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FREE VIBRATION OF A THIN SHELL

APPLICATION TO AN ARCH DAM (*)

(Vibration libre d'une coque fine - Application à un barrage-voûte)

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ABSTRACT

In this contribution we show how to define and to approximate the normal modes and the associated natural frequencies of free vibration of a thin shell. This study is done for a general definition of the thin shell by using curvilinear coordinates. We integrate the usual three-dimensional elasticity equations through the thickness in order to obtain a set of two-dimensional equations. Then, the approximation combines the techniques of conforming finite element methods with numerical integration as well as the classical characterizations through the Rayleigh quotient and the min-max principle. This work is illustrated by numerical experiments on an arch dam problem.

RESUME

Dans ce travail nous montrons comment définir et approcher les modes propres et les fréquences naturelles associées des vibrations libres d'une coque fine. Cette étude est réalisée pour une définition générale de la coque à l'aide de coordonnées curvilignes. Nous intégrons sur l'épaisseur les équations classiques de l'élasticité tridimensionnelle afin d'obtenir un système d'équations bidimensionnelles. Alors, l'approximation combine les techniques des méthodes d'éléments finis conformes avec prise en compte de l'intégration numérique d'une part, et les caractérisations classiques de valeurs propres à l'aide du quotient de Rayleigh et du principe du min-max, d'autre part. Des tests numériques sur un problème de barrage-voûte illustrent ce travail.

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SUMMARY

In this contribution we show how to define and to approximate the normal modes and the associated natural frequencies of free vibration of a thin shell. This study is done for a general definition of the thin shell by using curvilinear coordinates. We integrate the usual three-dimensional elasticity equations through the thickness in order to obtain a set of two-dimensional equations. Then, the approximation combines the techniques of conforming finite element methods with numerical integration as well as the classical characterizations through the Rayleigh quotient and the min-max principle. This work is illustrated by numerical experiments on an arch dam problem.

1. Introduction

According to KRAUS [19], page 289, the "knowledge of the free-vibration characteristics of thin elastic shells is important both to our general understanding of the fundamentals of the behavior of a shell and to the industrial application of shell structures. In connection with the latter, the natural frequencies of shell structures must be known in order to avoid the destructive effect of resonance with nearby rotating or oscillating equipment" as well as earthquake excitation.

In this work we show how to define and to approximate the normal modes and the associated natural frequencies of free vibration of a thin shell. The geometrical definition of the shell is defined in paragraph 2 by using a general system of curvilinear coordinates. Next, in paragraph 3, we integrate through the thickness of the shell the elasticity equations in order to derive a set of two-dimensional free vibration equations. In paragraph 4, we show that the data of this problem satisfy the classical assumptions of the spectral analysis of a compact symmetric operator in a Hilbert space. Corresponding theoretical results are used in paragraph 5 in order to approximate the frequencies by conforming finite element methods and numerical integration techniques. Finally, in paragraph 6, we illustrate these results by numerical experiments on an arch dam problem.

2. Geometrical definition of the shell

2.1. Geometry of the middle surface

Let Ω be a bounded open subset in a plane $\mathcal{E}^2$, with boundary Γ. Then the middle surface $\mathcal{S}$ of the shell is the image of the set $\bar{\Omega}$ by a mapping $\underline{\phi} : \bar{\Omega} \subset \mathcal{E}^2 \to \mathcal{E}^3$ where $\mathcal{E}^3$ is the usual Euclidean space. Subsequently, we shall assume that $\underline{\phi} \in (\mathcal{C}^3(\bar{\Omega}))^3$ and that all points of $\mathcal{S} = \underline{\phi}(\bar{\Omega})$ are regular, in the sense that the two vectors $\underline{a}_\alpha = \underline{\phi}_{,\alpha}$, $\alpha = 1,2$ are linearly independent for all points $\xi = (\xi^1,\xi^2) \in \bar{\Omega}$. With the covariant basis $(\underline{a}_\alpha)$ of the tangent plane, we associate the contravariant basis $(\underline{a}^\alpha)$, which is defined through the relations $\underline{a}^\alpha \cdot \underline{a}_\beta = \delta^\alpha_\beta$, where δ^α_β is the Kronecker's symbol. We also introduce the normal vector $\underline{a}_3 = \underline{a}_1 \times \underline{a}_2 / |\underline{a}_1 \times \underline{a}_2|$.

