A detailed analysis of a queueing model with alternating priority

Philippe Nain

CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:954 90 20

Rapports de Recherche

N° 290

A DETAILED ANALYSIS OF A QUEUEING MODEL WITH ALTERNATING PRIORITY

Philippe NAIN

Mai 1984

N° 290

A DETAILED ANALYSIS OF

A QUEUEING MODEL WITH ALTERNATING PRIORITY

Philippe NAIN

INRIA

Domaine de Voluceau - Rocquencourt

B.P. 105- 78153 LE CHESNAY CEDEX (France)

Résumé

Le modèle étudié se compose de deux files d'attente et un seul serveur. Les clients d'une même file sont servis sans interruption, et dans l'ordre de leur arrivée, jusqu'à ce qu'elle soit vide. Le serveur traite ensuite suivant le même mode les clients de l'autre file. La commutation de l'unité de service d'une file à l'autre est supposée instantanée. Suivant une approche par processus régénératifs, NEUTS et YADIN [8] ont obtenu des résultats caractérisant le comportement transitoire et stationnaire de ce système.

L'analyse markovienne développée ici et qui conduit à la résolution de deux équations fonctionnelles à deux variables, permet de retrouver comme cas particuliers les résultats de [8], et ce d'une façon plus agréable. De nouvelles caractérisations de processus d'intérêt sont aussi données.

Abstract

We consider a queueing model consisting of two queues and one single server working under the alternating priority rule with zero changeover times. Based on a regenerative processes approach, NEUTS and YADIN [8] have obtained results which characterize the transient and asymptotic behavior of this system. The markovian analysis we develop in this paper and which leads to the resolution of two functional equations of two variables, allows us to get, as particular cases, the results of [8] in a more tractable and self-contained way. New interesting queueing processes are also investigated.

INTRODUCTION

We consider a queueing model consisting of two queues and one single server. Arrivals at queue i ($i = 1,2$) form an homogeneous Poisson process with finite intensity λ_i. Service times of customers of queue i ($i = 1,2$) are i.i.d. random variables with an absolutely continuous but otherwise arbitrary distribution $B_i(t)$ ($t \geq 0$) and finite mean α_i. Let $\beta_i(\sigma)$ ($i = 1,2$), $\mathrm{Re}\ \sigma \geq 0$, be the Laplace-Stieltjes transform (L.S.T.) of $B_i(t)$. All these processes are supposed to be mutually independent.

In each queue the service discipline is first in - first out. The service priority called in the literature the "alternating priority with zero changeover times" is as follows : the server serves all the customers of a given queue until it is empty and switches to the other queue if it is non-empty and repeats the algorithm. If it is empty, the server will serve the first arriving customer, which will initiate a new busy period of the system.

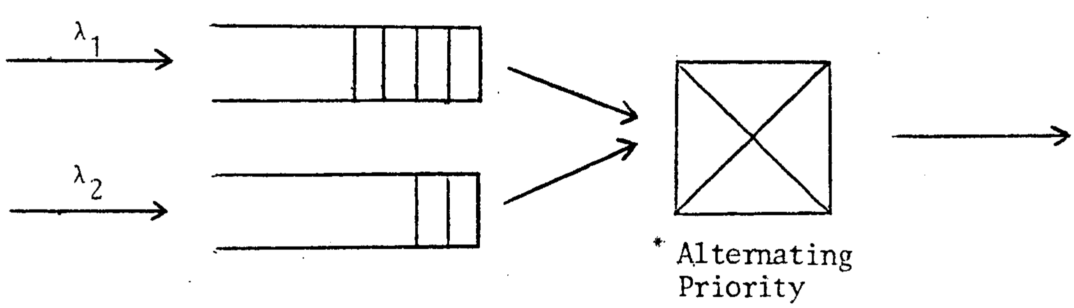
The switching from one queue to the other will be supposed to be instantaneous (For the alternative model with non-zero changeover times see EISENBERG [5], MILLER [7], SYKES [10], MEVERT [6]). The stationary behavior of this queueing model has been previously studied by TAKACS [11] and AVI-ITZHAK, MAXWELL and MILLER [1] for two queues and by COOPER and MURRAY [4] in the case of M queues.

NEUTS and YADIN [8] have investigated the transient and asymptotic behavior of this queueing model, in the case of two queues.

Their approach is based on a regenerative analysis of the system including imbedded semi-Markov processes and renewal theory.

The goal of our paper is mainly to provide an alternative to the regenerative analysis proposed by NEUTS and YADIN [8]. Indeed, it turns out that a direct approach, involving a Markov process, and leading to the resolution of functional equations, makes the analysis more tractable.

All the results contained in [8] are also derived by this approach. Moreover, some of the results we find are slightly more general than the ones obtained in [8] (for instance, including residual service time) or new (workload process). An illustration of the method we use can be found in [2], Chapter III.



Denote by $Y_i(t)$, $t \geq 0$, $i = 1,2$ the number of customers in queue i -also called the customers of type i- at time t. In order to investigate the stochastic process $\{(Y_1(t), Y_2(t)), t \geq 0\}$ we have to introduce two supplementary random variables.

At time t, let $Z(t)$ be the type of the customer in service and $R(t)$ the residual service time. It turns out that the stochastic process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ is a Markov process.

Writting the Chapman-Kolmorogov equations, we find (Section 1) a set of partial differential equations which are satisfied by the joint probability distribution $P(Y_1(t) = k_1,\ Y_2(t) = k_2,$ $Z(t) = i,\ R(t) < \tau\ /\ (Y_1(0),\ Y_2(0)))$ for $t \geq 0,\ (k_1,k_2) \in \mathbb{N}^2,\ \tau > 0,$ $i = 0,1,2,\ (Y_1(0),\ Y_2(0)) \in \mathbb{N}^2$.

Introducing Laplace-Stieljes transforms and generating functions, this set of partial differential equations is then transformed (Section 2) into a set of two functional equations which is solved (Section 3). Series of functions are involved in the solution of this set of functional equations.

Other queueing quantities are then derived (Section 4) as the virtual waiting time and the workload of the server at time t, $t \geq 0$. Finally, we give a description of the previous processes for $t \to +\infty$ (Section 5) and some mean queueing quantities (Section 6).

1 - Definitions and basic equations

Let us define for $i = 1,2$; $t \geq 0$,

$Y_i(t)$ = the number of customers in queue i at time t ,

$Z(t)$ a stochastic variable with state space $\{0,1,2\}$ and where

$Z(t) = 0$ if no customers are present in the system at time t,

$= i$ if a customer of type i is served at time t,

$R(t)$ = the residual service time of the customer served at time t.

We further assume that $t = 0$ can be considered as a service completion instant.

We denote $y = (y_1, y_2)$ and $Y(0) = (Y_1(0), Y_2(0))$ for $y_1, y_2 = 0,1,2 \ldots$ For $k_1, k_2, y_1, y_2 = 0,1,2\ldots$; $i = 1,2$; $\tau \geq 0$; $t \geq 0$, let

$$Q_i^y(t;k_1,k_2,\tau) = P(Y_1(t) = k_1, Y_2(t) = k_2, Z(t) = i, R(t) < \tau / Y(0) = y) \quad (1.1)$$

and

$$Q_0^y(t) = P(Y_1(t) = 0, Y_2(t) = 0 / Y(0) = y), \quad (1.2)$$

these probabilities being continuous from the right in the variable t.

We now introduce Laplace-Stieljes transforms and generating functions of the state probabilities defined in (1.1) and (1.2) (see [2]). In the following, we will assume that $(\rho, \sigma, p_1, p_2) \in \mathbb{C}^4$ satisfy

$$\text{Re } \rho > 0, \ \text{Re } \sigma \geq 0, \ |p_1| \leq 1, \ |p_2| \leq 1. \quad (1.3)$$

We define for $i = 1,2$; $k_1, k_2 = 0,1,2,\ldots$; $t > 0$,

$$\tilde{Q}_i^y(t;k_1,k_2) = \lim_{\tau \downarrow 0} \frac{\partial}{\partial \tau} Q_i^y(t;k_1,k_2,\tau). \quad (1.4)$$

assuming that this limit exists (which is the case if $B_i(t)$ possesses smoothness properties),

$$\Omega_i^y(\rho;p_1,p_2,\sigma) = \int_0^\infty e^{-\rho t} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty p_1^{k_1} p_2^{k_2} \int_{0^-}^\infty e^{-\sigma\tau} d_\tau Q_i^y(t,k_1,k_2,\tau)\, dt, \quad (1.5)$$

$$\Omega_0^y(\rho) = \int_0^\infty e^{-\rho t} Q_0^y(t)\, dt, \quad (1.6)$$

$$\tilde{\Omega}_i^y(\rho;p_1,p_2) = \int_0^\infty e^{-\rho t} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty p_1^{k_1} p_2^{k_2} \tilde{Q}_i^y(t,k_1,k_2)dt, \quad (1.7)$$

$$Z_1^Y(\rho;p_2) = \lim_{p_1 \downarrow 0} \frac{1}{p_1} \tilde{\Omega}_1^Y(\rho;p_1,p_2) \quad , \tag{1.8}$$

$$Z_2^Y(\rho;p_1) = \lim_{p_2 \downarrow 0} \frac{1}{p_2} \tilde{\Omega}_2^Y(\rho;p_1,p_2) \; . \tag{1.9}$$

From the previous definitions we note that

$$\Omega_1^Y(\rho;0,p_2,\sigma) = 0 \quad , \quad \Omega_2^Y(\rho;p_1,0,\sigma) = 0 \; , \quad \tilde{\Omega}_1^Y(\rho;0,p_2) = 0 \quad \text{and} \quad \tilde{\Omega}_1^Y(\rho;p_1,0) = 0.$$

These relations imply the existence of $Z_1^Y(\rho;p_2)$ and $Z_1^Y(\rho;p_1)$ defined in (1.8) and (1.9).

Finally the following function will be needed,

$$\Omega^Y(\rho;p_1,p_2,\sigma) = \int_0^\infty e^{-\rho t} \; E\{p_1^{Y_1(t)} \; p_2^{Y_2(t)} \; e^{-\sigma R(t)} \; / \; Y(0) = y\} \; dt \; . \tag{1.10}$$

We immediately notice that,

$$\Omega^Y(\rho;p_1,p_2,\sigma) = \Omega_0^Y(\rho) + \sum_{i=1}^{2} \Omega_i^Y(\rho;p_1,p_2,\sigma) \tag{1.11}$$

We will also assume the existence and the continuity of the following partial derivatives ,

$$\frac{\partial}{\partial t} Q_i^Y(t \; ; \; k_1,k_2,\tau) \quad , \quad \frac{\partial}{\partial t} Q_0^Y(t) \; , \quad \frac{\partial}{\partial \tau} Q_i^Y(t \; ; \; k_1,k_2,\tau) \; , \tag{1.12}$$

for $i = 1,2$; $k_1,k_2 = 0,1,2,\ldots$; $\tau > 0$; $t > 0$.

