INRIA

CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(3) 954 90 20

Rapports de Recherche

N° 312

DIFFERENTIAL STABILITY OF SOLUTIONS TO CONSTRAINED OPTIMIZATION PROBLEMS

Jan SOKOŁOWSKI

Juin 1984

Batch003 008

Differential Stability of Solutions to Constrained Optimization Problems*

Jan Sokołowski

Systems Research Institute

Polish Academy of Sciences

ul.Newelska 6, 01-447 Warszawa

Poland

(*) This research has been supported by INRIA, in the framework of a collaboration between the project "Théorie des Systèmes" and the Systems Research Institute of the Polish Academy of Sciences.

May 1984

Abstract

In this paper the differential stability of solutions to constrained optimization problems is investigated. The form of right-derivatives of optimal solutions to such problems, with respect to a real parameter, is derived. Several examples are provided.

Résumé

Dans cette article, on étudie la stabilité différentielle de solutions de problèmes d'optimisation avec contraintes. On donne la forme de la dérivée à droite de ces solutions par rapport à un paramètre réel. Plusieurs exemples sont fournis.

1. Introduction

The paper is devoted to the sensitivity analysis of constrained optimal control problems. The method presented here is used in [8] in the case of optimal control problems for linear distributed parameter systems with linear constraints and quadratic cost functional.

The differential stability of solutions to convex, constrained, optimal control problems is investigated in [5] for a system of ordinary differential equations.

Our approach is founded on the concept of conical differentiability of projection in Hilbert space onto a convex set [1,6,9]. We derive the form of right-derivatives of optimal solutions with respect to a parameter for a class of optimization problems. The outline of the paper is following.

Section 2 is devoted to the analysis of directional differentiability of projection in a Hilbert space onto a convex set.

Section 3 describes results obtained for an abstract optimization problem. An example of optimal control problem for a parabolic equation is presented.

Section 4 is devoted to the analysis of differential stability of right-derivatives of optimal solutions.

Throughout the paper standard notation is used [4].

2. Conical Differentiability of Projection in Hilbert Space

Let H be a Hilbert space, $a(.,.) : H \times H \mapsto R$ a continuous and coercive bilinear form, i.e.

$$a(v,v) \geq \alpha \|v\|_H^2 \quad , \quad \forall v \in H, \quad \alpha > 0 \tag{2.1}$$

$$|a(u,v)| \leq M \, \|u\|_H \, \|v\|_H \, , \quad \forall u,v \in H \tag{2.2}$$

where $\alpha > 0$, M are given constants.

We assume for simplicity, that the bilinear form $a(.,.)$ is symmetric: $a(u,v)=a(v,u)$, $\forall u,v \in H$. Let us denote by $P_K(f)$ a-projection in H of an element $f \in H$ onto a convex, closed set $K \subset H$. The element $y=P_K(f)$ satisfies variational inequality:

$$\begin{aligned} & y \in K \\ & a(y-f, v-y) \geq 0 \, , \quad \forall v \in K \end{aligned} \tag{2.3}$$

It can be shown that mapping $P_K(.) : H \to K$ is Lipschitz continuous

$$\|P_K(f_1)-P_K(f_2)\|_H \leq \frac{M}{\alpha} \|f_1-f_2\|_H \, , \quad \forall f_1, f_2 \in H \tag{2.4}$$

For a given element $u \in K$ we denote by:

$$N_K(u) = \{v \in H \mid a(v, \phi-u) \leq 0, \ \forall \phi \in K\} \tag{2.5}$$

$$C_K(u) = \{v \in H \mid \exists \tau > 0 \ \text{ such that } \ u+\tau v \in K\} \tag{2.6}$$

the normal cone and the tangent cone, respectively.

Furthermore for a given element $f \in H$ we denote:

$$S_K(f) = \{v \in \overline{C_K(P_K(f))} \mid a(f-P_K(f), v) = 0\} \tag{2.7}$$

where $\overline{C_K(u)}$ is the closure in H of tangent cone $C_K(u)$. It can be verified that the set $S_K(f)$ is a closed and convex cone.

Let us assume that there is given a continuous mapping $f(.) : [0,\delta) \to H$ which is right differentiable at 0, i.e. there exists an element $f'(0) \in H$ such that $\lim\limits_{\tau \downarrow 0} \|(f(\tau)-f(0))/\tau - f'(0)\|_H = 0$.

Denote $y(\tau)=P_K(f(\tau))$, $\gamma(\tau)=(y(\tau)-y(0))/\tau$ and observe that, in view of (2.4), $\|\gamma(\tau)\|_H \le C$ for all $\tau \in (0,\delta)$.

Proposition 1

Every weak limit-point γ of $\gamma(\tau)$ for $\tau\downarrow 0$ satisfies

$$\gamma \in S_K(f(0)) \tag{2.8}$$

The proof of Proposition 1 is given e.g. in [1].

Definition 1

The set K is called polyhedric if the following condition is verified:

$$S_K(f) = \overline{\{v \in C_K(P_K(f)) \,|\, a(f-P_K(f),v) = 0\}} \tag{2.9}$$

Theorem 1 [1,6]

Let us assume that the set K is polyhedric, then for $\tau > 0$, τ small enough:

$$P_K(f(\tau)) = P_K(f(0)) + \tau P_{S_K(f(0))}(f'(0)) + o(\tau) \tag{2.10}$$

where $\|o(\tau)\|_H/\tau \to 0$ with $\tau\downarrow 0$.

Definition 2

Projection $P_K(.)$ is conically differentiable at a point $f \in H$ if there exists a continuous and positively homogenuous mapping $Q : H \to H$ such that for $\tau > 0$, small enough

$$\forall h \in H : P_K(f+\tau h)=P_K(f)+\tau Q(h)+o(\tau;h) \tag{2.11}$$

where $\|o(\tau;h)\|_H/\tau \to 0$ with $\tau\downarrow 0$ uniformly with respect to h on compact subsets of H.

