Least square method to solve Euler equations

F. Angrand

IRIA

CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tel. (3) 954 90 20

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Françoise ANGRAND

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PLAN

ABSTRACT

We consider two dimensional steady flows around profiles, of a compressible inviscid fluid governed by the Euler equations. Our purpose is to study robust unsteady solvers with no parameter to be adjusted. Here we solve several least square formulations of the Galerkin discrete system with conjugate gradient algorithm with different preconditioning operators. Some comparisons are made between those solvers.

RESUME

On s'intéresse à des écoulements bidimensionnels stationnaires, autour de profils, modélisés par les équations d'Euler des fluides parfaits compressibles. L'objectif de ce travail c'est l'étude de solveurs instationnaires robustes sans paramètres à ajuster. Ici on resout des formulations moindres carrés du système de Galerkin discret, par un algorithme de gradient conjugué avec différents préconditionnements. On compare ces solveurs entre eux.

LEAST SQUARE METHOD TO SOLVE EULER EQUATIONS.

ANGRAND FRANCOISE

0. INTRODUCTION

The numerical solution of Euler equations have regained interest in the CFD community. Very efficient methods have been developped with FDM/FVM [1] to [13] and FEM solvers were also contructed [14] to [20]. More precisely for industrial use, mainly centered FDM, FVM and centered galerkin FEM were developed, including artifical viscosity terms. The role of these terms is easier to analyse when we consider the unsteady solution of a steady flow. Two purposes are to be distinguised :

(1) artifical viscosity is used for monotone like shock capturing
(2) artifical viscosity is used for convergence of the unsteady solution to the steady one.

In the litterature the second point proved to be crucial not only for usual explicit procedures, but also with multigrid and several implicit ones ; it may lead to costly parameter adjustment and also to some deterioration of the accuracy with respect to the one got with the first point alone.
In fact, it is the dream of many CFD people to construct methods with no parameter adjustment.
Many works have been motivated by the wish to get rid of the parameter with point (1) (monotone schemes). Although point (2) is somewhat better solved with respect to versability (external, internal flows...), convergence property generally does not give satisfaction. Our purpose is to study unsteady solvers when no parameter is adjusted for convergence purpose. However, in this paper, an artificial viscosity is still used, only for shock capturing (point (1)). The main technique of our no-parameter solver is the Least Square formulation of the Galerkin discrete system together with a non linear conjugate gradient algorithm with preconditioning [22]. Such techniques have been extensively used by the INRIA-AMDBA team for Navier-Stokes FEM models [23] and for a transonic full potential FEM solver [24]. The present paper is a first attempt to use these techniques with a purely hyperbolic system.
The key choice is then the choice of the preconditioning operator :

(A) Poisson preconditioning : In the above studies some success was obtained with Poisson operators. Although the system to be solved is not elliptic, several time stepping schemes containing Poisson solvers exist [6] . So the first preconditioning presented is a Poisson operator.

(B) First order preconditioning operator : Since the Euler system does not give symmetric Jacobian matrices, first order non symmetrical approximations of the Jacobian are also used for preconditioning. While in case (A) the idea is to smooth the variations of the unknowns during the algorithm ; the case (B) is analogous to a Newton-like approach.

As a last remark of this introduction we emphasize that we are not concerned here with the so-called Least Square Approximation. The discrete problem is derived from the classical Galerkin approach only. The solver uses the Least Square principle.

I. DEFINITION OF THE DISCRETE SYSTEM

I.1. Governing equations

The flow of a two dimensional inviscid, compressible fluid is governed by the Euler equations [21] :

$$(1.1)\quad \frac{\partial w}{\partial t} + \frac{\partial}{\partial x} F(w) + \frac{\partial}{\partial y} G(w) = 0$$

Where

$$w = \begin{cases} \rho \\ \rho u \\ \rho v \\ e \end{cases} \qquad F(w) = \begin{cases} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{cases} \qquad G(w) = \begin{cases} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{cases}$$

and with

$$P = 0.4\ (e - \frac{1}{2}\ \rho(u^2+v^2))$$

ρ is the density, p is the pressure, e is the total energy, u and v are the components of the velocity, t is the time.

I.2 Variational formulation

We consider a domain Ω, in which we want to solve (1.1). The conservative variational form of (1.1) is the following.

$$(1.2)\begin{cases} w(t) \in [H^1(\Omega)]^4 \\ \forall \varphi \in [H^1(\Omega)]^4 \quad \displaystyle\int_\Omega \{\frac{\partial w}{\partial t}\varphi - F(w)\ \frac{\partial \varphi}{\partial x} - G(w)\ \frac{\partial \varphi}{\partial y}\}\ dx\ dy = \\ \qquad\qquad \displaystyle -\int_{\partial\Omega} \varphi[F(w)n_x + G(w)n_y]\ d\sigma \end{cases}$$

(n_x, n_y) is the normal vector exterior to the frontier $\partial\Omega$ of Ω.

I.3 Space discretization

Given a triangulation $\mathcal{C}_h$ of Ω_h, polyhedric approximation of Ω, and a Lagrangian finite element, we can define a finite dimensional space V_h included in $[H^1(\Omega)]^4$ and approximating that space. Then the spatial semi-discretization of (1.2) is :

$$(1.3)\begin{cases} w_h(t) \in V_h \\ \forall \varphi \in V_h \quad \displaystyle\int^*_{\Omega_h} \{s_o \frac{\partial w_h}{\partial t}\ s_o\varphi - F(w_h)\ \frac{\partial \varphi}{\partial x} - G(w_h)\ \frac{\partial \varphi}{\partial y}\}\ dx\ dy = \\ \qquad\qquad \displaystyle -\int^*_{\partial\Omega_h} \varphi\ [F(w_h)n_x + G(w_h)\ n_y]\ d\sigma \end{cases}$$

The * symbol denotes numerical integration and some boundary procedure. S_o is the mass lumping operator.

I.4. Boundary conditions

Boundary conditions are plugged via the boundary integral $\int^*_{\partial\Omega_h}$ of (1.3) for wall and profile conditions as well as for the conditions at infinity. For the integrals at infinity values of $w_h(t)$ are substituted at each integration point by a value obtained from a choice of infinity and interior quantities in accordance to table (I.1), which gives a discretization of the conditions.