From the model assumptions, it is readily seen that the stochastic process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ is a Markov process with

(minimal) state space $\{0,0,0,0\} \cup \{\mathbb{N}^* \times \mathbb{N} \times \{1\} \times \mathbb{R}^{+*}\} \cup \{\mathbb{N} \times \mathbb{N}^* \times \{2\} \times \mathbb{R}^{+*}\}$
So the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ is irreducible.

The following theorem provides a set of partial differential equations which are satisfied by the state probabilities (1.1) and (1.2).

These equations fully describe the time-evolution of the system.

Theorem 1.1

Under the assumptions made in (1.4) and (1.12), the state probabilities (1.1), (1.2) satisfy for $t > 0$, $\tau > 0$ the following equations,

i) for $k_1 \geq 1$, $k_2 \geq 1$,

$$\frac{\partial}{\partial t} Q_1^Y(t;k_1,k_2,\tau) = \lambda_1 Q_1^Y(t;k_1-1, k_2,\tau) + \lambda_2 Q_1^Y(t,k_1,k_2-1,\tau)$$

$$- (\lambda_1+\lambda_2) Q_1^Y(t;k_1,k_2,\tau) + \frac{\partial}{\partial \tau} Q_1^Y(t,k_1,k_2,\tau) - \tilde{Q}_1^Y(t,k_1,k_2)$$

$$+ B_1(\tau) \tilde{Q}_1^Y(t;k_1 + 1, k_2) , \tag{1.13}$$

for $k_1 \geq 2$, $k_2 = 0$,

$$\frac{\partial}{\partial t} Q_1^Y(t;k_1,0,\tau) = \lambda_1 Q_1^Y(t;k_1 - 1,0,\tau) - (\lambda_1+\lambda_2) Q_1^Y(t;k_1,0,\tau)$$

$$+ \frac{\partial}{\partial \tau} Q_1^Y(t;k_1,0,\tau) - \tilde{Q}_1^Y(t;k_1,0) + B_1(\tau) [\tilde{Q}_1(t;k_1+1,0) + \tilde{Q}_2(t;k_1,1)], \tag{1.14}$$

for $k_1 = 1$, $k_2 = 0$,

$$\frac{\partial}{\partial t} Q_1^Y(t;1,0,\tau) = \lambda_1 Q_0^Y(t)\ B_1(\tau) - (\lambda_1+\lambda_2)\ Q_1^Y(t;1,0,\tau) + \frac{\partial}{\partial \tau}\ Q_1^Y(t;0,1,\tau)$$

$$- \tilde{Q}_1^Y(t;1,0) + B_1(\tau)\ \ [\tilde{Q}_1^Y(t;2,0) + \tilde{Q}_2^Y(t;1,1)]\ , \tag{1.15}$$

ii) an analogous set of equations holds for $Q_2^Y(t;k_1,k_2,\tau)$ for $k_1 \geq 0$, $k_2 \geq 0$,

iii) $$\frac{\partial}{\partial t}\ Q_0^Y(t) = -(\lambda_1+\lambda_2)\ Q_0^Y(t) + \tilde{Q}_1^Y(t;1,0) + \tilde{Q}_2^Y(t;0,1). \tag{1.16}$$

<u>Proof</u>

Let us consider a small time interval $(t-h, t]$ and the events (arrivals or departures) which occur in this interval. Because arrival processes are Poisson processes, we get for $k_1 \geq 1$, $k_2 \geq 1$, $\tau > 0$,

$$Q_1^Y(t;k_1;k_2,\tau) = \lambda_1 h Q_1^Y(t-h;k_1-1,k_2,\tau+h) + \lambda_2 h Q_1^Y(t-h;k_1,k_2-1,\tau+h)$$

$$+ [1-(\lambda_1+\lambda_2)h]\ \ [Q_1^Y(t-h;\ k_1,k_2,\ \tau+h) - Q_1^Y(t-h,\ k_1,k_2,h)$$

$$+ \int_0^h B_1(\tau+h-x)\,d_x\ Q_1^Y(t-h,\ k_1+1,\ k_2,\ x)] + o(h).$$

Substracting $Q_1^Y(t-h;\ k_1,k_2,\tau)$ from both sides of this equation, dividing it by h and letting $h \downarrow 0$ we obtain equation (1.13) using (1.4) and the assumptions on the existence of the partial derivatives (1.12).

The remaining equations (1.14), (1.15), (1.16) can be obtained in a similar way. □

2 - The functional equations

Using the Laplace-Stieljes transforms and the generating functions previously introduced, we transform the Chapman-Kolmogorov equations of Theorem 1.1 into a set of two functional equations.

Theorem 2.1

The transforms $\Omega_0^Y(\rho)$ and $\Omega_i^Y(\rho;p_1,p_2,\sigma)$, $i = 1,2$, of the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ satisfy for $\operatorname{Re}\rho > 0$, $\operatorname{Re}\sigma \geq 0$, $|p_1| \leq 1$, $|p_2| \leq 1$ the following functional equations,

$$[\rho - \sigma + \lambda_1(1-p_1)+\lambda_2(1-p_2)]\ \Omega_1^Y(\rho;p_1,p_2,\sigma) = -\tilde{\Omega}_1^Y(\rho;p_1,p_2)\ [1 - \frac{\beta_1(\sigma)}{p_1}]$$
$$+\beta_1(\sigma)\ [\lambda_1 p_1\ \Omega_0^Y(\rho) + Z_2^Y(\rho;p_1) - Z_1^Y(\rho;p_2) - Z_2^Y(\rho;0) + p_1^{y_1}\ \{I(y_1 > 0,\ y_2 = 0)$$
$$+ p_2^{y_2}\ I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 1)\}]\ , \qquad (2.1)$$

$$[\rho - \sigma + \lambda_1(1-p_1) + \lambda_2(1-p_2)]\ \Omega_2^Y(\rho;p_1,p_2,\sigma) = -\tilde{\Omega}_2^Y(\rho;p_1,p_2)\ [1 - \frac{\beta_2(\sigma)}{p_2}]$$
$$+ \beta_2(\sigma)\ [\lambda_2 p_2\ \Omega_0^Y(\rho) + Z_1^Y(\rho;p_2) - Z_2^Y(\rho;p_1) - Z_1^Y(\rho;0) + p_2^{y_2}\ \{I(y_1 = 0,\ y_2 > 0)$$
$$+ p_1^{y_1}\ I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 2)\}]\ , \qquad (2.2)$$

$$(\rho + \lambda_1 + \lambda_2)\ \Omega_0^Y(\rho) = I(y_1 = 0,\ y_2 = 0) + Z_1^Y(\rho;0) + Z_2^Y(\rho;0). \qquad (2.3)$$

where $I(A)$ is the indicator function of the event $\{A\}$.

Proof

The functional equation (2.1) is obtained from the partial differential equations (1.13), (1.14), (1.15) and by using the following results, (see [2])

for $k_1 \geq 1$, $k_2 \geq 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, $t > 0$, $\tau > 0$, $\mathrm{Re}\,\rho > 0$, $\mathrm{Re}\,\sigma \geq 0$ we have

$$\int_{0^-}^{\infty} e^{-\sigma\tau} Q_1^Y(t;k_1,k_2,\tau)d\tau = \frac{1}{\sigma} \int_{0^-}^{\infty} e^{-\sigma\tau} d_\tau Q_1^Y(t;k_1,k_2,\tau)$$

since $Q_1^Y(t;k_1,k_2,0) = 0$,

$$\int_0^{\infty} e^{-\rho t} \frac{\partial}{\partial t} Q_1^Y(t;k_1,k_2,\tau)dt = Q_1^Y(0;k_1,k_2,\tau) + \rho \int_0^{\infty} Q_1^Y(t;k_1,k_2,\tau)e^{-\rho t} dt,$$

$$\sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} \int_{0^-}^{\infty} e^{-\sigma\tau} d_\tau Q_1^Y(0;k_1,k_2,\tau) = \frac{\beta_1(\sigma)\, p_1^{Y_1}}{\sigma} [I(Y_1 > 0,\ Y_2=0)+ p_2^{Y_2} I(Y_1 > 0,$$

$Y_2 > 0,\ Z(0^+) = 1)]$,

$$\int_0^{\infty} e^{-\rho t} \sum_{k_1=1}^{\infty} \sum_{k_2=0}^{\infty} p_1^{k_1} p_2^{k_2} \tilde{Q}_1^Y(t;k_1+1,k_2)dt = \frac{\tilde{\Omega}_1^Y(\rho;p_1,p_2)}{p_1} - Z_1^Y(\rho;p_2)$$

and

$$\int_0^{\infty} e^{-\rho t} \sum_{k_1=1}^{\infty} p_1^{k_1} \tilde{Q}_2^Y(t;k_1,1)dt = Z_2^Y(\rho;p_1) - Z_2^Y(\rho;0) .$$

The functional equations (2.2), (2.3) are obtained in a similar way. □

We then notice from Theorem 2.1 the following important fact : the sought functions $\Omega_i^Y(\rho;p_1,p_2,\sigma)$, $i = 1,2$, and $\Omega_0^Y(\rho)$ will be completely determined by equations (2.1), (2.2) and (2.3) once we will know the intermediate functions $Z_1^Y(\rho;p_2)$, $Z_2^Y(\rho;p_1)$ and $\tilde{\Omega}_i^Y(\rho;p_1,p_2)$, $i=1,2$. So, in the following, we will be only concerned with the determination of the functions $Z_1^Y(\rho;p_2)$, $Z_2^Y(\rho;p_1)$, $\Omega_0^Y(\rho)$ and $\tilde{\Omega}_i^Y(\rho;p_1,p_2)$ for $i=1,2$.