Remark 1

Let us note that if a set K is polyhedric, then in view of (2.10), it follows that (2.11) holds with

$$Q(h) = P_{S_K(f)}(h) \ , \qquad \forall h \in H \tag{2.12}$$

Remark 2

It can be verified that if (2.11) holds then

$$P_K(f(\tau)) = P_K(f(0)) + \tau Q(f'(0)) + o(\tau) \tag{2.13}$$

where $\|o(\tau)\|_H/\tau \to 0$ with $\tau \downarrow 0$

for any Lipschitz continuous mapping $f(.) : [0,\delta) \to H$ which is right-differentiable at 0.

Conical Differentiability of Projection in R^n

We shall consider differentiability of projection in R^n onto a set K of the form:

$$K = \{x \in R^n \mid \phi_w(x) \le 0 \quad \text{for all} \quad w \in W\} \tag{2.14}$$

where W is a compact space with a Hausdorff topology, $\phi_w(.) \in C^2(R^n)$, $w \in W$ are given convex functions such that $\phi_w(x)$, $D\phi_w(x)$, $D^2\phi_w(x)$ depend continuously on $(w,x) \in W \times R^n$. Let us consider a point $x \in K$ and the corresponding set of active indices:

$$M(x) = \{w \in W \mid \phi_w(x) = 0\} \tag{2.15}$$

Suppose that there exists a vector $y \in R^n$ such that

$$y.D\phi_w(x) < 0 \quad \text{for all} \quad w \in M(x) \tag{2.16}$$

then the normal cone $N_K(x)$ is given by:

$$N_K(x) = \{v \in R^n \mid v = \sum_{w \in M(x)} \alpha_w D\phi_w(x),\ \alpha_w \geq 0 \quad \text{for all} \quad w \in M(x)\} \tag{2.17}$$

and similarly tangent cone $C_K(x)$ is given by:

$$C_K(x) = \{v \in R^n \mid D\phi_w(x).v \leq 0, \text{ for all } \quad w \in M(x)\} \tag{2.18}$$

Let us denote $u=P_K(f)$ for a given element $f \in R^n$. It is well known [7] that optimality condition say that the vector $f-u$ meets $N_K(u)$. Thus there exist multipliers $\lambda_w > 0$ such that $\lambda_w > 0$ for at most finitely many indices w, those all being in the active set $M(x)$, furthermore

$$f-u = \sum_{w \in M(u)} \lambda_w D\phi_w(u) \tag{2.19}$$

Let $\tau \in [0,\delta)$ be a parameter, denote $u_\tau=P_K(f_\tau)$ where $f_\tau=f+\tau h$, $h \in R^n$ is a given vector. We have

$$f_\tau - u_\tau = \Lambda_\tau \in N_K(u_\tau) \tag{2.20}$$

for all $\tau \in [0,\delta)$.

Since

$$\|u_\tau - u_o\|_{R^n} \leq \tau \|h\|_{R^n} \quad , \quad \forall h \in R^n,\ \forall \tau \in [0,\delta) \tag{2.21}$$

then there exists an element q and a subsequence $\{u_{\tau_n}\}$, $n=1,2,\ldots$ such that for $\tau_n > 0$, τ_n small enough:

$$u_{\tau_n} = u_o + \tau_n q + o(\tau_n) \tag{2.22}$$

where $\|o(\tau_n)\|_{R^n}/\tau \to 0$ with $\tau_n \downarrow 0$.

It follows by Proposition 1 that the element q verifies:

$$q \in S_K(u_o) \tag{2.23}$$

In the following examples the right-derivaties are uniquely determined.

Example 1

Let us consider convex set

$$K = \{x \in R^n \mid a_i^T \cdot x - b_i \leq 0 , \; i=1,\dots,N\} \tag{2.24}$$

where W is discret set $\{1,\dots,N\}$, $\phi_i(x)=a_i^T \cdot x-b_i$, $i=1,\dots,N$ where $a_i \in R^n$, $b_i \in R$ are given elements. We assume that the set (2.24) is nonempty. Since the set (2.24) is polyhedric it follows by Theorem 1 that the projection in R^n onto K is conically differentiable. In particular the mapping $Q(.) : R^n \to R^n$ is given by (2.12). Cone $S_K(f)$ takes on the form:

$$S_K(f) = \{v \in R^n \mid a_i^T \cdot v \leq 0, \; i \in M(P_K(f)), (f-P_K(f),v)_{R^n}=0\} \tag{2.25}$$

Example 2

Let $K \subset R^n$ be a compact, convex set with nonempty interior and with smooth boundary $\partial K \in C^2$.

We assume that there is given a convex function $\psi(.)$ $C^2(R^n)$ such that $\psi(\bar{x}) < 0$ for some $\bar{x} \in R^n$, $\partial K=\{x \in R^n \mid \psi(x)=0\}$ and

$$K = \{v \in R^n \mid \phi_w(v) \leq 0 \quad \text{for all} \quad w \in W \equiv \partial K\} \tag{2.26}$$

where $\phi_w(v)=D\psi(w) \circ (v-w)$, $v \in R^n$, $w \in W$.

It can be verified, using known results [1,2,5], that the projection in R^n onto the set (2.26) is conically differentiable. For given elements $f,h \in R^n$ the corresponding element $Q=Q(h) \in R^n$ satisfies variational inequality:

$$\begin{aligned} & Q \in S_K(f) \\ & (AQ,v-Q)_{R^n} \geq (h,v-Q)_{R^n}, \quad \forall v \in S_K(f) \end{aligned} \tag{2.27}$$

where

$$A = I + \lambda D^2\psi(u) \tag{2.28}$$

$$u = P_K(f)$$

$$\lambda = \begin{cases} \|f-u\|_{R^n} / \|D\psi(u)\|_{R^n}, & f \notin K \\ 0, & f \in K \end{cases} \tag{2.29}$$

$$S_K(f) = \begin{cases} v \in R^n : D\psi(u).v \leq 0,\ \lambda D\psi(u).v = 0 & \text{if } f \notin \text{int}\, K \\ R^n & \text{if } f \in \text{int}\, K \end{cases} \tag{2.30}$$

Conical Differentiability of Projection in Infinite Dimensional Space

Let V be a real Hilbert space, $U \subset V$ be a closed and convex set. Let (Ω, μ) be a positively measured space, we set

$$H = L^2(\Omega; V)$$

$$K = \{v \in H \mid v(\xi) \in U \quad \mu \text{ a.e. in } \Omega\} \tag{2.31}$$

