\{A / \mathcal{F}_n^x\}$. If $A \notin \mathcal{F}^x$ but $A \in \mathcal{G}$ then $P_m'\{A / \mathcal{G}_n\} = P_0\{A / \mathcal{G}_n\}$ for every m. Finally we will assume that given any two stopping times (s.t.) N_1 and N_2 and any p, $0 < p < 1$ there exists a sequence of events $A_n \in \mathcal{G}_n, n \geq 0$ independent of $\sigma(N_1, N_2, \mathcal{F}^x)$ such that $P_m'(A_m) = p$ for any m and n. If the probability space is not reach enough to support a randomization of this form we can embed it in a natural way in an appropriate larger space. From now on for simplicity with P_m we will denote the measures P_m' defined on .

We define optimality of a s.t. in the sense of Lorden [4]. That is, if N a s.t. define

$$D_m(N) = ess\ sup\ E_m\{[N - m + 1]^+ / \mathcal{G}_{m-1}\} \quad m \geq 1 \tag{1}$$

$$D(N) = \sup_{m \geq 1} D_m(N) \tag{2}$$

Thus we consider the conditional expectation of the delay over those events before the change occurs, that favor the least the detection of the change. We would like to minimize $D(N)$ over all s.t. that satisfy

$$E_0\{N\} \geq \gamma > 0 \tag{3}$$

This is a min-max approach, because we try to minimize the worst possible performance. Our goal in the next section will be to prove that Page's s.t. is optimum in the above sense. Let us first define this s.t. For simplicity we will assume that F_0 and F_1 are mutually absolutely continues. Let $l(x)$ denote the Radon-Nikodym derivative of F_1 with respect to F_0, we define the following sequence of random variables

$$\begin{aligned} T_0 &= 1 \\ T_n &= \max\{T_{n-1}l(X_n), 1\} \quad n \geq 1 \end{aligned} \tag{4}$$

Consider now the following two s.t. N_P^+, N_P^- defined as

$$N_P^+ \ (N_P^-) = \begin{cases} \inf\limits_{n\geq 1} \{ n : T_{n-1}l(X_n) > (\geq) \ \mu \} \\ \\ \infty \quad \text{otherwise} \end{cases} \tag{5}$$

where μ is a real. Let N_P denote any randomization of the two times, that is at every instant n with probability p we have $N_P = N_P^+$ and with probability $1-p$ we have $N_P = N_P^-$. Page's s.t. is defined a little differently here than it is in the literature. Disregarding the randomization, N_P is defined as the first n for which $T_n = \max\{T_{n-1}l(X_n), 1\}$ exceeds μ. Notice that the two definitions are equivalent when $\mu \geq 1$ but there is a big difference when $\mu < 1$. Clearly with the old definition we stop at $n = 1$ but with the definition in (5) this is not the case. As we will see in the next section there exists a nontrivial range of values of γ for which $\mu \in (0,1]$. With the following lemma we give some properties of the sequence T_n that will be used later.

Lemma 1. For any $n \geq m \geq 1$ we have that T_n is an a.s. nondecreasing convex function of T_m, also T_n can be written as

$$T_n = \sum_{j=1}^{n+1} \left[1 - T_{j-2}l(X_{j-1})\right]^+ \prod_{k=j}^{n} l(X_k) \tag{6}$$

where we define $T_{-1} = 0$, $l(X_0) = 1$ and $\prod_{k+1}^{k} = 1$.

Proof. The property that T_n is a nondecreasing convex function of T_m can be proved by induction and using the definition in (4). To prove (6), we can see from (4) that

$$T_k = T_{k-1}l(X_k) + \left[1 - T_{k-1}l(X_k)\right]^+ \tag{7}$$

If we use (7) and induction we can easily show (6).

A very important consequence of the monotonicity of T_n with respect to T_m is that on the event $N_P \geq m$ the s.t. N_P is decreasing with T_{m-1}, thus the essential supremum in (1) is achieved for $T_{m-1} = 1$. This means that by restarting the procedure at m gives the worst average delay. Since we have stationarity this means that all the $D_m(N_P)$ are equal. This equality is a very common characteristic of min-max procedures when they try to balance different performance measures. As we will see next, it plays an important role for the proof of optimality.