Theorem 2.2

The functions $\tilde{\Omega}_i^y(\rho;p_1,p_2)$, $i = 1,2$ satisfy for $\mathrm{Re}\ \rho > 0$, $|p_1| \le 1$, $|p_2| \le 1$ the following set of functional equations,

$$[p_1 - \beta_1(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2))]\ \tilde{\Omega}_1^y(\rho;p_1,p_2) = p_1\beta_1(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2))$$

$$[Z_2^y(\rho;p_1) - Z_1^y(\rho;p_2) - Z_2^y(\rho;0) + \lambda_1 p_1\ \Omega_0^y(\rho)$$

$$+ p_1^{y_1}\ \{I(y_1 > 0,\ y_2 = 0) + p_2^{y_2}\ I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 1)\}] , \qquad (2.4)$$

$$[p_2 - \beta_2(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2))]\ \tilde{\Omega}_2^y(\rho;p_1,p_2) = p_2\beta_2(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))$$

$$[Z_1^y(\rho;p_2) - Z_2^y(\rho;p_1) - Z_1^y(\rho;0) + \lambda_2 p_2\ \ \Omega_0^y(\rho) + p_2^{y_2}\ \{I(y_1 = 0,\ y_2 > 0) + p_1^{y_1}$$

$$I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 2)\}] . \qquad (2.5)$$

Proof

The functions $\Omega_i^y\ (\rho;p_1,p_2,\sigma)$ are sought analytic for $\mathrm{Re}\ \rho > 0$, $\mathrm{Re}\ \sigma \ge 0$, $|p_1| \le 1$, $|p_2| \le 1$.

Since $\mathrm{Re}\ \{\rho+ \lambda_1(1-p_1) + \lambda_2(1-p_2)\} > 0$ if $\mathrm{Re}\ \rho > 0$, $|p_1| \le 1$, $|p_2| \le 1$, then the right sides of equations (2.1) and (2.2) must vanish for $\sigma = \rho + \lambda_1(1-p_1) + \lambda_2(1-p_2)$.

Substituting this value of σ in equations (2.1) and (2.2), we readily get equations (2.4) and (2.5). □

Let us introduce the following notations,

$$Y_1^y(p_1,p_2) = p_1^{y_1}\left\{I(y_1 > 0,\ y_2 = 0) + p_2^{y_2}\ I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 1)\right\}$$

$$Y_2^y(p_1,p_2) = p_2^{y_2}\left\{I(y_1 = 0,\ y_2 > 0) + p_1^{y_1}\ I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 2)\right\} \quad (2.6)$$

Notice that the functions $Y_1^y(p_1,p_2)$ and $Y_2^y(p_1,p_2)$ are known functions.

The method we use for the resolution of the functional equations (2.4) and (2.5) generalizes the one used by TAKÁCS [11] (for $\rho = 0$ the kernels $p_i - \beta_i(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))$ for $i = 1,2$ of equations (2.4), (2.5) are those of the corresponding functional equations of TAKÁCS [11]).

First, we transform the set of functional equations of Theorem 2.2 into a new set of functional equations using the following lemma.

<u>Lemma 2.1</u>

For ρ fixed, Re $\rho > 0$, the set of functional equations,

$$\begin{cases} p_1 = \beta_1(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2)) \\ p_2 = \beta_2(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2)) \end{cases}$$

has exactly one solution $(p_1(\rho),\ p_2(\rho))$ in the region $|p_1| \leq 1$, $|p_2| \leq 1$.

<u>Proof</u>

A proof of this lemma can be found in [13]. □

For Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, let us define the following functions,

$$X_1^Y(\rho;p_2) = Z_1^Y(\rho;p_2) - Z_1^Y(\rho;p_2(\rho)) + \lambda_2\, \Omega_0^Y(\rho)(p_2 - p_2(\rho)), \tag{2.7}$$

$$X_2^Y(\rho;p_1) = Z_2^Y(\rho;p_1) - Z_2^Y(\rho;p_1(\rho)) + \lambda_1\, \Omega_0^Y(\rho)(p_1 - p_1(\rho)). \tag{2.8}$$

Note that $X_1^Y(\rho;p_2(\rho)) = X_2^Y(\rho;p_1(\rho)) = 0.$ (2.9)

Taking into account equations (2.7), (2.8) we can modify equations (2.4) and (2.5) of Theorem 2.2 as follows,

<u>Theorem 2.3</u>

The functions $\tilde{\Omega}_i^Y(\rho;p_1,p_2)$, $i = 1,2$, satisfy for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, the following set of functional equations,

$$[p_1 - \beta_1(\rho + \lambda_1(1-p_1) + \lambda_2(1-p_2))]\ \tilde{\Omega}_1^Y(\rho;p_1,p_2) = p_1\beta_1(\rho + \lambda_1(1-p_1)+\lambda_2(1-p_2))$$

$$[X_2(\rho;p_1) - X_1(\rho;p_2) + \lambda_2(p_2-p_2(\rho))\ \Omega_0^Y(\rho) + Y_1^Y(p_1,p_2) - Y_1^Y(p_1(\rho),p_2(\rho))], \tag{2.10}$$

$$[p_2 - \beta_2(\rho + \lambda_1(1-p_1) + \lambda_2(1-p_2))]\ \tilde{\Omega}_2^Y(\rho;p_1,p_2) = p_2\beta_2(\rho-\lambda_1(1-p_1) - \lambda_2(1-p_2))$$

$$[X_1^Y(\rho;p_2)-X_2^Y(\rho;p_1) + \lambda_1(p_1-p_1(\rho))\ \Omega_0^Y(\rho) + Y_2^Y(p_1,p_2) - Y_2^Y(p_1(\rho),p_2(\rho))]. \tag{2.11}$$

Proof

Introducing notations (2.7) and (2.8) into equation (2.4), we readily obtain,

$$[p_1 - \beta_1(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))]\ \tilde{\Omega}_1^Y(\rho;p_1,p_2) = p_1\beta_1(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2))$$

$$[X_2^Y(\rho;p_1)-X_1^Y(\rho;p_2) +\lambda_2(p_2-p_2(\rho))\ \Omega_0^Y(\rho) + Y_1^Y(p_1,p_2)+ C_1^Y(\rho)]\ , \tag{2.12}$$

where

$$C_1^Y(\rho) \stackrel{\text{def}}{=} \lambda_1 p_1(\rho)\ \Omega_0^Y(\rho) + Z_2^Y(\rho;p_1(\rho)) - Z_1^Y(\rho;p_2(\rho)) - Z_2^Y(\rho;0).$$

Since the function $\tilde{\Omega}_1^Y(\rho;p_1,p_2)$ is sought analytic for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, the right side of equation (2.12) must vanish for $p_1 = p_1(\rho)$ and $p_2 = p_2(\rho)$ from lemma 2.1. Then using relations (2.9), we see that $C_1^Y(\rho)$ must be equal to $-Y_1^Y(p_1(\rho);\ p_2(\rho))$.

The same treatment applies to equation (2.5).

□

3 - Analytic resolution

In this section, we solve the set of functional equations which has been established in Theorem 2.3.

Before proceeding with the solution of the set of equations (2.10) and (2.11), let us define,

$\gamma_1(\rho;p_2)$ [resp. $\gamma_2(\rho;p_1)$] as the unique root in the unit circle of the equation $p_1 = \beta_1(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))$ for $|p_2| \leq 1$ and $\mathrm{Re}\,\rho > 0$. [resp. $p_2 = \beta_2(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))$ for $|p_1| \leq 1$].

The existence and unicity of $\gamma_1(\rho;p_2)$ and $\gamma_2(\rho;p_1)$ can be found in [12]. Moreover, we know that [13]

$\gamma_1(\rho;p_2)$ [resp. $\gamma_2(\rho;p_1)$] is given for $\mathrm{Re}\,\rho > 0$, $|p_2| \leq 1$ [resp. $|p_1| \leq 1$]

by $$\gamma_1(\rho;p_2) = E\{e^{-(\rho+\lambda_2(1-p_2))\mathbb{P}_1}\}, \tag{3.1}$$

[resp. $\gamma_2(\rho;p_1) = E\{e^{-(\rho+\lambda_1(1-p_1))\mathbb{P}_2}\}$ where $\mathbb{P}_1$ [resp. $\mathbb{P}_2$] is the busy period of a M/G/1 queue with input parameter λ_1 [resp. λ_2] and L.S.T. of the service times distribution $\beta_1(s)$ [resp. $\beta_2(s)$], for $\mathrm{Re}\,s \geq 0$.

<u>*Remark 3.1.*</u> *: from (3.1) it is seen that $\gamma_1(\rho;p_2) \in\,]0,1[$ when $\rho \in \mathbb{R}^{+*}$ and $p_2 \in\,]0,1[$. Of course a similar result holds for $\gamma_2(\rho;p_1)$.*

The function $\tilde{\Omega}_1^Y(\rho;p_1,p_2)$ being defined for $\mathrm{Re}\,\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, it follows that the right side of equation (2.10) must vanish when $p_1 = \gamma_1(\rho;p_2)$ for fixed (ρ,p_2) with $\mathrm{Re}\,\rho > 0$, $|p_2| \leq 1$.

Hence we get the relation,

$$X_1^Y(\rho;p_2) = X_2^Y(\rho;\gamma_1(\rho;p_2)) + \lambda_2(p_2-p_2(\rho))\,\Omega_0^Y(\rho) + Y_1^Y(\gamma_1(\rho;p_2),p_2) - Y_1^Y(p_1(\rho),p_2(\rho)). \tag{3.2}$$

In a similar way, we have from equation (2.11),

$$X_2^Y(\rho,p_1) = X_1^Y(\rho;\gamma_2(\rho;p_1)) + \lambda_1(p_1-p_1(\rho))\,\Omega_0^Y(\rho) + Y_2^Y(p_1,\gamma_2(\rho;p_1)) - Y_2^Y(p_1(\rho),p_2(\rho)). \tag{3.3}$$

Taking $p_1 = \gamma_1(\rho;p_2)$ in (3.3) and using (3.2) we obtain for Re $\rho > 0$, $|p_2| \le 1$,

$$X_1^y(\rho;p_2) = [\lambda_2(p_2-p_2(\rho)) + \lambda_1(\gamma_1(\rho;p_2)-p_1(\rho))]\,\Omega_0^y(\rho) + Y_1^y(\gamma_1(\rho;p_2),p_2)$$

$$- Y_1^y(p_1(\rho),p_2(\rho)) + Y_2^y(\gamma_1(\rho;p_2),\ \gamma_2(\rho;\gamma_1(\rho;p_2)) - Y_2^y(p_1(\rho),\ p_2(\rho))$$

$$+ X_1^y(\rho;\gamma_2(\rho;\gamma_1(\rho;p_2)))\ . \tag{3.4}$$

For Re $\rho > 0$, $|z| \le 1$ we introduce the following notations,

$fg^{(n)}(\rho;z) = f(\rho;g(\rho;f(\rho;g(\dots\ ;\ g(\rho;z))\dots)$ for $n = 1,2,\dots$ where f and g each appear n times,

$$fg^{(0)}(\rho;z) = z\ . \tag{3.5}$$

For Re $\rho > 0$, $|z| \le 1$ we also define for $n \in \mathbb{N}$,

$$\varphi_{2n}(\rho;z) = \gamma_1\gamma_2^{(n)}(\rho;z),$$
$$\varphi_{2n+1}(\rho;z) = \gamma_1(\rho;\gamma_2\gamma_1^{(n)}(\rho;z))$$

and

$$\Psi_{2n}(\rho;z) = \gamma_2\gamma_1^{(n)}(\rho;z),$$
$$\Psi_{2n+1}(\rho;z) = \gamma_2(\rho;\gamma_1\gamma_2^{(n)}(\rho;z)). \tag{3.6}$$

This can be rewritten as

$$\varphi_{n+1}(\rho;z) = \gamma_1(\rho;\ \Psi_n(\rho;z)),$$
$$\Psi_{n+1}(\rho;z) = \gamma_2(\rho;\varphi_n(\rho;z)),\quad n \ge 0.$$

We set $\varphi_{-1}(\rho;z) = \Psi_{-1}(\rho;z) = 0.$

We shall assume in what follows that $(\sigma,\rho,p_1,p_2) \in \mathbb{R}^4$, with $\sigma \ge 0$, $\rho > 0$, $0 \le p_1 \le 1$ and $0 \le p_2 \le 1$.