Let us consider differentiability of projection in H onto K. We assume that projection in V onto U is conically differentiable, i.e. for $\tau > 0$, τ small enough

$$P_U(g + \tau v) = P_U(g) + \tau F(g; v) + o(\tau; v), \quad \forall v \in V \tag{2.32}$$

where $\|o(\tau;v)\|_V / \tau \to 0$ with $\tau \downarrow 0$ and $F(g,.) : V \to V$ is continuous and positively homogenuous mapping for any $g \in V$. Given elements $f(.), h(.) \in L^2(\Omega; V)$, we denote

$$q(\xi) = F(f(\xi); h(\xi)) \quad \mu \text{ a.e. in } \Omega \tag{2.33}$$

Proposition 2

Let us assume that $q(.) \in L^2(\Omega; V)$. Then for $\tau > 0$, τ small enough

$$P_K(f+\tau h) = P_K(f) + \tau q + o(\tau;h) \tag{2.34}$$

where $\|o(\tau;h)\|_H/\tau \to 0$ with $\tau \downarrow 0$.

Proof:

Observe that

$$P_K(f)(\xi) = P_U(f(\xi)) \quad \mu \text{ a.e. in } \Omega \tag{2.35}$$

Since

$$\|P_K(f+\tau h) - P_K(f)\|_H \le \tau \|h\|_H, \quad \forall h \in H \tag{2.36}$$

then for fixed elements $f,h \in H$ there exists an element $\tilde{q} \in H$ such that

$$\frac{1}{\tau}(P_K(f+\tau h) - P_K(f)) \longrightarrow \tilde{q} \tag{2.37}$$

weakly in H with $\tau \downarrow 0$

furthermore $\tilde{q} \in S_K(f)$ by Proposition 1. On the other hand

$$P_U(f(\xi)+\tau h(\xi)) = P_U(f(\xi)) + \tau F(f(\xi);h(\xi)) + r(f(\xi),h(\xi);\tau) \tag{2.38}$$

μ a.e. in Ω

where $\|r(f(\xi), h(\xi);\tau)\|_V/\tau \to 0$ with $\tau \downarrow 0$ (2.39)

μ a.e. in Ω

thus

$$\tilde{q}(\xi) = F(f(\xi);h(\xi)) \quad \mu \text{ a.e. in } \Omega.$$

Since

$$\frac{1}{\tau}\|r(f(\xi),h(\xi);\tau)\|_V \le \|F(f(\xi);h(\xi))\|_V + \frac{1}{\tau}\|P_U(f(\xi)+h(\xi)) - P_U(f(\xi))\|_V \le \|q(\xi)\|_V + \|h(\xi)\|_V \tag{2.40}$$

hence

$$\frac{1}{\tau} r(f(.),h(.);\tau) \in H \tag{2.41}$$

We can conclude the proof using (2.39), since by Lesbeque theorem it follows that

$$\|\frac{1}{\tau} r(f(.),h'(.);\tau)\|_H \to 0 \quad \text{with} \quad \tau \downarrow 0$$

therefore from (2.36), in view of (2.35), it follows (2.34).

q.e.d.

We present two examples.

Example 3

We assume that $V=R^n$ and we take set U of the form (2.24), thus the set (2.31) is given by

$$K = \{v(.) \in L^2(\Omega;R^n) \mid a_i^T.v(\xi)-b_i \leq 0, \quad i=1,\dots,N, \quad \mu \text{ a.e. in } \Omega\} \tag{2.42}$$

Given elements $f(.)$, $h(.) \in L^2(\Omega;R^n)$, denote $u(\xi)=P_U(f(\xi))$ and observe that

$$P_K(f)(\xi) = P_U(f(\xi)) \quad \mu \text{ a.e. in } \Omega \tag{2.43}$$

Let us denote

$$q(\xi) \stackrel{def}{=} P_{S_U(f(\xi))}(h(\xi)) \quad \mu \text{ a.e. in } \Omega \tag{2.44}$$

i.e. for a fixed $\xi \in \Omega$, the element $q(\xi) \in R^n$ satisfies variational inequality:

$$\begin{aligned} & q(\xi) \in S_U(f(\xi)) \\ & (q(\xi)-h(\xi), v-q(\xi))_{R^n} \geq 0, \; \forall v \in S_U(f(\xi)) \end{aligned} \tag{2.45}$$

μ a.e. in Ω.

Using standard argument it can be verified that

$$\|q(\xi)\|_{R^n} \leq \|h(\xi)\|_{R^n} \quad \mu \text{ q.e. in } \Omega \tag{2.46}$$

We set $v=v(\xi)$ in (2.43) and integrate the inequality (2.43) over Ω hence

$$q \in S_K(f)$$
$$\int_\Omega (q(\xi)-h(\xi), v(\xi)-q(\xi))_{R^n} d\mu \geq 0, \ \forall v \in S_K(f) \tag{2.47}$$

where $S_K(f)=\{v \in L^2(\Omega;R^n) \,|\, v(\xi) \in S_U(f(\xi)) \quad \mu \text{ a.e. in } \Omega\}$

By Proposition 2 it follows that projection in $L^2(\Omega;R^n)$ onto the set (2.42) is directionally differentiable, the right-derivative $q \in L^2(\Omega;R^n)$ is given by (2.47).

Example 4

In this example we shall consider a set U of the form:

$$U = \{v \in R^n \,|\, \frac{1}{2} \sum_{i=1}^{n} a_i v_i^2 \leq 1\} , \tag{2.48}$$

where $a_i > 0$, $i=1,\ldots,n$ are given constants. We denote $\psi(v) = \frac{1}{2}\sum_{i=1}^{n} a_i v_i^2 - 1$, $v \in R^n$. Given element $f(.) \in L^\infty(\Omega;R^n)$, we shall use the following notation:

$$u(\xi) = P_U(f(\xi))$$

$$\lambda(\xi) = \begin{cases} \|f(\xi)-u(\xi)\|_{R^n} / \|D\psi(u(\xi))\|_{R^n} , & f(\xi) \notin U \\ 0 , & f(\xi) \in U \end{cases}$$

$$A(\xi) = [(1 + \lambda(\xi)a_i)\delta_{ij}]_{n \times n}$$

$$S_U(f(\xi))=\{v \in R^n \,|\, D\psi(u(\xi)).v \leq 0 \quad \lambda(\xi)D\psi(u(\xi)).v = 0\}$$

for a.a. $\xi \in \Omega$.