Boundary \ quantities	entropy	enthalpy	Velocity direction	pressure
inflow	infinity	infinity	infinity	interior
outflow	interior	interior	interior	infinity

Table (I.1) Conditions at infinity

This discretization introduces no diffusion, no upwinding is used, neither extrapolation since with Lagrangian finite elements the dependant variables are defined along the boundaries.

I.5. Time discretization

Let Δt the time step. Between time $n\,\Delta t$ and time $(n+1)\,\Delta t$ we have the following relation

$$
(1.4)\quad \begin{cases} (E(w_h^{n+1}),\varphi)^* = \displaystyle\int^*_{\Omega_h} (s_o\, w_h^{n+1} - s_o\, w_h^n)\; s_o\, \varphi \;\; dx\, dy \\ - \Delta t \displaystyle\int^*_{\Omega_h} \left[F(w_h^{n+1}) \;\frac{\partial \varphi}{\partial x} + G(w_h^{n+1}) \;\frac{\partial \varphi}{\partial y} \right] dx\, dy \\ - \Delta t \displaystyle\int^*_{\partial\Omega_h} \varphi \,[F(w_*^{n+1})\; n_x + G(w_*^{n+1})\; n_y] \;\; d\sigma \\ + \Delta t \;((D(w_h^n),\, \varphi)) \end{cases}
$$

$((D(w_h^*),\, \varphi))$ is an artifical viscosity term to have monotone like shock capturing. This term is calculate as an explicit one. Vector w_*^{n+1} is obtained from the values of w_h^{n+1} and the boundary conditions, using the Table (1.1) rules.

II. LEAST SQUARE FORMULATION AND PRECONDITIONERS

II.1 Least Square formulations

From (1.4), $(E(w_h^{n+1}),\varphi)^*$ has to be a small term, since it is a time approximation of (1.3). Moreover, we want to minimize $E(w^{n+1})$ for a norm which we have to define, for each time step, so we will solve the time approximation of (1.3). So we have, for each time step, an optimal control problem. We want to find for each integer n, w^n which satisfies

$$(2.1)\quad \min_{z\in V_h} \{J(z) = \| E(z)\| ,\ (E(z),\varphi)^* = \text{RHS (1.4)}\}$$

J(z) is the cost function, $(E(z),\varphi)^*$ = RHS (1.4) is the state equation, $(\varphi,\psi)^*$ is an integration formula in $[L^2(\Omega_h)]$, $\| \ \|$ is a norm.

II.2 Choice of the norm

II.2.1 Sobolev norm

We use an idea from the implicit version of the Lax Wendroff scheme, see [17]. It's a H^1 norm.

$$J(w) = \frac{1}{2}\{(s_o E(w),\ s_o E(w)) + \lambda\ (\nabla E\ (w),\ \nabla E(w))\}$$

$(\ , \ )$ is the scalar product in $[L^2(\Omega_h)]^4$

For the integration formula $(\ )^*$ of the state equation, we use the scalar product associated with the norm J(w).

This formulation has two main interests. First, the equations of the Euler system are uncoupled, secondly the introduced operator is an elementary one. We use Neumann conditions at the boundary, so we have a limitative condition for the CFL number.

II.2.2 Splitted first order linearization

The one dimensional Euler equations of gas dynamics in conservation form is written as $\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} F(w) = 0$.

The flux vector F is an homogeneous function of degree one

$$F(\alpha w) = \alpha F(w) \quad \forall\ \alpha \in \mathbb{R},\ w \in \mathbb{R}^3$$

As a consequence we have

$$F(w) = F'(w)\, w \qquad \forall\ w \in \mathbb{R}^3$$

The Jacobian matrix $A = F'(W)$ is diagonalisable and can be put in the form

$$A(w) = T(w)\,\Lambda(w)\,T^{-1}(w)$$
$$A(w) = T(w)\,\Lambda^{+}(w)\,T^{-1}(w) + T(w)\,\Lambda^{-}(w)\,T^{-1}(w)$$
$$A(w) = A^{+}(w) + A^{-}(w)$$

$\Lambda(w)$ is a diagonal matrix

$$\Lambda(w) = (\lambda_j(w))$$
$$\Lambda^{+}(w) = (\max(0,\lambda_j(w))) \quad , \quad \Lambda^{-}(w) = (\min(0,\lambda_j(w)))$$

A two dimensional generalization of this, [20] , gives the matrix $A = A^{+}+A^{-}$, where A^{+} an A^{-} are triangular matrices. It seems, to us, interesting to use the operator $\mathscr{B} = I + \Delta t$ to compare our results with Stoufflet's Newton implicit approach. We have to define J and $(\ )^{*}$.

(2.1) $J(w) = \frac{1}{2}\,(E(w),\ E(w))$

(2.2) $(E(w),\varphi)^{*} = (\mathscr{B}\,E(w),\varphi)$

II.2.3 Symmetrised first order linearization

In II.2.2 the norm is only the $[L^2(\Omega_h)]^4$ norm. We want to have a norm which looks like more the Jacobian matrix of the system. So we define

$$J(w) = \frac{1}{2}\,(\mathscr{B}\,E(w),\ \mathscr{B}\,E(w))$$
$$(E(w),\varphi)^{*} = ({}^{t}\mathscr{B}\,\mathscr{B}\,E(w),\varphi)$$

$(\ )^{*}$ corresponds to the associated scalar product of the norm J

III. CONJUGATE GRADIENT METHOD

To solve the optimal control problem, at each step, we use gradient method. So we have to calculate the gradient, see the enclosure. To accelerate the gradient method, we use Polak-Ribière's [25] conjugate gradient algorithm. The used norm is the norm defined by the cost function. We note $((,))$ the associated scalar product and $\frac{\partial J}{\partial w}.\varphi$ the bilinear form between H^1 and its dual space.

Algorithm III.1

$$T = n\,\Delta t$$

① let $z_o = w^n$

we compute the g_o direction, which satisfies

$$((g_o,\varphi)) = \frac{\partial J}{\partial w}(z_o).\varphi \quad \forall\varphi$$

let $\zeta_o = g_o$

(2) z_p, g_p, ζ_p are known

We minimize $J(w)$ in the g_p direction.