Theorem 3.1

The functions $X_1^Y(\rho;p_2)$ and $X_2^Y(\rho;p_1)$ (defined in equations (2.7), (2.8)) are given for $\rho > 0$, $0 \le p_1 \le 1$, $0 \le p_2 \le 1$ by,

$$X_1^Y(\rho;p_2) = \Omega_0^Y(\rho)\ \{\lambda_1 \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;p_2)-p_1(\rho)] + \lambda_2 \sum_{n=0}^{\infty} [\Psi_{2n}(\rho;p_2)-p_2(\rho)]\}$$

$$+ I_1 \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;p_2)^{y_1}\ \Psi_{2n}(\rho;p_2)^{y_2} - p_1(\rho)^{y_1}\ p_2(\rho)^{y_2}]$$

$$+ I_2 \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;p_2)^{y_1}\ \Psi_{2n+2}(\rho;p_2)^{y_2} - p_1(\rho)^{y_1}\ p_2(\rho)^{y_2}], \qquad (3.7)$$

$$X_2^Y(\rho;p_1) = \Omega_0^Y(\rho)\ \{\lambda_1 \sum_{n=0}^{\infty} [\varphi_{2n}(\rho;p_1) - p_1(\rho)] + \lambda_2 \sum_{n=0} [\Psi_{2n+1}(\rho;p_1)-p_2(\rho)]\}$$

$$+ I_1 \sum_{n=0}^{\infty} [\Psi_{2n+1}(\rho;p_1)^{y_2}\ \varphi_{2n+2}(\rho;p_1)^{y_1} - p_1(\rho)^{y_1}\ p_2(\rho)^{y_2}]$$

$$+ I_2 \sum_{n=0}^{\infty} [\Psi_{2n+1}(\rho;p_1)^{y_2}\ \varphi_{2n}(\rho;p_1)^{y_1} - p_1(\rho)^{y_1}\ p_2(\rho)^{y_2}], \qquad (3.8)$$

where

$$\begin{aligned} I_1 &= I(\ y_1 > 0,\ y_2 = 0) + I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 1), \\ I_2 &= I(y_1 = 0,\ y_2 > 0) + I(y_1 > 0,\ y_2 > 0,\ Z(0^+) = 2). \end{aligned} \qquad (3.9)$$

Proof

First, let us verify that $X_1^Y(\rho;p_2)$ given by (3.7) vanishes when $p_2 = p_2(\rho)$.

We have (lemma 2.1) $p_1(\rho) = \gamma_1(\rho;p_2(\rho))$ and $p_2(\rho) = \gamma_2(\rho,p_1(\rho))$. Hence $p_2(\rho) = \gamma_2\gamma_1^{(1)}(\rho;p_2(\rho))$ using definition (3.5). Applying repeatedly this last relation, we get for $n \in \mathbb{N}$,

$$p_2(\rho) = \gamma_2\gamma_1^{(n)}(\rho;p_2(\rho)) . \tag{3.10}$$

Hence for $n \in \mathbb{N}$, $\Psi_{2n}(\rho;p_2(\rho)) = \gamma_2\gamma_1^{(n)}(\rho;p_2(\rho)) = p_2(\rho)$ and $\varphi_{2n+1}(\rho;p_2(\rho)) = \gamma_1(\rho;\gamma_2\gamma_1^{(n)}(\rho;p_2(\rho))) = \gamma_1(\rho;p_2(\rho)) = p_1(\rho)$ using definitions (3.6).

Then $X_1^y(\rho;p_2(\rho)) = 0$ since each term of the infinite sums of (3.7) vanishes when $p_2 = p_2(\rho)$.

In what follows ρ is fixed, $\rho > 0$.

Applying repeatedly (3.4) and coming back to the definition of $Y_1^y(p_1,p_2)$ and $Y_2^y(p_1,p_2)$ given in (2.6) we get,

$$\begin{aligned}
X_1^y(\rho;p_2) - X_1^y(\rho;\gamma_2\gamma_1^{(n)}(\rho;p_2)) &= \Omega_0^y(\rho)\ \{\lambda_1 \sum_{i=0}^{n-1} [\gamma_1(\rho;\gamma_2\gamma_1^{(i)}(\rho;p_2)-p_1(\rho)] \\
&+ \lambda_2 \sum_{i=0}^{n-1} \gamma_2\gamma_1^{(i)}(\rho;p_2)-p_2(\rho)]\} \quad + I(y_1>0,y_2=0)\sum_{i=0}^{n-1} [\gamma_1(\rho;\gamma_2\gamma_1^{(i)}(\rho;p_2))^{y_1} - p_1(\rho)^{y_1}] \\
&+ I(y_1>0,\ y_2>0,\ Z(0^+)=1)\sum_{i=0}^{n-1} [\gamma_1(\rho;\gamma_2\gamma_1^{(i)}(\rho;p_2))^{y_1}\gamma_2\gamma_1^{(i)}(\rho;p_2)^{y_2}-p_1(\rho)^{y_1}p_2(\rho)^{y_2} \\
&+ I(y_1>0,\ y_2>0,Z(0^+)=2)\sum_{i=0}^{n-1} [\gamma_1(\rho;\gamma_2\gamma_1^{(i)}(\rho;p_2))^{y_1}\gamma_2\gamma_1^{(i+1)}(\rho;p_2)^{y_2}-p_1(\rho)^{y_1}p_2(\rho)^{y_2}] \\
&+ I(y_1=0,y_2>0)\sum_{i=0}^{n-1} [\gamma_2\gamma_1^{(i+1)}(\rho;p_2)^{y_2} - p_2(\rho)^{y_2}] \qquad \text{for } n=1,2,\ldots
\end{aligned} \tag{3.11}$$

i) $\quad 0 \le p_2 \le p_2(\rho)$

$\gamma_1(\rho;p_2)$ and $\gamma_2\gamma_1(\rho;p_2)$ being non-decreasing functions in p_2 for $0 \le p_2 \le 1$ (see lemma 3.1), it follows from (3.10) that each term of the finite sums involved in (3.11) is non-positive.

Then $\{X_1^Y(\rho;\gamma_2\gamma_1^{(n)}(\rho;p_2)) - X_1^Y(\rho;p_2)\}_{n \in \mathbb{N}^*}$ is a sequence of non-decreasing and bounded functions for $0 \le p_2 \le p_2(\rho)$. Hence the left side of equation (3.11) has a finite limit when $n \uparrow +\infty$, which in turn implies the convergence of all the infinite sums of the right side.

In particular, we have $\lim\limits_{n \to \infty} \gamma_2\gamma_1^{(n)}(\rho;p_2) = p_2(\rho)$ for $0 \le p_2 \le p_2(\rho)$.

Letting $n \uparrow +\infty$ in equation (3.11) and using the continuity of $X_1^Y(\rho;p_2)$ in the variable p_2 and (2.9), we obtain the result (3.7) including notations (3.6).

ii) $\quad p_2(\rho) \le p_2 \le 1$

The proof is the same except that in this case each term of the finite sums involved in (3.11) is non-negative.

The same results can be obtained for $X_2^Y(\rho;p_1)$. □

It remains to determine the last unknown function $\Omega_0^Y(\rho)$ for $\rho > 0$. This can be done directly using a theorem of TAKÁCS ([12] p. 59 eq. 9) which gives the LST of the probability distribution that a M/G/1 queueing system be empty at time t (t > 0) given the initial workload is known.

Let us consider a busy period of the system. Since the lenght of a busy period is independent of the service discipline, it follows that the duration of a busy period has the same probability distribution as the one of a M/G/1 queueing system with input parameter $\lambda_1 + \lambda_2$ and with LST of the service times distribution given by

$$\frac{\lambda_1\beta_1(s) + \lambda_2\beta_2(s)}{\lambda_1 + \lambda_2} \qquad \text{for} \quad \mathrm{Re}\ s \geq 0.$$

At time t = 0, the LST of the workload distribution is $\beta_1(s)^{Y_1}\ \beta_2(s)^{Y_2}$ for Re s ≥ 0.

From TAKACS' theorem we get for Re $\rho > 0$,

$$\Omega_0^Y(\rho) = \frac{\beta_1(\rho+(\lambda_1+\lambda_2)(1-\nu(\rho)))^{Y_1}\ \beta_2(\rho+(\lambda_1+\lambda_2)(1-\nu(\rho)))^{Y_2}}{\rho + (\lambda_1+\lambda_2)(1-\nu(\rho))}, \tag{3.12}$$

where $\nu(\rho)$ is the unique root in the unit circle of the equation

$$z = \frac{\lambda_1\beta_1(\rho+(\lambda_1+\lambda_2)(1-z)) + \lambda_2\beta_2(\rho+(\lambda_1+\lambda_2)(1-z))}{\lambda_1 + \lambda_2}. \tag{3.13}$$

We also have [3],

$\nu(\rho) = E\{e^{-\rho\mathbb{P}}\}$ where $\mathbb{P}$ is the duration of a busy period of a M/G/1 queueing system with input parameter $\lambda_1+\lambda_2$ and with LST of service times distribution given by

$$\frac{\lambda_1\beta_1(s) + \lambda_2\beta_2(s)}{\lambda_1 + \lambda_2} \quad \text{for Re } s \geq 0 .$$

Hence if $\rho \in \mathbb{R}^{+*}$ then $\nu(\rho) \in \mathbb{R}^+$ with $0 < \nu(\rho) < 1$.