It can be verified that $\lambda(.) \in L^\infty(\Omega)$, hence $A(.) \in [L^\infty(\Omega)]^{n^2}$.

Let us denote by $q(\xi)$, $\xi \in \Omega$ a unique solution of variational inequality:

$$q(\xi) \in S_U(f(\xi))$$
$$(A(\xi)q(\xi)-h(\xi), v-q(\xi))_{R^n} \geq 0, \ \forall v \in S_U(f(\xi)) \tag{2.49}$$

where $h(.) \in L^2(\Omega;R^n)$ is a given element. By standard argument it follows that

$$||q(\xi)||_{R^n} \leq ||h(\xi)||_{R^n} \quad \mu \text{ a.e. in } \Omega \tag{2.50}$$

Hence the assumption of Proposition 2 is verified and the projection onto the set

$$K = \{v(.) \in L^2(\Omega;R^n) \mid \psi(v(\xi)) \leq 0 \quad \mu \text{ a.e. in } \Omega\}$$

is differentiable in every direction $h(.) \in L^2(\Omega;R^n)$. The right-derivative $q(.) \in L^2(\Omega;R^n)$ is given by:

$$q(.) \in S_K(f) = \{v(.) \in L^2(\Omega;R^n) \mid D\psi(u(\xi)).v(\xi) \leq 0$$
$$\lambda(\xi)D\psi(u(\xi)).v(\xi) = 0 \quad \mu \text{ a.e. in } \Omega\}$$

$$\int_\Omega (A(\xi)q(\xi)-h(\xi), v(\xi)-q(\xi))_{R^n} d\mu \geq 0 \qquad \forall v(.) \in S_K(f)$$

For further examples we refer the reader to [1,5,6].

3. An Abstract Optimization Problem

This section is devoted to the sensitivity analysis of an abstract optimization problem. We shall prove existence of right-derivative of an optimal solution of this problem with respect to a parameter.

Let H,Y be Hilbert spaces, $\tau \in [0,\delta)$ be a real parameter, $\delta > 0$ is a given constant. We assume that there are given linear operators $L_\tau \in L(H;Y)$, $\tau \in [0,\delta)$ such that the following condition is verified:

(H1) There exists a linear mapping $L'_o \in L(H;Y)$ such that if there are given elements

$$v_\tau = v_o + \tau v'_o + r(\tau) \ , \quad \tau \in [0,\delta) \tag{3.1}$$

where $v_o, v'_o \in H$, $r(\tau)/\tau \to 0$ weakly in H with $\tau \downarrow 0$ then

$$L_\tau v_\tau = L_o v_o + \tau(L'_o v_o + L_o v'_o) + o(\tau) \tag{3.2}$$

where $\|o(\tau)\|_Y/\tau \to 0$ with $\tau \downarrow 0$.

Let $K \subset H$ be a convex, closed set and

$$I_\tau(.) : Y \mapsto R \ , \quad \tau \in [0,\delta) \tag{3.3}$$

be a family of functionals.

Let us consider the following optimization problem:

(P_τ) Find an element $u_\tau \in K$ which minimizes functional

$$J_\tau(v) = I_\tau(L_\tau v) + \frac{\alpha}{2}\|v - z_\tau\|_H^2 \tag{3.4}$$

over the set $K \subset H$.

where $z_\tau \in H$, $\tau \in [0,\delta)$ are given elements, K is a convex and closed subset of space H.

We need the following notation. We assume that for a fixed parameter the functional $I_\tau(.)$ is continuously differentiable, furthermore we assume that the gradient $DI_\tau(.) \in Y'$ is continuously differentiable, i.e. for any $v \in Y$ there exists a continuous mapping $D^2 I_\tau(v;.) \in L(Y;Y')$ such that

$$\forall u \in Y : \lim_{\varepsilon \downarrow 0} \|(DI_\tau(v+\varepsilon u) - DI_\tau(v))/\varepsilon - D^2 I_\tau(v;u)\|_{Y'} = 0 \tag{3.5}$$

We denote by $\dot{I}_\tau(v)$ the derivative with respect to the parameter:

$$\forall v \in Y : \quad \dot{I}_\tau(v) = \lim_{\varepsilon \downarrow 0}(I_{\tau+\varepsilon}(v) - I_\tau(v))/\varepsilon \tag{3.6}$$

It is well known, that for a fixed parameter $\tau \in [0,\delta)$, an optimal solution $u_\tau \in K$ of (P_τ) satisfies variational inequality:

$$
\begin{aligned}
& u_\tau \in K \\
& (L_\tau^* DI_\tau(L_\tau u_\tau) + \alpha(u_\tau - z_\tau), v - u_\tau)_H \geq 0, \quad \forall v \in K
\end{aligned} \tag{3.7}
$$

In order to assure uniqueness of an optimal solution we assume that there exists a positive constant $\sigma > 0$ such that

$$
(L_\tau^* DI_\tau(L_\tau v) - L_\tau^* DI_\tau(L_\tau u), v-u)_H + \alpha \|v-u\|_H^2 \geq \sigma \|v-u\|_H^2 , \quad \forall u,v \in H \tag{3.8}
$$

<u>Proposition 3</u>

Let us assume that (3.8) holds for any $\tau \in [0,\delta)$ and the following conditions are verified for any $\varepsilon,\tau \in [0,\delta)$

$$
\|DI_\tau(L_\tau v) - DI_\tau(L_\varepsilon v)\|_{Y'} \leq C \|L_\tau v - L_\varepsilon v\|_Y \leq C|\tau-\varepsilon| \ \|v\|_H , \quad \forall v \in H \tag{3.9}
$$

$$
\|DI_\tau(y) - DI_\varepsilon(y)\|_{Y'} \leq C|\tau-\varepsilon| \ \|y\|_Y , \quad \forall y \in Y \tag{3.10}
$$

$$
\|z_\tau - z_\varepsilon\|_H \leq C|\tau-\varepsilon| \tag{3.11}
$$

where C is a generic constant.