We compute μ such that

$$J(z_p - \mu\,\zeta_p) \leq J(z_p - k\,\zeta_p) \quad \forall\, k$$

We get $z_{p+1} = z_p - \mu\,\zeta_p$

We compute the new direction g_{p+1}

$$((g_{p+1},\varphi)) = \frac{\partial J}{\partial w}(z_{p+1}).\,\varphi$$

$$\gamma = \frac{((g_{p+1},\, g_{p+1} - g_p))}{((g_p,g_p))}$$

$$\zeta^{p+1} = g_{p+1} + \gamma\,\zeta^p$$

if $((g_{p+1},\, g_{p+1}))$ is not small enough, $p = p+1$ and we go to (2)
if $((g_{p+1},\, g_{p+1}))$ is small enough, $w^{n+1} = z_{p+1}$, $n = n+1$ then we go to (1)

IV. NUMERICAL RESULTS

The Sobolev norm has important good properties. Equations are uncoupled, the operator matrix is constructed only one time. There is no co-state equation (terminology of Optimal Control theory) equation to solve. The system resolution is quite fast. We notice only one main drawback. We have some restriction on the choice of CFL and λ(here CFL = 10, λ= 1). This is due to the Neumann boundary conditions used for the operator. To solve II.2.2 and II.2.3 problems, we have first factorized the $\mathscr{B}$ operator into two triangular matrices. We approximate with the product $(I + \Delta t\, A^+)\,(I + \Delta t\, A^-)$. So we can solve directly the systems. $(I + \Delta t\, A^+)$ and $(I + \Delta t\, A^+)$matrices couple equations, so memory core are requirements are more important than for Sobolev norm. We can't use great CFL number, because in the $\mathscr{B}$ approximation we get the term $\Delta t^2\, A^+\, A^-$ which is not neglectable when Δt is a great number.

So we use also an incomplete relaxation method to inverse the operator $\mathscr{B}$. We have the same drawback, CFL number can't be too great because the relaxation iterations number increases while the CFL does. The $\mathscr{B}$ operator looks like the hyperbolic system to solve, so we could hope that it should be good preconditioner for our problem. But the number of iterations which are required by II.2.1 and II.2.2 norms are equivalent. As CPU time to inverse $\mathscr{B}$ is much more important, we have no gain.

We expected to have a good comparison of the II.2.2 problem with a quasi-Newton method, see VI c. The argument was that, if RHS (1.4) is small, $\frac{\partial}{\partial w}$ RHS (1.4)$\cdot\mathcal{B}^{-1}$ looked like the identity operator. In fact, for the used coarse triangulation comparison of the matrices showed large differences. So it has been quite normal if Least Squares method does not work as a quasi Newton method.

Our numerical test is the computation of the steady-state for a channel transonic flow past a circular bump. We choose a test problem proposed at the GAMM workshop held in 1979 at Stockholm. The bump is a 4.2 % thick circular arc with length 1 and the canal is of height 2.073. Free stream values correspond to a Mach of .85. We have a coarse triangulation with only 264 triangles. Figure 4.1 presents the triangulation. Figure 4.2 corresponds to the isomach lines obtained by solver II.2.1