2. Optimal Stopping Time. Notice first that for $\gamma > 0$ we have $D(N) \geq 1$. This is true because with $E_0\{N\} \geq \gamma > 0$ it is not possible to stop at $n = 0$ a.s. and thus we will take at least one sample. With this remark we can see that when $1 \geq \gamma > 0$, the optimum s.t. (say N_o) is: {stop at $n = 0$ with probability $1 - \gamma$ otherwise stop at $n = 1$}. This yields $D(N_o) = 1$ and $E_0\{N_o\} = \gamma$. We now consider the case $1 < \gamma < \infty$. With the next lemma we will show that in order to find the optimum s.t. it is enough to limit ourselves to a smaller class of s.t.

Lemma 2. In order to minimize $D(N)$ over the s.t. that satisfy (3) it is enough to consider only the s.t. that satisfy (3) with equality.

Proof. The proof goes as follows, if $E_0\{N\} = \infty$ we can always find a large enough integer K such that if we define $N' = \min\{N, K\}$ to have $E_0\{N'\} \geq \gamma$. Since $N' \leq N$ a.s. we also have $D(N') \leq D(N)$. Thus it is enough to consider s.t. with finite $E_0\{N\}$. If $\gamma < E_0\{N\} < \infty$ we can define a new s.t. N' by defining a randomization at $n = 0$ as follows, N' is equal to N with probability $\gamma / E_0\{N\}$ and otherwise equal to zero. Again $N' \leq N$ a.s. thus $D(N') \leq D(N)$ but $E_0\{N'\} = \gamma$. And this concludes the proof.

In the following lemma we introduce a lower bound for $D(N)$ which we will use as our performance measure instead of $D(N)$.

Lemma 3. For any s.t. N satisfying $0 < E_0\{N\} < \infty$ we have that

$$D(N) \geq \frac{E_0\{ \sum_{j=0}^{N-1} T_j \}}{E_0\{ \sum_{j=0}^{N-1} [1 - T_{j-1} l(X_j)]^+ \}} = \bar{D}(N) \tag{8}$$

where we define $\sum_{j=k}^{k-1} = 0$. We have equality in (8) when $N = N_P$.

Proof. Let $I(A)$ denote the index function of the event A, then we define

$$B_m(N) = E_m\{ [N - m + 1]^+ / \mathcal{Y}_{m-1} \} = E_0\{ [\sum_{k=m}^{N} \prod_{j=m}^{k-1} l(X_j)] I(N \geq m) / \mathcal{Y}_{m-1} \} \tag{9}$$

Notice that in (1) $D_m(N)$ was defined as the essential supremum of $B_m(N)$. Since $D(N) \geq D_m(N)$ for every $m \geq 1$ we have

$$E_0\{ [1 - T_{m-2} l(X_{m-1})]^+ I(N \geq m) \} D(N) \geq$$

$$E_0\{ I(N \geq m) \sum_{k=m}^{N} [1 - T_{k-2} l(X_{k-1})]^+ \prod_{j=m}^{k-1} l(X_j) \} \tag{10}$$

When $N = N_P$ we have equality in (10). This is true because $D(N_P) = D_m(N_P)$ for every m and because as we said in the introduction the essential supremum of $B_m(N_P)$ is achieved on the event $\{N_P \geq m\} \cap \{T_{m-1} = 1\}$. We can see that $[1 - T_{m-2}l(X_{m-1})]^+ I(N \geq m)$ is nonzero on this event. Summing now (10) for all $m \geq 1$ after interchanging summations and expectations and using (6), the right hand side gives

$$\sum_{m=1}^{\infty} E_0\{ I(N \geq m) \sum_{k=m}^{N} [1 - T_{m-2}l(X_{m-1})]^+ \prod_{j=m}^{k-1} l(X_j) \} =$$

$$E_0\{ \sum_{m=1}^{N} \sum_{k=m}^{N} [1 - T_{m-2}l(X_{m-1})]^+ \prod_{j=m}^{k-1} l(X_j) \} =$$

$$E_0\{ \sum_{k=1}^{N} [\sum_{m=1}^{k} [1 - T_{m-2}l(X_{m-1})]^+ \prod_{j=m}^{k-1} l(X_j)] \} =$$

$$E_0\{ \sum_{k=1}^{N} T_{k-1} \} = E_0\{ \sum_{k=0}^{N-1} T_k \} \tag{11}$$

For the left hand side we have that

$$\sum_{m=1}^{\infty} E_0\{ I(N \geq m)[1 - T_{m-2}l(X_{m-1})]^+ \} = E_0\{ \sum_{m=0}^{N-1} [1 - T_{m-1}l(X_m)]^+ \} \tag{12}$$

The quantity in (12) is less than $E_0\{N\}$ thus finite. For $N \geq 1$ we also have that

$$\sum_{k=0}^{N-1} [1 - T_{k-1}l(X_k)]^+ \geq 1 \tag{13}$$

thus the quantity in (12) is greater than the probability $P_0\{N \geq 1\}$ which is nonzero since by assumption we have $E_0\{N\} > 0$. And thus we have shown (8). We have equality for $N = N_P$ because, as we said before all the $D_m(N_P)$ are equal to $D(N_P)$.

Let us denote by N_P Page's s.t. for which μ and p have been defined in such a way that (3) is satisfied with equality. In order now to show that this s.t. is the optimum it is enough to show that among all s.t. that satisfy $E_0\{N\} = \gamma$ it is the one that minimizes $\bar{D}(N)$. This is shown in the following theorem.

Theorem. Let $\infty > \gamma > 1$. Among all s.t. that satisfy $E_0\{N\} = \gamma$ Page's s.t. N_P minimizes $\bar{D}(N)$ by simultaneously minimizing its numerator and maximizing its denominator.

Proof. Let $J(N)$ be the denominator of $\bar{D}(N)$. We would like to find its supremum for the class $E_0\{N\} = \gamma$. We can see here why it was necessary to limit ourselves to this class, if we had instead the class $E_0\{N\} \geq \gamma$, this gives as optimum $N = \infty$ which is unwanted. In order now to apply existing results, we consider optimization over the class $E_0\{N\} \leq \gamma$. Notice that since $[1 - x]^+ \leq 1$, for $x \geq 0$, we have that $0 \leq J(N) \leq \gamma$. Thus the constrained optimum $\bar{J} = \sup J(N)$ exists. From [3] we then have that there exist a Lagrange multiplier $\lambda \geq 0$ and a s.t. N_o that satisfy: $E_0\{N_o\} \leq \gamma$, $\lambda[\, E_0\{N_o\} - \gamma\,] = 0$ and N_o is optimum for the unconstrained problem

$$\bar{J} = \sup_N \{ \sum_{j=0}^{N-1} [1 - T_{j-1} l(X_j)]^+ - \lambda \} \tag{14}$$

and also $\bar{J} = J + \lambda\gamma$. If $\lambda = 0$ then the optimum s.t. for (14) is $N_o = \infty$ which does not satisfy $E_0\{N_o\} \leq \gamma$ thus we have $\lambda > 0$, but then we will have that $E_0\{N_o\} = \gamma$. Thus we see that the optimum s.t. for the larger class we have considered is also optimum for the smaller class $E_0\{N\} = \gamma$ we had in the beginning. We also have $\lambda < 1$ because if $\lambda \geq 1$ then the optimum time for (14) is $N_o = 0$ which does not satisfy $E_0\{N_o\} = \gamma$. In order now to find explicitly N_o notice that $z_n = T_{n-1} l(X_n)$ is a Markov sequence, thus we can apply the methods in [1] and [6]. If we consider that $T_0 = z \geq 1$ then $\bar{J}$ becomes a function of z, $\bar{J}(z)$ is nonnegative because for $N = 0$ we have $\bar{J}(N) = 0$. Also it is decreasing in z because $[1 - T_{j-1} l(X_j)]^+$ is decreasing a.s. in T_0. Thus $\bar{J}(z)$ exists for every z and is decreasing. We would like to show now that it is also continuous. This is done in the following lemma.

Lemma 4. If $z_1 \geq z_2$ then $\bar{J}(z)$ satisfies

$$\bar{J}(z_2) - \bar{J}(z_1) \leq z_1 - z_2 \tag{15}$$

Proof. Following [6], the function $\bar{J}(z)$ is the limit of the sequence $\bar{J}_n(z)$ defined by

$$\begin{aligned} &\bar{J}_0(z) = 0 \\ &\bar{J}_n(z) = E_0\{ \left[[1 - zl(X_1)]^+ - \lambda + \bar{J}_{n-1}(\max[zl(X_1),1]) \right]^+ \} \end{aligned} \tag{16}$$

We will show that $\bar{J}_n(z_2) - \bar{J}_n(z_1) \leq z_1 - z_2$. We use induction. It is true for $n = 0$. If it is true for n, using the fact that $a^+ - b^+ \leq a - b$ for $a \geq b$ and that $\max[a,1] = a + [1 - a]^+$ we have that

$$\bar{J}_{n+1}(z_2) - \bar{J}_{n+1}(z_1) \leq$$

$$E_0\{ [1 - z_2 l(X_1)]^+ - [1 - z_1 l(X_1)]^+ + \max[z_1 l(X_1),1] - \max[z_2 l(X_1),1] \} =$$

$$E_0\{ (z_1 - z_2) l(X_1) \} = z_1 - z_2 \tag{17}$$

Thus we have proved that $0 \leq \bar{J}(z_2) - \bar{J}(z_1) \leq z_1 - z_2$, which means that $\bar{J}(z)$ is continuous.

To find now the optimum s.t. we consider the function $A(z) = [1 - z]^+ - \lambda + \bar{J}(\max[z,1])$. We are interested in finding the set of points for which $A(z)$ is nonpositive. Notice that $A(z)$ is decreasing, thus if there is a μ such that $A(z) \leq 0$ for $z \geq \mu$ then the optimum s.t. is to stop the first time we have $z_n = T_{n-1} l(X_n) > \mu$ and any randomization when $z_n = \mu$. If no μ exists, then $A(z) > 0$ for every z and then the optimum s.t. is $N_o = \infty$, but this last situation is not possible since we have $E_0\{N_o\} = \gamma$.

For the numerator (say $G(N)$) we follow a similar approach. We consider the class $E_0\{N\} \geq \gamma$. The constrained optimization problem has a solution because from (6) we have $E_0\{T_n\} \leq n$ and thus $G = inf\, G(N) \leq \sum_{j=0}^{K-1} E_0\{T_j\} < \infty$, where K is an integer greater than γ. Extending the results in [3] to this case, we can show as before that there exist a Lagrange multiplier $\lambda > 0$ and a s.t. N_o satisfying $E_0\{N_o\} = \gamma$ and N_o being optimum for the unconstrained problem

$$\bar{G} = \inf_N E_0\{ \sum_{j=0}^{N-1} T_j - \lambda \} \tag{18}$$

Again assuming $T_0 = z \geq 1$ we can show that $\bar{G}(z)$ exists, and it is increasing and nonpositive. To show the continuity it is easier than before. From Lemma 1 we have that T_j is an a.s. convex function of T_0 thus $\bar{G}(z)$ will be also convex and since it is increasing it will be continuous on $[1,\infty)$. In order now to find the optimum s.t. we consider the function $B(z) = \max[1,z] - \lambda + \bar{G}(\max[1,z])$ and we look for the set of values for which $B(z) \geq 0$. Since $B(z)$ is increasing this set is of the form $z \geq \mu$ and thus we recover again that the optimum s.t. is Pages s.t. And this concludes the proof.

Remark It is very difficult in general to relate explicitly γ with μ and the randomization probability p. There is though a range of values of γ where this is possible. Let us consider the case $\mu \leq 1$. For this case N_P is equivalent (disregarding the randomization) to: {stop at the first n for which $l(X_n) \geq \mu$ }. In other words, given that there is no stop before n we have that $T_m = 1$ for $m < n$. Indeed if for some m we had $T_m > 1$ then $T_m = T_{m-1} l(X_m) > 1 \geq \mu$ thus having a stop at m, contradiction. For this case we can compute the expectation of N_P under P_0 and P_1.

$$E_i\{N_P\} = \frac{1}{1 - P_i\{l(X_1) < \mu\} - pP_i\{l(X_1) = \mu\}} \quad i = 0,1 \tag{19}$$

Thus for $1 < \gamma \leq [\, P_0\{l(X_1) > 1\}\,]^{-1}$ the relation between γ, μ and p is given by

$$\gamma = \frac{1}{P_0\{l(X_1 > \mu\} + (1-p)P_0\{l(X_1) = \mu\}} \tag{20}$$

For the other values of γ the integral equation defined in Page's paper [5] can be used, but clearly this is a more complicated situation. For large values of γ the approximation $\gamma = \mu$ (see [4]) can be used.

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