Then

$$
\|u_\tau - u_\varepsilon\|_H \leq C|\tau-\varepsilon| , \quad \tau,\varepsilon \in [0,\delta) \tag{3.12}
$$

<u>Proof</u>:

The element $u_\varepsilon \in K$ satisfies variational inequality:

$$
(L_\varepsilon^* DI_\varepsilon(L_\varepsilon u_\varepsilon) + \alpha(u_\varepsilon - z_\varepsilon), v - u_\varepsilon)_H \geq 0, \quad \forall v \in K \tag{3.13}
$$

We set $v=u_\varepsilon$ in (3.7) and $v=u_\tau$ in (3.13) and we add (3.7) to (3.13). After simple calculations we obtain:

$$\alpha \|u_\tau - u_\varepsilon\|_H^2 + (L_\tau^* DI_\tau(L_\tau u_\tau) - L_\tau^* DI_\tau(L_\tau u_\varepsilon), u_\tau - u_\varepsilon)_H \le (L_\tau^* DI_\tau(L_\tau u_\varepsilon) -$$

$$L_\tau^* DI_\tau(L_\varepsilon u_\varepsilon), u_\varepsilon - u_\tau)_H + (L_\tau^* DI_\tau(L_\varepsilon u_\varepsilon) - L_\tau^* DI_\varepsilon(L_\varepsilon u_\varepsilon), u_\varepsilon - u_\tau)_H$$

$$+ \; L_\tau^* DI_\varepsilon(L_\varepsilon u_\varepsilon) - L_\varepsilon^* DI_\varepsilon(L_\varepsilon u_\varepsilon), u_\varepsilon - u_\tau)_H + \alpha(z_\tau - z_\varepsilon), u_\tau - u_\varepsilon)_H \tag{3.14}$$

It can be shown that from (3.14), using (3.8)÷(3.11), we obtain

$$\sigma \|u_\tau - u_\varepsilon\|_H \le C \sup\{ \|u_\tau\|_H , \|u_\varepsilon\|_H \} |\tau - \varepsilon| \tag{3.15}$$

therefore (3.12) follows from (3.15).

q.e.d.

Let us consider the existence of the right-derivative at $\tau=0$ of optimal solution u_τ with respect to the parameter τ. It follows by Proposition 3, that for any sequence $\{\varepsilon_n\}$, $n=1,2,\ldots$, $\varepsilon_n' \downarrow 0$ with $n \to \infty$, there exists a subsequence, still denoted $\{\varepsilon_n\}$ and an element $q_\tau \in H$ such that for $\tau > 0$, small enough

$$u_{\tau+\varepsilon_n} = u_\tau + \varepsilon_n q_\tau + o(\varepsilon_n) \tag{3.16}$$

where $o(\varepsilon_n)/\varepsilon_n \longrightarrow 0$ weakly in H with $n \to \infty$. On the other hand from (3.7) it follows that

$$u_\tau = P_K(F_\tau) \tag{3.17}$$

where

$$F_\tau \overset{\text{def}}{=} z_\tau - \frac{1}{\alpha} L_\tau^* DI_\tau(L_\tau u_\tau) \tag{3.18}$$

furthermore, using (H1) for $\tau \in [0,\delta)$, it follows that

$$F_{\tau+\varepsilon_n} = F_\tau + \varepsilon_n F_\tau' + o(\varepsilon_n) \tag{3.19}$$

where $\|o(\varepsilon_n)\|_H / \varepsilon_n \to 0$ with $n \to \infty$, and

$$F_\tau' = z_\tau' - \frac{1}{\alpha}(L_\tau')^* DI_\tau(L_\tau u_\tau) - \frac{1}{\alpha} L_\tau^* D^2 I(L_\tau u_\tau ; L_\tau' u_\tau - L_\tau q_\tau) \tag{3.20}$$

provided the element $z_\tau' \in H$ verifies:

$$\lim_{n\to\infty} \|(z_{\tau+\varepsilon_n} - z_\tau)/\varepsilon_n - z'_\tau\|_H = 0$$

If the projection in H onto K is conically differentiable then by (2.11), (2.13), (3.19) it follows that

$$P_K(F_{\tau+\varepsilon_n}) = P_K(F_\tau) + \varepsilon_n Q(F'_\tau) + o(\varepsilon_n) \tag{3.21}$$

Thus we are in position to characterize an element q_τ in (3.16) in the following way.

Proposition 4

The right-derivative q_τ verifies the following conditions

$$q_\tau \in S_K(F_\tau) \tag{3.22}$$

$$q_\tau = Q(F'_\tau) \tag{3.23}$$

Remark:

Let us note that element F'_τ in (3.23) depends on q_τ, i.e. by (3.23) an element q_τ is the fixed point. We shall use the condition (3.23) in order to prove uniqueness of the element q_τ.

We will need the following notation:

$$b_\tau(u,v) \overset{\text{def}}{=} \frac{1}{\alpha}(L_\tau^* D^2 I_\tau(L_\tau u_\tau; L_\tau u), v)_H + (u,v)_H \;, \quad \forall u,v \in H \tag{3.24}$$

$$\eta_\tau \overset{\text{def}}{=} z'_\tau - \frac{1}{\alpha}(L'_\tau)^* DI_\tau(L_\tau u_\tau) - \frac{1}{\alpha} L_\tau^* D^2 I_\tau(L_\tau u_\tau; L'_\tau u_\tau) \tag{3.25}$$

Let us consider the case of polyhedric set K.

Theorem 2

Assume that

(i) set $K \subset H$ is polyhedric

(ii) bilinear form $b_\tau(.,.)$ is continuous and coercive uniformly with respect to $\tau \in [0,\delta)$, i.e. there exists a positive constant $\varepsilon > 0$ such that

$$b_\tau(u,u) \geq \varepsilon \, \|u\|_H^2 \;, \quad \forall u \in H, \; \forall \tau \in [0,\delta) \tag{3.26}$$

then right-derivative $q_\tau \in H$ is given by

$$\begin{aligned} & q_\tau \in S_K(F_\tau) \\ & b_\tau(q_\tau, v_\tau - q_\tau) \geq (\eta_\tau, v_\tau - q_\tau)_H, \quad \forall v_\tau \in S_K(F_\tau) \end{aligned} \tag{3.27}$$

and therefore the element q_τ is uniquely determined for any $\tau \in [0,\delta)$.