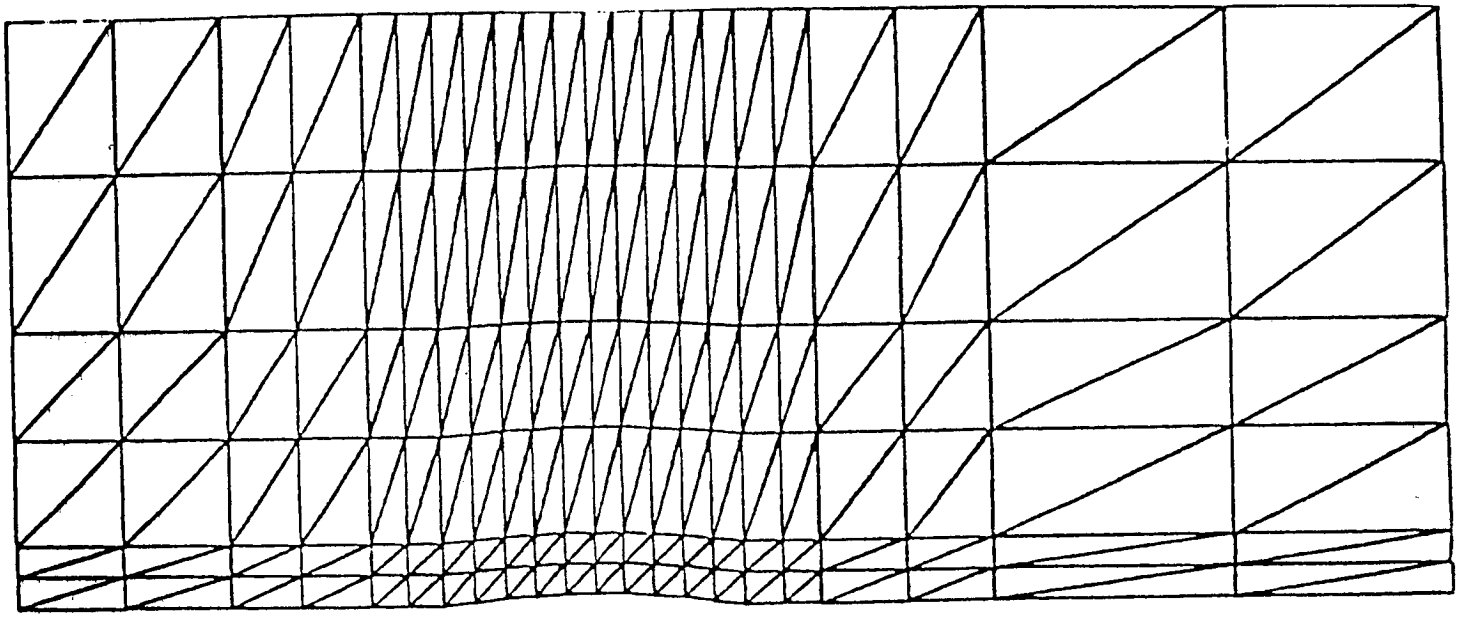


FIGURE 4.1 : Coarse triangulation around the GAMM Channel with circular bump