<u>Theorem 3.2</u>

The transforms $\Omega_1^Y(\rho;p_1,p_2,\sigma)$ and $\Omega_2^Y(\rho;p_1,p_2,\sigma)$ of the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ are given for $\rho > 0$, $0 \leq p_1 \leq 1$, $0 \leq p_2 \leq 1$, $\sigma \geq 0$ by,

$$\Omega_1^Y(\rho;p_1,p_2,\sigma) = \frac{p_1[\beta_1(\sigma) - \beta_1(\rho+\lambda_1(1-p_1) + \lambda_2(1-p_2))]}{[\rho- \sigma+ \lambda_1(1-p_1) + \lambda_2(1-p_2)][p_1-\beta_1(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2)]}$$

$$\{\Omega_0^Y(\rho) \sum_{n=0}^{\infty} (\lambda_1 [\varphi_{2n}(\rho;p_1)- \varphi_{2n+1}(\rho;p_2)] + \lambda_2 [\Psi_{2n+1}(\rho;p_1)-\Psi_{2n+2}(\rho;p_2))])$$

$$+ I_1(p_1^{Y_1} p_2^{Y_2} + \sum_{n=0}^{\infty} [\varphi_{2n}(\rho;p_1)^{Y_1} \Psi_{2n-1}(\rho;p_1)^{Y_2}- \varphi_{2n+1}(\rho;p_2)^{Y_1} \Psi_{2n}(\rho;p_2)^{Y_2}])$$

$$+ I_2 \sum_{n=0}^{\infty} [\varphi_{2n}(\rho;p_1)^{Y_1} \Psi_{2n+1}(\rho;p_1)^{Y_2}- \varphi_{2n+1}(\rho;p_2)^{Y_1} \Psi_{2n+2}(\rho;p_2)^{Y_2}]\}, \tag{3.14}$$

$$\Omega_2^Y(\rho;p_1,p_2,\sigma) = \frac{p_2[\beta_2(\sigma) - \beta_2(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2))]}{[\rho-\sigma+\lambda_1(1-p_1)+\lambda_2(1-p_2)][p_2-\beta_2(\rho+\lambda_1(1-p_1)+\lambda_2(1-p_2)]}$$

$$\{\Omega_0^Y(\rho) \sum_{n=0}^{\infty} (\lambda_1 [\varphi_{2n+1}(\rho;p_2)- \varphi_{2n+2}(\rho;p_1)]+\lambda_2[\Psi_{2n}(\rho;p_2)-\Psi_{2n+1}(\rho;p_1)])$$

$$+ I_1 \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;p_2)^{Y_1} \Psi_{2n}(\rho;p_2)^{Y_2}- \varphi_{2n+2}(\rho;p_1)^{Y_1} \Psi_{2n+1}(\rho;p_1)^{Y_1}]$$

$$+ I_2(p_1^{Y_1} p_2^{Y_2} + \sum_{n=0}^{\infty} [\varphi_{2n-1}(\rho;p_2)^{Y_1}\Psi_{2n}(\rho;p_2)^{Y_2}- \varphi_{2n}(\rho;p_1)^{Y_1}\Psi_{2n+1} (\rho;p_1)^{Y_2}])\} \tag{3.15}$$

where

$\Omega_0^Y(\rho)$ is given by equation (3.12),

I_1 and I_2 are given by (3.9).

Proof

These relations follow from Theorem 2.1, Theorem 2.3 and Theorem 3.1 □

4 - Workload and virtual waiting time

Once determined the transform of the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ (Section 3), we easily can derived other interesting queueing quantities, as for example , the Laplace-Stieltjes transform of the joint distribution of the workload [resp. virtual waiting time] .

Let us define for $i = 1,2$,

$q_i(t)$	the number of type i customers waiting for service at time t,
$\eta_i(t)$	the workload of the server w.r.t. type i customers at time t,
$v_i(t)$	the virtual waiting time in queue i at time t.

For $\rho > 0$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, we introduce the following transforms of the processes $\{(\eta_1(t), \eta_2(t)), t \geq 0\}$ and $\{(v_1(t), v_2(t)), t \geq 0\}$,

$$W^y(\rho;\sigma_1,\sigma_2) = \int_0^\infty e^{-\rho t} \; E\{e^{-\sigma_1\eta_1(t) - \sigma_2\eta_2(t)} \,|\, Y(0) = y\} \, dt \tag{4.1}$$

and

$$V^y(\rho;\sigma_1,\sigma_2) = \int_0^\infty e^{-\rho t} \; E\{e^{-\sigma_1 v_1(t) - \sigma_2 v_2(t)} \,|\, Y(0) = y\} \, dt. \tag{4.2}$$

First of all, let us notice that,

$$\begin{aligned}
&q_1(t) = Y_1(t) - 1 , \quad q_2(t) = Y_2(t) && \text{if } Z(t) = 1, \\
&q_1(t) = Y_1(t) , \quad q_2(t) = Y_2(t) - 1 && \text{if } Z(t) = 2, \\
&q_1(t) = q_2(t) = 0 && \text{if } Z(t) = 0, \text{for } t \geq 0.
\end{aligned} \tag{4.3}$$

Theorem 4.1

The Laplace-Stieljes transform of the joint distribution of the workload $\eta_1(t)$ and $\eta_2(t)$ is given for $\rho > 0$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ by,

$$W^y(\rho;\sigma_1,\sigma_2) = \Omega_0^y(\rho) + \beta_1(\sigma_1)^{-1}\,\Omega_1^y(\rho;\beta_1(\sigma_1),\beta_2(\sigma_2),\sigma_1)$$
$$+ \beta_2(\sigma_2)^{-1}\,\Omega_2^y(\rho;\beta_1(\sigma_1),\beta_2(\sigma_2),\sigma_2). \qquad (4.4)$$

Ω_0^y, Ω_1^y and Ω_2^y are respectively given by (3.12), (3.14), (3.15).

Proof

For $t \geq 0$, we have,

$$\eta_1(t) = \tau_1^1 + \ldots + \tau_{q_1(t)}^1 + R(t),$$
$$\eta_2(t) = \tau_1^1 + \ldots + \tau_{q_2(t)}^2 \qquad \text{if } Z(t) = 1$$

and

$$\eta_1(t) = \tau_1^1 + \ldots + \tau_{q_1(t)}^1,$$
$$\eta_2(t) = \tau_1^1 + \ldots + \tau_{q_2(t)}^2 + R(t) \qquad \text{if } Z(t) = 2$$

and

$$\eta_1(t) = \tau_1(t) = 0 \qquad \text{if } Z(t) = 0,$$

where τ_j^i denotes the service time required by the customer in position j in queue i, $i = 1,2$, $j = 1,\ldots, q_i(t)$.

Then

$$W^y(\rho;\sigma_1,\sigma_2) = \int_0^\infty e^{-\rho t} \; E\{(Z(t)=0) / Y(0)=y\}\, dt$$

$$+ \int_0^\infty e^{-\rho t} \; E\{\beta_1(\sigma_1)^{q_1(t)} \; \beta_2(\sigma_2)^{q_2(t)} \; e^{-\sigma_1 R(t)} \; (Z(t)=1) / Y(0)=y\}\, dt$$

$$+ \int_0^\infty e^{-\rho t} \; E\{\beta_1(\sigma_1)^{q_1(t)} \; \beta_2(\sigma_2)^{q_2(t)} \; e^{-\sigma_2 R(t)} \; (Z(t)=2) / Y(0)=y\}\, dt$$

from the independence of the service times.

Using relations (2.3) we get,

$$W^y(\rho;\sigma_1,\sigma_2) = \Omega_0^y(\rho) + \beta_1(\sigma_1)^{-1} \; \Omega_1^y(\rho;\beta_1(\sigma_1),\beta_2(\sigma_2),\sigma_1)$$

$$+ \; \beta_2(\sigma_2)^{-1} \; \Omega_2^y(\rho;\beta_1(\sigma_1),\beta_2(\sigma_2),\sigma_2)$$ from definitions (1.5) and (1.6). □

We are now concerned with the joint distribution of the virtual waiting time $v_1(t)$, $v_2(t)$.

We define for $i = 1, 2$,

$\xi_i(u)$ the duration of a busy period of queue i given that at time $t = 0$ the workload is u, $u > 0$.

We know (see [12], p. 63, rq 4) that for $\sigma \geq 0$, $i = 1,2$,

$$E\{ e^{-\sigma \xi_i(u)} \} = e^{-(\sigma+\lambda_i(1-\nu_i(\sigma)))u} \tag{4.5}$$

where $\nu_i(\sigma)$ is the smallest root of the equation

$$x = \beta_i(\sigma+\lambda_i(1-x)) \qquad (0 < \nu_i(\sigma) \leq 1). \tag{4.6}$$

Theorem 4.2

The Laplace-Stieljes transform of the joint distribution of the virtual waiting time $v_1(t)$, $v_2(t)$ is given for $\rho > 0$, $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ by,