Proof:

From Theorem 1, using (3.23), it follows that

$$q_\tau = P_{S_K(F_\tau)}(F'_\tau) \;, \quad \tau \in [0,\delta) \tag{2.28}$$

then simple calculations show, in view of (3.20), (3.24), (3.25), that the element q_τ satisfies (3.27). Uniqueness of the element q_τ follows by standard argument.

q.e.d.

Let us consider an optimal control problem for a partial differential equation of parabolic type. We derive the form of right-derivative of an optimal control with respect to the parameter.

Example 5

Let $\Omega \subset R^n$ be a given domain with smooth boundary $\Gamma = \partial\Omega$, we denote $Q = \Omega \times (0,T)$ where $T > 0$ is a given constant, $\tau \in [0,\delta)$ is parameter. The optimal control problem considered here consists of state equation, cost functional and constraints of the form:

State equation:

$$\frac{\partial y^\tau}{\partial t}(x,t)-\Delta y^\tau(x,t) = \sum_{i=1}^{N} u_i(t)\psi_i(\tau,x) \quad \text{in } Q$$

$$y(x,t) = 0 \qquad \text{on } \Sigma = \Gamma \times (0,T) \tag{3.29}$$

$$y(x,o) = 0 \qquad \text{in } \Omega$$

where $\psi_i(\cdot,\cdot) \in C(0,\delta;L^2(\Omega))$, $i=1,\ldots,N$ are given elements, $u=(u_1,\ldots,u_N) \in [L^2(0,T)]^N$ is a control.

Cost functional:

$$J_\tau(u) = \frac{1}{2}\int_\Omega (y^\tau(x,T)-z_d(x))^2 dx + \frac{\alpha}{2}\sum_{i=1}^{N}\int_o^T (u_i(t))^2 dt, \quad \alpha > 0 \tag{3.30}$$

where $z_d(\cdot) \in L^2(\Omega)$ is a given element.

Constraints:

Set of admissible controls is given by:

$$K = \{u \in [L^2(0,T)]^N \mid \frac{1}{2}\sum_{i=1}^{N} a_i u_i^2(t) \le 1 \quad \text{for a.a.} \quad t \in (0,T)\} \tag{3.31}$$

By standard argument it follows that for any $\tau \in [0,\delta)$ there exists a unique element $u^\tau \in [L^2(0,T)]^N$ which minimizes the cost functional (3.30) over the set (3.31). An optimal control satisfies the following optimality system [3]:

Adjoint-state equation:

$$\frac{\partial p^\tau}{\partial t}(x,t)-\Delta p^\tau(x,t)=0 \qquad \text{in } Q$$

$$p^\tau(x,t)=0 \qquad \text{on } \Sigma \tag{3.32}$$

$$p^\tau(x,T)=z_d(x)-y^\tau(x,T) \qquad \text{in } \Omega$$

Optimality conditions:

$$u^\tau \in K$$

$$\int_o^T \langle u^\tau(t)-\eta^\tau(t), v(t)-u^\tau(t) \rangle_{R^N} dt \ge 0, \quad \forall v \in K \tag{3.33}$$

where $\eta^{\tau}=(\eta_1^{\tau},\dots,\eta_N^{\tau}) \in [C(0,T)]^N$,

$$\eta_i^{\tau}(t)=\frac{1}{\alpha}\int_{\Omega} p^{\tau}(x,t)\psi_i(\tau,x)dx\ ,\quad i=1,\dots,N.$$

In order to prove the existence of right-derivative $q=q_o$ of an optimal control u^{τ} with respect to the parameter τ we need the following assumption:

(H2) functions $\psi_i(.,.) \in C(0,\delta;L^2(\Omega))$, $i=1,\dots,N$ are right-differentiable with respect to τ at $\tau=0$, i.e. there exist elements $\psi_i'(.) \in L^2(\Omega)$ such that for $\tau > 0$, τ small enough:

$$\psi_i(\tau,.)=\psi_i(0,.)+\tau\psi_i'(.)+o(\tau) \quad \text{in} \quad L^2(\Omega) \qquad (3.34)$$

where $\|o(\tau)\|_{L^2(\Omega)}/\tau \to 0$ with $\tau \downarrow 0$, $i=1,\dots,N$.

It can be shown that in this case the assumptions of Proposition 3 are verified, thus the optimal control u^{τ} is Lipschitz continuous with respect to the parameter:

$$\|u^{\tau}-u^{o}\|_{[L^2(0,T)]^N} \le C\tau\ , \quad \tau \in [0,\delta) \qquad (3.35)$$

hence there exists an element $q \in S_K(\eta^o)$ such that for $\tau > 0$, τ small enough:

$$u^{\tau} = u^{o} + \tau q + r_o(\tau) \quad \text{in} \quad [L^2(0,T)]^N \qquad (3.36)$$

where $r_o(\tau)/\tau \to 0$ weakly in $[L^2(0,T)]^N$ with $\tau\downarrow 0$.

Using (3.26), in view of (3.29), (3.32), it can be verified that

$$y^{\tau} = y^{o} + \tau w + r_1(\tau) \quad \text{in} \quad W(0,T) \qquad (3.37)$$

$$p^{\tau} = p^{o} + \tau z + r_2(\tau) \quad \text{in} \quad W(0,T) \qquad (3.38)$$

where $\|r_i(\tau)\|_{L^2(Q)}/\tau \to 0$, $i=1,2$ with $\tau \to 0$ and elements $w,z \in W(0,T)$ are given by:

$$\frac{\partial w}{\partial t}(x,t)-\Delta w(x,t) = \sum_{i=1}^{N} (u_i^o(t)\psi_i'(x)+q_i(t)\psi_i(0,x)) \quad \text{in } Q$$

$$w(x,t)=0 \quad \text{on } \Sigma \qquad (3.39)$$

$$w(x,0)=0 \quad \text{in } \Omega$$

$$-\frac{\partial z}{\partial t}(x,t) - \Delta z(x,t) = 0 \quad \text{in } Q$$

$$z(x,t)=0 \quad \text{on } \Sigma \qquad (3.40)$$

$$z(x,T)=-w(x,T) \quad \text{in } \Omega$$

respectively.

Since $u^\tau=P_K(\eta^\tau)$, furthermore by (3.34),(3.38) there exists an element $\eta'\in[L^2(0,T)]^N$ such that

$$\eta^\tau=\eta^o+\tau\eta'+o(\tau) \quad \text{in} \quad [C(0,T)]^N \qquad (3.41)$$

where $\|o(\tau)\|_{[L^2(0,T)]^N}/\tau \to 0$ with $\tau\downarrow 0$,

$$\eta'=(\eta_1',\dots,\eta_N')$$

$$\eta_i'=\frac{1}{\alpha}\int_\Omega(p^o(x,t)\psi_i'(x)+z(x,t)\psi_i(0,x))dx \qquad (3.42)$$

then taking into account Example 4, it can be shown that right-derivative q is given by:

$$q\in S_K(\eta^o)$$

$$\int_0^T \langle A(t)q(t)-\eta'(t),v(t)-q(t)\rangle_{R^N} dt \geq 0 \qquad (3.43)$$

$$\forall v\in S_K(\eta^o)$$

where

$$A(t)=[(1+\lambda(t)a_i)\delta_{ij}]_{N\times N}\quad,\quad t\in(0,T) \qquad (3.44)$$

$$\lambda(t)=\begin{cases}\|\eta^o(t)-u^o(t)\|_{R^n}/\|b(t)\|_{R^n}, & \text{if } \eta^o(t)\notin U\\ 0 & \text{if } \eta^o(t)\in U\end{cases} \qquad (3.45)$$

$$t\in(0,T)$$

here $b(t)=(a_1u_1^o(t),\dots,a_Nu_N^o(t))$, the set U is given by (2.48) and the cone $S_K(\eta^o)$ is given by:

$$S_K(\eta^o)=\{v(.)\in\left[L^2(0,T)\right]^N \mid \langle b(t),v(t)\rangle_{R^N}\le 0$$
$$\lambda(t)\langle b(t),v(t)\rangle_{R^N}=0 \quad \text{for a.a. } t\in(0,T)\} \tag{3.46}$$

Using (3.39), (3.40) and (3.43) it can be verified that the right-derivative q is a unique solution of an optimal control problem. The optimal control problem consists of state equation (3.39), cost functional of the form:

$$I(q)=\frac{1}{2}\int_Q (w(x,t))^2 dQ+\frac{\alpha}{2}\int_o^T \|A^{1/2}(t)q(t)-A^{-1/2}(t)\theta(t)\|^2_{R^N}dt$$

and the set of admissible controls given by (3.46).

We denote here $\theta(t)=(\theta_1(t),\dots,\theta_N(t))$,

$$\theta_i(t)=\int_\Omega p^o(x,t)\psi_i'(x)dx\ ,\quad t\in(0,T)\ ,\ i=1,\dots,N$$

4. Differential Stability of the Right-Derivative

In this section we present a method which can be used in order to determine the second right-derivative of the optimal solution of an optimization problem.

Let us consider the following simple example.

Example 6

Let us consider projection $P_K(.)$ in R^n onto the set (2.26). We take for simplicity $\psi(v)=\frac{1}{2}\sum_{i=1}^n a_iv_i^2-1$, $a_i>0$, $i=1,\dots,n$. Given a C^2 function $f(.):[0,\delta)\to R^n$, we denote $u^\tau=P_K(f(\tau))$, $h=f'(0)$, $\omega=f''(0)$. For any $\tau\in[0,\delta)$ the right-derivative q_τ of u_τ is given by:

$$q_\tau \in S_K(f(\tau))$$
$$< A_\tau q_\tau - f'(\tau), v_\tau - q_\tau >_{R^n} \geq 0, \ \forall v_\tau \in S_K(f(\tau)) \tag{4.1}$$

where A_τ, λ_τ', $S_K(f(\tau))$ are defined by (2.28), (2.29) and (2.30), respectively.

In order to determine the second right-derivative let us suppose that $f(\tau) \notin K$, $\tau \in [0,\delta)$, furthermore components of vector $u_o=(u_o^1,\ldots,u_o^n)$ verify $u_o^i \neq 0$, $i=1,\ldots,n$.

We denote $q^\tau \overset{\text{def}}{=} L^{-1} q_\tau$, where $L_\tau^{-1}=[(u^i/u_o^i)\delta_{ij}]_{xn}$. The element q^τ is given by:

$$q^\tau \in S_K(f(0))$$
$$< A_\tau L_\tau q^\tau - f'(\tau), L_\tau(v-q^\tau) >_{R^n} \geq 0 \ , \ \ \forall v \in S_K(f(0)) \tag{4.2}$$

It can be verified, using (4.2), that $\|q^\tau - q^o\|_{R^n} \leq C\tau$ therefore [9] there exists a unique element

$$\dot{q} \overset{\text{def}}{=} \lim_{\tau \downarrow 0}(q^\tau - q^o)/\tau \tag{4.3}$$

given by

$$\dot{q} \in S_K(f(0))$$
$$< A_o\dot{q}, v-\dot{q} >_{R^n} \geq < L_o'h + \omega - (A_o'+2L_o'A_o)q_o, v-q >_{R^n} \tag{4.4}$$
$$\forall v \in S_K(f(0))$$

hence there exists a unique element

$$q' \overset{\text{def}}{=} \lim_{\tau \downarrow o}(q_\tau - q_o)/\tau \tag{4.5}$$

given by

$$q' = \dot{q} - (L_o^{-1})' q_o \tag{4.6}$$

Simple calculations show, in view of (4.4),(4.6), that the second right-derivative q' is a unique solution of the following variational inequality:

$$q' \in M = \{v \in R^n \mid D\psi(u_o).v = -q_o^T.D^2\psi(u_o).q_o\}$$
$$\langle A_o q' - \chi, v-q' \rangle_{R^n} \geq 0, \quad \forall v \in M \tag{4.7}$$

here $\chi_i = \omega_i + 2q_o^i(h_i + (1+\lambda_o a_i)q_o^i)/u_o^i$, $i=1,\ldots,n$. ■

Let us consider an abstract variational inequality:

$$q_\tau \in S_\tau$$
$$a_\tau(q_\tau, v_\tau - q_\tau) \geq (\xi_\tau, v_\tau - q_\tau), \quad \forall v_\tau \in S_\tau \tag{4.8}$$

here $\tau \in [0,\delta)$ is a parameter, $S_\tau \subset H$ is a convex and closed cone e.g. $S_\tau = S_K(f(\tau))$, $f(\tau) \in H$, $a_\tau(.,.) : H \times H \mapsto R$ is a continuous and coercive bilinear form, $\xi_\tau \in H, \tau \in [0,\delta)$ are given elements.