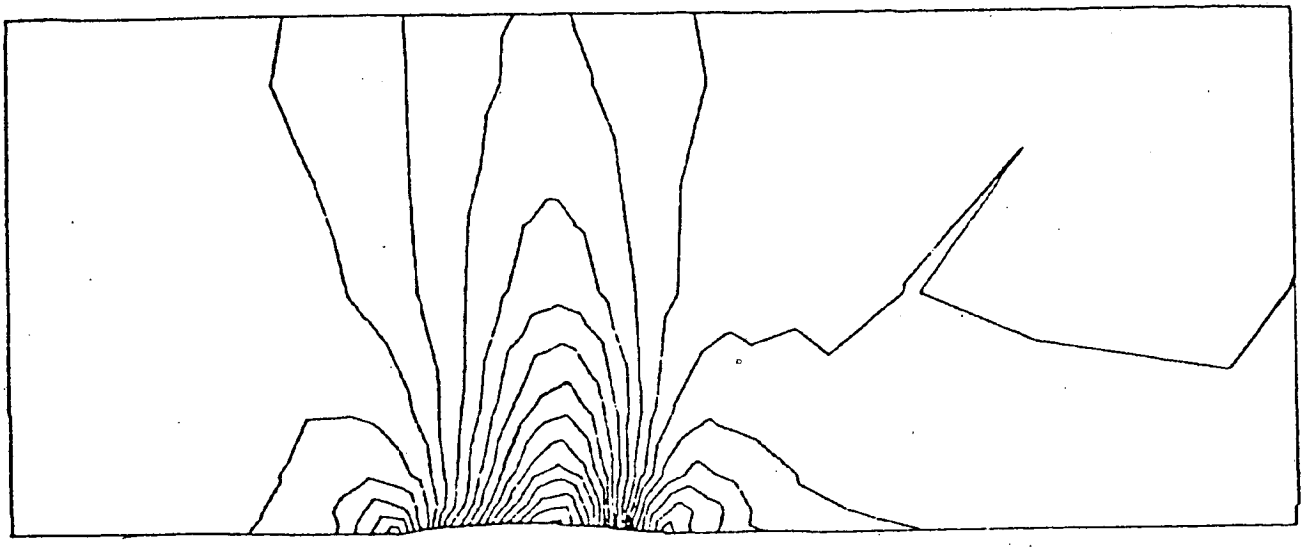
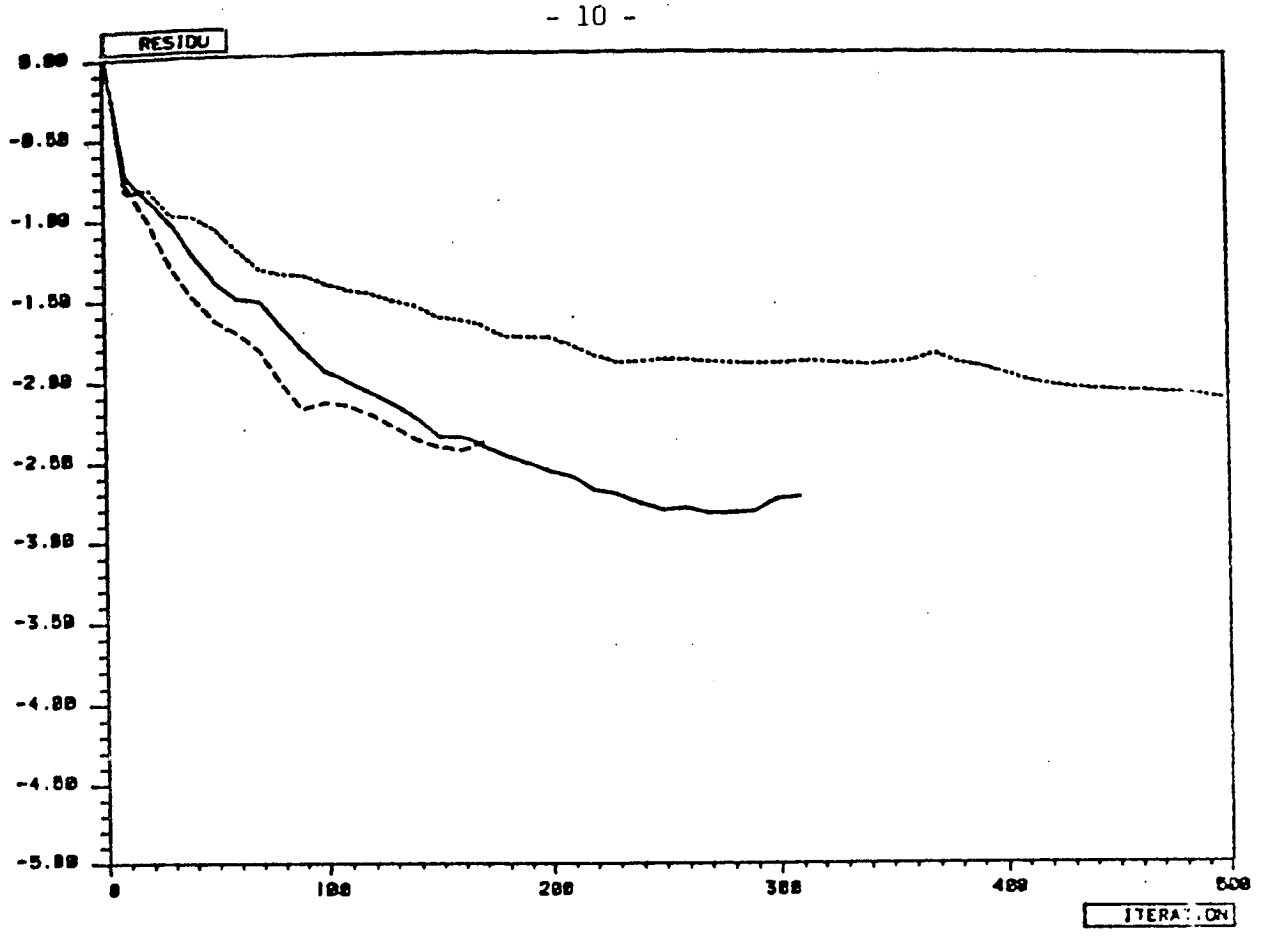
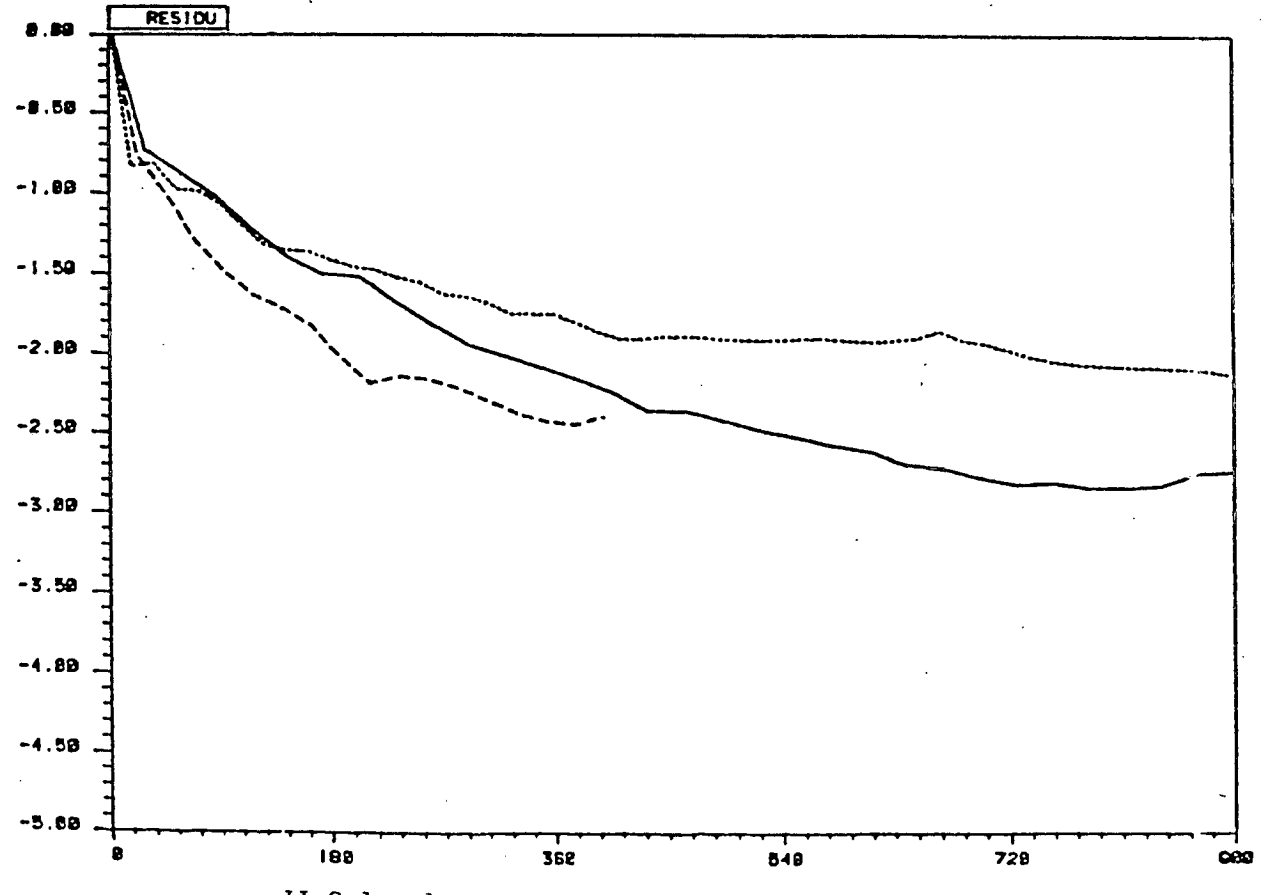


FIGURE 4.2 : Gamm channel with circular bump, isomach lines. M_∞ = .85 solver II.2.1

In figure 4.3 we compare the 3 solvers presented in this paper. We use the logarithm of the root mean square value of $\frac{\partial \rho}{\partial t}$ as a measure of the residual. Computational costs are supported by the GCCVR, and CPU time is given in seconds on CRAY-1-S. We can see that the II.2.2 solver is more efficient than the two other solvers. We have also compared II.2.1 and II.2.2 solvers forthe subsonic case, without viscosity. We have the same comparison than for transonic flow.

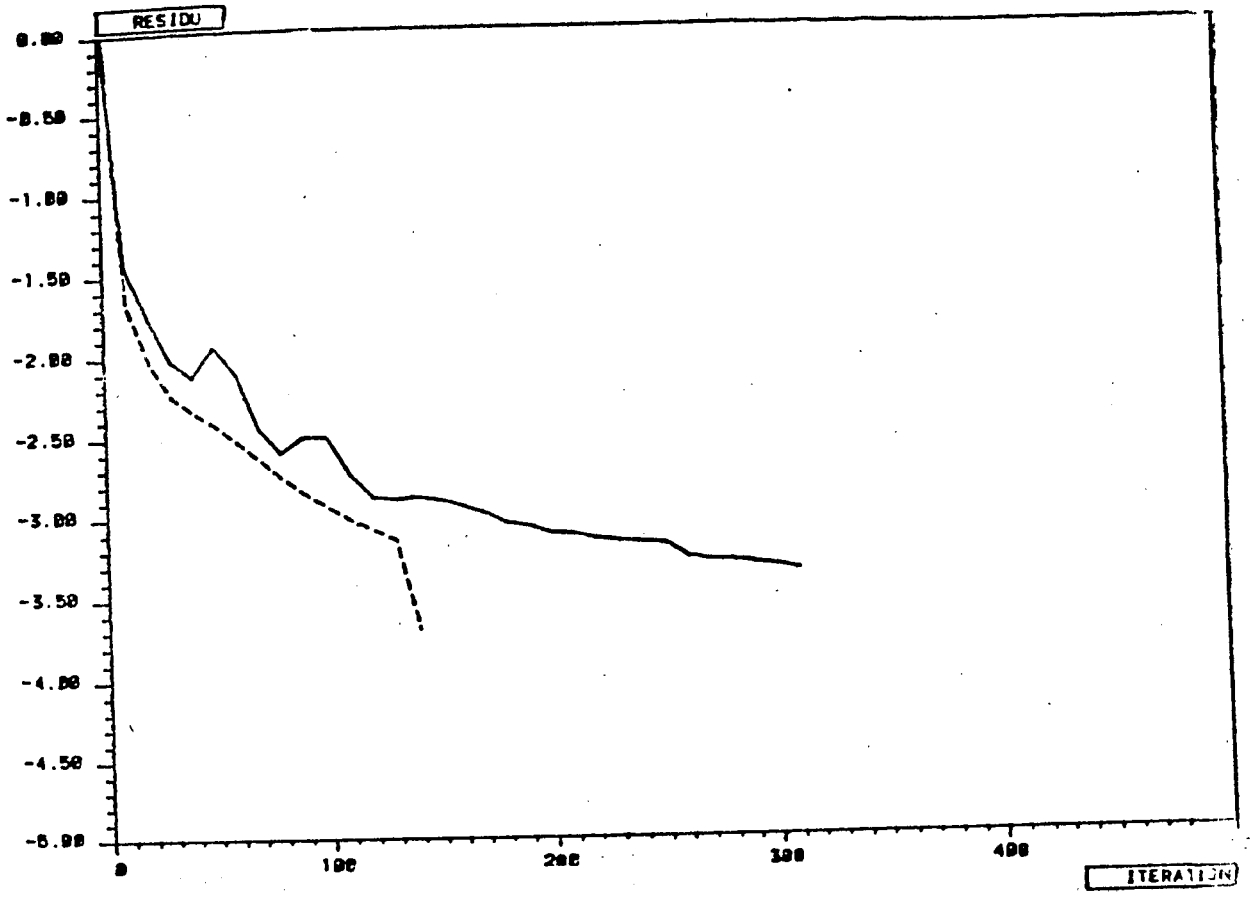
the explicit treatment of the viscosity is not responsible of the slow rate of convergence.

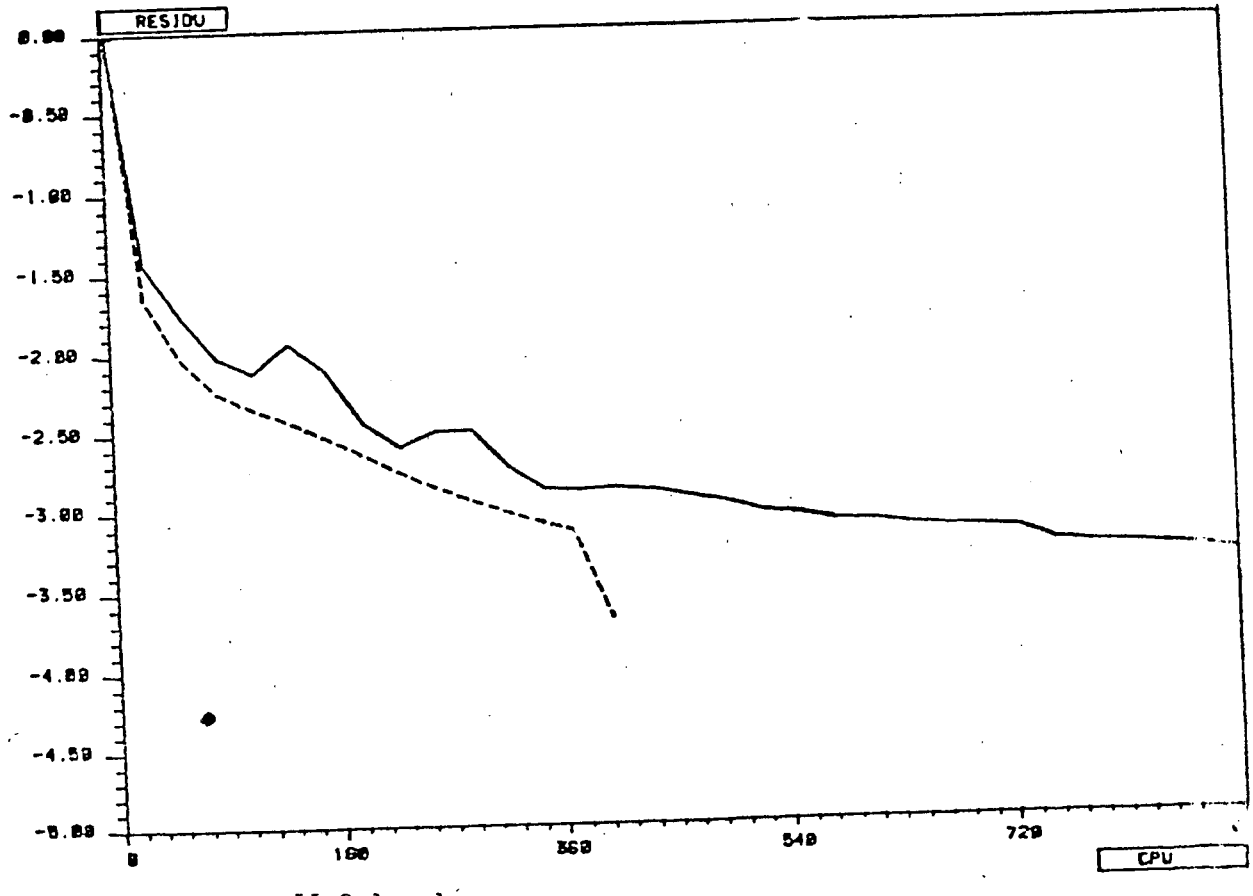




——— II.2.1 solver
——— II.2.3 solver
------ II.2.2 solver

FIGURE 4.3 : Comparison of convergence curves, M_∞ =.85





II.2.1 solver
II.2.2 solver

FIGURE 4.4 : Comparison of convergence curves, subsonic case

V. CONCLUSION

Several solvers have been constructed ; they are free of parameter at the convergence level. Solver (1) is more reliable and is cheaper. Solver (2) converges with less iterations, but each iteration is expensive. These methods converges even when artifical viscosity is not added (subsonic case).

However the efficiency problem is critical. The three presented methods are at least one order more expensive than classical methods as explicit time stepping! Since the desired extrema are reached, we deduce that the formulation with least squares is not responsible at the accuracy level (may be this formulation is ill conditioned) or the gradient algorithm also is not well adapted to stiff hyperbolic problems. This was also observed with the incompressible Navier-Stokes solver with very high Reynolds number, i.e when the hyperbolic part becomes strongly dominating. Then a next step in the study could be to look for a more efficient optimization algorithm.

It could be also efficient to use some better preconditioning : Instead of first order accurate approximations, we could use a second order centered one ; this approach may be interesting if the memory core requirement could be reduced, fo example using domain decomposition [26] .

VI. ENCLOSURES

A Global gradient

- For the Sobolev norm we have

$$(1) \qquad J(w) = \frac{1}{2}\left\{\int_{\Omega_h} s_o\, E(w).\, s_o\, E(w)\, dx\, dy + \lambda \int_{\Omega_h} \nabla E(w).\, \nabla E(w)\, dx\, dy\right\}$$

$$(2) \qquad \int_{\Omega_h} s_o\, E(w).\, s_o\, \varphi\, dx\, dy + \lambda \int_{\Omega_h} \nabla E(w)\, \nabla \varphi\, dx\, dy = \text{RHS (1.4)}$$

So we derive (1)

$$(3) \qquad \frac{\partial J}{\partial w}.\delta w = \int_{\Omega_h} s_o\, E(w).\left(s_o \frac{\partial E}{\partial w}.\delta w\right) dx\, dy + \lambda \int_{\Omega_h} \nabla E(w).\left(\nabla \frac{\partial E}{\partial w}.\delta w\right) dx\, dy$$

we derive (2)

$$(4) \qquad \int_{\Omega_h} s_o \frac{\partial E}{\partial w} \cdot s_o\, \varphi\, dx\, dy + \lambda \int_{\Omega_h}\left(\nabla \frac{\partial E}{\partial w}.\delta w\right) \nabla \varphi\, dx\, dy =$$

$$\int_{\Omega_h} s_o\, \delta w^{n+1}.\, s_o\, \varphi\, dx\, dy - \Delta t \int_{\Omega_h}^{*}\left\{\frac{\partial E}{\partial w}\delta w \frac{\partial}{\partial x}\varphi + \frac{\partial G}{\partial w}\delta w \frac{\partial}{\partial y}\varphi\right\} dx\, dy$$

$$+ \Delta t \int_{\partial\Omega}^{*}\left\{\frac{\partial F}{\partial w_*}\frac{\partial w^*}{\partial w}\delta w\, n_x + \frac{\partial G}{\partial w^*}\frac{\partial w^*}{\partial w}\delta w\, n_y\right\}\varphi\, d\sigma$$

The viscosity term is an explicit one, so we have no gradient to compute for it, this is why test with $\chi = 0$ is interesting. (4) is available for each φ. So we choose $\varphi = E(w)$ in (4). Then we obtain the complete computation of $\frac{\partial J}{\partial w}.\delta w$. We can remark that no co-state is needed.

- For the splitted first order linearization we have

$$J(w) = \frac{1}{2}\int_{\Omega_h} E(w)\, .\, E(w)\, dx\, dy$$

$$\int_{\Omega_h} \mathcal{B}\, E(w).\, \varphi\, dx\, dy = \text{RHS (1.4)}$$

We consider the operator $\mathscr{B}$ as an explicit one, we compute it at the time step before. So no derivation is required for this operator.

$$(5) \qquad \frac{\partial J}{\partial w}.\ \delta w \ = \ \int_{\Omega_h} E(w).\ \frac{\partial \dot{E}}{\partial w} \quad \delta w \quad dx\ dy$$

From the state equation we have

$$\int_{\Omega_h} \left(\mathscr{B}\ \frac{\partial E}{\partial w}.\ \delta w \right).\ \varphi \quad dx\ dy = \text{RHS } (4) = \int_{\Omega_h} \left(\frac{\partial E}{\partial w}\ .\delta w \right) \mathscr{B}^{*} \varphi \ dx\ dy$$

Define the co-state p as

$$\int_{\Omega_h} \mathscr{B}^{*}\ p.\ \varphi \quad dx\ dy = \quad \int_{\Omega_h} E(w).\ \varphi \ dx\ dy$$

$\mathscr{B}^{*}$ is the adjoint operator of $\mathscr{B}$.

Using the result of the adjoint equation in (5) we get the gradient. Contrary to the previous case, we have to solve co-state equation.

- Symmetrised first order linearization

$$J(w) = \frac{1}{2} \int_{\Omega_h} \mathscr{P}\, E(w).\ \mathscr{B} E(w)\ dx\ dy$$

$$\int_{\Omega_h} \mathscr{B} E(w).\ \mathscr{B}\ \varphi \ dx\ dy = \text{RHS } (1.4)$$

We have to solve the adjoint problem

$$\int_{\Omega_h} \mathscr{B}^{*} \mathscr{B} \quad p.\ \varphi \ dx\ dy = \int_{\Omega_h} E(w).\ \varphi \ dx\ dy$$

Computations are similar than for the previous case.

(B) Partial derivatives

We see that we have to compute RHS (4) for every case. So we have to calculate $\frac{\partial F}{\partial w}$, $\frac{\partial G}{\partial w}$ and $\frac{\partial w^*}{\partial w}$, $\frac{\partial F}{\partial w^*}$, $\frac{\partial G}{\partial w^*}$

Let $q_x = \rho u$, $q_y = \rho v$

$$F = \begin{cases} q_x \\ \frac{3-\gamma}{2} \frac{q_x^2}{\rho} + \frac{1-\gamma}{2} \frac{q_y^2}{\rho} + (\gamma-1)e \\ q_x\, q_y\, \rho^{-1} \\ \gamma\, e\, q_x \rho^{-1} - \frac{\gamma-1}{2} q_x^3\, \rho^{-2} - \frac{\gamma-1}{2} q_y^2\, q_x\, \rho^{-2} \end{cases}$$

$$\frac{\partial F}{\partial \rho} = \begin{cases} 0 \\ -\frac{3-\gamma}{2} \frac{q_x^2}{\rho^2} - \frac{1-\gamma}{2} \frac{q_y^2}{\rho^2} \\ -q_x\, q_y\, \rho^{-2} \\ -\gamma\, e\, q_x \rho^{-2} + (\gamma-1)\, q_x^3\, \rho^{-3} + (\gamma-1)\, q_y^2\, q_x \rho^{-3} \end{cases}$$

$$\frac{\partial F}{\partial q_x} = \begin{cases} 1 \\ (3-\gamma)\, q_x \rho^{-1} \\ q_y \rho^{-1} \\ \gamma e \rho^{-1} - \frac{3}{2}(\gamma-1)\, q_x^2\, \rho^{-2} - \frac{\gamma-1}{2} q_y^2\, \rho^{-2} \end{cases}$$

$$\frac{\partial F}{\partial q_y} = \begin{cases} 0 \\ (1-\gamma)\, q_y\, \rho^{-1} \\ q_x\, \rho^{-1} \\ -(\gamma-1)\, q_y\, q_x\, \rho^{-2} \end{cases}$$

$$\frac{\partial F}{\partial e} = \begin{cases} 0 \\ \gamma - 1 \\ 0 \\ \gamma\, q_x\, \rho^{-1} \end{cases}$$

We have similar formula for $\frac{\partial G}{\partial w}$

Now we have to compute $\frac{\partial w^*}{\partial w}$

w^* depends of the pressure p, the enthalpy, the entropy and the velocity direction. Here we consider that the angle of attack is equal to zero ($q_y = 0$ at infinity)

$$p = (\gamma-1)\left[e - \frac{1}{2} q_x^2 \rho^{-1}\right]$$

$$h = \gamma p (\gamma-1) + \frac{1}{2} q_x^2 \rho^{-1}$$

$$s = p \rho^{-\gamma}$$

$$\frac{\partial p}{\partial w} = \begin{cases} \frac{\gamma-1}{2} q_x^2 \rho^{-2} \\ -(\gamma-1) q_x \rho^{-1} \\ 0 \\ (\gamma-1) e \end{cases} \qquad \frac{\partial s}{\partial w} = \begin{cases} -\gamma \frac{p}{\rho^{\gamma+1}} + \frac{\partial p}{\partial p} \rho^{-\gamma} \\ \frac{\partial p}{\partial q_x} \rho^{-\gamma} \\ 0 \\ \frac{\partial p}{\partial e} \rho^{-\gamma} \end{cases}$$

$$\frac{\partial h}{\partial w} = \begin{cases} \gamma \frac{\partial p}{\partial p} ((\gamma-1)p)^{-1} - \frac{\gamma}{\gamma-1} p \rho^{-2} \frac{1}{2} q_x^2 \rho^{-2} \\ \frac{\gamma}{\gamma-1} \frac{\partial p}{\partial q_x} \rho^{-1} + q_x \rho^{-1} \\ 0 \\ \frac{\gamma}{\gamma-1} \frac{\partial p}{\partial e} \rho^{-1} \end{cases}$$

$$E(w^*) = \begin{cases} 0 \\ 0 \\ P \\ 0 \end{cases}$$

$$F(w^*) = \begin{cases} q_x \\ q_x^2 \rho^{-1} + p \\ 0 \\ (e+p) q_x \rho^{-1} \end{cases}$$

$$p = \left(\frac{p}{s}\right)^{1/\gamma}$$

$$q_x = \left[2 h \left(\frac{p}{s}\right)^{1/\gamma} - \frac{2\gamma}{\gamma-1} p\right]^{1/\gamma}$$

$$e = \frac{p}{\gamma-1} + h - \frac{\gamma}{\gamma-1} p \left(\frac{p}{s}\right)^{-1/\gamma}$$

So we can compute $\frac{\partial F(w^*)}{\partial w^*}$

Implementation

With double precision, it has been verified that the gradients obtained by the previous formula were computed precisely. This verification was made by comparison of $J(w+\delta w) - J(w)$ and $\frac{\partial J}{\partial w} \cdot \delta w$

(C) Motivation for comparison of splitted first order linearization method with quasi-Newton scheme.

We want to solve $E(w) = 0$

Define $X(w) = \mathscr{B}(w)\, E(w)$

(2.1) becomes

$$J(w) = \frac{1}{2}\ (\mathscr{B}^{-1} X(w),\ \mathscr{B}^{-1} X(w))$$

Differentiate this equations

$$\frac{\partial J}{\partial w} \cdot \delta w = (\mathscr{B}^{-1} X(w), \mathscr{B}^{-1} \frac{\partial X}{\partial w}\ \delta w + \frac{\partial}{\partial w} \mathscr{B}^{-1}\ \delta w\ \ X(w))$$

$$\simeq (\mathscr{B}^{-1} X(w), \mathscr{B}^{-1} \frac{\partial X}{\partial w}\ \delta w)$$

$$\simeq ((\mathscr{B}^{-1} \frac{\partial X}{\partial w})^* \mathscr{B}^{-1}\ \ X(w),\ \delta w)$$

So the gradient method gives the following algorithm

$$w^{n+1} = w^n - (\mathscr{B}^{-1} \frac{\partial X}{\partial w})^*\ \mathscr{B}^{-1}\ X(w)$$

When we are near the stationnary state and the convergence of the algorithm, the derivation of the equation (1.4) gives $\frac{\partial E}{\partial w} \simeq Id$

But $X(w) = \mathscr{B}(w).\ E(w)$,

So we have $\frac{\partial X}{\partial w} \simeq \mathscr{B}(w)\ \frac{\partial E}{\partial w}$ since E is near from zero. So $\mathscr{B}^{-1}\ \frac{\partial X}{\partial w} \simeq Id$

So the gradient method, near the convergence to the stationnary solution, can be written as

$$w^{n+1} = w^n - \mathscr{B}^{-1} X(w)$$

So we could think that the method will converge as a Newton one.

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