In the following, we shall use Greek letters $\alpha,\beta,\ldots$, for indices which take their values in the set $\{1,2\}$, while Latin letters $i,j,\ldots$, will be used for indices which take their values in the set $\{1,2,3\}$. Also, we shall employ the summation convention for a repeated index, occuring once as a subscript and once as a superscript.

2.2. Geometrical definition of the shell

In addition to the two curvilinear coordinates ξ^1,ξ^2 which permit to define the middle surface, we introduce a third curvilinear coordinate, ξ^3, which is measured along the normal $\underline{a}_3$ to the surface $\mathcal{S}$ at point $\underline{\phi}(\xi^1,\xi^2)$. This system (ξ^1,ξ^2,ξ^3) of curvilinear coordinates is, at least locally, a system of curvilinear coordinates of $\mathcal{E}^3$, generally called normal coordinates system.

The thickness e of the shell is defined through an application

(1) $\quad e : (\xi^1,\xi^2) \in \bar{\Omega} \to \{x \in \mathbb{R} \; ; \; x > 0\}$.

Then, the shell $\mathcal{E}$ is the closed subset of $\mathcal{E}^3$ defined by

$$
(2) \quad \begin{cases} \mathcal{E} = \{M \in \mathcal{E}^3 \; ; \; \underline{OM} = \underline{\phi}(\xi^1,\xi^2) + \xi^3 \underline{a}_3 \, , \\ \qquad (\xi^1,\xi^2) \in \bar{\Omega}, \; -\frac{1}{2}\, e(\xi^1,\xi^2) \le \xi^3 \le \frac{1}{2}\, e(\xi^1,\xi^2)\} \, . \end{cases}
$$

The differentiation of relation $\underline{OM} = \underline{\phi}(\xi^1,\xi^2) + \xi^3 \underline{a}_3$ allows us to define at every point M a covariant basis $(\underline{g}_i)$ of $\mathcal{E}^3$, i.e.,

$$
(3) \quad \begin{cases} \underline{g}_\alpha = \underline{OM}_{,\alpha} = (\delta^\nu_\alpha - \xi^3 b^\nu_\alpha)\underline{a}_\nu \\ \underline{g}_3 = \underline{OM}_{,3} = \underline{a}_3 \end{cases}
$$

where b_α^ν are the mixed components of the second fundamental form of the surface. To this basis (g_i), we associate the contravariant basis (g^i) such that

(4) $\underline{g}^i \cdot \underline{g}_j = \delta^i_j$.

The element of volume $d\mathscr{E}$ is given by

$$d\mathscr{E} = \sqrt{g}\, d\xi^1\, d\xi^2\, d\xi^3 \ , \ g = \det\,(\underline{g}_i \cdot \underline{g}_j) \ ,$$

or equivalently, if we note by dS the area element of the middle surface

(5) $d\mathscr{E} = [1 - \xi^3 b^\alpha_\alpha + (\xi^3)^2 \det\,(b^\alpha_\beta)]\, d\xi^3\, dS.$

3. Free vibration equations of a thin shell of arbitrary shape

In the following, the geometry of the shell $\mathscr{E}$ defined in (2) is used as a reference configuration. Since we are concerned in the free-vibration analysis, it is appropriate to set down the dynamic equations of motion of a thin elastic shell in the absence of distributed loads. We consider the three following steps :

3.1. Three-dimensional equations

First, we record the dynamic equations of the shell in the absence of distributed loads as derived from elasticity theory. Let ρ be the mass density of the material and let $\underline{U}(\xi^1,\xi^2,\xi^3\ ;\ t)$ denotes the displacement field at time t from the reference configuration. Then, according to [14], chapter 3 and [25], p. 173, the displacement field $\underline{U}$ satisfies the variational equation

(6) $$\begin{cases} \displaystyle\int_{\mathscr{E}} \rho\, \frac{\partial^2 \underline{U}}{\partial t^2}\, \underline{V}\, d\mathscr{E} + \int_{\mathscr{E}} E^{*ijkl}\, \gamma^*_{ij}(\underline{U})\, \gamma^*_{kl}(\underline{V})\, d\mathscr{E} = 0, \\ \forall \underline{V} \in \mathscr{V} . \end{cases}$$

where

(7) $\mathscr{V} = \{\underline{v} \in (H^1(\mathring{\mathscr{E}}))^3 \ , \ \underline{v}|_{\partial\mathscr{E}_0} = \underline{0}\}$;