$$
\begin{aligned}
V^{y}(\rho;\sigma_1,\sigma_2) = \Omega_0^{y}(\rho) \Big\{ & 1+\lambda_1 S_1(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\varphi_{2n+1}(\rho;\beta_2(\sigma_2))-\varphi_{2n}(\rho;f_1(\sigma_1,\sigma_2))\Big] \\
& + \lambda_1 S_2(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\varphi_{2n+2}(\rho;\beta_1(\sigma_1))-\varphi_{2n+1}(\rho;f_2(\sigma_1,\sigma_2))\Big] \\
& + \lambda_2 S_1(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\psi_{2n+2}(\rho;\beta_2(\sigma_2))-\psi_{2n+1}(\rho;f_1(\sigma_1,\sigma_2))\Big] \\
& + \lambda_2 S_2(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\psi_{2n+1}(\rho;\beta_1(\sigma_1))-\psi_{2n}(\rho;f_2(\sigma_1,\sigma_2))\Big]\Big\} \\
& + I_1 \Big\{ S_1(\sigma_1,\sigma_2) \Big[f_1(\sigma_1,\sigma_2)^{y_1} \beta_2(\sigma_2)^{y_2} \sum_{n=0}^{\infty} \Big[\varphi_{2n+1}(\rho;\beta_2(\sigma_2))^{y_1} \psi_{2n}(\rho;\beta_2(\sigma_2))^{y_2} \\
& - \varphi_{2n}(\rho;f_1(\sigma_1,\sigma_2))^{y_1} \psi_{2n-1}(\rho;f_1(\sigma_1,\sigma_2))^{y_2}\Big]\Big] \\
& + S_2(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\varphi_{2n+2}(\rho;\beta_1(\sigma_1))^{y_1} \psi_{2n+1}(\rho;\beta_1(\sigma_1))^{y_1} - \varphi_{2n+1}(\rho;f_2(\sigma_1,\sigma_2))^{y_1} \\
& \psi_{2n}(\rho;f_2(\sigma_1,\sigma_2))^{y_2}\Big]\Big\} \\
& + I_2 \Big\{ S_2(\sigma_1,\sigma_2) \Big[\beta_1(\sigma_1)^{y_1} f_2(\sigma_1,\sigma_2)^{y_2} + \sum_{n=0}^{\infty} \Big[\varphi_{2n}(\rho;\beta_1(\sigma_1))^{y_1} \psi_{2n+1}(\rho;\beta_1(\sigma_1))^{y_2} \\
& - \varphi_{2n-1}(\rho;f_2(\sigma_1,\sigma_2))^{y_1} \psi_{2n}(\rho;f_2(\sigma_1,\sigma_2))^{y_2}\Big]\Big] \\
& + S_1(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} \Big[\varphi_{2n+1}(\rho;\beta_2(\sigma_2))^{y_1} \psi_{2n+1}(\rho;\beta_2(\sigma_2))^{y_2} \\
& - \varphi_{2n}(\rho;f_1(\sigma_1,\sigma_2))^{y_1} \psi_{2n+1}(\rho;f_1(\sigma_1,\sigma_2))^{y_2}\Big]\Big\},
\end{aligned}
\tag{4.7}
$$

where

$$S_1^{-1}(\sigma_1,\sigma_2) = \sigma_1+\sigma_2 -\rho - \lambda_2(1-\beta_2(\sigma_2))+\lambda_1 \, [\beta_1(\sigma_1+\sigma_2+\lambda_1(1-\nu_1(\sigma_2)))-\nu_1(\sigma_2)] \; ,$$

$$S_2^{-1}(\sigma_1,\sigma_2) = \sigma_1+\sigma_2 -\rho - \lambda_1(1-\beta_1(\sigma_1))+\lambda_2 \, [\beta_2(\sigma_1+\sigma_2+\lambda_2(1-\nu_2(\sigma_1)))-\nu_2(\sigma_1)] \; , \quad (4.8)$$

$$f_1(\sigma_1,\sigma_2) = \beta_1(\sigma_1+\sigma_2+\lambda_1(1-\nu_1(\sigma_2))),$$

$$f_2(\sigma_1,\sigma_2) = \beta_2(\sigma_1 +\sigma_2+\lambda_2(1-\nu_2(\sigma_1))).$$

ν_1,ν_2 are given by (4.6).

Proof

For $t \geq 0$ we have,

$$v_1(t) = \tau_1^1 + \dots + \tau_{q_1}^1(t) + R(t),$$
$$v_2(t) = \tau_1^2 + \dots + \tau_{q_2}^2(t) + \xi_1(v_1(t)) \qquad \text{if } Z(t) = 1$$

and

$$v_1(t) = \tau_1^1 + \dots + \tau_{q_1}^1(t) + \xi_2(v_2(t)),$$
$$v_2(t) = \tau_1^2 + \dots + \tau_{q_2}^2(t) + R(t) \qquad \text{if } Z(t) = 2$$

and

$$v_1(t) = v_2(t) = 0 \qquad \text{if } Z(t) = 0.$$

Then,

$$V^y(\rho;\sigma_1,\sigma_2) = \int_0^\infty e^{-\rho t} \; E\,\{e^{-\sigma_1(\tau_1^1+\ldots+\tau^1_{q_1(t)}+R(t))-\sigma_2\xi_1(\tau_1^1+\ldots+\tau^1_{q_1(t)}+R(t))}\,\beta_2(\sigma_2)^{q_2(t)}$$

$$(Z(t) = 1) \;/\; Y(0) = y\,\}dt$$

$$+ \int_0^\infty e^{-\rho t} \; E\{\beta_1(\sigma_1)^{q_1(t)-\sigma_1\xi_2(\tau_1^2+\ldots\tau^2_{q_2(t)}+R(t))-\sigma_2(\tau_1^2+\ldots+\tau^2_{q_2(t)}+R(t))}$$

$$(Z(t) = 2) \;/\; Y(0) = y\,\}dt \; + \Omega_0^y(\rho).$$

Hence, from the independence of the service times and formula (4.5) we easily see that,

$$V^y(\rho;\sigma_1,\sigma_2) = \Omega_0^y(\rho) + \frac{\Omega_1^y\,(\rho;\beta_1(\sigma_1+\sigma_2+\lambda_1(1-\nu_1(\sigma_2))),\beta_2(\sigma_2),\sigma_1+\sigma_2+\lambda_1(1-\nu_1(\sigma_2)))}{\beta_1(\sigma_1+\sigma_2+\lambda_1(1-\nu_1(\sigma_2)))}$$

$$+ \frac{\Omega_2^y\,(\rho;\beta_1(\sigma_1),\beta_2(\sigma_1+\sigma_2+\lambda_2(1-\nu_2(\sigma_1))),\sigma_1+\sigma_2+\lambda_2(1-\nu_2(\sigma_1)))}{\beta_2(\sigma_1+\sigma_2+\lambda_2(1-\nu_2(\sigma_1)))} \; . \qquad (4.9)$$

Using equations (3.14) and (3.15) we finally obtain (4.7).

5- Description of the process for $t \to +\infty$

If the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)),\ t \geq 0\}$ is ergodic, that is if for $(p_1,p_2) \in [0,1]$, $\sigma \geq 0$, $i = 0, 1, 2,$

$$\lim_{t\to+\infty} E\{p_1^{y_1(t)}\,p_2^{y_2(t)}\,e^{-\sigma R(t)}\,(Z(t)=i)/Y(0)=y\,\}$$ exists and is independent of every initial state $y = (y_1,y_2)$, $y_1 \geq 0$, $y_2 \geq 0$, then

$$\lim_{t\to+\infty} E\,\{p_1^{y_1(t)}\,p_2^{y_2(t)}\,e^{-\sigma R(t)}\,(Z(t)=i)/Y(0)=y\} = \lim_{\rho\downarrow 0}\; \rho\Omega_i^y(\rho;p_1,p_2,\sigma)$$

applying an Abelian theorem.

Theorem 5.1

The limiting probability ($t \to +\infty$) of an empty system is zero if $a_1 + a_2 \geq 1$ and if $a_1 + a_2 < 1$ it is given by $\Omega_0 \overset{\text{def}}{=} \lim_{\rho \downarrow 0} \rho \Omega_0^y(\rho) = 1 - a_1 - a_2$ for all initial state $y = (y_1, y_2)$, where $a_i \overset{\text{def}}{=} \lambda_i \alpha_i$ for $i = 1, 2$.

Proof

From (3.12) we have for $\mathrm{Re}\, \rho > 0$,

$$\Omega_0^y(\rho) = \int_0^\infty e^{-\rho t} P(Y_1(t) = Y_2(t) = 0 \,/\, Y(0) = y)\, dt$$

$$= \frac{\beta_1(\rho + \lambda(1-\nu(\rho)))^{y_1} \; \beta_2(\rho + \lambda(1-\nu(\rho)))^{y_2}}{\rho + \lambda(1-\nu(\rho))}$$

where $\nu(\rho)$ is the unique root in the unit circle of the equation

$$z = \frac{\lambda_1}{\lambda} \beta_1(\rho + \lambda(1-z)) + \frac{\lambda_2}{\lambda} \beta_2(\rho + \lambda(1-z)) \quad (\lambda = \lambda_1 + \lambda_2).$$

The result is obtained applying a TAKÁCS' theorem ([12] Theorem 8, p. 66). □

By theorem 5.1 the Markov process $\{(Y_1(t), Y_2(t), Z(t), R(t)), t \geq 0\}$ is ergodic iff. $a_1 + a_2 < 1$.

From now on, it is assumed that this condition holds.

We define for $(p_1, p_2) \in [0,1]$, $\sigma \geq 0$, $i = 1,2$,

$$\Omega_i(p_1, p_2, \sigma) = \lim_{\rho \downarrow 0} \rho \Omega_i^y(\rho; p_1, p_2, \sigma), \tag{5.1}$$

$$\Omega(p_1, p_2, \sigma) = \lim_{\rho \downarrow 0} \rho \Omega(\rho; p_1, p_2, \sigma). \tag{5.2}$$

These limits exist under the condition $a_1 + a_2 < 1$.

Theorem 5.2

For $(p_1,p_2) \in [0,1]$, $\sigma \geq 0$, $a_1 + a_2 < 1$ the transform $\Omega(p_1,p_2,\sigma)$ of the stationary distribution of the Markov process $\{ (Y_1(t), Y_2(t), R(t)), t \geq 0 \}$ is given by,

$$\Omega(p_1,p_2,\sigma) = \Omega_0 + \sum_{i=1}^{2} \Omega_i(p_1,p_2,\sigma), \tag{5.3}$$

where

$$\Omega_0 = 1 - a_1 - a_2, \tag{5.4}$$

$$\Omega_1(p_1,p_2,\sigma) = \frac{p_1\Omega_0(\beta_1(\sigma)-\beta_1(\lambda_1(1-p_1)+\lambda_2(1-p_2)))}{[p_1-\beta_1(\lambda_1(1-p_1)+\lambda_2(1-p_2))][\lambda_1(1-p_1)+\lambda_2(1-p_2)-\sigma]}$$

$$\sum_{n=0}^{\infty} \{\lambda_1 [\varphi_{2n}(0;p_1)-\varphi_{2n+1}(0;p_2)] + \lambda_2[\Psi_{2n+1}(0;p_1)-\Psi_{2n+2}(0;p_2)]\}, \tag{5.5}$$

$$\Omega_2(p_1,p_2,\sigma) = \frac{p_2\Omega_0(\beta_2(\sigma)-\beta_2(\lambda_1(1-p_1)+\lambda_2(1-p_2)))}{[p_2-\beta_2(\lambda_1(1-p_1)+\lambda_2(1-p_2))][\lambda_1(1-p_1)+\lambda_2(1-p_2)-\sigma]}$$

$$\sum_{n=0}^{\infty} \{\lambda_1 [\varphi_{2n+1}(0;p_2)-\varphi_{2n+2}(0;p_1)] + \lambda_2[\Psi_{2n}(0;p_2)-\Psi_{2n+1}(0;p_1)]\}. \tag{5.6}$$

Proof

i) The functions $\varphi_.(\rho;x)$ and $\Psi_.(\rho;x)$ which have been defined by (3.6) for $\rho > 0$, $x \in [0,1]$, are also continuous for $\rho = 0$, $\forall x \in [0,1]$ under the condition $a_1 + a_2 < 1$.