In order to determine the right-derivative q' of q_τ with respect to the parameter τ we the need following assumptions which allows us to obtain from (4.8) a variational inequality defined on the cone S_o.

(H3) We assume that for any $\tau \in [0,\delta)$ there exists a linear mapping $\Phi_\tau \in L(H;H)$ such that

$\Phi_o = I$, I is identity mapping in H

$\Phi_\tau^{-1} \in L(H;H)$, $\forall \tau \in [0,\delta)$

$\Phi_\tau v \in S_\tau$ iff $v \in S_o$

and there exists an element $\Phi_o' \in L(H;H)$ such that

$\Phi_o' = \lim_{\tau \downarrow 0}(\Phi_\tau - \Phi_o)/\tau$ in $L(H,H)$.

We denote

$$q' = \lim_{\tau \downarrow 0}(q_\tau - q_o)/\tau \quad \text{in} \quad H$$

$$\dot{q} = \lim_{\tau \downarrow 0}(\Phi_\tau q_\tau - q_o)/\tau \quad \text{in } H$$

The element $q^\tau \overset{\text{def}}{=} \Phi_\tau q_\tau$ satisfies variational inequality:

$$q_\tau \in S_o$$
$$a^\tau(q^\tau, v-q^\tau) \geq (\xi_\tau, \Phi_\tau^{-1}(v-q^\tau))_H \qquad \forall v \in S_o$$

where $a^\tau(u,v) = a_\tau(\Phi_\tau^{-1}u, \Phi_\tau^{-1}v)$, $\forall u,v \in H$.

Existence of the element $\dot{q} \in H$ can be shown, using standard argument e.g. if the set S_o is polyhedric. Then the element q' can be determined from the following formula:

$$q' = \mathring{q} - \Phi_o' q_o$$

The method described here is used in shape sensitivity analysis of unilateral problems [10]. We present here an application of the method in the case of a simple example.

Example 7

Let us consider the following optimization problem

$$(\Pi_\tau) \qquad \min\{ \int_\Omega (\tfrac{1}{2}(v(x))^2 - (g(x)-\tau)v(x))dx \mid v(x) \geq 0 \text{ a.e. in } \Omega,\ v(.) \in L^2(\Omega)\}$$

where $\Omega \subset R^2$, $g(.)$ is a smooth function, strictly positive on Ω, $g(x)=0$, $x \in \partial\Omega$ and $\|\nabla g(x)\|_{R^2} > 0$ in Ω except for a one point $\bar{x}$ of Ω, $\tau \in [0,\delta)$.

Optimal solutions is given by:

$$u_\tau(x) = \max\{0, g(x)-\tau\}, \quad x \in \Omega$$

The variation q_τ of the optimal solution minimizes the functional $I(v) = \int_{\Omega \setminus \Omega_\tau} \{\frac{1}{2}v^2(x) - v(x)\}dx$ subject to $\{v \in S_\tau = v \in L^2(\Omega) \mid v(x)=0$ a.e. on $\Omega_\tau\}$ where $\Omega_\tau = \{x \in \Omega \mid g(x) \leq \tau\}$. It can be verified that element q_τ is given by:

$$q_\tau(x) = \begin{cases} -1 & \text{if} \quad x \in \Omega \setminus \Omega_\tau \\ 0 & \text{if} \quad x \in \Omega_\tau \end{cases}$$

Using the method described here and the results of J.P.Zolesio [11], it can be shown that the second variation of the optimal solution q' is a distribution of the form:

$$< q'_\tau, \phi > = \int_{\partial\Omega_\tau} \|\nabla g(x)\|^{-1}_{R^n} \frac{\partial\phi}{\partial n}(x)d\sigma(x) \qquad \forall\phi\in\mathcal{D}'(\Omega)$$

for $\tau > 0$, τ small enough.

References

[1] A. Haraux, How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities. J. Math. Soc. Japan, Vol.29, No.4 (1977), 615-631.

[2] K. Jittorntrum, Solution point differentiability without strict complementarity in nonlinear programming, Mathematical Programming Studies, A.V.Fiacco (ed.), North-Holland , Amsterdam.

[3] J.L. Lions, Sur le contrôle optimal de systèmes gouvernes par des équations aux dérivées partielles, Paris, Dunod, 1968.

[4] J.L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, Vol.2, Paris, Dunod, 1968.

[5] K. Malanowski, Differential stability of solutions to convex, control constrained optimal control problems (to be published).

[6] F. Mignot, Controle dans les inequations variationnelles, J. Functional Analysis, 22 (1976).

[7] R.T. Rockafellar, La theorie des sous-gradients et ses applications à l'optimisation, les Presses de l'Université de Montréal, 1978.

[8] J. Sokołowski, Sensitivity analysis of control constrained optimal control problems for distributed parameter systems (to be published).

[9] J. Sokołowski, Conical differentiability of projection on convex sets - an application to sensitivity analysis of Signorini V.I., Technical Report, Institute of Mathematics of the University of Genova (Italy), 1981.

[10] J. Sokołowski, J.P. Zolesio, Derivation par rapport au domaine dans les problemes unilateraux, INRIA, Rapport de Recherche No.132, 1982.

[11] J.P. Zolesio, The material derivative (or speed) method for shape optimization, in: Optimization of Distributed Parameter Structures, Vol.2, eds. E.J.Haug, J.Cea, Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands, 1981.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

Batch003 031