(8) $\partial\mathscr{E}_0$ = clamped part of the boundary $\partial\mathscr{E}$; part $\partial\mathscr{E}_0$ is assumed such that meas $(\partial\mathscr{E}_0) > 0$;

(9) γ^*_{ij} = covariant component of spatial strain tensor referred to local basis $\{g^1, g^2, g^3\}$ as defined in (3)(4),

(10) E^{*ijkl} = contravariant tensor of elastic moduli in $\mathscr{E}^3$. ∎

3.2. Derivation of two-dimensional thin shell equations

In [17,18] it is shown that a good approximation of the second term in equation (6) is given by

(11) $$\int_{\mathscr{E}} E^{*ijkl}\, \gamma^*_{ij}(\underline{U})\, \gamma^*_{kl}(\underline{V})\, d\mathscr{E} \simeq a(\underline{u},\underline{v})$$

with

(12) $$\begin{cases} a(\underline{u},\underline{v}) = \displaystyle\int_\Omega eE^{\alpha\beta\lambda\mu}\{\gamma_{\alpha\beta}(\underline{u})\gamma_{\lambda\mu}(\underline{v}) + \\ \qquad + \dfrac{e^2}{12}\, \bar\rho_{\alpha\beta}(\underline{u})\bar\rho_{\lambda\mu}(\underline{v})\}\, \sqrt{a}\, d\xi^1\, d\xi^2 \end{cases}$$

where the surface contravariant tensor of elastic moduli for plane stresses is given by

(13) $$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\, [a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}\, a^{\alpha\beta}a^{\lambda\mu}]$$

and where the middle surface strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of curvature $\bar\rho_{\alpha\beta}$ are defined by

(14) $\gamma_{\alpha\beta}(\underline{u}) = \frac{1}{2}(u_{\beta|\alpha} + u_{\alpha|\beta}) - b_{\alpha\beta}u_3$,

(15) $\bar\rho_{\alpha\beta}(\underline{u}) = u_{3|\alpha\beta} + b^\lambda_{\beta|\alpha}u_\lambda + b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha u_{\lambda|\beta} - b^\lambda_\alpha b_{\lambda\beta}u_3$

In the expressions (11),(12),(14) and (15), we note $\underline{u}$ the displacement field of the middle surface $\mathscr{S}$ of the shell. Its components u_i are given by $\underline{u} = u_i\underline{a}^i$, where the contravariant basis $\{\underline{a}^i\}$ is defined in section 2.1. Covariant derivatives are defined as follows

(16) $$\begin{cases} u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta}u_\lambda \\ u_{3|\beta} = u_{3,\beta}\ ,\ u_{3|\alpha\beta} = u_{3,\alpha\beta} - \Gamma^\lambda_{\alpha\beta}u_{3,\lambda} \end{cases}$$

where a comma denotes partial derivatives and $\Gamma^\lambda_{\alpha\beta}$ denote Christoffel's coefficients.

A convenient approximation of the displacement $\underline{U}$ using surface displacement $\underline{u}$ is given by [18]

(17) $\underline{U} = \underline{u} - \xi^3(u_{3|\alpha} + b^\lambda_\alpha u_\lambda)\underline{a}^\alpha$

We assume that the mass density ρ is independent of ξ^3. Then, by substituting (5) and (17) into the first integral of (6) and integrating through the thickness we find (a dot means partial derivative with respect to time t) :

(18) $$\int_{\mathscr{E}} \rho\, \frac{\partial^2 \underline{U}}{\partial t^2}\, \underline{V}\, d\mathscr{E} \simeq b(\underline{u},\underline{v})$$

with

(19) $$\begin{cases} b(\underline{u},\underline{v}) = \displaystyle\int_\Omega \rho e\{[1 + \frac{e^2}{12}(b^1_1 b^2_2 - b^2_1 b^1_2)][a^{\alpha\beta}\ddot u_\alpha v_\beta + \ddot u_3 v_3] \\ \quad + \dfrac{e^2}{12}\, a^{\alpha\beta}[(\ddot u_{3|\alpha} + b^\lambda_\alpha \ddot u_\lambda)(v_{3|\beta} + b^\mu_\beta v_\mu) \\ \quad + (\ddot u_\alpha v_{3|\beta} + v_\beta \ddot u_{3|\alpha} + 2b^\lambda_\alpha \ddot u_\lambda v_\beta)b^\eta_\eta]\}\sqrt{a}\, d\xi^1\, d\xi^2. \end{cases}$$

Then, relations (6),(11) and (18) involve that the displacement field $\underline{u}$ of the middle surface satisfies the variational equation

(20) $a(\underline{u},\underline{v}) + b(\underline{u},\underline{v}) = 0 \ ,\ \forall \underline{v} \in \mathscr{W}$,

where the bilinear forms $a(.,.)$ and $b(.,.)$ are given by relations (12) and (19), and where

(21) $\mathscr{W} = \{\underline{w} \in (H^1(\Omega))^2 \times H^2(\Omega)\ ,\ \underline{w}|_{\Gamma_0} = \underline{0}\ ,\ \frac{\partial w_3}{\partial n}|_{\Gamma_0} = 0\}$ ∎

In (21), Γ_0 stands for the part of boundary $\Gamma = \partial\Omega$ which is associated to the clamped part $\partial\mathscr{E}_0$ of the boundary $\partial\mathscr{E}$, i.e., $\partial\mathscr{E}_0 = \{\underline{\phi}(\Gamma_0) \times\,]-\frac{e}{2}, \frac{e}{2}[\}$.

3.3. Free vibration equations

Now, we consider vibrations which may take place in a shell free of the effect of any external loads at any time, and subject to time-independent kinematic boundary conditions. Vibrations of this kind are called free vibrations and then can be represented by expressions of the type [19], page 291 :

(22) $u_j(\xi^1,\xi^2,t) = \tilde u_j(\xi^1,\xi^2)\cos\omega t\ ,\ j = 1,2,3$

and so on, where ω is the frequency of the vibration. The time-independent kinematic boundary conditions have to be satisfied by the functions $\tilde u_j$. Let us substitute

(22) into (20) ; we obtain free vibration equations in the following variational form (for simplicity, we denote $\underline{u}$ instead of $\tilde{\underline{u}}$; moreover we set $\mathbb{R}^+ = \{x \in \mathbb{R}\ ,\ x > 0\}$) :

$$(23)\quad \begin{cases} \text{Find couples } (\omega,\underline{u}) \in \mathbb{R}^+ \times \underline{\mathcal{W}} \text{ such that} \\ a(\underline{u},\underline{v}) - \omega^2 \tilde{b}(\underline{u},\underline{v}) = 0\ ,\ \forall \underline{v} \in \underline{\mathcal{W}} \end{cases}$$

where $a(.,.)$ and $\underline{\mathcal{W}}$ are defined by (12) and (21) and

$$(24)\quad \begin{cases} \tilde{b}(\underline{u},\underline{v}) = \int_\Omega \rho e\{[1 + \frac{e^2}{12}(b_1^1 b_2^2 - b_1^2 b_2^1)][a^{\alpha\beta}u_\alpha v_\beta + u_3 v_3] \\ \qquad + \frac{e^2}{12} a^{\alpha\beta}[(u_{3|\alpha} + b_\alpha^\lambda u_\lambda)(v_{3|\beta} + b_\beta^\mu v_\mu) \\ \qquad + (u_\alpha v_{3|\beta} + v_\beta u_{3|\alpha} + 2b_\alpha^\lambda u_\lambda v_\beta) b_\eta^\eta]\}\sqrt{a}\, d\xi^1 d\xi^2 \end{cases}$$

Thus, free vibration equations (23) represent an eigenvalue problem. From now on, we set

$$(25)\quad \lambda = \omega^2$$

so that we have to study the eigenvalue problem :

Problem 1 : Find couples $(\lambda,\underline{u}) \in \mathbb{R}^+ \times \underline{\mathcal{W}}$ such that

$$(26)\quad a(\underline{u},\underline{v}) = \lambda \tilde{b}(\underline{u},\underline{v})\ ,\ \forall \underline{v} \in \underline{\mathcal{W}}. \qquad \blacksquare$$

4. Some results concerning problem 1

4.1. Abstract setting of the problem

Consider the following abstract problem :

Problem 2 : Let V and H two real Hilbert spaces such that

(i) $V \subsetneq H$, the inclusion is dense ;

(ii) the canonical injection of V into H is compact. The scalar products (resp. the norms) on the spaces V and H are denoted $((.,.))_V$ and $(.,.)_H$ (resp. $\|\cdot\|_V$ and $|\cdot|_H$).

Let $a(.,.)$ a continuous, symmetric, V-elliptic bilinear form on $V \times V$.

Then, the problem is to find couples (λ,u), $\lambda \in \mathbb{R}$, $u \in V - \{0\}$ such that

$$(27)\quad a(u,v) = \lambda(u,v)_H\ ,\ \forall v \in V\ . \qquad \blacksquare$$

Then, we have the following theorem :

Theorem 1 (see [24]) :

The eigenvalues of Problem 2 form an increasing sequence

$$(28)\quad 0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_m \le \dots ,$$

growing to $+\infty$ when the space V is of infinite dimension, each of these eigenvalues having a finite multiplicity.

Moreover, there exists an orthonormal basis of the space H constituted by the eigenvectors associated to the eigenvalues λ_j , i.e.,

$$(29)\quad \begin{cases} a(u_j,v) = \lambda_j (u_j,v)_H, \quad \forall v \in V\ , \\ (u_j,u_i)_H = \delta_{ij} \end{cases} \qquad \blacksquare$$

Let us come back to Problem 1. We are going to prove that this problem enters in the class of Problem 2. So, we shall be able to apply Theorem 1.

With notations of statement of Problem 2 we have successively :

$$(30)\quad V = \underline{\mathcal{W}}\ ,\ \underline{\mathcal{W}} \text{ given by (21)}$$

$$(31)\quad H = \underline{\mathcal{H}}\ ,\ \underline{\mathcal{H}} = (L^2(\Omega))^2 \times H^1(\Omega)$$

$$(32)\quad (\tilde{\underline{u}},\tilde{\underline{v}})_{\underline{\mathcal{H}}} = \tilde{b}(\tilde{\underline{u}};\tilde{\underline{v}})\ .$$

Then the properties (i) and (ii) of statement of Problem 2 are true thanks to Sobolev's and Kondrasov's theorems [1,20,22] as also to Theorems 1 and 2.

Theorem 2 (see [8])

The bilinear form $a(.,.)$ defined by (12) is $\underline{\mathcal{W}}$-elliptic. $\blacksquare$

Theorem 3 (see [7]) :

The expression (32) defines a scalar product on the space $\underline{\mathcal{H}}$. Corresponding norm is equivalent to the usual product norm on the space $\underline{\mathcal{H}}$, i.e., there exists constants C_1 and C_2 , $0 < C_1 < C_2$, such that

$$(33)\quad \begin{cases} C_1|\underline{u}|^2 \le \tilde{b}(\underline{u},\underline{u}) \le C_2|\underline{u}|^2\ ,\ \forall \underline{u} \in \underline{\mathcal{H}}\ , \\ |\underline{u}|^2 = |u_1|^2_{0,\Omega} + |u_2|^2_{0,\Omega} + \|u_3\|^2_{1,\Omega}\ . \end{cases}$$

5. Approximation by conforming finite element methods

5.1. The discrete space $\underline{V}_h$

In the sequel, we assume that the plane domain $\bar{\Omega}$ is a polygonal bounded set. So, to any regular triangulation of $\bar{\Omega}$, we associate a discrete space $\underline{V}_h$ such that

$$(34)\quad \underline{V}_h \subset \underline{\mathcal{W}}.$$

Examples of construction and implementation of such spaces are given in [3,6,9,10] by using $\mathcal{C}^1$-elem nts.

5.2. The first discrete problem

Inclusion (34) allows us to associate to problem (26) the first discrete problem :

Find couples $(\tilde{\lambda}_h,\tilde{\underline{u}}_h) \in \mathbb{R}^+ \times \underline{V}_h$ such that

$$(35)\quad a(\tilde{\underline{u}}_h,\tilde{\underline{v}}_h) = \tilde{\lambda}_h\, \tilde{b}(\tilde{\underline{u}}_h,\tilde{\underline{v}}_h)\ ,\ \forall \underline{v}_h \in \underline{V}_h\ .$$

Moreover, taking into account the inclusion (34), the discrete problem (35) is of type of Problem 2.

5.3. The second discrete problem

But, if we consider the expressions (12) and (24) we can check that is not generally possible to compute exactly the integrals. So, we use a numerical integration scheme, i.e.,

$$(36)\quad \int_\Omega \psi(\xi^1,\xi^2)\, d\xi^1\, d\xi^2 = \sum_{K \in \mathcal{C}_h} \sum_{l=1}^{L} \omega_{l,K}\, \psi(b_{l,K})$$

where $\mathcal{C}_h$ is the triangulation of $\bar{\Omega}$, $\omega_{l,K}$ are the weights and $b_{l,K}$ are the nodes of the scheme.

Using numerical integration schemes of type (36) to compute the integrals which appear in (12) and (24), we define the approximate bilinear forms $a_h(.,.)$ and

$b_h(.,.)$. Hence the new discrete problem is stated as follows :

$$(37)\quad \begin{cases} \underline{\text{Find couples}}\ (\lambda_h, \underline{u}_h) \in \mathbb{R}^+ \times \underline{V}_h\ \underline{\text{such that}} \\ a_h(\underline{u}_h, \underline{v}_h) = \lambda_h b_h(\underline{u}_h, \underline{v}_h)\ , \quad \forall \underline{v}_h \in \underline{V}_h\ . \end{cases}$$

5.4. Convergence

The first discrete problem enters exactly in the frame of studies developped in [4,26]. There, it has been proved that

(i) the discrete problem has an increasing finite sequence of eigenvalues λ_{hj} , $j = 1,\dots,M_h = \dim \underline{V}_h$ such that $0 < \lambda_j \leq \lambda_{hj}$ where λ_j are the first M_h eigenvalues of problem 1.

(ii) for j fixed, $j = 1,\dots,M_h$, we have

$$\lim_{h\to 0} (\lambda_{hj} - \lambda_j) = 0.$$

The extension of these results to the second discrete problem is possible by using techniques similar to those developped in [15,16].

6. Numerical experiments

In this paragraph, we illustrate previous considerations by some numerical experiments realized on an arch dam problem. We just record the main data of the problem and we refer to [5,6] for more details.

6.1. Geometrical definition of the dam

In this section we give the geometrical definition of the arch dam of the project of GRAND'MAISON studied by [13].

Step 1 : Definition of the middle surface of the dam

With notations of Figure 1 and 2, the coordinates x^i of any point M of the middle surface $\mathcal{S}$ of the dam are given as the components of the mapping

$$\Phi : (\xi^1,\xi^2) \in \bar{\Omega} \to \underline{OM} = \Phi(\xi^1,\xi^2) = x^i(\xi^1,\xi^2)\underline{e}_i$$

by the relations

$$\begin{cases} x^1(\xi^1,\xi^2) = \rho_o(\xi^2)\left[e^{\alpha\theta_o|\xi^1|}\cos(\theta_o|\xi^1|+40°)-\cos 40°\right] \\ \qquad + 0.269\ z_o\xi^2 - 0.0000085\ z_o^3(\xi^2)^3 \\ x^2(\xi^1,\xi^2) = \dfrac{|\xi^1|}{\xi^1}\rho_o(\xi^2)\left[e^{\alpha\theta_o|\xi^1|}\sin(\theta_o|\xi^1|+40°) - \sin 40°\right] \\ x^3(\xi^1,\xi^2) = z_o\xi^2 \end{cases}$$

where

$$\begin{cases} \alpha = \mathrm{tg}\ 40°\ ,\ \theta_o = 48°\ 178\ ,\ z_o = 157\ \mathrm{m} \\ \rho_o(\xi^2) = 200 - 0.008233(z_o)^2(\xi^2)^2 + \\ \qquad + 0.000029(z_o)^3(\xi^2)^3 \end{cases}$$

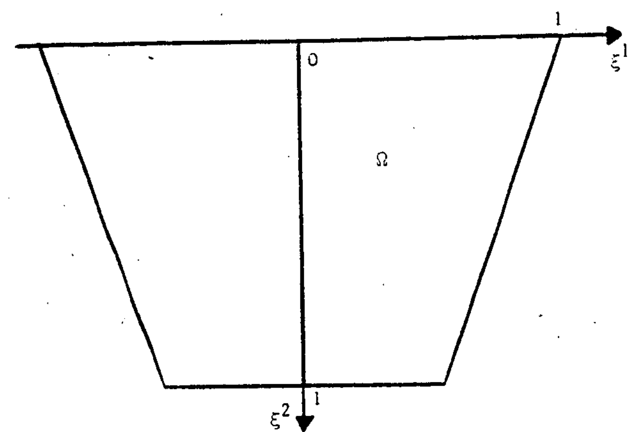

Figure 1 : the reference domain Ω

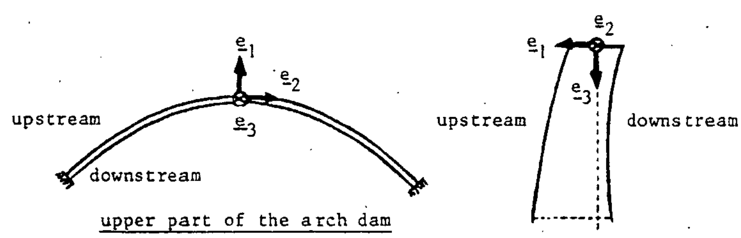

Figure 2 : the orthonormal reference system of the space $\mathcal{E}^3$.

Step 2 : Definition of the thickness of the dam

The thickness of the dam at the point (ξ^1,ξ^2) is given by

$$\begin{cases} e(\xi^1,\xi^2) = 8 + 0.248\ z_o\xi^2 - 0.000003(z_o\xi^2)^3 \\ \qquad + 2.10^{-8}(z_o\xi^2)^2\ [1 + 0.003\ z_o\xi^2]. \\ \qquad \left[\dfrac{e^{\alpha\theta o|\xi^1|}-1}{\sin 40°}\ \rho_o(\xi^2)\right]^2 \end{cases}$$

Step 3 : The free vibration problem

The dam is assumed to be free on the upper part of its boundary and clamped everywhere else.

6.2. Numerical experiments

These numerical experiments have been obtained by combined use of

(i) a space $\underline{V}_h$ constructed from ARGYRIS [2] or reduced - H.C.T. [9,11] triangles ;

(ii) the simultaneous iteration method [23] in order to solve the discrete problems (37) ;

In Tableau 1 and 2, we list the results concerning the approximation of the first five frequencies. In both cases we observe an excellent convergence for the first frequency. As usual for this kind of approximation we observe that the higher eigenvalues are progressively more difficult to approximate (for instance, see [26], page 232, for theoretical considerations and [12], pages 1962-1963, for numerical experiments on a plate).

Finally, in Tableau 3 we check that these results are in good agreement with experimental results obtained on similar arch dams by [21,27] (we record that usually the frequencies are decrasing when the general dimensions of the structure increase).

Number of elements / Frequencies	8 $(h=\frac{1}{2})$	18 $(h=\frac{1}{3})$	32 $(h=\frac{1}{4})$	50 $(h=\frac{1}{5})$
1st freq.	2.491	2.485	2.483	2.483
2nd freq.	3.688	3.674	3.670	3.670
3rd freq.	5.691	5.503	5.494	
4th freq.	5.729	5.668	5.662	
5th freq.	8.570	7.871	7.581	

Tableau 1 : Approximation of the first five frequencies by using ARGYRIS triangle (global computing time : 183 s. on CRAY 1)

Number of elements / Frequencies	32 $(h=\frac{1}{4})$	72 $(h=\frac{1}{6})$	128 $(h=\frac{1}{8})$	200 $(h=\frac{1}{10})$
1st freq.	2.609	2.545	2.521	2.508
2nd freq.	4.366	4.038	3.900	3.823
3rd freq.	6.121	5.863	5.777	5.733
4th freq.	6.812	6.735		
5th freq.	8.969	8.916		

Tableau 2 : Approximation of the first five frequencies by using reduced H.C.T.-element. (global computing time : 426 s on CRAY 1)

Arch dam / characteristics	Sazanami-gawa dam[27]	Russian arch dam[21]	Kamishiiba dam [27]	Projet of Grand' Maison arch dam [13]
Max.height	67.4 m	103 m	110 m	157 m
Crest length	127 m	340 m	310 m	670 m
Crest width	2.4 m	5 m	7 m	8 m
Base width	8.8 m	31 m	27.7 m	36 m
Radius of crest arch	74.74 m	150 m	142.4 m	261 m
First frequency	5.5 hertz (cycle par sec.)	3.57 hz.	3.83 hz.	2.48 hz.
Second frequency	6.83 hz.		5.83 hz.	3.67 hz.
Third frequency			8.67 hz.	5.49 hz.
Fourth frequency				5.66 hz.
Fifth frequency				≃ 6.8 hz.

Tableau 3 : Comparison with experimental results issued from existing arch dams.

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