This follows form the fact that $\gamma_1(\rho;x)$ and $\gamma_2(\rho;x)$ (see eq.(3.1)) are continuous for $\rho = 0$, $\forall x \in [0,1]$ when $a_1 + a_2 < 1$ (if $a_i < 1$ then Γ_i [illegible] p.s and $\gamma_i(0;0) = 1$, $\forall i = 1,2$ [3]).

ii) For $a_1 + a_1 < 1$, each series of functions involved in expressions (3.14) and (3.15) of $\Omega_1^Y(\rho;p_1,p_2,\sigma)$ and $\Omega_2^Y(\rho;p_1,p_2,\sigma)$ converges uniformly in $\mathbb{R}^+ \times [0,1]$. This is shown in Appendix.

Hence, multiplying (3.14) and (3.15) by ρ, letting $\rho \downarrow 0$, we obtain (5.5) and (5.6) using i), ii) and Theorem 5.1.

□

This result is slightly more general than the one found by NEUTS and YADIN [8] since it also provides the transform of the residual service time -(see also Corollary 5.3)- If $\sigma = 0$ this is exactly the result obtained in [8] (eqs. (74), (75)).

Similarly, let us define the following limits, for $\sigma_1 \geq 0$, $\sigma_2 \geq 0$,

$$W(\sigma_1,\sigma_2) = \lim_{\rho \downarrow 0} \rho\, W^Y(\rho;\sigma_1,\sigma_2), \tag{5.7}$$

$$V(\sigma_1,\sigma_2) = \lim_{\rho \downarrow 0} \rho\, V^Y(\rho;\sigma_1,\sigma_2). \tag{5.8}$$

These limits exist under the condition $a_1 + a_2 < 1$. From Theorems 4.1 and 5.2 and equation (5.7), we deduce the following corollary, which is a new result.

<u>Corollary 5.1</u>

The Laplace-Stieljes transform of the joint stationary distribution of the workload is given for $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ by,

$$W(\sigma_1,\sigma_2) = (1-a_1-a_2)\Bigg[1 + \frac{\sum_{n=0}^{\infty}\{\lambda_1[\varphi_{2n}(0;\beta_1(\sigma_1)) - \varphi_{2n+1}(0;\beta_2(\sigma_2))] + \lambda_2[\Psi_{2n+1}(0;\beta_1(\sigma_1)) - \Psi_{2n+2}(0;\beta_2(\sigma_2))]\}}{\sigma_1 + \lambda_1(1-\beta_1(\sigma_1)) + \lambda_2(1-\beta_2(\sigma_2))}$$

$$+ \frac{\sum_{n=0}^{\infty}\{\lambda_1[\varphi_{2n+1}(0;\beta_2(\sigma_2)) - \varphi_{2n+2}(0;\beta_1(\sigma_1))] + \lambda_2[\Psi_{2n}(0;\beta_2(\sigma_2)) - \Psi_{2n+1}(0;\beta_1(\sigma_1))]\}}{\sigma_2 + \lambda_1(1-\beta_1(\sigma_1)) + \lambda_2(1-\beta_2(\sigma_2))}\Bigg]$$

$\square$ (5.9)

If $\sigma = \sigma_1 = \sigma_2 \geq 0$, then from (5.9) we have,

$$W(\sigma,\sigma) = (1-a_1-a_2)\Big[1 + \frac{\lambda_1[\beta_1(\sigma) - \lim_{n\to+\infty}\varphi_{2n}(0;\beta_1(\sigma))] + \lambda_2[\beta_2(\sigma) - \lim_{n\to+\infty}\Psi_{2n}(0;\beta_2(\sigma))]}{-\sigma + \lambda_1(1-\beta_1(\sigma)) + \lambda_2(1-\beta_2(\sigma))}\Big].$$

Both limits appearing in this identity are equal to one, as shown in the proof of Theorem 3.1 (since $p_1(0)=p_2(0) = 1$). Hence for $\sigma \geq 0$ we obtain,

$$W(\sigma,\sigma) = \frac{(1-a_1-a_2)\sigma}{\sigma - \lambda_1(1-\beta_1(\sigma)) - \lambda_2(1-\beta_2(\sigma))} \tag{5.10}$$

which is the L.S.T. of the workload (virtual waiting time) distribution of a M/G/1 queueing system with input parameter $\lambda_1 + \lambda_2$ and with L.S.T. of service times distribution $(\lambda_1\beta_1(\sigma) + \lambda_2\beta_2(\sigma)) / (\lambda_1 + \lambda_2)$ for $\mathrm{Re}\ \sigma \geq 0$.

This is not a surprising result from the derivation of $\Omega_0^y(\rho)$ -(Section 3).

Similarly from Theorems 4.2 and 5.2 and equation (5.8) we get,

Corollary 5.2

The Laplace-Stieljes transform of the joint stationary distribution of the virtual waiting time is given for $\sigma_1 \geq 0$, $\sigma_2 \geq 0$ by,

$$V(\sigma_1,\sigma_2)=(1-a_1-a_2)\ \{1+\lambda_1 S_1(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} [\varphi_{2n+1}(0;\beta_2(\sigma_2))-\varphi_{2n}(\rho;f_1(\sigma_1,\sigma_2))]$$

$$+\lambda_1\ S_2(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} [\varphi_{2n+2}(0;\beta_1(\sigma_1))-\varphi_{2n+1}(0;f_2(\sigma_1,\sigma_2))]$$

$$+\lambda_2\ S_1(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} [\Psi_{2n+2}(0;\beta_2(\sigma_2))-\Psi_{2n+1}(0;f_1(\sigma_1,\sigma_2))]$$

$$+\lambda_2\ S_2(\sigma_1,\sigma_2) \sum_{n=0}^{\infty} [\Psi_{2n+1}(0;\beta_1(\sigma_1))-\Psi_{2n}(0;f_2(\sigma_1,\sigma_2))]\} . \square$$

Let us notice that since the arrival processes are Poisson processes, the limiting distribution of the virtual waiting time of type i customers is equal to the limiting distribution of the actual waiting time of type i customers -(i = 1,2)- [9]. Then the L.S.T. of the joint stationary distribution of the waiting time is also given by $V(\sigma_1,\sigma_2)$ for $\sigma_1 \geq 0$, $\sigma_2 \geq 0$.

Corallary 5.3

The Laplace-Stieljes transform $R(\sigma)$ of the stationary distribution of the residual service time, is given for $\sigma \geq 0$ by,

$$R(\sigma) = [\lambda_1(1-\beta_1(\sigma)) + \lambda_2(1-\beta_2(\sigma))] / \sigma.$$

Proof

We have $R(\sigma) = \sum_{i=1}^{2} \Omega_i(1,1,\sigma)$ for $\sigma \geq 0$. (5.11)

On the other hand, the following relations can be deduced from equations (3.6), $\varphi'_{.}(0;1)$ [resp. $\Psi'_{.}(0,1)$] denoting the derivative of $\varphi_{.}(0,z)$ [resp. $\Psi_{.}(0;z)$] at point $z = 1$),

$$\varphi'_{2n}(0,1) = \Psi'_{2n}(0;1) = \left(\frac{a_1 a_2}{(1-a_1)(1-a_2)}\right)^n ,$$

$$\varphi'_{2n+1}(0,1) = \frac{\lambda_2 \alpha_1}{1-a_1}\left(\frac{a_1 a_2}{(1-a_1)(1-a_2)}\right)^n ,$$

$$\Psi'_{2n+1}(0;1) = \frac{\lambda_1 \alpha_2}{1-a_2}\left(\frac{a_1 a_2}{(1-a_1)(1-a_2)}\right)^n \quad \text{for } n \geq 0. \qquad (5.12)$$

Applying l'Hôpital's rule to equations (5.5) and (5.6) and using relations (5.12), we obtain,

$$\Omega_i(1,1,\sigma) = \lambda_i(1-\beta_i(\sigma))/\sigma \quad \text{for } i = 1,2 \text{ and } \sigma \geq 0 . \qquad (5.13)$$

Hence, from (5.11) we have,

$$R(\sigma) = \sum_{i=1}^{2} \lambda_i(1-\beta_i(\sigma))/\sigma \quad \text{for } \sigma \geq 0. \qquad \square$$

Corollary 5.4

The probability Π_i that at steady state a type i customer is being served is given by $\Pi_i = a_i$, for i = 1,2.

Proof

We have $\Pi_i = \Omega_i(1,1,0)$ for i = 1,2. Applying l'Hôpital's rule in equation (5.13) we obtain

$$\Pi_i = a_i \quad , \text{ for } i = 1, 2. \qquad \square$$

6 - Mean queueing quantities

The mean waiting time in each queue can be computed using Corollary 5.2.

We do not recall the result which can be found in [11] and [1] (the result given in [7] is incorrect due to minor errors in its derivation).

Let $E(W_i)$ be the mean workload or backlog in queue i, i = 1,2.

Differentiating equation (5.9) w.r.t. σ_1 with $\sigma_2 = 0$, then using first or second order expansions, we find

$$E(W_2) = \frac{1}{2[(1-a_1)^2(1-a_2)^2 - a_1^2 a_2^2]}[\lambda_2 \mu_2''\{(1-a_2)(1-a_1)^2 + a_2 a_1^2\} + \lambda_1 \mu_1''(1-a_2)a_2] \tag{6.1}$$

where $\mu_i'' \overset{\text{def}}{=} \int_0^{+\infty} x^2 \, dB_i(x)$ is supposed finite for $i = 1,2$.

$E(W_1)$ has an analogous expression with the indices 1 and 2 interchanged. This result is obtain using relation (5.12) and the following formula,

$$\sum_{n=0}^{\infty} \varphi''_{2n+1}(0;1) = [1 - \frac{a_1^2 a_2^2}{(1-a_1)^2(1-a_2)^2 - a_1^2 a_2^2}]^{-1} \frac{(1-a_2)\lambda_2^2}{(1-a_1)^2(1-a_1-a_2)}\{\mu_1'' + \frac{a_1^2 \alpha_1 \lambda_2 \mu_2''}{(1-a_2)^3}\}$$

and

$$\sum_{n=0}^{\infty} \Psi''_{2n}(0;1) = [1 - \frac{a_1^2 a_2^2}{(1-a_1)^2(1-a_2)^2 - a_1^2 a_2^2}]^{-1} \frac{\lambda_1 \lambda_2^2}{(1-a_1)(1-a_1-a_2)}\{\frac{\alpha_2 \mu_1''}{1-a_1} + \frac{a_1 \alpha_1 \mu_2''}{(1-a_2)^2}\}$$

and corresponding expressions for $\sum_{n=0}^{\infty} \varphi''_{2n}(0;1)$ and $\sum_{n=0}^{\infty} \Psi''_{2n+1}(0;1)$, where $\varphi''_{\bullet}(0;z)$ [resp. $\Psi''_{\bullet}(0;z)$] denotes the second order derivative of $\varphi_{\bullet}(0;z)$ [resp. $\Psi_{\bullet}(0;z)$].

These formula are obtained from relations (3.6) following the same procedure as the one given in [7] (Nevertheless the expression of $\sum_{n=0}^{\infty} \Psi''_{2n}(0;1)$ given in [7] is not correct as well as the expressions of $\varphi''_{2n+1}(0;1)$ and $\varphi''_{2n}(1;0)$).

It is easily checked that $E(W_1) + E(W_2)$ is the mean waiting time of a M/G/1 queue with input parameter $\lambda_1 + \lambda_2$ and with L.S.T. of the service times distribution $(\lambda_1 \beta_1(\sigma) + \lambda_2 \beta_2(\sigma))/(\lambda_1 + \lambda_2)$ for $\mathrm{Re}\, \sigma \geq 0$.

APPENDIX

We show that under the condition $a_1 + a_2 < 1$ the series contained in the expressions of $X_1^y(\rho;z)$ and $X_2^y(\rho;z)$ (cf. Theorem 3.1) uniformly converge in ρ and z in the domain $\{\rho/\text{Re }\rho > 0\} \times \{z/|z| \leq 1\}$.

Lemma A.1

For fixed ρ (Re $\rho > 0$) and under the ergodicity condition $a_1 + a_2 < 1$, the functions of z, $\gamma_1(\rho;z)$ and $\gamma_2\gamma_1^{(1)}(\rho;z)$ [resp. $\gamma_2(\rho;z)$ and $\gamma_1\gamma_2^{(1)}(\rho;z)$] satisfy the Lipchitz condition, respectively with coefficients k_1 and k_{21} ($k_{21} < 1$) [resp. k_2 and k_{12} ($k_{12} < 1$)].

Proof

Let us define for Re $\rho > 0$, $|z| \leq 1$,

$$f(\rho;z) = \rho + \lambda_1(1-\gamma_1(\rho;z)) + \lambda_2(1-z)$$

and

$$g(\rho;z) = \rho + \lambda_1(1-\gamma_1(\rho;z)) + \lambda_2(1-\gamma_2\gamma_1^{(1)}(\rho;z)).$$

From Section 3, we know that

$$\gamma_1(\rho;z) = \beta_1(f(\rho;z)) \qquad \text{(a.1)}$$

and

$$\gamma_2\gamma_1^{(1)}(\rho;z) = \beta_2(g(\rho;z)). \qquad \text{(a.2)}$$

Differentiating (a.1) and (a.2) with respect to z, we obtain

$$\frac{\partial}{\partial z}\gamma_1(\rho;z) = \frac{\lambda_2\beta_1'(f(\rho;z))}{1+\lambda_1\beta_1'(f(\rho;z))} \tag{a.3}$$

and

$$\frac{\partial}{\partial z}\gamma_2\gamma_1^{(1)}(\rho;z) = \frac{\lambda_1\lambda_2\beta_1'(f(\rho;z))\ \beta_2'(g(\rho;z))}{(1+\lambda_1\beta_1'(f(\rho;z))(1+\lambda_2\beta_2'(g(\rho;z)))}, \tag{a.4}$$

where $\beta_i'(\sigma)$ denotes the derivative of $\beta_i(\sigma)$ for Re $\sigma \geq 0$, $i = 1,2$.

We have $|\beta_i'(\sigma)| < \alpha_i$ for Re $\sigma > 0$, $i = 1,2$. Hence for $a_1 + a_2 < 1$, $|1+\lambda_i\beta_i'(\sigma)| > 1-a_i > 0$ for Re $\sigma > 0$.

Applying these inequalities to (a.3) and (a.4), we readily get for Re $\rho > 0$, $|z| \leq 1$,

$$\left|\frac{\partial}{\partial z}\gamma_1(\rho;z)\right| < \frac{\lambda_2\alpha_1}{(1-a_1)} \overset{\text{def}}{=} k_1$$

and

$$\left|\frac{\partial}{\partial z}\gamma_2\gamma_1^{(1)}(\rho;z)\right| < \frac{a_1a_2}{(1-a_1)(1-a_2)} \overset{\text{def}}{=} k_{21} < 1,$$

which concludes the proof. □

Let us denote for Re $\rho > 0$, $|z| \leq 1$,

$$S^{y}(\rho;z) = \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;z)^{y_1} \Psi_{2n}(\rho;z)^{y_2} - p_1(\rho)^{y_1} p_2(\rho)^{y_2}], \tag{a.5}$$

which is one of the four series contained in $X_1^{y}(\rho;z)$ (see eq. (3.7)). Note that one of the three remaining series is obtained making $y = (1,0)$ in (a.5).

From the identity $ab-cd=(a-c)(b+d)+bc-ad$ and the inequality $|\Psi_{2n}(\rho;z)^{y_2} + p_2(\rho)^{y_2}| < 2$ for Re $\rho > 0$, $|z| \leq 1$, we obtain,

$$|S^{y}(\rho;z)| < 2\,|S_1^{y}(\rho;z)| + |S_2^{y}(\rho;z)|, \tag{a.6}$$

where $S_1^{y}(\rho;z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;z)^{y_1} - p_1(\rho)^{y_1}]$

and

$$S_2^{y}(\rho;z) = \sum_{n=0}^{\infty} [\Psi_{2n}(\rho;z)^{y_2} p_1(\rho)^{y_1} - \varphi_{2n+1}(\rho;z)^{y_1} p_2(\rho)^{y_2}].$$

<u>First let us consider $S_1^{y}(\rho;z)$</u>.

We have for Re $\rho > 0$, $|z| \leq 1$,

$$|S_1^{y}(\rho;z)| = \Big| \sum_{n=0}^{\infty} [\varphi_{2n+1}(\rho;z) - p_1(\rho)] \sum_{j=0}^{y_1} \varphi_{2n+1}(\rho;z)^{j} p_1(\rho)^{y_1-j} \Big|$$

$$\leq (y_1+1) \sum_{n=0}^{\infty} |\varphi_{2n+1}(\rho;z) - p_1(\rho)| .$$

Now from the relation $p_1(\rho) = \gamma_1(\rho;p_2(\rho))$ (see Section 3), definition (3.6) and Lemma A.1, we deduce that

$$|S_1^Y(\rho;z)| \le (y_1+1)k_1 \sum_{n=0}^{\infty} |\gamma_2\gamma_1^{(n)}(\rho;z) - p_2(\rho)| .$$

Using equation (3.10) and Lemma A.1, we finally obtain,

$$|S_1^Y(\rho;z)| \le \frac{2k_1(y_1+1)}{1-k_{21}} \quad \text{for Re } \rho > 0, \ |z| \le 1 . \tag{a.7}$$

<u>Let us now consider $S_2^Y(\rho;z)$</u>.

We have

$$|S_2^y(\rho;z)| < |p_1(\rho)|^{y_1} \sum_{n=0}^{\infty} |\Psi_{2n}(\rho;z)^{y_2} - p_2(\rho)^{y_2}| + |p_2(\rho)|^{y_2} |S_1^Y(\rho;z)|. \tag{a.8}$$

As above, one easily show that

$$\sum_{n=0}^{\infty} |\Psi_{2n}(\rho;z)^{y_2} - p_2(\rho)^{y_2}| < \frac{2(y_2+1)}{1-k_{21}} \quad \text{for Re } \rho > 0, \ |z| \le 1. \tag{a.9}$$

Finally form (a.6), (a.7), (a.8) and (a.9) we find

$$|S^Y(\rho;z)| < \frac{5k_1(y_1+1)+2(y_2+1)}{1-k_{21}} \quad \text{for Re } \rho > 0, \ |z| \le 1. \tag{a.10}$$

The proof of the uniform convergence of the last two series contained in $X_1^Y(\rho;z)$ is analogous to the one above. Similar methods apply to the series involved in $X_2^Y(\rho;z)$.

REFERENCES

[1] B. AVI-ITZHAK, W.L. MAXWELL and L.W. MILLER, "*Queueing with Alternating Priorities*", Opns. Res 13, 306-318 (1965).

[2] J.P.C. BLANC, "*Application of the Theory of Boundary Value Problems in the Analysis of a Queueing Model with Paired Services*", Mathematical Centre Tracts 153, Amsterdam, 1982.

[3] J.W. COHEN, "*The Single Server Queue*", North-Holland Publ. Co., Amsterdam. 2nd ed., 1982.

[4] R.B. COOPER and G. MURRAY, "*Queues Served in Cyclic Order*", Bell System Techn. J. 48. 675-689 (1969).

[5] M. EISENBERG. "*Two Queues with Changeover Times*", Opns. Res. 19, 386-401 (1971).

[6] P. MEVERT, "*A Priority System with Setup Times*", Opns. Res. 16, 602-612 (1968).

[7] L.W. MILLER. "*Alternating Priorities in Multiclass Queues*", Ph. D. Dissertation, Dept. of IE, Cornell University (1964).

[8] M.F. NEUTS and M. YADIN, "*The Transient Behavior of the Queue with Alternating Priorities, with Special Reference to Waiting Times*", Bull. Soc. Mathématique de Belgique 20, 343-376 (1968).

[9] S. STIDHAM, "*Regenerative Processes in the Theory of Queues, with Application to the Alternating Priority Queue*", Adv. Appl. Prob. 4, 542-577 (1972).

[10] J.S. SYKES, "*Simplified Analysis of an Alternating Priority Queueing Model with Setup Times*", Opns. Res. 18, 1182-1192 (1970).

[11] L. TAKÁCS, "*Two Queues Attended by a Single Serve* , Opns. Res. 16, 639-650 (1968).

[12] L. TAKÁCS, "*Introduction to the Theory of Queues*", Oxford University Press, New York. 1962.

[13] P.D. WELCH, "*On the Busy Period of a Facility, which Serves Customers of Several Types*", J.R.S.S. Ser. B. 27,

Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique