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Roland Glowinski, H.B. Keller, L. Reinhart

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Roland GLOWINSKI
Herbert KELLER
Laure REINHART

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: 954 90 20

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*Roland GLOWINSKI**, *Herbert KELLER***, *Laure REINHART****

RESUME

On étudie dans cet article une méthode de résolution de problèmes aux limites non linéaires dépendant d'un paramètre. Cette méthode de type continuation fait appel de façon essentielle à une formulation par moindres carrés associée à un algorithme de gradient conjugué avec préconditionnement et à une approximation par éléments finis.

On peut ainsi calculer des branches de solutions avec points limites, points de bifurcation, etc...

De nombreux essais numériques illustrent les possibilités de la méthode étudiée dans cet article.

ABSTRACT

We discuss in this paper a new method for solving nonlinear boundary value problems containing a parameter. This method of the continuation type combines also a least squares formulation, a preconditioned conjugate gradient algorithm and finite element approximations.

We can compute then branches of solutions with limit points, bifurcation points, etc...

Several numerical tests illustrate the possibilities of the methods discussed in the present paper.

* Université Paris VI, LA 189, Tour 55.65, 4 Place Jussieu, 75230 Paris Cedex 05 and INRIA

** California Institute of Technology, Applied Mathematics 217-50, Pasadena California 91125, U.S.A.

*** INRIA, Domaine de Voluceau Rocquencourt, B.P. 105, 78153 Le Chesnay Cedex

1. - INTRODUCTION.

We present in this paper a powerful combination of techniques that are used to solve a variety of nonlinear boundary value problems containing a parameter. Indeed the resulting method can be employed to study a large class of nonlinear eigenvalue problems. The individual techniques include : arclength or pseudo-arclength continuation, least squares formulation in an appropriate Hilbert space setting, a conjugate gradient iterative method for solving the least squares problem and finite element approximations to yield a finite dimensional problem for computation.

In Sec. 2 the solution techniques are described in some details. Specifically in Sec. 2.1 the least squares formulation of a broad class of nonlinear problems, say in the form

$$(1.1) \qquad AU = T(U),$$

are formulated in an appropriate Hilbert space setting. Then in Sec. 2.2 a conjugate gradient iterative solution technique for solving such least squares problems is presented. In Sec. 2.3 a pseudo-arc length continuation method for nonlinear eigenvalue problems in the form

$$(1.2) \qquad Lu = G(\lambda,u)$$

is discussed. This involves adjoining a scalar linear constraint, say

$$(1.3) \qquad \ell(u,\lambda,s) = 0,$$

and with $U \equiv \{u,\lambda\}$ the previous least squares and conjugate gradient techniques can be applied to the system (1.2), (1.3). One big advantage of our specific continuation method is that simple limit points of the original problem (1.2) are just regular points for our reformulation in the form (1.1). The entire procedure thus enables us to determine large arcs or branches of solutions of (1.2) with no special precautions or change of methods near limit points.

These techniques, as described in Section 2, apply to the analytical problem. However they go over extremely well when various discrete approximations are applied to yield computational methods of great power and practicality. We illustrate this by considering several nonlinear boundary value problems of some difficulty and current interest. In each of these problems the discretization is obtained by some finite element formulation. The well-known Bratu problem on a square domain is treated in Section 3. Several ordinary differential equation examples displaying bifurcation and the effects of perturbed bifurcation are treated in Section 4. Finally in Section 5 the Navier-Stokes equations are solved for the driven cavity problem.

2. - SOLUTION TECHNIQUES.

We introduce in this section the methods that we shall apply in Secs. 3, 4, 5, to the solution of nonlinear boundary value problems. We shall consider thus the solution of a *nonlinear problem*, given in a quite general form, by *least squares, conjugate gradient* and *arc length continuation* methods.

Let V be an *Hilbert space* (real for simplicity) equipped with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$; we denote by V' the dual space of V, by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V, and by $\|\cdot\|_*$ the corresponding dual norm, i.e.

$$(2.1) \qquad \|f\|_* = \operatorname*{Sup}_{v\in V-\{0\}} \frac{|\langle f,v\rangle|}{\|v\|} \quad \forall f \in V'.$$

The problem that we consider is to find $u \in V$ such that

$$(2.2) \qquad S(u) = 0,$$

where S is a nonlinear operator from V to V'.

2.1. Least squares formulation of problem (2.2).

A least squares formulation of (2.2) is obtained by observing that any solution of (2.2) is also a *global minimizer* over V of the functional $J : V \to \mathbb{R}$ defined by

$$(2.3) \qquad J(v) = \frac{1}{2}\|S(v)\|_*^2$$

(for which J should vanish) ; hence a leastsquares formulation of (2.2) is

$$(2.4) \qquad \begin{cases} \underline{\text{Find}}\ u \in V\ \underline{\text{such that}} \\ J(u) \le J(v) \quad \forall v \in V. \end{cases}$$

In practice we should proceed as follows :
Let A be the *duality isomorphism* corresponding to $(\cdot,\cdot)$ and $<\cdot,\cdot>$, i.e. obeying

(2.5) $\quad <Av,w> = (v,w) \quad \forall v,w \in V,$

(2.6) $\quad \|v\| = \|Av\|_* \quad \forall v \in V$ (*or equivalently* $\|f\|_* = \|A^{-1}f\| \quad \forall f \in V'$) ;

we have then

(2.7) $\quad J(v) = \frac{1}{2} <A\xi,\xi> \quad (= \frac{1}{2} \|\xi\|^2),$

where ξ is a (nonlinear) function of v obtained via the solution of the *well-posed linear problem*

(2.8) $\quad A\xi = S(v).$

Hence, (2.4) has the structure of an *Optimal Control* problem, where

(i) v is the *control vector*,
(ii) ξ is the *state vector*,
(iii) (2.8) is the *state equation*,
(iv) J is the *cost function*.

As a final remark we observe that any solution of the minimization problem (2.4) for which J *vanishes* is also a solution of the original problem (2.2).

2.2 Solution of the least squares problem (2.4) by a conjugate gradient algorithm.

We suppose from now on that S is *differentiable* implying in turn the differentiability of J over V ; we denote by S' and J' the differentials of S and J, respectively.

From the differentiability of J it is quite natural to solve the minimization problem (2.4) by a *conjugate gradient* algorithm ; among the possible conjugate gradient algorithms we have selected the *Polak-Ribière* variant (cf. POLAK [1]) whose very good performances (in general) have been discussed by POWELL [2]. The Polak-Ribière method applied to the solution of (2.4) provides the following algorithm

Step 0 : Initialization

(2.9) $u^o \in V$, *given,*

compute then $g^o \in V$ *as the solution of*

(2.10) $$Ag^o = J'(u^o)$$

and set

(2.11) $$z^o = g^o.$$

Then, for $n \geq 0$, *assuming that* u^n, g^n, z^n *are known, compute* $u^{n+1}, g^{n+1}, z^{n+1}$ *by*

Step 1 : Descent

(2.12) *Compute* $\rho_n = \underset{\rho \in \mathbb{R}}{\text{Arg Min}}\ J(u^n - \rho z^n)$,

set then

(2.13) $$u^{n+1} = u^n - \rho_n z^n.$$

Step 2 : Construction of the new descent direction

Define $g^{n+1} \in V$ *as the solution of*

(2.14) $$Ag^{n+1} = J'(u^{n+1}) \ ;$$

compute γ_n *by*

(2.15) $$\gamma_n = \frac{\langle A(g^{n+1} - g^n), g^{n+1} \rangle}{\langle Ag^n, g^n \rangle} \ \left(= \frac{(g^{n+1} - g^n, g^{n+1})}{\|g^n\|^2}\right)$$

and set

(2.16) $$z^{n+1} = g^{n+1} + \gamma_n z^n.$$

Do then $n = n+1$ *and go to* (2.12).

The two nontrivial steps of algorithm (2.9)-(2.16) are :

(i) The solution of the one-dimensional minimization problem (2.12) to obtain ρ_n ; we have done the corresponding line search by *dichotomy* and *quadratic interpolation*, using ρ_{n-1} as starting value (see [3] for more details). We observe that each evaluation of $J(v)$, for a given argument v, requires the solution of the linear problem (2.8) to obtain the corresponding ξ.

(ii) The calculation of g^{n+1} from u^{n+1} which requires the solution of two linear problems associated to A (namely (2.8) with $v = u^{n+1}$ and (2.14)).

Calculation of $J(u^n)$ and g^n : Owing to the importance of Step (ii), let us detail the calculation of $J'(u^n)$ and g^n :

Let $v \in V$, then $J'(v)$ may be defined by

$$(2.17) \qquad \langle J'(v),w\rangle = \lim_{\substack{t\to 0\\ t\neq 0}} \frac{J(v+tw)-J(v)}{t}, \quad \forall w \in V.$$

We obtain from (2.7), (2.8), (2.17) that

$$(2.18) \qquad \langle J'(v),w\rangle = \langle A\xi,\eta\rangle$$

where ξ and η are the solutions of (2.8) and

$$(2.19) \qquad A\eta = S'(v)\cdot w,$$

respectively. Since A is *self-adjoint* (from (2.5)) we also have from (2.18), (2.19) that

$$(2.20) \qquad \langle J'(v),w\rangle = \langle A\xi,\eta\rangle = \langle A\eta,\xi\rangle = \langle S'(v)\cdot w,\xi\rangle .$$

Therefore $J'(v) \in V'$ may be identified with the *linear functional*

$$(2.21) \qquad w \to \langle S'(v)\cdot w,\xi\rangle .$$

It follows then from (2.14), (2.20), (2.21) that g^n is the solution of the following *linear variational problem*

$$(2.22) \qquad \begin{cases} \textit{Find } g^n \in V \textit{ such that } \forall w \in V \\ \langle Ag^n,w\rangle = \langle S'(u^n)\cdot w,\xi^n\rangle \quad \forall w \in V, \end{cases}$$

where ξ^n is the solution of (2.8) corresponding to $v=u^n$.

Remark 2.1 : It is clear from the above observations that an efficient solver for linear problems related to operator A (in fact to a *finite dimensional* approximation of A) will be a fundamental tool for the solution of problem (2.2) by the conjugate gradient algorithm (2.9)-(2.16).

Remark 2.2 : The fact that $J'(v)$ is known through (2.20) is not at all a drawback if a *Galerkin* or a *finite element* method is used to approximate (2.2). Indeed we only need to know the value of $\langle J'(v),w\rangle$ for w belonging to a *basis* of the *finite dimensional subspace* of V corresponding to the Galerkin or finite element approximation under consideration. ■

Convergence of algorithm (2.9)-(2.16) :

We introduce the concept of *regular solution* of problem (2.2) by

Definition 2.1 : *A solution* u *of* (2.2) *is said to be regular if the operator* $S'(u)$ $(\in \mathcal{L}(V,V'))$ *is an isomorphism from* V *onto* V'.

Using a modification of the finite dimensional techniques of POLAK [1], it has been proved in REINHART [3] that if problem (2.2) has a *finite number* of solutions and if these solutions are *regular,* then the conjugate gradient algorithm (2.9)-(2.16) converges to a solution of (2.2) depending upon the initial solution u^o in (2.9) (this convergence result supposes that u^o is well chosen, like in Newton's method).

2.3. Arc length continuation methods.

Consider now the solution of nonlinear problems depending upon a real parameter λ ; we would like to follow in the space $V \times \mathbb{R}$ branches of solutions $\{u(\lambda),\lambda\}$ when λ belongs to a compact interval of $\mathbb{R}$.

These nonlinear eigenvalue problems can be written as follows

$$(2.23) \qquad S(\lambda,u) = 0, \ \lambda \in \mathbb{R}, \ u \in V.$$

Equation (2.23) reduces quite often to

$$(2.24) \qquad Lu = G(\lambda,u), \ \lambda \in \mathbb{R}, \ u \in V,$$

where $L : V \to V'$ is a linear elliptic operator, and where G is a *nonlinear Fredholm operator* (see e.g. BERGER [4] for the definition of Fredholm operators).

A classical approach is to use λ as the parameter defining arcs of solutions ; if for $\lambda = \lambda_o$ problem (2.23) has a unique solution $u=u_o$ and if that solution is *isolated,* that is

(2.25) $$S_u^o = \frac{\partial S}{\partial u}(\lambda_o,u_o) \textit{ is an isomorphism from } V \textit{ onto } V',$$

and if $\{\lambda,u\} \to S(\lambda,u)$ is C^1 in some ball around $\{\lambda_o,u_o\}$, then the *implicit function theorem* implies the existence of a smooth arc of regular solutions $u=u(\lambda)$ for $|\lambda-\lambda_o| < \rho$. Therefore, for λ given *sufficiently close* to λ_o we may solve problem (2.23) just as problem (2.2).

These procedures, however, may fail or encounter difficulties (slow convergence for example) close to a *non isolated* solution.

To overcome these difficulties we replace problem (2.23) by the following system

(2.26) $$S(\lambda,u) = 0,$$

(2.27) $$\ell(u,\lambda,\bar{s}) = 0,$$

where $\ell : V \times \mathbb{R} \times \mathbb{R}$ is chosen such that s is some *arc length* (or a convenient *approximation* of it) on the solution branch. We look then for a solution $\{u(s), \lambda(s)\}$, s *being given* (but not λ). If in addition to $\{u_o,\lambda_o\}$ we know exactly or approximately, $\{\dot{u}(s_o),\dot{\lambda}(s_o)\}$ (where $\dot{v}$ denotes the derivative of v with respect to s) satisfying

(2.28) $$\|\dot{u}(s_o)\|^2 + |\dot{\lambda}(s_o)|^2 = 1 \textit{ at } s=s_o,$$

then we can use

(2.29) $$\ell(u,\lambda,s) \equiv (\dot{u}(s_o),u(s)-u(s_o)) + \dot{\lambda}(s_o)(\lambda(s)-\lambda(s_o))-(s-s_o) = 0,$$

for $|s-s_o|$ sufficiently small.

Let us define $U \in V \times \mathbb{R}$ by $U = \{u,\lambda\}$; then problem (2.26), (2.27) can be written as

$$(2.30) \qquad T_s(U) = 0,$$

where

$$(2.31) \qquad T_s(U) = \begin{pmatrix} T_{1s}(U) \\ T_{2s}(U) \end{pmatrix}$$

with

$$(2.32) \qquad T_{1s}(U) = S(\lambda,u), T_{2s}(U) = \ell(u,\lambda,s).$$

The main interest of this new formulation is that the ordinary *limit points* of (2.26) become *regular* solutions of (2.30) (see H.B. KELLER [5],[6] for more details).

Using the notation of Sec. 2.1 a least squares formulation of (2.30) (generalizing (2.4)) is given by

$$(2.33) \qquad \begin{cases} \textit{Find } U(s) = \{u(s),\lambda(s)\} \textit{ such that} \\ J_s(U(s)) \leq J_s(W) \quad \forall W = \{w,\mu\} \in V \times \mathbb{R}, \end{cases}$$

with

$$(2.34) \qquad J_s(W) = \frac{1}{2} \langle A\tilde{w},\tilde{w} \rangle + \frac{1}{2} |\tilde{\mu}|^2$$

where, in (2.34), $\tilde{w}$ and $\tilde{\mu}$ are (nonlinear) functions of $\{w,\mu\}$ via the solution of the linear problems

$$(2.35) \qquad A\tilde{w} = T_{1s}(w,\mu),$$

$$(2.36) \qquad \tilde{\mu} = T_{2s}(w,\mu),$$

respectively.

We consider now a conjugate gradient algorithm (in fact, a variation of algorithm (2.9)-(2.16)) to solve the least squares problem (2.33) ; this algorithm is defined as follows

<u>Step 0</u> : <u>Initialization</u>

(2.37) $U^o = \{u^o, \lambda^o\}$ *is given ;*

compute then $G^o = \{g_u^o, g_\lambda^o\} \in V \times \mathbb{R}$ *as the solution of*

(2.38) $$A g_u^o = \frac{\partial J_s}{\partial u}(U^o),$$

(2.39) $$g_\lambda^o = \frac{\partial J_s}{\partial \lambda}(U^o),$$

and set

(2.40) $$Z^o = G^o \;. \blacksquare$$

Then for $n \geq 0$, *assuming that* U^n, G^n, Z^n *are known, compute* $U^{n+1}, G^{n+1}, Z^{n+1}$ *by*

<u>Step 1</u> : <u>Descent</u>

(2.41) $$\rho_n = \operatorname*{Arg\,Min}_{\rho \in \mathbb{R}} J_s(U^n - \rho Z^n),$$

(2.42) $$U^{n+1} = U^n - \rho_n Z^n$$

(i.e. $u^{n+1} = u^n - \rho_n z_u^n$, $\lambda^{n+1} = \lambda^n - \rho_n z_\lambda^n$).

<u>Step 2</u> : <u>Calculation of the new descent direction</u>

Define $G^{n+1} = \{g_u^{n+1}, g_\lambda^{n+1}\} \in V \times \mathbb{R}$ *by*

(2.43) $$A g_u^{n+1} = \frac{\partial J_s}{\partial u}(U^{n+1}),$$

(2.44) $$g_\lambda^{n+1} = \frac{\partial J_s}{\partial \lambda}(U^{n+1}),$$

compute

$$(2.45)\quad \begin{cases} \gamma_n = \dfrac{\langle A(g_u^{n+1}-g_u^n), g_u^{n+1}\rangle + (g_\lambda^{n+1}-g_\lambda^n)g_\lambda^{n+1}}{\langle Ag_u^n, g_u^n\rangle + |g_\lambda^n|^2} = \\ = \dfrac{(g_u^{n+1}-g_u^n, g_u^{n+1}) + (g_\lambda^{n+1}-g_\lambda^n)g_\lambda^{n+1}}{\|g_u^n\|^2 + |g_\lambda^n|^2}, \end{cases}$$

and set

$$(2.46)\qquad Z^{n+1} = G^{n+1} + \gamma_n Z^n. \blacksquare$$

Do then n=n+1 *and go to* (2.41).

The various comments done in Sec. 2.2, concerning algorithm (2.9)-(2.16), still hold for algorithm (2.37)-(2.46) ; we pointed out in Remark 2.1 the importance of efficient methods for solving linear problems related to operator A ; this last remark still holds, indeed, since in the context of Sec. 2.3, A is replaced by the *block-diagonal* isomorphism $\mathcal{A} : V\times\mathbb{R} \to V'\times\mathbb{R}$ defined by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

We do not go into the details of the calculation of $\dfrac{\partial J_s}{\partial u}$, $\dfrac{\partial J_s}{\partial \lambda}$ since it is just a trivial modification of the calculation done in Sec. 2.2 to obtain J'.

A most important step of the continuation method is a "good" *initialization* choice in (2.37). The natural choice

$$(2.47)\qquad \{u^o,\lambda^o\} = \{u(s_o),\lambda(s_o)\}$$

is usually too naive and a better choice is provided by the following simple *extrapolation* technique

$$(2.48)\qquad \begin{cases} u^o = u(s_o) + (s-s_o)\dot{u}(s_o) \\ \lambda^o = \lambda(s_o)+(s-s_o)\dot{\lambda}(s_o), \end{cases}$$

resulting in a *much faster* convergence, close to the limit points particularly. We shall return to this initialization problem in Sec. 2.4.

Convergence of algorithm (2.37)-(2.46) :

The fundamental advantage of the continuation approach is that it provides an efficient solution method in the neighborhood of the so-called *limit* (or *turning*) *points* solutions of problem (2.23). Let us precise that concept of limit point ; we have the following

Definition 2.2 : *Let* $\{u_o,\lambda_o\} \in V \times \mathbb{R}$ *be a solution of problem* (2.23). *We say that* $\{u_o,\lambda_o\}$ *is a normal limit point if*

(2.49) $$\frac{\partial S}{\partial u}(u_o,\lambda_o)\cdot\dot{u}_o + \frac{\partial S}{\partial \lambda}(u_o,\lambda_o)\dot{\lambda}_o = 0,$$

(2.50) $$\dim N\left(\frac{\partial S}{\partial u}(u_o,\lambda_o)\right) \equiv \mathrm{Codim}\, R\left(\frac{\partial S}{\partial u}(u_o,\lambda_o)\right) = 1,$$

(2.51) $$\frac{\partial S}{\partial \lambda}(u_o,\lambda_o) \notin R\left(\frac{\partial S}{\partial u}(u_o,\lambda_o)\right)$$

(*we recall that if* $\Lambda \in \mathcal{L}(X,Y)$, *then* $R(\Lambda)$ = *range of* Λ, $N(\Lambda)$ = *null space of* $\Lambda = \mathrm{Ker}\,(\Lambda)$, *respectively*).

We have shown on Figure 2.1, below, an example of such limit point (located on a curve of equation $S(u,\lambda) = 0$).

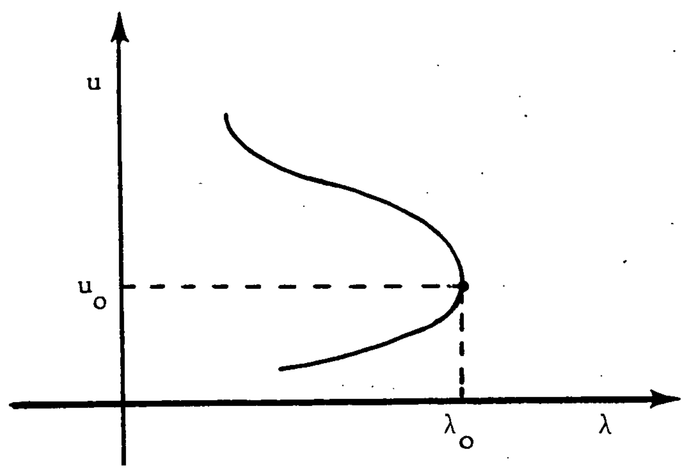


Figure 2.1

The main justification of *arc length continuation methods* follows from

Proposition 2.1 : *Any normal limit point solution of problem* (2.23) *is a regular solution of problem* (2.30).

For a proof, see KELLER [5]. ■

From a practical point of view, Proposition 2.1 is of fundamental importance for the following reasons :

(i) Since normal limit points for problem (2.23) are regular points for problem (2.30), the conjugate gradient algorithm can be used to compute these limit points via the least squares formulation (2.33). This property is a direct consequence of the convergence properties of the conjugate gradient algorithm mentioned in Sec. 2.2 and discussed in details in [3].

(ii) Using convenient *perturbation* technique, genuine *bifurcation* points can be approximated by normal limit points for which the solution methods described in the present Sec. 2 can be applied ; several examples of such situations will be discussed in Sec. 4.

2.4. On the practical implementation of the arc length continuation method.

To help potential users of those continuation methods discussed in Sec. 2.3, we summarize here what we consider as the essential of these methods :

To solve the nonlinear problem (2.23), i.e. $S(u,\lambda) = 0$, we associate to it the "continuation" equation (other choices are possible)

$$(2.52) \qquad \|\dot{u}\|^2 + \dot{\lambda}^2 = 1$$

where $\dot{u} = \frac{du}{ds}$, $\dot{\lambda} = \frac{d\lambda}{ds}$, we have

$$(2.53) \qquad \delta s = \dot{\lambda}\delta\lambda + (\dot{u},\delta u),$$

or equivalently

$$(2.54) \qquad (\delta s)^2 = (\delta\lambda)^2 + (\delta u,\delta u).$$

As discussed in Sec. 2.3 we solve the nonlinear problem (2.23) via the solution of the family (parametrized by s) of nonlinear systems (2.23),(2.52). In practice we approximate (2.23),(2.52) by the *discrete* family of nonlinear systems described below, where Δs is an *arc length step, positive or negative* (possibly varying with n) and where $u^n \simeq u(n\Delta s)$, $\lambda^n \simeq \lambda(n\Delta s)$:

Initialization : We suppose that we know a solution $\{u^o,\lambda^o\}$ of (2.23) ; we take it as origine of the arc of solutions, i.e. $u^o = u(0)$, $\lambda^o = \lambda(0)$. We suppose also that we know $\dot{\lambda}(0),\dot{u}(0)$ (or at least an approximation of it ; see Remark 2.3).

Continuation : Then for $n \geq 0$, assuming u^n, λ^n known and also u^{n-1}, λ^{n-1} (resp. $\dot{u}(0), \dot{\lambda}(0)$) if $n \geq 1$ (resp. $n=0$), we obtain $\{u^{n+1}, \lambda^{n+1}\} \in V \times \mathbb{R}$ as the solution of

$$(2.55) \qquad S(u^{n+1}, \lambda^{n+1}) = 0$$

and

$$(2.56) \qquad (u^1-u^o, \dot{u}(0)) + (\lambda^1-\lambda^o)\dot{\lambda}(0) = \Delta s \ \textit{if} \ n=0,$$

$$(2.57) \qquad (u^{n+1}-u^n, \frac{u^n-u^{n-1}}{\Delta s}) + (\lambda^{n+1}-\lambda^n)(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}) = \Delta s \ \textit{if} \ n \geq 1.$$

Remark 2.3 : It may occur that obtaining $\dot{u}(0)$, $\dot{\lambda}(0)$ is by itself a complicated problem ; however obtaining a second solution of (2.23), close to $\{u^o, \lambda^o\}$, may be easy (using the nonlinear least squares-conjugate gradient methods of Secs. 2.1, 2.2, for example) ; this supposes that we are sufficiently far from a singular point). Let us denote this second solution by $\{u^{-1}, \lambda^{-1}\}$; to approximate $\{\dot{u}(0), \dot{\lambda}(0)\}$ we compute first Δs^o by

$$(2.58) \qquad (\Delta s^o)^2 = \|u^o-u^{-1}\|^2 + |\lambda^o-\lambda^{-1}|^2,$$

and approximate $\dot{u}(0), \dot{\lambda}(0)$ by

$$(2.59) \qquad \frac{u^o-u^{-1}}{\Delta s^o}, \ \frac{\lambda^o-\lambda^{-1}}{\Delta s^o},$$

respectively. The sign of Δs^o is depending upon the orientation chosen for the arc of solutions and of the relative positions of $\{u^o, \lambda^o\}$ and $\{u^{-1}, \lambda^{-1}\}$ on it.

Remark 2.4 : Relations (2.55)-(2.57) look like clearly a discretization scheme for solving the Cauchy problem for first order Ordinary Differential Equations ; from that analogy we can derive many other discretization schemes for the approximation of (2.23),(2.52) (Runge-Kutta, Multisteps, etc...) and also methods for the automatic adjustment of Δs. ∎

A least squares-conjugate gradient method for solving in $V \times \mathbb{R}$ systems very close to (2.55)-(2.57) has been discussed previously in Sec. 2.3 ; a fundamental (and practical) observation is the following :

To initialize the conjugate gradient algorithm solving (2.55)-(2.57), we have used $\{2u^n-u^{n-1}, 2\lambda^n-\lambda^{n-1}\}$ as initial guess to compute $\{u^{n+1}, \lambda^{n+1}\}$; with such a choice we obtain a much faster convergence than by taking $\{u^n, \lambda^n\}$ as initial guess.

3. - APPLICATION TO THE SOLUTION OF THE BRATU PROBLEM.

3.1. Formulation of the problem, properties of its solutions.

As a first test problem for the solution techniques discussed in Sec. 2 we consider the numerical solution of the so-called *Bratu problem*, i.e. find a function u solution of the following nonlinear boundary value problem

$$(3.1)\qquad \begin{cases} -\Delta u = \lambda e^u + f \ \textit{in}\ \Omega, \\ u=0 \ \textit{on}\ \partial\Omega, \end{cases}$$

where Ω is a *bounded* domain of $\mathbb{R}^N$ and $\partial\Omega$ its boundary. We denote by $x = \{x_i\}_{i=1}^N$ the generic point of $\mathbb{R}^N$ and define dx by $dx = dx_1 \ldots dx_N$. The (quite classical) Sobolev-Hilbert space

$$(3.2)\qquad H_o^1(\Omega) = \{v \mid v \in L^2(\Omega),\ \frac{\partial v}{\partial x_i} \in L^2(\Omega) \quad \forall i=1,\ldots N,\ v=0 \ \textit{on}\ \partial\Omega\},$$

equipped with the scalar product

$$(3.3)\qquad (u,v)_{H_o^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v \, dx$$

and the corresponding norm

$$(3.4)\qquad \|v\|_{H_o^1(\Omega)} = (\int_\Omega |\nabla v|^2 \, dx)^{1/2},$$

provides a functional framework well-suited to the solution of (3.1) by variational methods, and most particularly by those discussed in Sec. 2.

For simplicity we consider only situations for which f is a *nonnegative constant*. We suppose also that $\lambda \geq 0$, since problem (3.1) has a unique solution in $H_o^1(\Omega)$ if $\lambda \leq 0$; such a result can be proved using *monotonicity* methods, like those discussed in e.g. LIONS [7], and based on the fact that the operator

$$v \to -\Delta v - \lambda e^v - f$$

is *monotone* over $H_o^1(\Omega)$ if $\lambda < 0$. If $\lambda > 0$, problem (3.1) (or closely related nonlinear problems) has been considered by many authors ; with regard to recent publications let us mention among others CRANDALL-RABINOWITZ [8],[9], AMANN [10], MIGNOT-PUEL [11], MIGNOT-MURAT-PUEL [12] ; in particular we may find in [12] an interesting discussion showing the relationships between (3.1) and *combustion phenomena :*

About the existence of solutions for (3.1), the following has been proved if $\lambda > 0$:

There exists a critical value of λ , *say* λ^*, *such that*

(i) *If* $\lambda > \lambda^*$, *then problem* (3.1) *has no solution.*

(ii) If $\lambda \in]0,\lambda^*]$ *(resp.* $\lambda \in]0,\lambda^*[$ *), then problem* (3.1)*has at least one solution (resp. two solutions) belonging to* $H_0^1(\Omega) \cap W^{2,p}(\Omega)$ $\forall p \geq 1$ *(where*

$$W^{2,p}(\Omega) = \{v | v , \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i \partial x_j} \in L^p(\Omega), \forall\ 1 \leq i,j \leq N\}).$$

(iii) If $\lambda = \lambda^*$ *there exists a unique* $u^* \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, $\forall p \geq 1$, *moreover* $\{u^*,\lambda^*\}$ *is a normal limit point for the equation*

$$S(u,\lambda) = 0$$

where the operator S *is defined over* $H_0^1(\Omega) \times \mathbb{R}$ *by*

$$S(v,\mu) = -\Delta v - \mu e^v - f. \quad \blacksquare$$

In the above theoretical references it is also proved that these solutions which are not limit points are regular solutions. It follows from all these properties that the solution techniques discussed in Sec. 2 can be applied to the solution of (3.1) if $\lambda > 0$; their application to the computer solution of (3.1) requires however a *finite dimensional approximation* of this last problem ; such an approximation - by *finite element methods* - is considered in the following sections. Actually problem (3.1) has been investigated numerically by, among others, KIKUCHI [13], SIMPSON [14], MOORE-SPENCE [15], CHAN-KELLER [16] (by arc length continuation and miltigrid finite difference methods), REINHART [3], for which we refer for more details and further references.

3.2. Finite element approximation of the Bratu problem.

3.2.1. Variational formulation of the Bratu problem. Triangulation of Ω. Fundamental discrete spaces.

A variational formulation of the Bratu problem (3.1), well-suited to finite element approximations and to the solution techniques of Sec. 2, is given by

$$(3.5)\qquad \begin{cases} \textit{Find } \{u,\lambda\} \in H_0^1(\Omega) \times \mathbb{R} \textit{ such that} \\ \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega (\lambda e^u + f) \, v dx \quad \forall v \in H_0^1(\Omega). \end{cases}$$

We describe only the approximation of problem (3.1) if N=2 (the one dimensional case N=1 is much simpler) ; we suppose also for simplicity that Ω is a polygonal domain of $\mathbb{R}^2$. We consider now a standard family of finite element triangulations $\{\mathcal{C}_h\}_h$ of Ω, i.e. for a given h, $\mathcal{C}_h$ is a finite collection of (*closed*) subtriangles of Ω, such that

(i) $\bigcup_{T \in \mathcal{C}_h} T = \bar{\Omega}$,

(ii) $\forall T, T' \in \mathcal{C}_h$, $T \neq T'$, we have either

(*) $T \cap T' = \emptyset$,

(**) *or* T,T' *have only one vertex in common,*

(***) *or* T,T' *have only a whole edge in common,*

(iii) h is the maximal length of the edges of the $T \in \mathcal{C}_h$.

An example of such a triangulation is shown on Figure 3.1, for $\Omega =]0,1[\times]0,1[$.

We approximate then $H_0^1(\Omega)$ by the following finite dimensional space

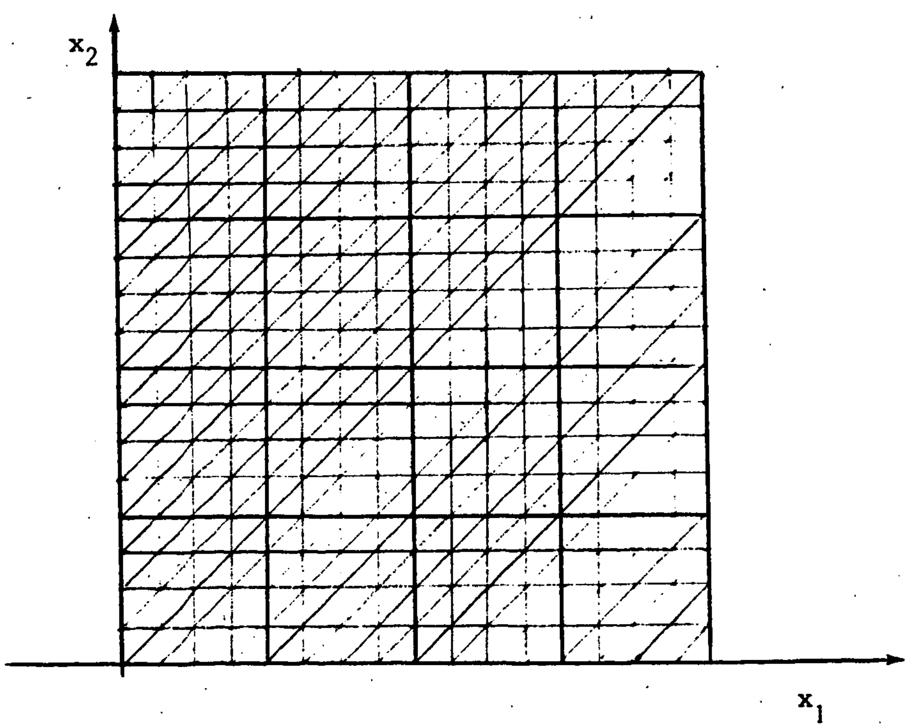


Figure 3.1

$$(3.6)\qquad V_{oh} = \{v_h \mid v_h \in C^0(\bar{\Omega}),\ v_h|_T \in P_1 \quad \forall T \in \mathcal{T}_h,\ v_h = 0 \text{ on } \partial\Omega\},$$

where P_1 is the space of the polynomials in x_1, x_2 of degree ≤ 1 ; we have $\dim V_{oh} = N_{oh}$ - where N_{oh} is the number of vertices of $\mathcal{T}_h$ interior to Ω - and $V_{oh} \subset H_o^1(\Omega)$.

3.2.2. Formulation of the approximate problems.

As approximate problem it is quite natural to take

$$(3.7)\qquad \begin{cases} \textit{Find } \{u_h, \lambda\} \in V_{oh} \times \mathbb{R} \textit{ such that} \\ \int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega (\lambda e^{u_h} + f) v_h \, dx \quad \forall v_h \in V_{oh}. \end{cases}$$

Problem (3.7) is equivalent, actually, to a system of nonlinear equations in $\mathbb{R}^{N_{oh}+1}$; to obtain such a system we suppose that the set Σ_{oh} of the vertices of $\mathcal{T}_h$ has been ordered, so that

$$(3.8)\qquad \Sigma_{oh} = \{P_i\}_{i=1}^{N_{oh}} \quad ,$$

and then that to each P_i of Σ_{oh} we have associated *the* function w_i satisfying

$$(3.9)\qquad w_i \in V_{oh} \ , \ w_i(P_j) = \delta_{ij} \quad \forall\ 1 \leq i,j \leq N_{oh}.$$

The set $\mathcal{B}_{oh} = \{w_i\}_{i=1}^{N_{oh}}$ is a basis of the vector space V_{oh} and we clearly have the important following relation

$$(3.10)\qquad \forall v_h \in V_{oh},\ v_h = \sum_{i=1}^{N_{oh}} v_h(P_i) w_i.$$

Back to (3.7) we observe that this last problem is equivalent to the nonlinear system in $\mathbb{R}^{N_{oh}} \times \mathbb{R}$, below,

$$(3.11)\qquad \begin{cases} \sum_{j=1}^{N_{oh}} \left(\int_\Omega \nabla w_i \cdot \nabla w_j \, dx\right) u_h(P_j) = \int_\Omega \left(\lambda \exp\left(\sum_{j=1}^{N_{oh}} u_h(P_j) w_j\right) + f\right) w_i \, dx, \\ 1 \leq i \leq N_{oh}, \end{cases}$$

where the unknown vector is $\{\{u_h(P_i)\}_{i=1}^{N_{oh}}, \lambda\}$.

Since $\nabla w_i, \nabla w_j$ are *piecewise constant* functions, the calculation of the left hand sides in (3.11) is an easy task. The integrals occuring in the right hand sides of (3.11) can be calculated exactly ; however in order to reduce the computational work we suggest the following possibilities to calculate the above integrals approximately :

(i) Calculate $\int_\Omega e^{u_h} w_i \, dx$ using the *two-dimensional Simpson rule* on each $T \in \mathcal{T}_h$, i.e.

$$(3.12) \qquad \int_T \phi(x)\, dx \simeq \frac{1}{3} \text{ meas.}(T) \sum_{j=1}^{3} \phi(m_{jT}),$$

where m_{1T}, m_{2T}, m_{3T} are the mid-points of the three edges of T. Formula (3.12) is exact if $\phi \in P_2$ (P_2 = space of polynomials of degree ≤ 2). To calculate $\int_\Omega e^{u_h} w_i \, dx$ we should apply Simpson rule, only on those triangles of $\mathcal{T}_h$ with P_i as a common vertex.

(ii) Apply to $\int_\Omega e^{u_h} w_i \, dx$ the *two dimensional trapezoidal rule,* i.e.

$$(3.13) \qquad \int_T \phi(x) dx \simeq \frac{1}{3} \text{ meas.}(T) \sum_{j=1}^{3} \phi(P_{jT}),$$

where P_{jT}, $j=1,2,3$ are the vertices of T ; (3.13) is exact if $\phi \in P_1$. If $\mathcal{T}_h$ is a regular triangulation, like the one of Figure 3.1, using (3.13) to calculate the right hand sides of (3.11) we recover classical *finite difference* schemes for the discretization of (3.1). ■

Other numerical integration techniques are available for the approximate calculation of $\int_\Omega e^{u_h} w_i \, dx$.

3.3. Numerical solution of the discrete Bratu problem by arc length continuation methods.

We apply now the continuation methods of Secs. 2.3, 2.4 to the solution of the discrete Bratu problem (3.7) (we have considered problem (3.7) only, for its formalism is simpler, but the following methods can be (and have been) applied to the solution of approximations of (3.1) obtained using numerical integration).

In the particular case of problem (3.7), the continuation techniques of Secs. 2.3, 2.4 lead to the following algorithm :

(a) Initialization.

(3.14) *Take* $\lambda^o = 0$;

the corresponding u_h^o is the *unique* solution of the following discrete linear Dirichlet problem (given in variational form)

$$(3.15) \quad \begin{cases} \textit{Find } u_h^o \in V_{oh} \textit{ such that} \\ \int_\Omega \nabla u_h^o \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \quad \forall v_h \in V_{oh} \ ; \end{cases}$$

(3.15) is equivalent to a linear system (obtained by taking $\lambda=0$ in (3.11)) whose matrix is *symmetric* and *positive definite*.

We take $\{u_h^o, 0\}$ as the origin of the arc of solutions passing through it and we define the arclength s by

$$(3.16) \quad (\delta s)^2 = \int_\Omega |\nabla \delta u_h|^2 \, dx + (\delta\lambda)^2.$$

Denote $\frac{dX}{ds}$ by $\dot{X}$; then by differentiation of (3.7), with respect to s, we obtain at s=0

$$(3.17) \quad \begin{cases} \int_\Omega \nabla \dot{u}_h(0) \cdot \nabla v_h \, dx = \dot{\lambda}(0) \int_\Omega e^{u_h^o} \, v_h \, dx \quad \forall v_h \in V_{oh}, \\ \dot{u}_h(0) \in V_{oh} \ ; \end{cases}$$

we also have by definition of s

$$(3.18) \quad \int_\Omega |\nabla \dot{u}_h(0)|^2 \, dx + \dot{\lambda}^2(0) = 1.$$

Define $\hat{u}_h$ as the solution of

$$(3.19) \quad \begin{cases} \hat{u}_h \in V_{oh}, \\ \int_\Omega \nabla \hat{u}_h \cdot \nabla v_h \, dx = \int_\Omega e^{u_h^o} \, v_h \, dx \quad \forall v_h \in V_{oh} \ ; \end{cases}$$

we clearly have from (3.17)-(3.19) that

$$(3.20) \qquad \dot{u}_h(0) = \dot{\lambda}(0)\hat{u}_h,$$

$$(3.21) \qquad \dot{\lambda}^2(0) = (1+\int_\Omega |\nabla \hat{u}_h|^2 \, dx)^{-1}.$$

Since we are mostly interested by solving the Bratu problem for $\lambda > 0$, we shall suppose that the arc of solutions is oriented in the neighborhood of s=0 in such a way that $\frac{d\lambda}{ds}$ $(= \dot{\lambda}) \geq 0$; from that choice, and from (3.21), we have

$$(3.22) \qquad \dot{\lambda}(0) = (1+\int_\Omega |\nabla \hat{u}_h|^2 \, dx)^{-1/2}.$$

(b) Continuation.

With $\Delta s (> 0)$ an elementary arc length, we define for $n \geq 0$ an approximation $\{u_h^{n+1}, \lambda^{n+1}\}$ $(\in V_{oh} \times \mathbb{R})$ of $\{u_h(n+1)\Delta s), \lambda((n+1)\Delta s)\}$ as the solution of the following nonlinear variational system :

Find $\{u_h^{n+1}, \lambda^{n+1}\} \in V_{oh} \times \mathbb{R}$ *such that*

$$(3.23)_1 \qquad \int_\Omega \nabla u_h^{n+1} \cdot \nabla v_h \, dx = \int_\Omega (\lambda^{n+1} e^{u_h^{n+1}} + f) \, v_h \, dx \quad \forall v_h \in V_{oh},$$

$$(3.23)_2 \qquad \int_\Omega \nabla(u_h^1 - u_h^0) \cdot \nabla \dot{u}_h(0) \, dx + (\lambda^1 - \lambda^0)\dot{\lambda}(0) = \Delta s \text{ } \textit{if} \text{ } n=0,$$

$$(3.23)_3 \qquad \int_\Omega \nabla(u_h^{n+1} - u_h^n) \cdot \nabla(\frac{u_h^n - u_h^{n-1}}{\Delta s}) \, dx + (\lambda^{n+1} - \lambda^n)(\frac{\lambda^n - \lambda^{n-1}}{\Delta s}) = \Delta s \text{ } \textit{if} \text{ } n \geq 1. \blacksquare$$

To solve the nonlinear system (3.23) we may use the *nonlinear least squares conjugate gradient* techniques of Sec. 2.3. Since the solution of the discrete Bratu problem (3.7) is, in this paper, the first application of the above methods, we shall give a detailed description of the operations involved in the solution process :

We suppose that $V_{oh} \times \mathbb{R}$ is equipped with the inner product corresponding to the euclidian norm

$$(3.24) \qquad \{v_h, \mu\} \to (\int_\Omega |\nabla v_h|^2 \, dx + \mu^2)^{1/2}.$$

A convenient nonlinear least squares formulation of (3.23) is then

$$(3.25) \qquad \begin{cases} \textit{Find } \{u_h^{n+1}, \lambda^{n+1}\} \in V_{oh} \times \mathbb{R} \textit{ such that} \\ J_{n+1}(u_h^{n+1}, \lambda^{n+1}) \leq J_{n+1}(w_h, \mu) \quad \forall \{w_h, \mu\} \in V_{oh} \times \mathbb{R}, \end{cases}$$

where in (3.25) the functional $J_{n+1}(\cdot,\cdot)$ is defined by

$$(3.26) \qquad J_{n+1}(w_h,\mu) = \frac{1}{2}\int_\Omega |\nabla\tilde{w}_h|^2\,dx + \frac{1}{2}|\tilde{\mu}|^2 \;;$$

in (3.26), $\tilde{w}_h$ and $\tilde{\mu}$ are nonlinear functions of $\{w_h,\mu\}$, obtained as the solutions of the *linear* problems

$$(3.27) \qquad \begin{cases} \tilde{w}_h \in V_{oh}, \\ \int_\Omega \nabla\tilde{w}_h\cdot\nabla v_h\,dx = \int_\Omega \nabla w_h\cdot\nabla v_h\,dx - \int_\Omega (\mu e^{w_h}+f)v_h\,dx \quad \forall v_h \in V_{oh}, \end{cases}$$

$$(3.28) \qquad \tilde{\mu} = \int_\Omega \nabla(w_h-u_h^n)\cdot\nabla\left(\frac{u_h^n-u_h^{n-1}}{\Delta s}\right)dx + (\mu-\lambda^n)\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) - \Delta s,$$

respectively.

In that particular case the conjugate gradient algorithm (2.37)-(2.46) reduces to :

Step 0 : Initialization

$$(3.29) \qquad \{u_h^o,\lambda^o\} \in V_{oh}\times\mathbb{R} \textit{ is given.}$$

Compute then $\{g_u^o,g_\lambda^o\} \in V_{oh}\times\mathbb{R}$ *as the solution of*

$$(3.30) \qquad \begin{cases} \int_\Omega \nabla g_u^o\cdot\nabla v_h\,dx = \langle\frac{\partial J_{n+1}}{\partial u_h}(u_h^o,\lambda^o),v_h\rangle \quad \forall v_h \in V_{oh}, \\ g_u^o \in V_{oh}, \end{cases}$$

$$(3.31) \qquad g_\lambda^o = \frac{\partial J_{n+1}}{\partial\lambda}(u_h^o,\lambda^o),$$

and set

$$(3.32) \qquad \{z_u^o,z_\lambda^o\} = \{g_u^o,g_\lambda^o\} \;. \blacksquare$$

Then for $m \geq 0$, *assuming that* $\{u_h^m,\lambda^m\},\{g_u^m,g_\lambda^m\}$, $\{z_u^m,z_\lambda^m\}$ *are known, compute* $\{u_h^{m+1},\lambda^{m+1}\}$, $\{g_u^{m+1},g_\lambda^{m+1}\}$, $\{z_u^{m+1},z_\lambda^{m+1}\}$ *by*

Step 1 : Descent

$$(3.33)\qquad \begin{cases} \textit{Find } \rho_m \in \mathbb{R} \textit{ such that, } \forall \rho \in \mathbb{R} \\ J_{n+1}(u_h^m - \rho_m z_u^m, \lambda^m - \rho_m z_\lambda^m) \leq J_{n+1}(u_h^m - \rho z_u^m, \lambda^m - \rho z_\lambda^m) \end{cases},$$

and set

$$(3.34)\qquad u_h^{m+1} = u_h^m - \rho_m z_u^m \,,\; \lambda^{m+1} = \lambda^m - \rho_m z_\lambda^m \,. \blacksquare$$

Step 2 : Calculation of the new descent direction

Define $\{g_u^{m+1}, g_\lambda^{m+1}\}$ *as the solution of*

$$(3.35)\qquad \begin{cases} \int_\Omega \nabla g_u^{m+1} \cdot \nabla v_h \, dx = \langle \frac{\partial J_{n+1}}{\partial u_h}(u_h^{m+1}, \lambda^{m+1}), v_h \rangle \quad \forall v_h \in V_{oh}, \\ g_u^{m+1} \in V_{oh}, \end{cases}$$

$$(3.36)\qquad g_\lambda^m = \frac{\partial J_{n+1}}{\partial \lambda}(u_h^{m+1}, \lambda^{m+1}),$$

compute

$$(3.37)\qquad \gamma_m = \frac{\int_\Omega \nabla(g_u^{m+1} - g_u^m) \cdot \nabla g_u^{m+1} \, dx + (g_\lambda^{m+1} - g_\lambda^m) g_\lambda^{m+1}}{\int_\Omega |\nabla g_u^m|^2 \, dx + |g_\lambda^m|^2}$$

and set

$$(3.38)\qquad z_u^{m+1} = g_u^{m+1} + \gamma_m z_u^m \,,\; z_\lambda^{m+1} = g_\lambda^{m+1} + \gamma_m z_\lambda^m \,.$$

Do then $m=m+1$ *and go to* (3.33). ■

According to Sec. 2.4 we should use $\{2u_h^n - u_h^{n-1}, 2\lambda^n - \lambda^{n-1}\}$ in (3.29), to compute $\{u_h^{n+1}, \lambda^{n+1}\}$ by the conjugate gradient algorithm (3.29)-(3.38).

About the partial derivatives $\frac{\partial J_{n+1}}{\partial u_h}$, $\frac{\partial J_{n+1}}{\partial \lambda}$ (occuring in (3.35),(3.36), respectively) we should easily prove (using the derivative calculation technique of Sec. 2.2) that at $\{w_h, \mu\}$ we have :

$$(3.39)\quad \begin{cases} \left\langle \dfrac{\partial J_{n+1}}{\partial u_h}(w_h,\mu), v_h \right\rangle = \displaystyle\int_\Omega \nabla\tilde{w}_h \cdot \nabla v_h \, dx - \mu \int_\Omega e^{w_h} \tilde{w}_h v_h \, dx \\ \qquad + \tilde{\mu} \displaystyle\int_\Omega \nabla\left(\frac{u_h^n - u_h^{n-1}}{\Delta s}\right) \cdot \nabla v_h \, dx \quad \forall v_h \in V_{oh}, \end{cases}$$

$$(3.40)\quad \frac{\partial J_{n+1}}{\partial \lambda}(w_h,\mu) = \tilde{\mu}\left(\frac{\lambda^n - \lambda^{n-1}}{\Delta s}\right) - \int_\Omega e^{w_h} \tilde{w}_h \, dx,$$

where, in (3.39),(3.40), $\{\tilde{w}_h,\tilde{\mu}\}$ is obtained from $\{w_h,\mu\}$, through the solution of (3.27),(3.28).

3.4. Numerical experiments.

3.4.1. Formulation of the test problems.

We consider the numerical solution of the particular Bratu problems below

(i) First test problem :

$$(3.41)\quad \begin{cases} -\dfrac{d^2u}{dx^2} = \lambda e^u \ \textit{in}\ ]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

(ii) Second test problem :

$$(3.42)\quad \begin{cases} -\dfrac{d^2u}{dx^2} = \lambda e^u + 1 \ \textit{in}\ ]0,1[\\ u(0) = u(1) = 0. \end{cases}$$

(iii) Third test problem :

$$(3.43)\quad \begin{cases} -\Delta u = \lambda e^u \ \textit{in}\ ]0,1[\times]0,1[, \\ u=0 \ \textit{on}\ \partial\Omega. \ \blacksquare \end{cases}$$

Problems (3.41),(3.42) have been discretized by *one dimensional finite elements* using a space discretization step h=0.1. Problem (3.43) has been approximated by the *finite element method* discussed in Sec. 3.2, using the triangulation $\mathcal{T}_h$ shown on Figure 3.1 and consisting of 512 triangles ; the unknowns are then, the values taken by the approximate solution u_h at the interior nodes of $\mathcal{T}_h$; we have 225 such nodes. The continuation algorithm described in Sec. 3.3 has been applied with $\Delta s = 0.1$.

3.4.2. Numerical results. Further comments.

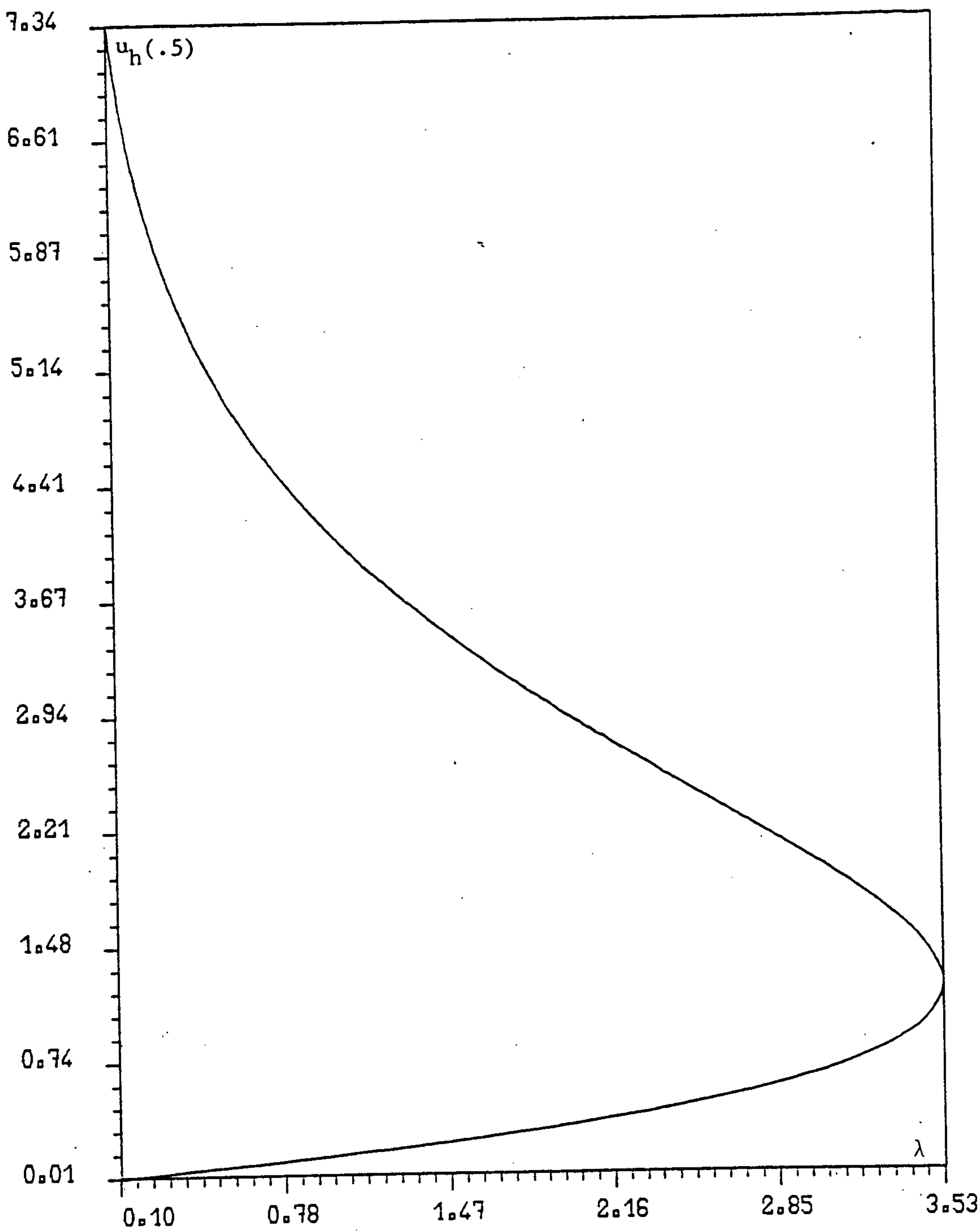
With u_h the approximate solution of (3.41)(resp. (3.42)) we have shown on Figure 3.2 (resp. 3.3) the variation of $u_h(0.5)$ (maximal value of u_h) as a function of λ.

For f=0 (problem (3.41)) the critical value of λ (i.e. the value λ_c corresponding to the *limit point)* is known explicitely and we have λ_c = 3.51 3830... ; with h=0.1 we obtain 3.53... .

Let u_h be the approximate solution of (3.43) ; we have shown on Figure 3.4 the variation of $u_h(0.5,0.5)$ (maximal value of u_h) as a function of λ. The numerical results agree quite very well with those of, e.g., KIKUCHI [13], obtained by quite different methods.

With Δs = 0.1, the solution of the above three test problems never required more than 3 to 4 iterations of the conjugate gradient algorithm (3.29)-(3.38) to obtain $\{u_h^{n+1},\lambda^{n+1}\}$ from $\{u_h^n,\lambda^n\}$, $\{u_h^{n-1},\lambda^{n-1}\}$ via the solution of the least squares problem (3.25) ; such an efficiency is partly due to the good initialization of algorithm (3.29)-(3.38) provided by $\{2u_h^n-u_h^{n-1},2\lambda^n-\lambda^{n-1}\}$. Using the above methods there was no particular difficulties, close to and at the limit point.

To conclude Sec. 3 (but a similar conclusion holds also for the test problems of the following sections) we would like to point out that each iteration of the conjugate gradient algorithm (3.29)-(2.28), requires the solution of several discrete linear problems, equivalent to linear systems, with the same matrix independent of n and m. Since this matrix is symmetric and positive definite we should use a Cholesky factorization done once and for all (taking into account the sparsity of the matrix) ; thus solving the above discrete Dirichlet problems requires only the solution of sparse triangular linear systems, which is a rather cheap operation.



<u>FIGURE 3.2</u>

$-u'' = \lambda\, e^{u}$, $u(o) = u(1) = 0$, $h = 0.1$

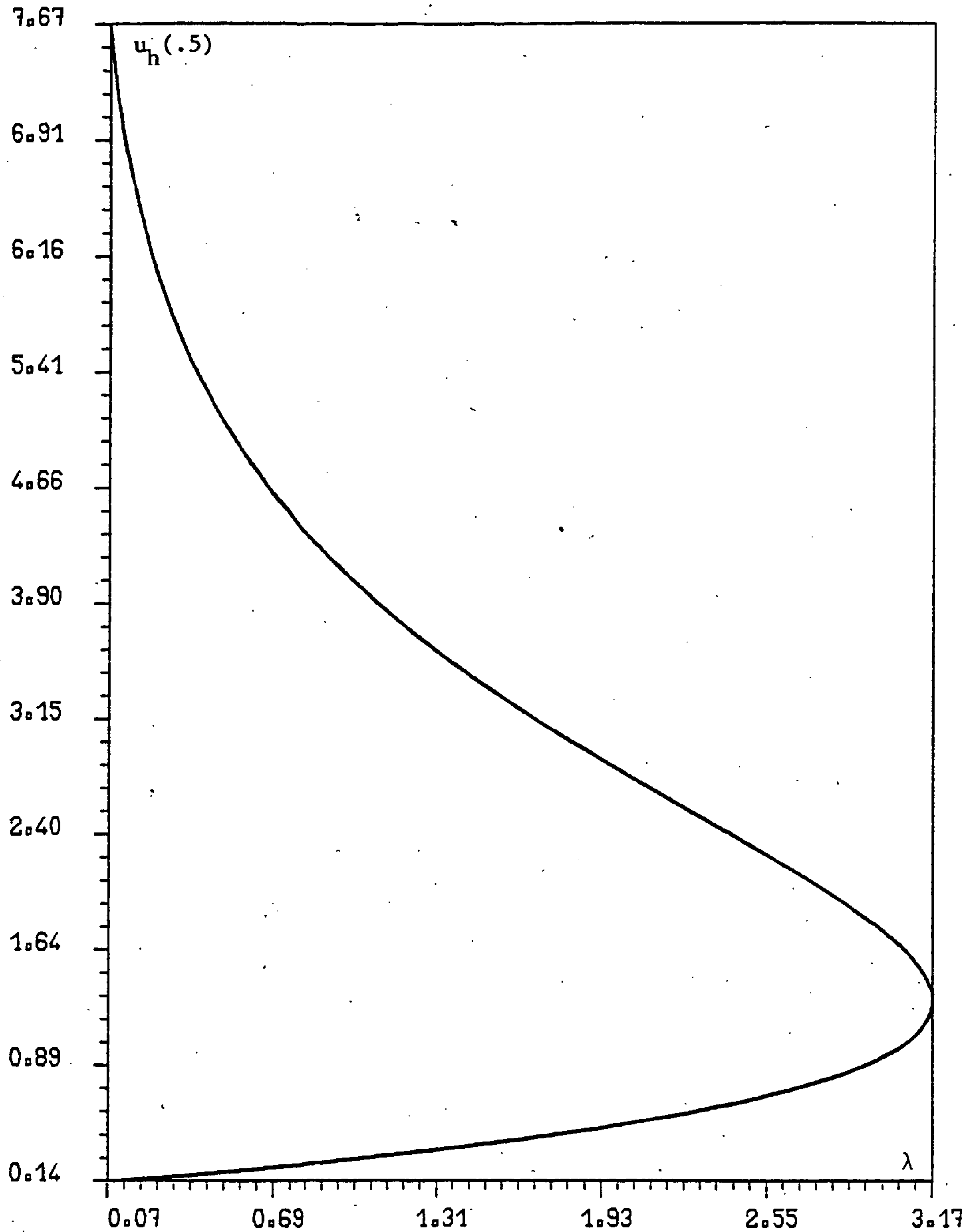


FIGURE 3.3

$-u'' = \lambda e^{u} + 1 , \qquad u(o) = u(1) = 0 , \quad h = 0.1$

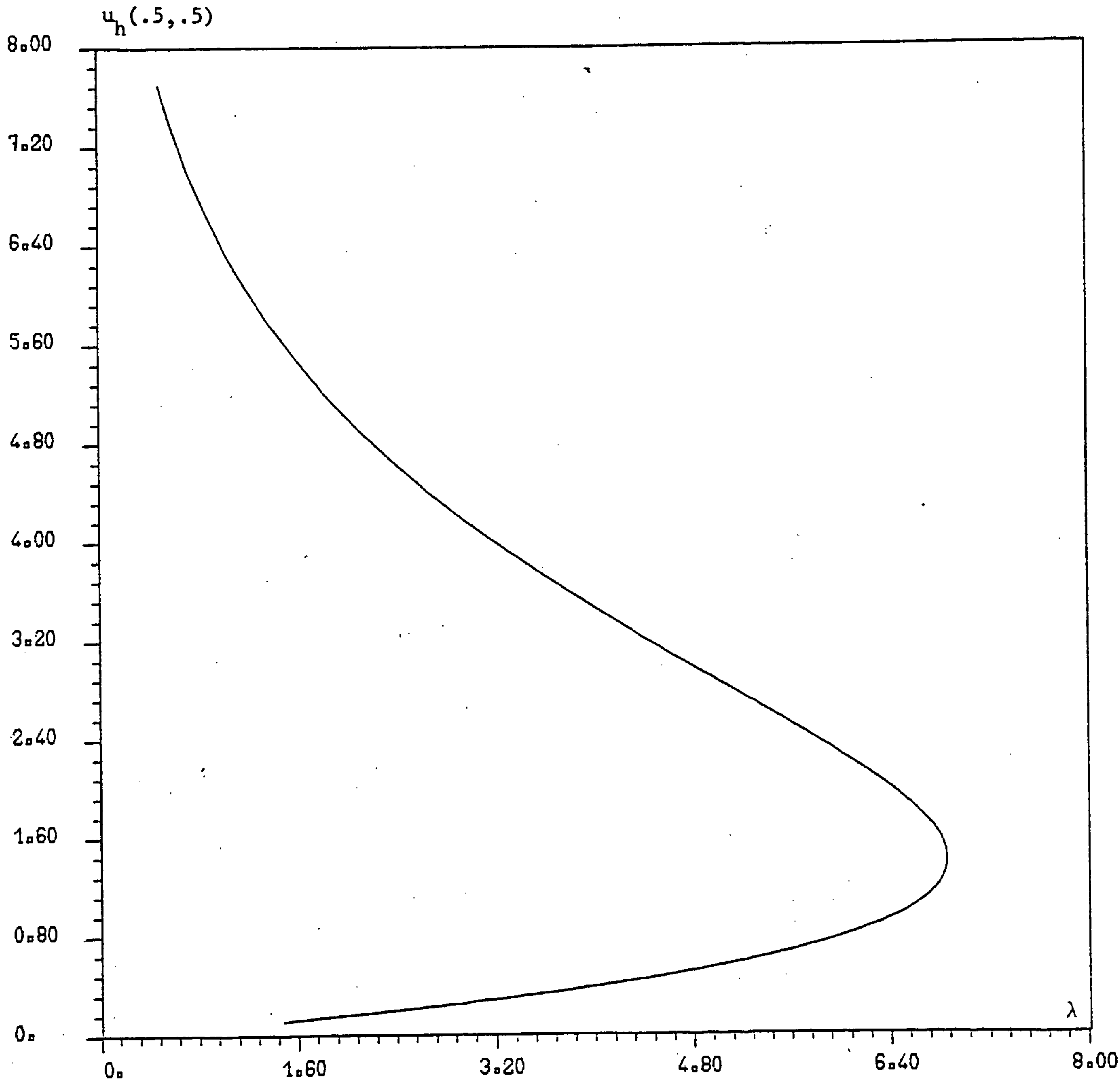


FIGURE 3.3

$$-\Delta u = \lambda e^{u} \ , \ u|_{\Gamma} = 0$$

4. - APPLICATION TO THE SOLUTION OF BIFURCATION AND PERTURBED BIFURCATION PROBLEMS.

4.1. Synopsis. Generalities.

In this section we shall discuss the numerical treatment of *nonlinear second order boundary value problems* whose branches of solutions exhibit *genuine bifurcation* points. To be more precise we shall approximate the original problem by a new one whose branches of solutions have *limit points* only (in the sense of Sec. 2.3) ; they can be computed therefore by the continuation methods of Sec. 2.3. The approximation process is founded on the use of a convenient perturbation method related to the concept of *perturbed bifurcation.*

The problems to be considered in this section have as general formulation

$$(4.1) \qquad (P_\delta) \quad Au = G(u,\lambda,\delta)$$

where (with the notation of Sec. 2) :

V is a real Hilbert space, $\{u,\lambda\} \in V \times \mathbb{R}$, A is an elliptic operator from V to its dual space V', δ is the perturbation parameter, and the range of G is contained in V'.

From (P_δ) we define the *non perturbed problem* (P_o) by

$$(4.2) \qquad (P_o) \quad Au = G(u,\lambda,0),$$

and the *linearized problem* by

$$(4.3) \qquad Av = G_u(u,\lambda,\delta)v,$$

where G_u denotes the Frechet derivative of G with respect to u.

We recall that bifurcation phenomena occur in the case of *singular solution* only (i.e. when $A-G_u(u,\lambda,\delta)$ is a singular operator).

4.2. First example : a nonlinear Dirichlet problem.

4.2.1. Formulation of the problem.

With Ω a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, we consider as first example the solution in $V = H_o^1(\Omega)$, of the nonlinear Dirichlet problem

$$(4.4) \quad \begin{cases} -\Delta u = \lambda u^2 + \delta \ \mathit{in}\ \Omega, \\ u = 0 \ \mathit{on}\ \partial\Omega, \end{cases}$$

with $\delta \in \mathbb{R}$ in (4.4).

The non perturbed problem ($\delta=0$ in (4.4)) has two solution branches for $\lambda \geq 0$:

(i) The trivial branch $\{0,\lambda\}$, $\lambda \in \mathbb{R}_+$,

(ii) A non trivial branch which never crosses the trivial one (see Figure 4.1 below).

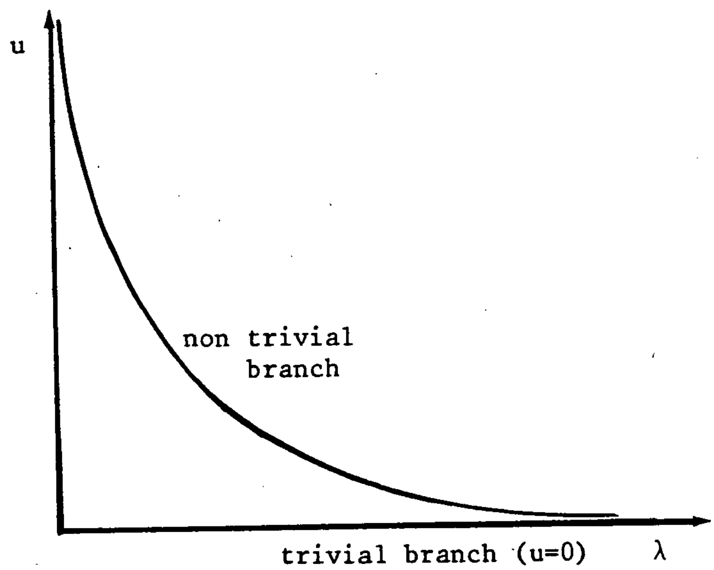


Figure 4.1

Branches of solutions for the non perturbed problem.

By symmetry with respect to $\{0,0\}$ we should obtain the solutions of (4.4) corresponding to $\lambda \leq 0$.

Concerning the perturbed problem it follows from MIGNOT-PUEL [11] that one has a branch of solutions whose points are all regular and one with a normal limit point (Figure 4.2, below, shows - for $\delta > 0$ - the behavior of the solutions of (4.4)).

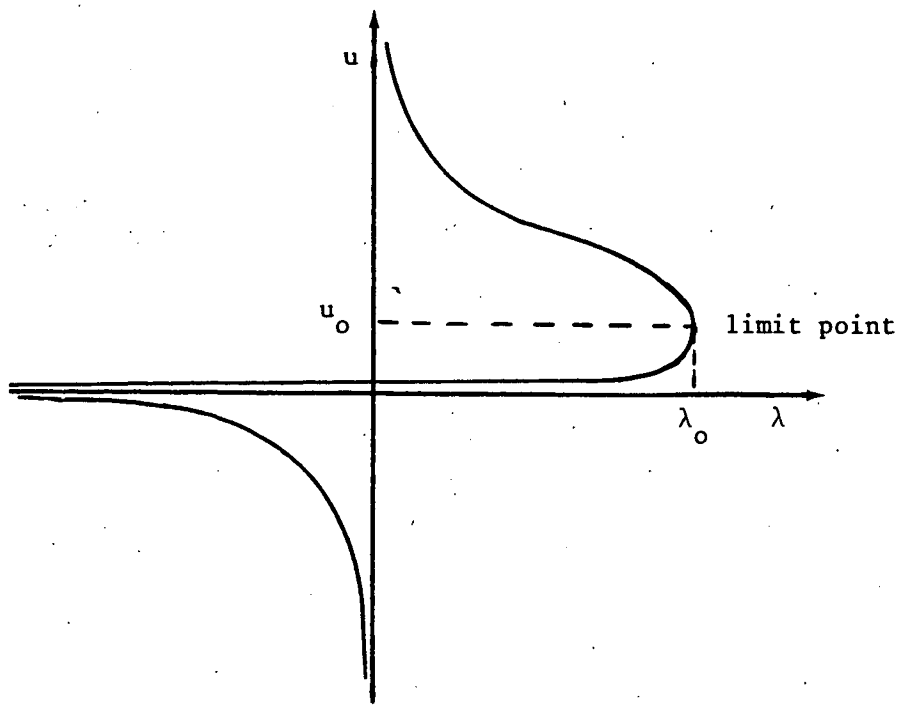


Figure 4.2

Branches of solutions for the perturbed problem ($\delta > 0$) .

Suppose that $\lambda \geq 0$; a possible strategy to compute the nontrivial branch of solutions of the non perturbed problem (4.4) is the following :

Take $\delta > 0$ "sufficiently small", and solve (4.4) by a continuation method, like those described in Sec. 2.3, using $\{u_\delta^o, 0\}$ as starting point, where $u_\delta^o \in H_o^1(\Omega)$ is *the* solution of

$$(4.5)\qquad \begin{cases} -\Delta u_\delta^o = \delta \ \text{in} \ \Omega, \\ u_\delta^o = 0 \ \text{on} \ \partial\Omega. \end{cases}$$

For $\delta > 0$ and sufficiently small, that part C_δ^+ of the branch of solutions of the perturbed problem, beyond the limit point $\{u_{\delta c}, \lambda_{\delta c}\}$ (the upper part in fact), is a good approximation of the nontrivial branch of solutions of the non perturbed problem ; take then two distinct points of C_δ^+, say $\{u_{\delta 1}, \lambda_{\delta 1}\}$ and $\{u_{\delta 2}, \lambda_{\delta 2}\}$, and compute the non trivial solutions of the non perturbed problem corresponding to $\lambda = \lambda_{\delta 1}$ and $\lambda = \lambda_{\delta 2}$ (these solutions can be obtained using simply the least squares-conjugate gradient method of Secs. 2.1, 2.2 (i.e. without continuation), taking $u_{\delta 1}$ and $u_{\delta 2}$ as starting points ; if necessary continuation with respect to δ can be used to reach the value $\delta=0$).

Once two distinct points (sufficiently close one to each other) of the non trivial branch of solutions of the non perturbed problem have been obtained, we can use continuation, again, to describe the whole branch.

Figure 4.3, below, illustrates the above process.

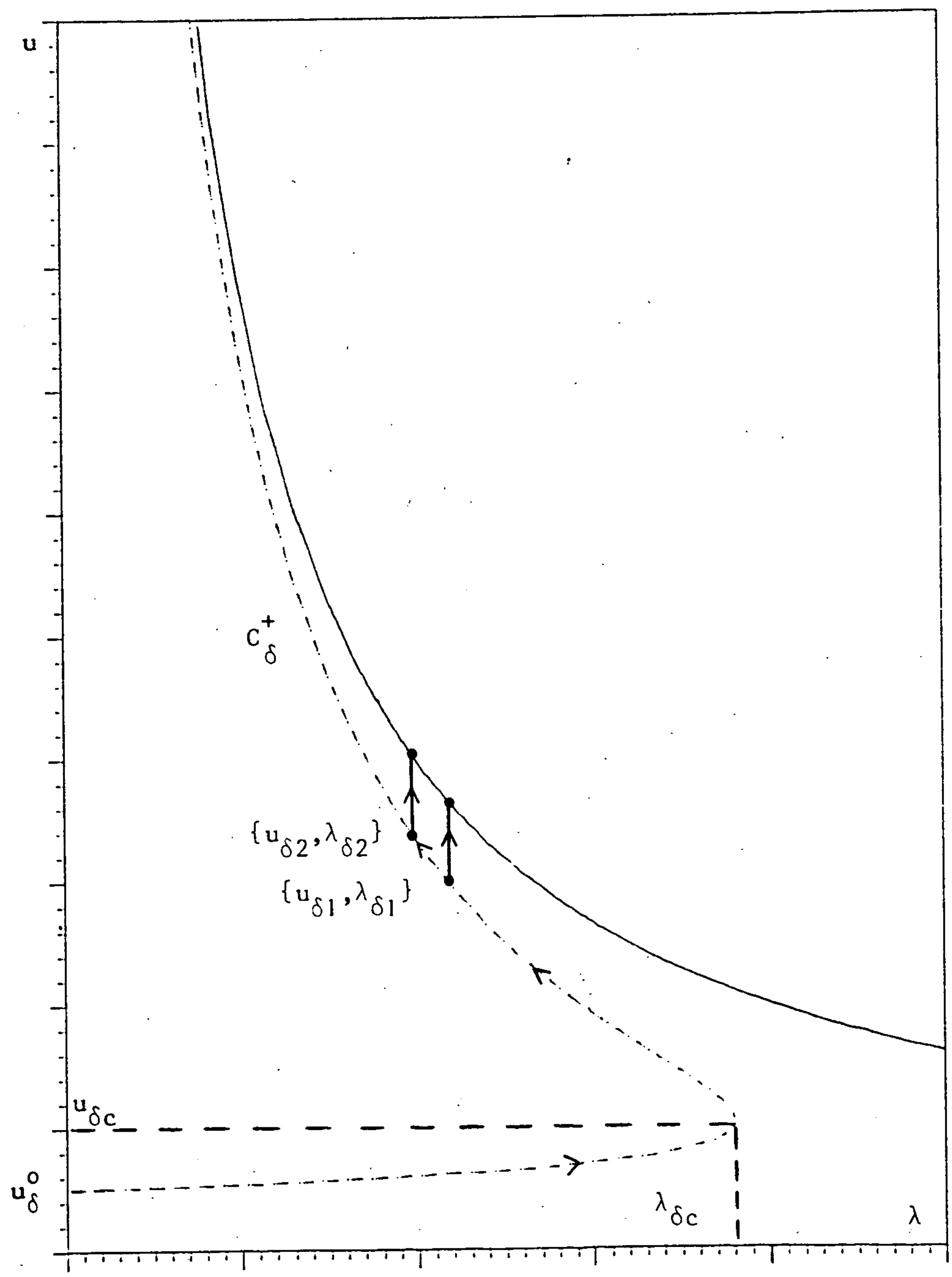


Figure 4.3

The above technique has been applied to compute the non trivial solutions of nonlinear boundary value problems more complicated than (4.4), like those considered in Secs. 4.3, 4.4, for example, and also the *Von Karman equations* for nonlinear plates, for which we refer to REINHART [3], [17] for a numerical treatment by the methods of this paper.

We shall discuss in Secs. 4.2.2, 4.2.3 the numerical solution of (4.4) for the perturbed and non perturbed problems, and take also (4.4) as a test problem to study the influence of several parameters (such as Δs, the approximation of $\{\dot{u},\dot{\lambda}\}$, ...) on the behavior of the continuation process.

4.2.2. Finite element approximation of problem (4.4).

Problem (4.4) has clearly the following *variational formulation*

$$(4.6)\qquad \begin{cases} \textit{Find } \{u,\lambda\} \in H_0^1(\Omega)\times\mathbb{R} \textit{ such that} \\ \displaystyle\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega (\lambda u^2+\delta)v\,dx \quad \forall v\in H_0^1(\Omega). \end{cases}$$

With the *finite element space* V_{oh} still defined by (3.6) we should approximate (4.4),(4.6) by

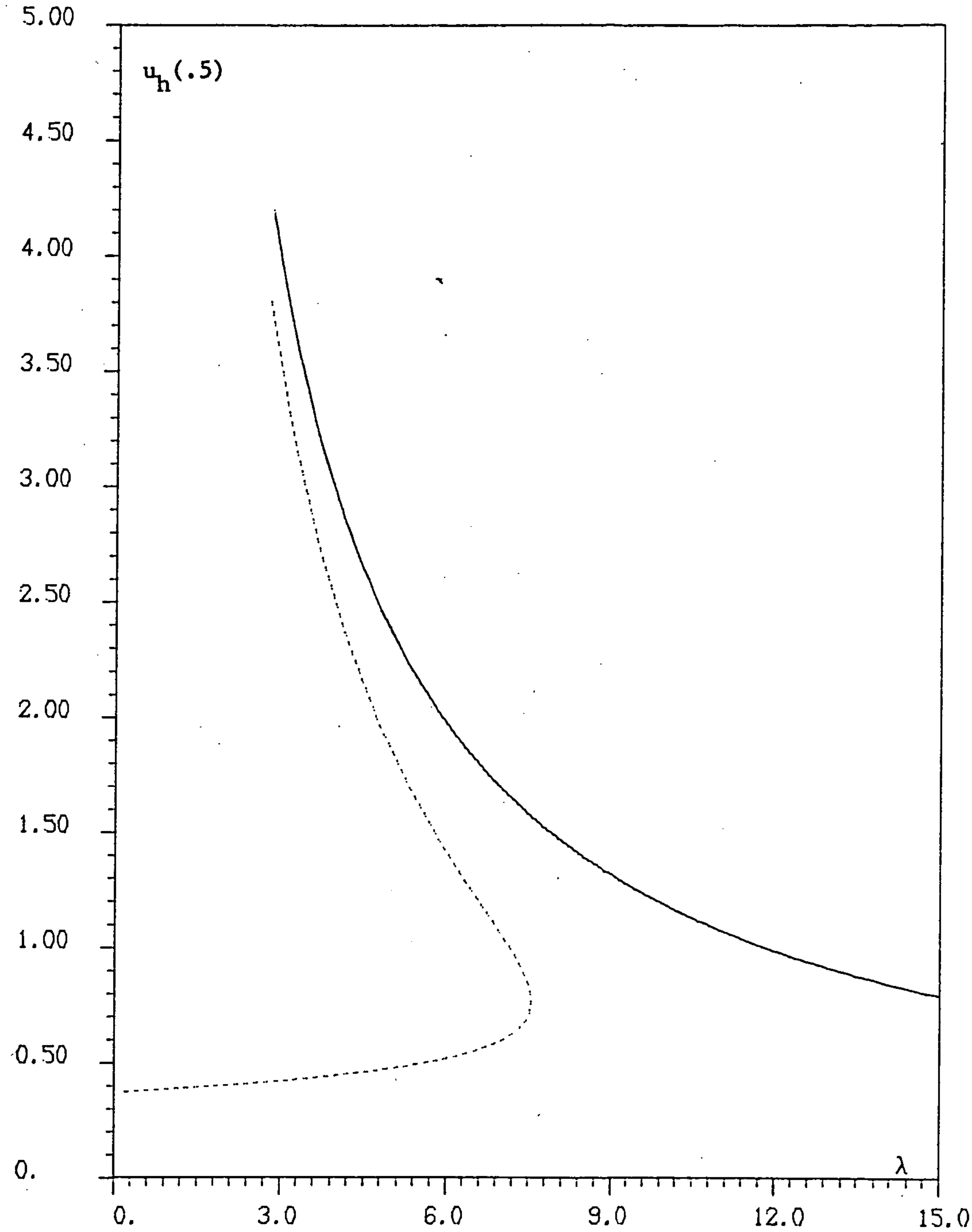
$$(4.7)\qquad \begin{cases} \textit{Find } \{u_h,\lambda\} \in V_{oh}\times\mathbb{R} \textit{ such that} \\ \displaystyle\int_\Omega \nabla u_h\cdot\nabla v_h\,dx = \int_\Omega (\lambda u_h^2+\delta)v_h\,dx \quad \forall v_h\in V_{oh}\ ; \end{cases}$$

problem (4.7) is equivalent to a nonlinear system of equations in $\mathbb{R}^{N_{oh}+1}$ (we recall that $N_{oh} = \dim V_{oh}$). Since the nonlinearity is a *polynomial* one, the various integrals occuring in (4.7) can be calculated exactly ; however, like in Sec. 3.3.2, the computational work can be reduced substantially if one uses *numerical integration* to calculate the contribution of the nonlinear term.

4.2.3. Numerical solution of test problems. Comparisons.

As test problems, for the numerical methods described previously, we have considered

$$(4.8)\qquad \begin{cases} -u'' = \lambda u^2+\delta \textit{ on }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$



<u>Figure 4.4</u>

$$\begin{cases} -u'' = \lambda u^2+\delta \ , \\ u(0) = u(1) = 0 \end{cases}$$

——— $\delta=0$

- - - - - $\delta=2$

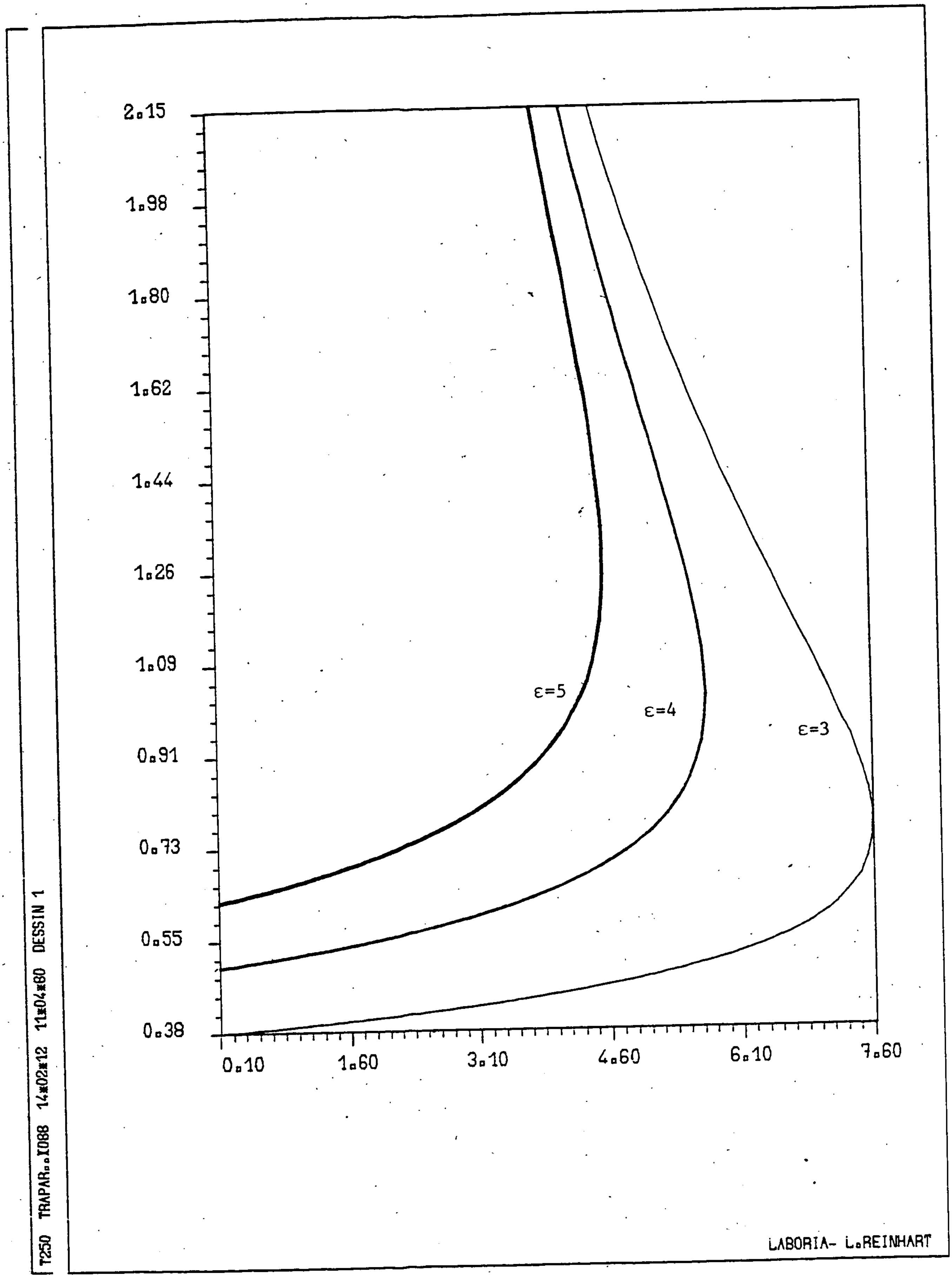


FIGURE 4.5

$$-u'' = \lambda u^2 + \varepsilon \qquad \varepsilon = 5,4,3$$

for δ = 0,2,3,4,5 ; the non trivial branch of solutions corresponding to δ=0 is obtained by the method summarized on Fig. 4.3. We have chosen h = 1/10 and Δs = 1/10 for the space discretization and for the continuation algorithm, respectively. The numerical results are shown on Figs. 4.4 (for δ=0,2) and 4.5 (for δ=3,4,5), where we have plotted $\max_{x\in[0,1]} u_h(x) = u_h(.5)$ versus λ. The computed results agree with those obtained elsewhere by other methods.

Taking (4.8) with δ=5 as test problem we have indicated on Fig. 4.6 the number of *conjugate gradient iterations* necessary to solve the least squares problem encountered at each step of the continuation process ; we observe the very good behavior of our methods at regular points and close to the limit point (for Δs=1/10, at least).

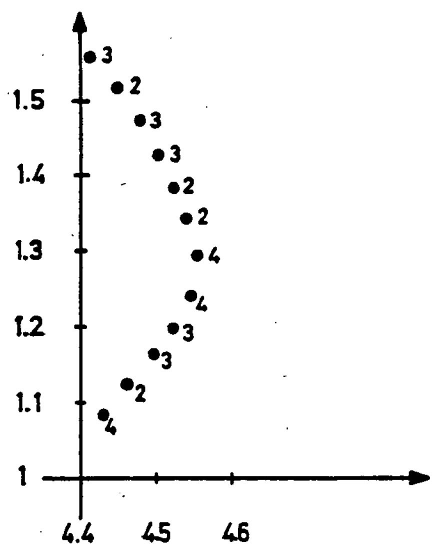


Figure 4.6

If one takes γ_n=0 in algorithm (2.37)-(2.46) (instead of γ_n given by (2.45)) we recover a *steepest descent* algorithm for solving the least squares problem (2.33). In the particular case of (4.8) with δ=4 we have done a comparison between the performances of the steepest descent and conjugate gradient algorithms when applied to the continuation solution of (4.8). The computed results are summarized on Figs. 4.7 (steepest descent) and 4.8 (conjugate gradient) ; they show clearly the superiority of the conjugate gradient variant in the neighborhood of the limit point.

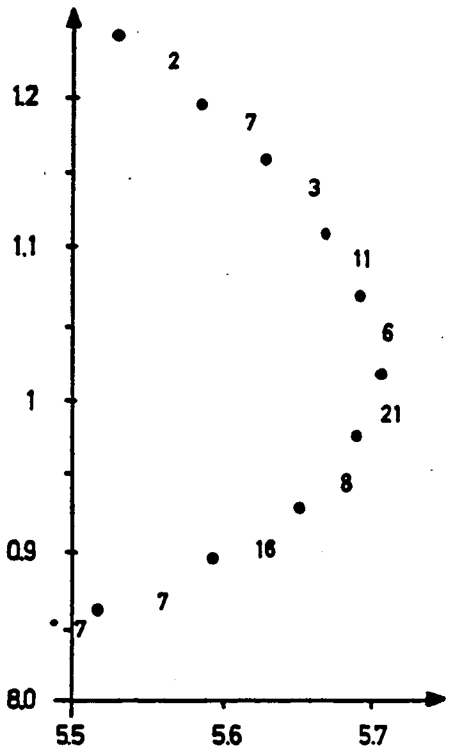


Figure 4.7

Steepest descent

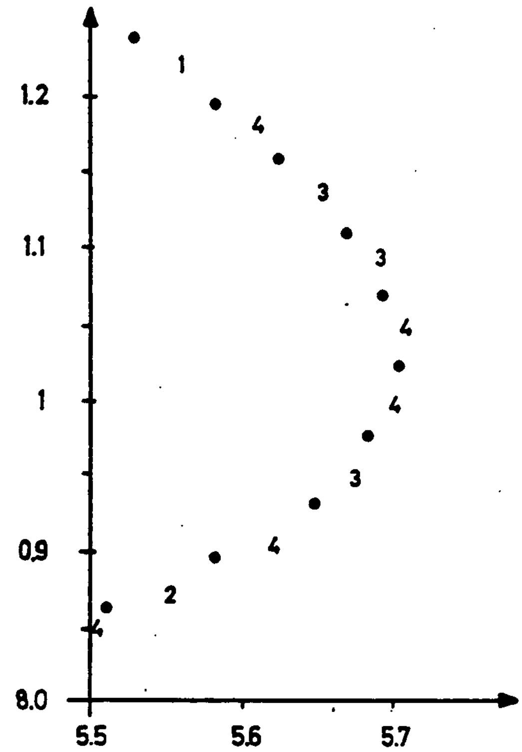


Figure 4.8

Conjugate gradient

To conclude Sec. 4.2, we shall discuss the effect of the size of Δs upon the convergence of our continuation method, *particularly in the neighborhood of the limit point* ; the test problem is (4.8) with $\delta=5$, again. We observed the following phenomena :

(a) If Δs is too large, the algorithm does not converge close to the limit point. We can explain this behavior by the fact that the initial solution provided by $\{2u_h^n - u_h^{n-1}, 2\lambda^n-\lambda^{n-1}\}$ (or $\{u_h^n,\lambda^n\}$) is too far from the branch of solutions.

(b) The smaller is Δs, the smaller is the number of iterations close to the limit points ; however if we are sufficiently far from the limit point the number of iterations is quite small and practically independent of Δs

(c) The smaller is Δs, better is the approximation of the location of the limit point . .

In conclusion we should use large Δs if we are sufficiently far from the limit point, and decrease Δs if we are close to the limit point (further details concerning the choice of Δs may be found in [3]).

4.3. Bifurcation from the trivial branch of solutions.

4.3.1. Synopsis. Generalities.

The purpose of this section is to study simple nonlinear eigenvalue problems, with genuine bifurcations, whose solutions behave like those of the *Von Karman equations* for nonlinear plates.

Using the formalism of Sec. 4.1 we consider problems (4.1) such that

$$(4.9) \qquad G(0,\lambda,0) = 0 \quad \forall\, \lambda \in \mathbb{R}.$$

It follows from (4.9) that $\{0,\lambda\}$ is a *(trivial)* solution of the non perturbed problem (4.2). The particular class of problems (4.1) satisfying (4.9) that we consider here is defined by

$$(4.10) \qquad \begin{cases} \textit{Find } u \in H_o^1(\Omega) \quad (\Longrightarrow u=0 \textit{ on } \partial\Omega) \textit{ such that} \\ -\Delta u = \lambda u + f(u,\lambda) + \delta \end{cases}$$

with f obeying

$$(4.11) \qquad f(0,\lambda) = 0 \quad \forall\, \lambda \in \mathbb{R} \ (\textit{from } (4.9))$$

and

$$(4.12) \qquad f'_u(0,\lambda) = 0 \quad \forall\, \lambda \in \mathbb{R} \ ;$$

we suppose Ω *bounded*, again.

From (4.3), (4.10),(4.12) the linearized problem at u=0, reduces to

$$(4.13) \qquad \begin{cases} -\Delta w = \lambda w \textit{ in } \Omega, \\ w = 0 \textit{ on } \partial\Omega. \end{cases}$$

It follows, from CRANDALL-RABINOWITZ [18], that if λ_i is an eigenvalue of *multiplicity one* in (4.13), then the pair $\{0,\lambda_i\}$ is a *bifurcation point* for the solutions of the non perturbed problem

$$(4.14)\qquad \begin{cases} -\Delta u = \lambda u + f(\lambda,u) \ \mathit{in}\ \Omega, \\ u = 0 \ \mathit{on}\ \Gamma. \end{cases}$$

We recall that if $\Omega =]0,1[$ (resp. $]0,1[\times]0,1[$) the eigenvalues in (4.13) are $\lambda_i = i^2\pi^2$, $i=1,2,\ldots$ (resp. $\lambda_{mn} = (m^2+n^2)\pi^2$, $m,n \in \mathbb{N}$, $m,n \geq 1$), with the corresponding eigenfunctions proportional to $\sin i\pi x$ (resp. $\sin m\pi x_1 \sin n\pi x_2$). We observe that λ_i, $\forall\, i=1,2\ldots$, and λ_{11} are of multiplicity one. Since Ω is bounded, $-\Delta$ is an isomorphism form $H_o^1(\Omega)$ onto $H^{-1}(\Omega)$ ($H^{-1}(\Omega)$: dual space of $H_o^1(\Omega)$) ; we denote by L the *inverse isomorphism* of $-\Delta$ (i.e. $L = (-\Delta)^{-1}$) and we define $S : H_o^1(\Omega) \times \mathbb{R} \times \mathbb{R} \to H_o^1(\Omega)$ by

$$(4.15)\qquad S(u,\lambda,\delta) = u - L(\lambda u + f(u,\lambda) + \delta),$$

implying for (4.10) the equivalent formulation $S(u,\lambda,\delta) = 0$. We have as first derivatives of S

$$(4.16)\qquad \begin{cases} S'_u(u,\lambda,\delta) = I - L(\lambda I + f'_u(u,\lambda)), \\ S'_\lambda(u,\lambda,\delta) = -L(u + f'_\lambda(u,\lambda)), \end{cases}$$

and then as second derivatives

$$(4.17)\qquad \begin{cases} S''_{u^2}(u,\lambda,\delta)\cdot(v,w) = -L(f''_{u^2}(u,\lambda)vw), \\ S''_{\lambda u}(u,\lambda,\delta)\cdot v = -L(v + f''_{\lambda u}(u,\lambda)v), \\ S''_{\lambda^2}(u,\lambda,\delta) = -L(f''_{\lambda^2}(u,\lambda)). \end{cases}$$

Suppose that λ_i is an eigenvalue of multiplicity one in (4.13) ; using (4.11), (4.12) we obtain from (4.16),(4.17) that

$$(4.18)\qquad \begin{cases} S'_u(0,\lambda_i,0) \ \textit{is a singular operator}, \\ S'_\lambda(0,\lambda_i,0) = 0, \end{cases}$$

and

$$(4.19)\qquad \begin{cases} S''_{u^2}(0,\lambda_i,0)\cdot(v,w) = -L(f''_{u^2}(0,\lambda_i)vw), \\ S''_{\lambda u}(0,\lambda_i,0)\cdot v = -Lv\ , \\ S''_{\lambda^2}(0,\lambda_i,0) = -L(f''_{\lambda^2}(0,\lambda_i)). \end{cases}$$

Suppose that $S''_{\lambda^2}(0,\lambda_i,0) = 0$; it follows from BREZZI-RAPPAZ-RAVIART [19] that, *locally* (i.e. in the neighborhood of $\{0,\lambda_i\}$) the solutions of the non perturbed problem (4.14) consist of two branches (one of them being the trivial one) crossing at $\{0,\lambda_i\}$. To λ_i we associate now α_i defined by

$$(4.20)\qquad \alpha_i = (w_i, S''_{u^2}(0,\lambda_i,0)\cdot(w_i,w_i))_{H^1_o(\Omega)}$$

where w_i ($\neq 0$) is an eigenfunction corresponding to λ_i in (4.13) ; it follows from [19] that if $\alpha_i = 0$ (resp. $\alpha_i \neq 0$) then the non trivial branch behaves (locally) as indicated on Fig. 4.9 , *(symmetric bifurcation)*, (resp. Fig. 4.10, *(asymmetric bifurcation))*.

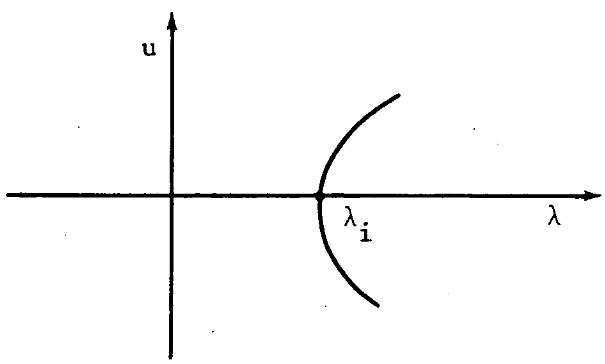


Figure 4.9
Symmetric bifurcation

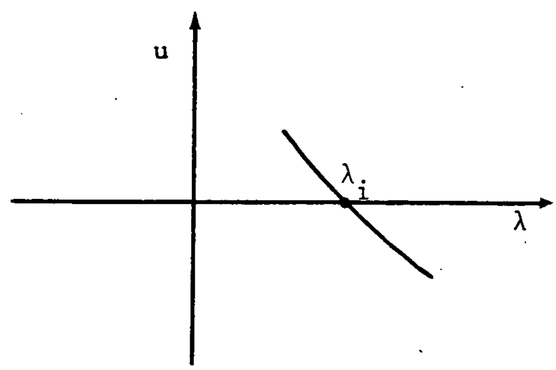
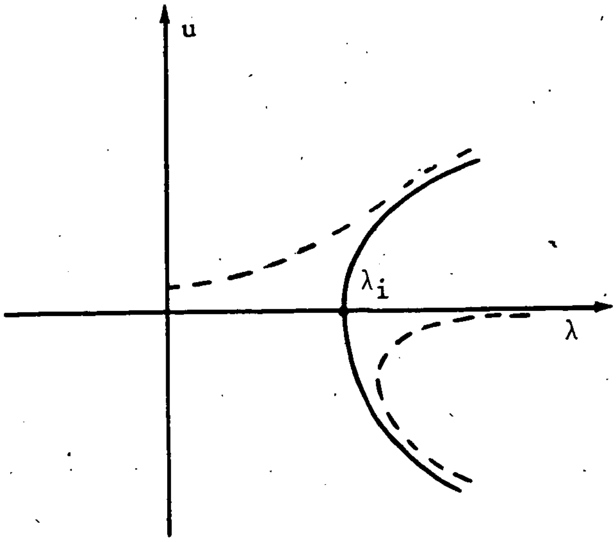


Figure 4.10
Asymmetric bifurcation

If $\delta \neq 0$ we have the local behavior indicated on Figures 4.11, 4.12 (for $\delta > 0$) for the solutions of the perturbed problem (4.10).



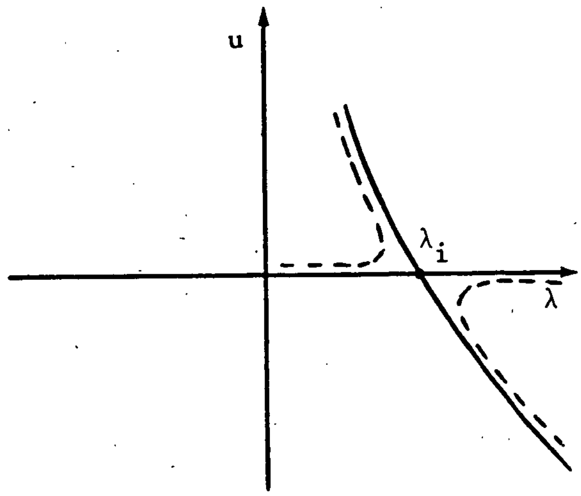


Figure 4.11
Perturbed symmetric bifurcation ($\delta>0$)

Figure 4.12
Perturbed asymmetric bifurcation ($\delta>0$),

4.3.2. Applications.

We apply now the general results of Sec. 4.3.1 to the two functions $f(\cdot,\cdot)$ defined by

(4.21) $$f(u,\lambda) = \frac{\lambda}{2} u^2,$$

and

(4.22) $$f(u,\lambda) = -u^3.$$

For both of them we have $S''_{\lambda^2} = 0$; let us evaluate the α_i :

(i) If $\Omega =]0,1[$, we have $\lambda_i = i^2\pi^2$, $i=1,2,\ldots$, as eigenvalues in (4.13), and we can take $w_i = \sin i\pi x$ as corresponding eigenfunction ; all these eigenvalues are of *multiplicity one*. If f is defined by (4.22), we clearly have $f''_{u^2}(0,\lambda) = 0,\ \forall\ \lambda$, implying

(4.23) $$\alpha_i = 0 \quad \forall i=1,2,\ldots$$

It follows then from (4.23) that the pairs $\{0,\lambda_i\}$ are *symmetric bifurcation points* for the solution branches of the non perturbed problem (4.14). If f is defined by (4.21) we have

$$\alpha_i = -\lambda_i \int_0^1 \sin^2 i\pi x \, dx,$$

implying

$$(4.24) \qquad \begin{cases} \alpha_i = 0 \textit{ if } i=2,4,\ldots \,, \\ \alpha_i \neq 0 \textit{ if } i=1,3,\ldots \,; \end{cases}$$

thus the $\{0,\lambda_i\}$'s are *symmetric (resp. asymmetric) bifurcation* points if i is *even (resp. odd)*.

(ii) If $\Omega =]0,1[\times]0,1[$ the first (i.e. the smallest) eigenvalue in (4.13) is $\lambda_{11} = 2\pi^2$; it is of multiplicity one and we can take $w_{11}(x_1,x_2) = \sin \pi x_1 \times \sin \pi x_2$ as corresponding eigenfunction. If f is defined by (4.22) we have $\alpha_{11} = 0$, implying that $\{0,\lambda_{11}\}$ is a symmetric bifurcation point for the solutions of the non perturbed problem. If f is defined by (4.21) we have $\alpha_{11} \neq 0$ corresponding to an asymmetric bifurcation at $\{0,\lambda_{11}\}$.

Using *finite element approximations* and *continuation methods* like those discussed in Sec. 4.2 we obtain approximate solutions of the non perturbed and perturbed problems, whose behavior at the *first eigenvalue* of the linearized approximate problem (the discrete analogue of (4.13)) is like the one predicted for the continuous problems.

Figure 4.13 is concerned with $\Omega =]0,1[$ and $f(u,\lambda) = \lambda \frac{u^2}{2}$; it shows perturbed and non perturbed *asymmetric* bifurcation phenomena at the first eigenvalue of the linearized approximate problem.

For $f(u,\lambda) = -u^3$ we have a *symmetric* bifurcation phenomenon at the first eigenvalue of the linearized approximate problem as shown on Figure 4.14 (for $\Omega =]0,1[$) and Figure 4.15 (for $\Omega =]0,1[\times]0,1[$).

For more details about the numerical procedure we refer to [3]. We refer also to [17] where it is shown (theoretically *and* computationally) that the solutions of the Von Karman equations for nonlinear plates, have the same qualitative behavior than the one observed for $f(u) = -u^3$ (for the first eigenvalue of the linearized problem, at least).

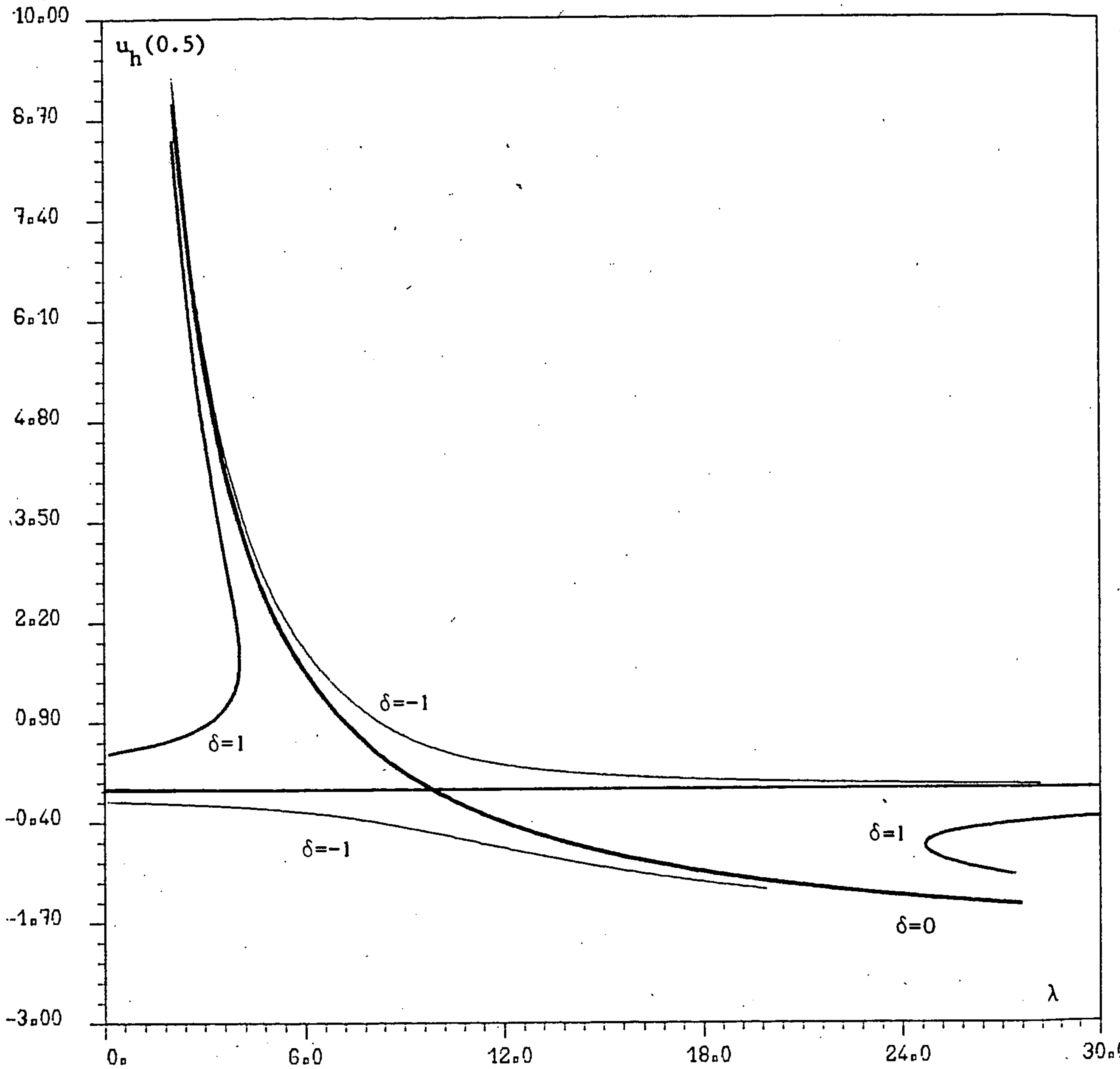
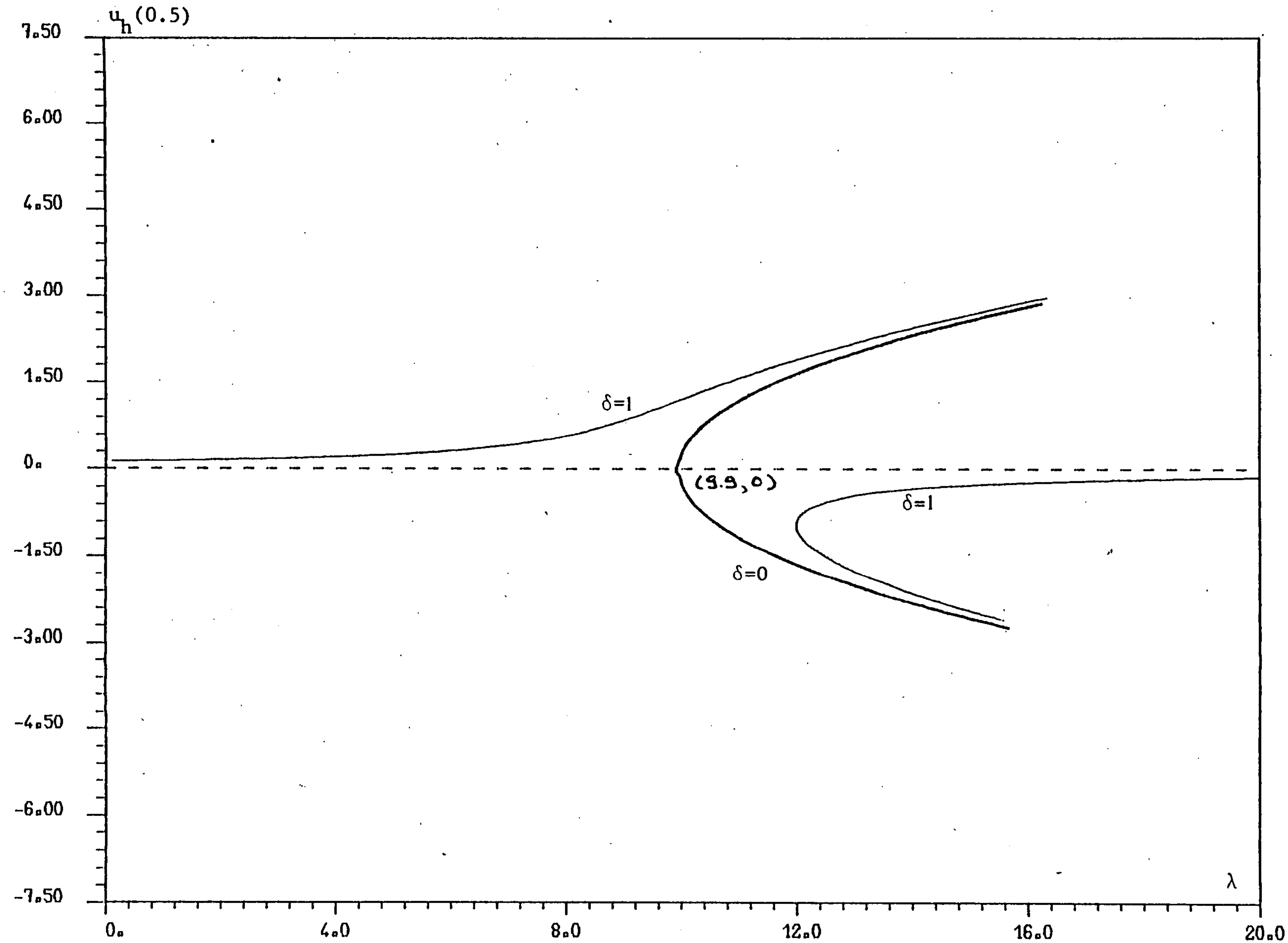


Figure 4.13

$$\begin{cases} -u'' = \lambda u + \lambda \dfrac{u^2}{2} + \delta \text{ in }]0,1[, \\ u(0) = u(1) = 0 \end{cases}$$



<u>Figure 4.14</u>

$$\begin{cases} -u'' = \lambda u - u^3 + \quad \underline{\text{in}}\ ]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

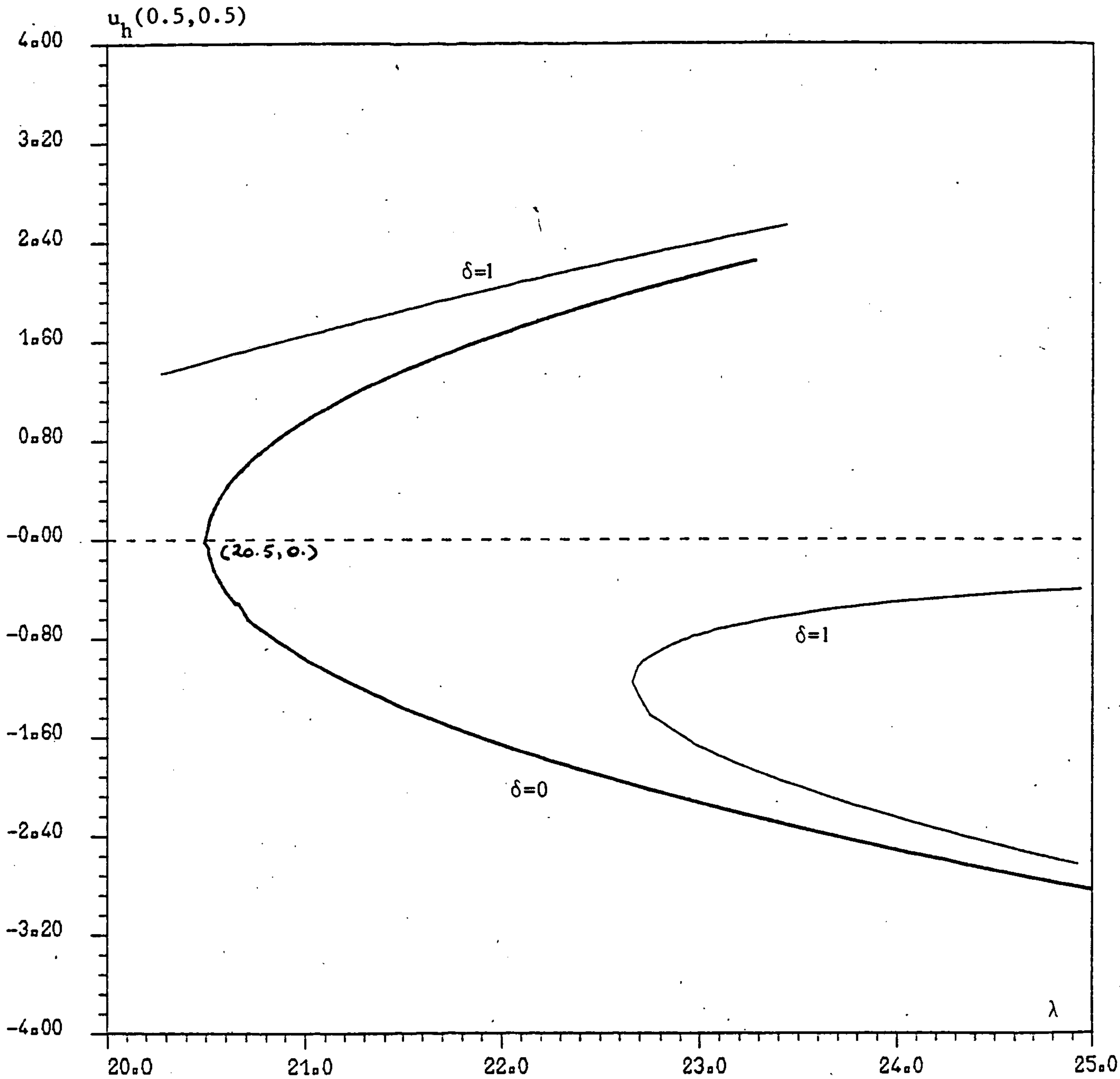


Figure 4.15

$$\begin{cases} -\Delta u = \lambda u - u^3 + \delta \quad \underline{\text{in}} \ ]0,1[\times]0,1[\ (= \Omega) \\ u = 0 \ \underline{\text{on}} \ \partial\Omega. \end{cases}$$

4.4. An example of bifurcation from a non trivial branch of solutions.

4.4.1. Formulation of the problem. Properties of the solutions.

We discuss in this section the solution of the following nonlinear boundary value problem (of *Neumann* type)

$$(4.25)\qquad \begin{cases} \textit{Find } \{u,\lambda\} \in H^1(0,1) \times \mathbb{R} \textit{ such that} \\ -u'' + u = \lambda e^u \textit{ on }]0,1[, \\ u'(0) = u'(1) = 0. \end{cases}$$

Problem (4.25) has the following *equivalent* variational formulation

$$(4.26)\qquad \begin{cases} \textit{Find } \{u,\lambda\} \in H^1(0,1) \times \mathbb{R} \textit{ such that} \\ \int_0^1 u'v' \, dx + \int_0^1 uv \, dx = \lambda \int_0^1 e^u v \, dx \quad \forall v \in H^1(0,1). \end{cases}$$

Problem (4.25),(4.26) has a branch of solutions $\{u,\lambda\}$ with u = const. on $]0,1[$ and therefore solution of

$$(4.27)\qquad u = \lambda e^u \text{ (i.e. } \lambda = ue^{-u}).$$

More precisely, if $\lambda \in]-\infty,0] \cup \{e^{-1}\}$ (resp. $\lambda \in]0,e^{-1}[$), equation (4.27) has a *unique* solution (resp. two *distinct* solutions) as indicated on Figure 4.16.

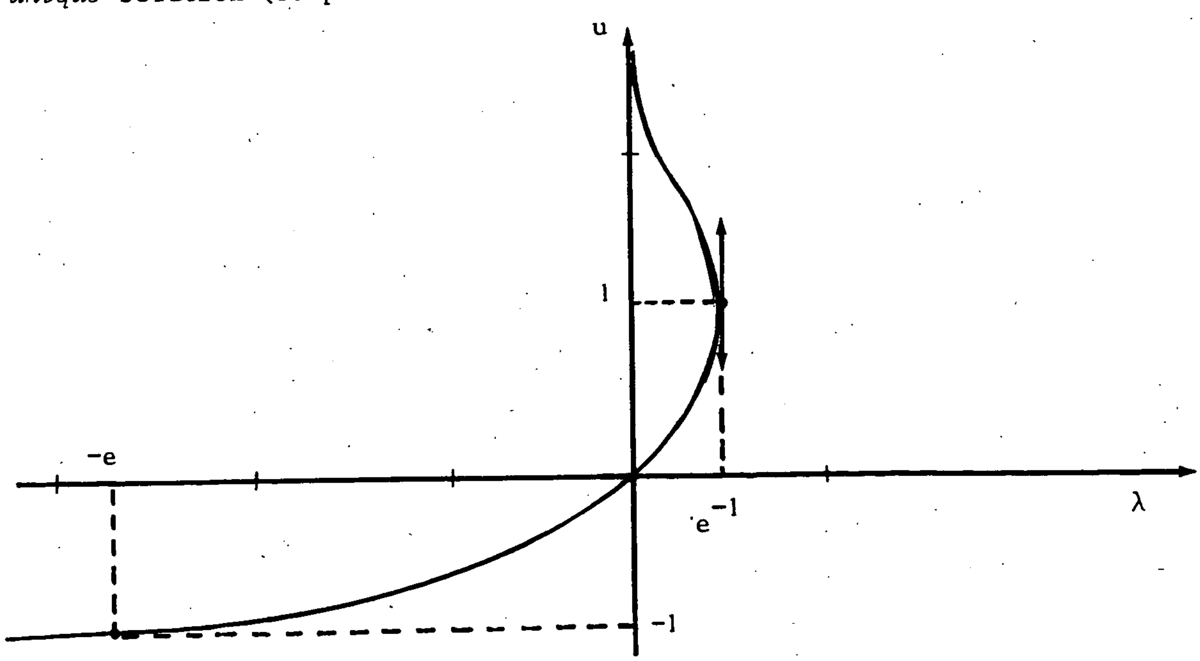


Figure 4.16 : Constant solutions of $u = \lambda e^u$.

Let us discuss now the existence of bifurcated solutions from the above (almost trivial) branch of solutions of (4.25),(4.26). From the simplicity of the problem under consideration, the discussion can be done directly, using (4.25). However, in view of more complicated problems ($\Omega \subset \mathbb{R}^N$ with $N \geq 2$ instead of $\Omega =]0,1[$, for example) we shall use the variational formulation (4.26) for its greater flexibility, and also because it allows the application of the solution methods described in Sec. 2.

A pair $\{v,\lambda\} \in H^1(0,1) \times \mathbb{R}$ being given, we associate to it the continuous linear functional defined over $H^1(0,1)$ by

$$(4.28) \qquad w \to \int_0^1 v'w'\,dx + \int_0^1 vw\,dx - \lambda\int_0^1 e^v w\,dx.$$

For simplicity we use the notation $V = H^1(0,1)$, $V' = (H^1(0,1))'$ where $(H^1(0,1))'$ is the dual space of $H^1(0,1)$; by (4.28) we have defined actually a mapping, denoted by S, from $V \times \mathbb{R}$ to V'. Let $\{u,\lambda\}$ be a solution of (4.25),(4.26) with u = const. on $]0,1[$ (i.e. satisfying (4.27)) ; the linearized problem of (4.26) at u is defined by

$$(4.29) \qquad \begin{cases} \textit{Find } w \in V \textit{ such that,} \\ \displaystyle\int_0^1 w'v'\,dx + \int_0^1 wv\,dx = \lambda e^u \int_0^1 wv\,dx \quad \forall v \in V, \end{cases}$$

or equivalently

$$(4.30) \qquad \begin{cases} w \in V, \\ \langle S'_u(u,\lambda)\cdot w, v\rangle = 0 \quad \forall v \in V, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V' and V. The linearized operator $S'_u(u,\lambda)$ is singular if and only if (4.29),(4.30) has non trivial solutions w, implying that λe^u is an eigenvalue of $v \to -v''+v$ for the (Neumann) boundary conditions $v'(0) = v'(1) = 0$; in that case there exists a function $w \in V-\{0\}$ such that

$$(4.31) \qquad \begin{cases} -w''+w = \lambda e^u w \textit{ in }]0,1[\,, \\ w'(0) = w'(1) = 0. \end{cases}$$

From standard results we know that λe^u has to satisfy

$$\lambda e^u = 1+k^2\pi^2, \ k=0,1,\dots ; \tag{4.32}$$

since the relation $u = \lambda e^u$ holds we obtain, eventually, that the set of singular pairs $\{u,\lambda\}$ is the *discrete* set defined by

$$\begin{cases} \{u_k,\lambda_k\}_{k\geq 0} \ ; \textit{ with for } k=0,1,\dots \\ u_k = 1+k^2\pi^2 \ , \ \lambda_k = (1+k^2\pi^2)e^{-(1+k^2\pi^2)} \ ; \end{cases} \tag{4.33}$$

we can take as eigenfunction w_k in (4.31) the function defined by

$$w_k(x) = \cos k\pi x, \ k=0,1,\dots . \tag{4.34}$$

The first singular pair (obtained by taking k=0 in (4.33)) is $\{1,e^{-1}\}$. Let us verify that $\{1,e^{-1}\}$ is a *normal limit point* for problem (4.25),(4.26) (in the sense of Sec. 2.3, Definition 2.2). Since conditions (2.49),(2.50) are obviously satisfied, let us concentrate on the verification of (2.51) ; proving (2.51) is *equivalent* to prove that the following problem

$$\begin{cases} \textit{Find } w \in V \ (= H^1(0,1)) \textit{ such that} \\ S'_u(1,e^{-1})\cdot w = S'_\lambda(1,e^{-1}), \end{cases} \tag{4.35}$$

has no solution. An equivalent (and more practical) *variational* formulation of (4.35) is given by

$$\begin{cases} \textit{Find } w \in V \textit{ such that} \\ \int_0^1 w'v' \, dx = -e \int_0^1 v \, dx \quad \forall v \in V. \end{cases} \tag{4.36}$$

Suppose the existence of w solution of (4.36) ; since the function v defined by $v(x)=1$ $\forall x$ $]0,1[$, belongs to V we have from (4.36) that $0=-e$, which is an absurdity. Hence $\{1,e^{-1}\}$ is a normal limit point for problem (4.25),(4.26).

On the contrary $\{u_1,\lambda_1\}$ $(= \{1+\pi^2, (1+\pi^2)\ e^{-(1+\pi^2)}\})$ is a *genuine bifurcation* point and it can be proved (using, e.g., BREZZI-RAPPAZ-RAVIART [19]) that the bifurcation at $\{u_1,\lambda_1\}$ is a *symmetric* one.

4.4.2. Approximation methods. Numerical experiments.

Instead of solving (4.25),(4.26) we have chosen for simplicity the following problem

$$(4.37)\qquad \begin{cases} -\dfrac{u''}{\pi^2} + u = \lambda e^u \ \textit{on}\ ]0,1[, \\ u'(0) = u'(1) = 0, \end{cases}$$

whose solutions behave, *qualitatively*, exactly like those of (4.25),(4.26), but which is much easier to handle numerically, since the singular pairs of (4.37) are given by

$$(4.38)\qquad \begin{cases} \{u_k,\lambda_k\}_{k\geq 0}\ ;\ \textit{with for}\ k=0,1,\ldots, \\ u_k = 1+k^2,\ \lambda_k = (1+k^2)\ e^{-(1+k^2)}. \end{cases}$$

We note that for this new problem we have $\{u_1,\lambda_1\} = \{2,e^{-2}\}$ instead of $\{1+\pi^2,(1+\pi^2)e^{-(1+\pi^2)}\}$ for (4.25) (with $(1+\pi^2)e^{-(1+\pi^2)} \simeq 11\times e^{-11} \simeq 5\times 10^{-4}$). Computing the branch of the *constant* solutions of (4.37) is quite easy since it can be identified with that curve of $\mathbb{R}^2$ whose equation is $\lambda = ue^{-u}$.

To compute the *non constant* solutions of (4.37) we use that combination of *finite element approximation* and *continuation techniques* already used in the previous sections. To avoid troubles close to the bifurcation points, during the continuation process, we introduce a *perturbation* of problem (4.37), defined as follows (with ε as perturbation parameter, instead of δ in the above sections)

$$(4.39)\qquad \begin{cases} -\dfrac{u''}{\pi^2} + u = \lambda e^u \ \textit{on}\ ]0,1[, \\ -u'(0) = u'(1) = \varepsilon, \end{cases}$$

whose *variational formulation* is given (with $V = H^1(0,1)$) by

$$(4.40)\qquad \begin{cases} \textit{Find}\ \{u,\lambda\}\in V\times\mathbb{R}\ \textit{such that} \\ \dfrac{1}{\pi^2}\displaystyle\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \lambda\int_0^1 e^u v\,dx + \varepsilon v(0) + \varepsilon v(1)\quad \forall v\in V, \end{cases}$$

(making $\varepsilon=0$ in (4.40) we recover a variational formulation of problem (4.37)).

Remark 4.1 : We observe that if $\{u,\lambda\}$ is solution of (4.39) and if u^* is defined from u by $u^*(x) = u(1-x)$, then $\{u^*,\lambda\}$ is solution of (4.39) with ε replaced by $-\varepsilon$; this property still holds if $\varepsilon=0$ (it holds also for the approximate problems described below).

Finite element approximations of problem (4.39),(4.40) :

With N a positive integer and $h = 1/N$, we define x_i and e_i by

$$x_i = ih \ , \ i=0,\dots N,$$

and

$$e_i = [x_{i-1}, x_i] \ , \ i=1,\dots N,$$

respectively. We approximate $V = H^1(0,1)$ by

$$V_h = \{v_h \mid v_h \in C^0[0,1] \ , \ v_h|_{e_i} \in P_1 \quad \forall i = 1,\dots N\} \ , \tag{4.41}$$

where P_1 = space of the polynomials in one variable, of degree ≤ 1.

Since $V_h \subset V$, it is quite natural to approximate (4.40) (and therefore (4.39)) by

$$\begin{cases} \textit{Find } \{u_h,\lambda\} \in V_h \times \mathbb{R} \textit{ such that} \\ \frac{1}{\pi^2}\int_0^1 u_h' v_h' \, dx + \int_0^1 u_h v_h \, dx = \lambda \int_0^1 e^{u_h} v_h \, dx + \varepsilon v_h(0) + \varepsilon v_h(1) \ \forall v_h \in V_h . \end{cases} \tag{4.42}$$

The various integrals occuring in (4.42) can be computed exactly. However, in practice, we should use the trapezoidal rule to compute the second and the third integral in (4.42) ; doing so we obtain as approximate problem

$$\begin{cases} \textit{Find } \{u_h,\lambda\} \in V_h \times \mathbb{R} \textit{ such that} \\ \frac{1}{\pi^2}\int_0^1 u_h' v_h' \, dx + h \sum_{j=0}^{N} \omega_j u_h(x_j) v_h(x_j) = \lambda h \sum_{j=0}^{N} \omega_j e^{u_h(x_j)} v_h(x_j) + \varepsilon v_h(0) + \varepsilon v_h(1) \\ \forall v_h \in V_h, \end{cases} \tag{4.43}$$

with $\omega_0 = \omega_N = \frac{1}{2}$ and $\omega_i = 1 \quad \forall \ i=1,\dots N-1$.

Both problems (4.42),(4.43) are in fact equivalent to a nonlinear system in $\mathbb{R}^{N+2}$ with $\{\{u_h(x_j)\}_{j=0}^{N},\lambda\}$ as unknown vector. To obtain such system we should take $v_h = w_i$ in (4.42) or (4.43), for i=0,1,...N, and use the following expansion of u_h

$$u_h = \sum_{j=0}^{N} u_h(x_j)w_j, \tag{4.44}$$

where $\mathcal{B}_h = \{w_i\}_{i=0}^{N}$ is that basis of V_h defined by

$$\begin{cases} w_i \in V_h \quad \forall i=0,\dots N, \\ w_i(x_j) = 0 \text{ if } i \neq j,\ w_i(x_i) = 1,\ \forall\ 0 \leq i,j \leq N. \end{cases} \tag{4.45}$$

Using (4.43) we recover in fact a standard finite difference approximation of the nonlinear Neumann problem (4.39).

<u>Continuation solution of the approximate problems</u> : with $V_h \times \mathbb{R}$ equipped with the following norm

$$\{v_h,\mu\} \to \left[\frac{1}{\pi^2}\int_0^1 |v_h'|^2\,dx + \int_0^1 |v_h|^2\,dx + \mu^2\right]^{1/2},$$

(and the corresponding scalar product) the solution of (4.43) or (4.44) by the continuation methods of Sec. 2.3 yields an algorithm which is a trivial modification of the one described in Sec. 3.3 for the solution of the Bratu problem.

<u>Numerical results</u> : Using again the continuation strategy summarized on Figure 4.3 we have computed branches of solutions of the perturbed problem (4.39) and also the first branch of non constant solutions of the non perturbed problem (4.37) ; the following results have been obtained using N=20 (i.e. h= 0.05) for the approximate problems, and ε=0.01 as perturbation parameter.

The variations of $u_h(0)$ and $u_h(0.5)$ as functions of λ have been described on Figure 4.17 and 4.18, respectively, for the perturbed ($\varepsilon \neq 0$) and non perturbed (ε=0) problems (from Remark 4.1, the behavior of $u_h(1)$ is described by Figure 4.17, also). Since the first bifurcation is *symmetric*, the tangent at the branch of non constant solutions, at this first bifurcation point, has to be vertical ; it is so with a good precision. Actually, using smaller Δs and amplifying the vertical variations, we have shown on Fig. 4.19 the variations of $u_h(0)$; the above property of vertical tangent appears even more clearly on Fig. 4.19.

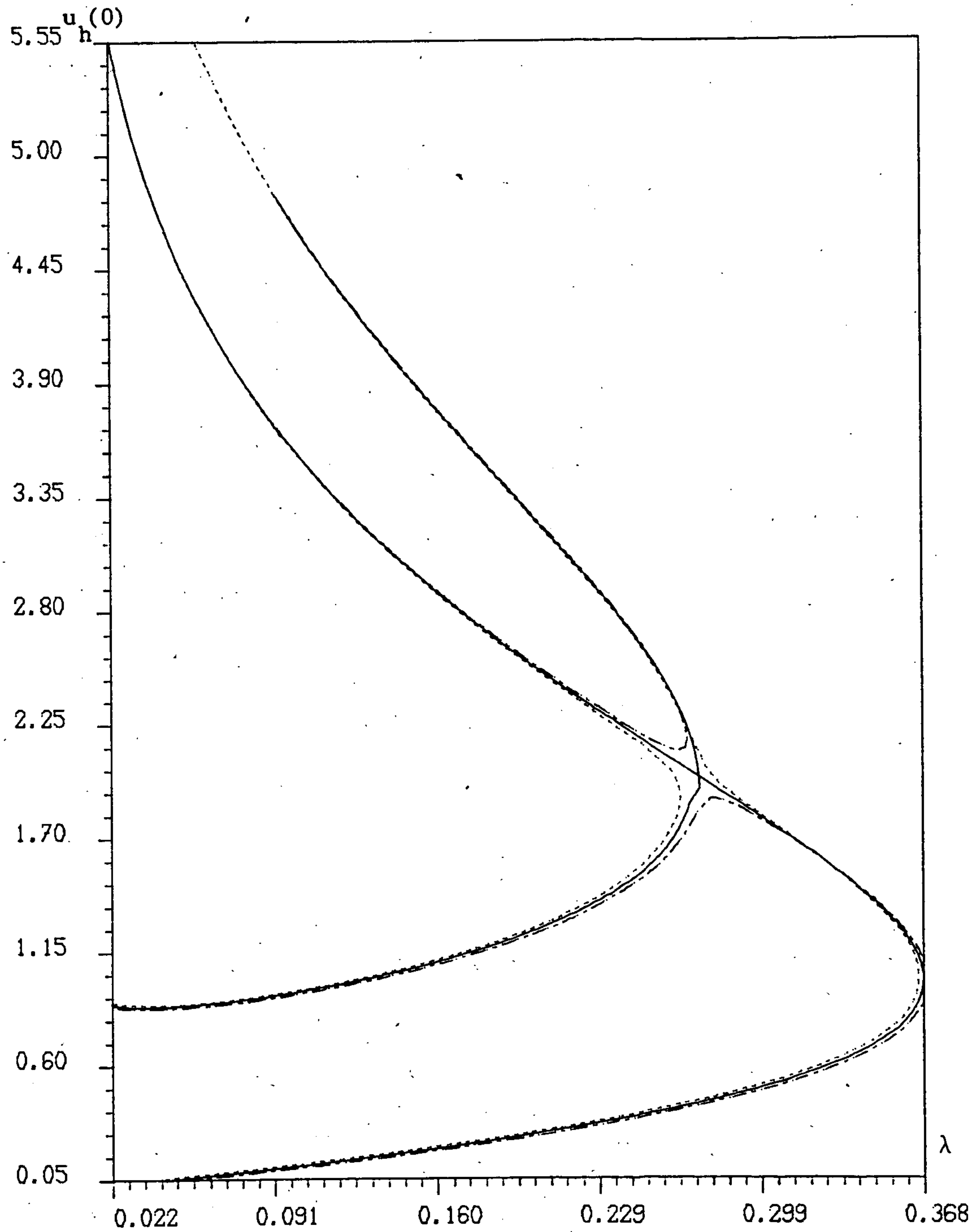


Figure 4.17

$$\begin{cases} -\dfrac{u''}{\pi^2} + u = \lambda e^u \\ -u'(0) = u'(1) = \varepsilon \end{cases}$$

--- ε = 0.01

—— ε = 0

-·- ε = -0.01

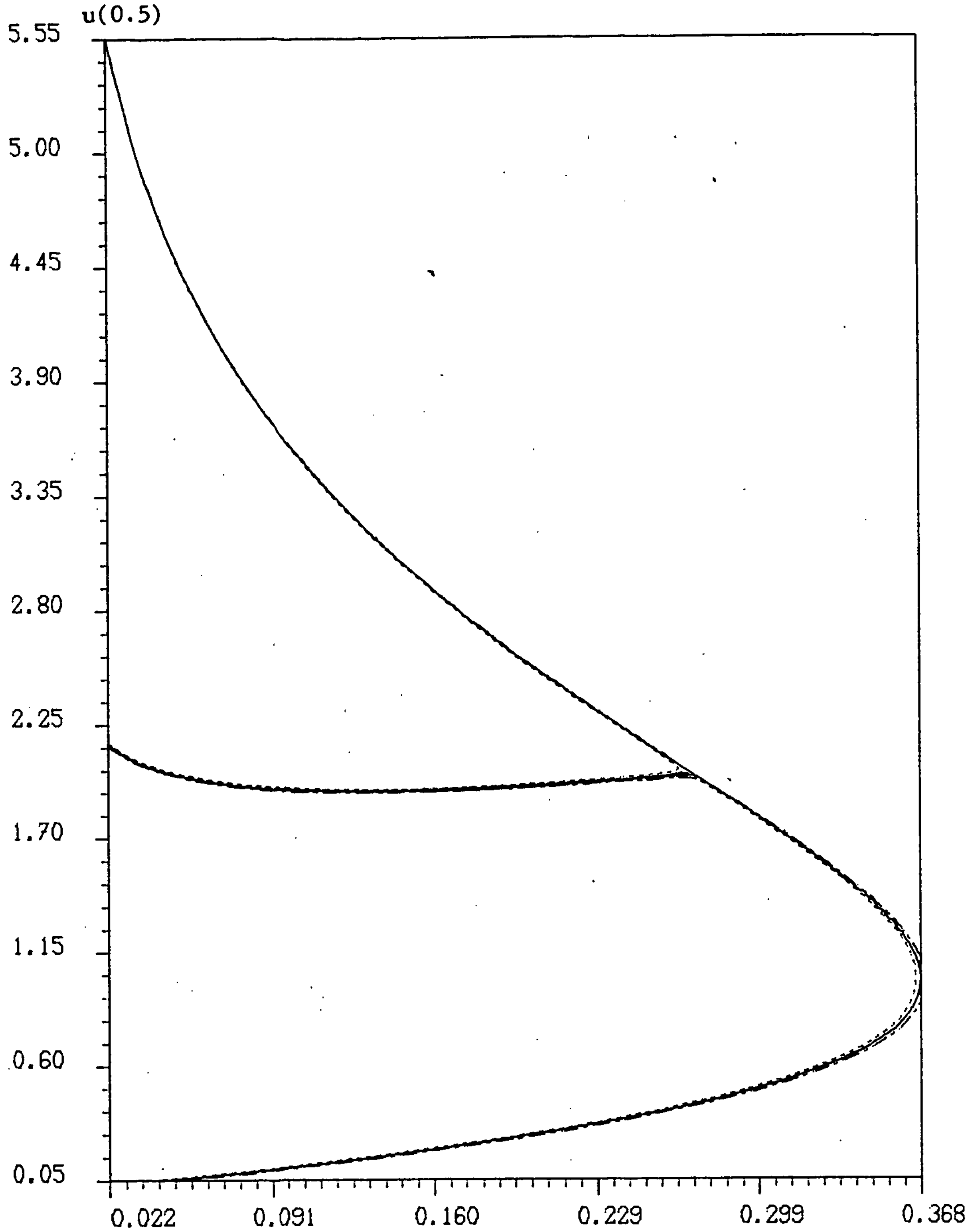


Figure 4.18

$$\begin{cases} -\dfrac{u''}{\pi^2} + u = \lambda e^u \\ -u'(0)=u'(1)=\varepsilon \end{cases}$$

--- $\varepsilon = 0.01$

—— $\varepsilon = 0$

-·- $\varepsilon = -0.01$

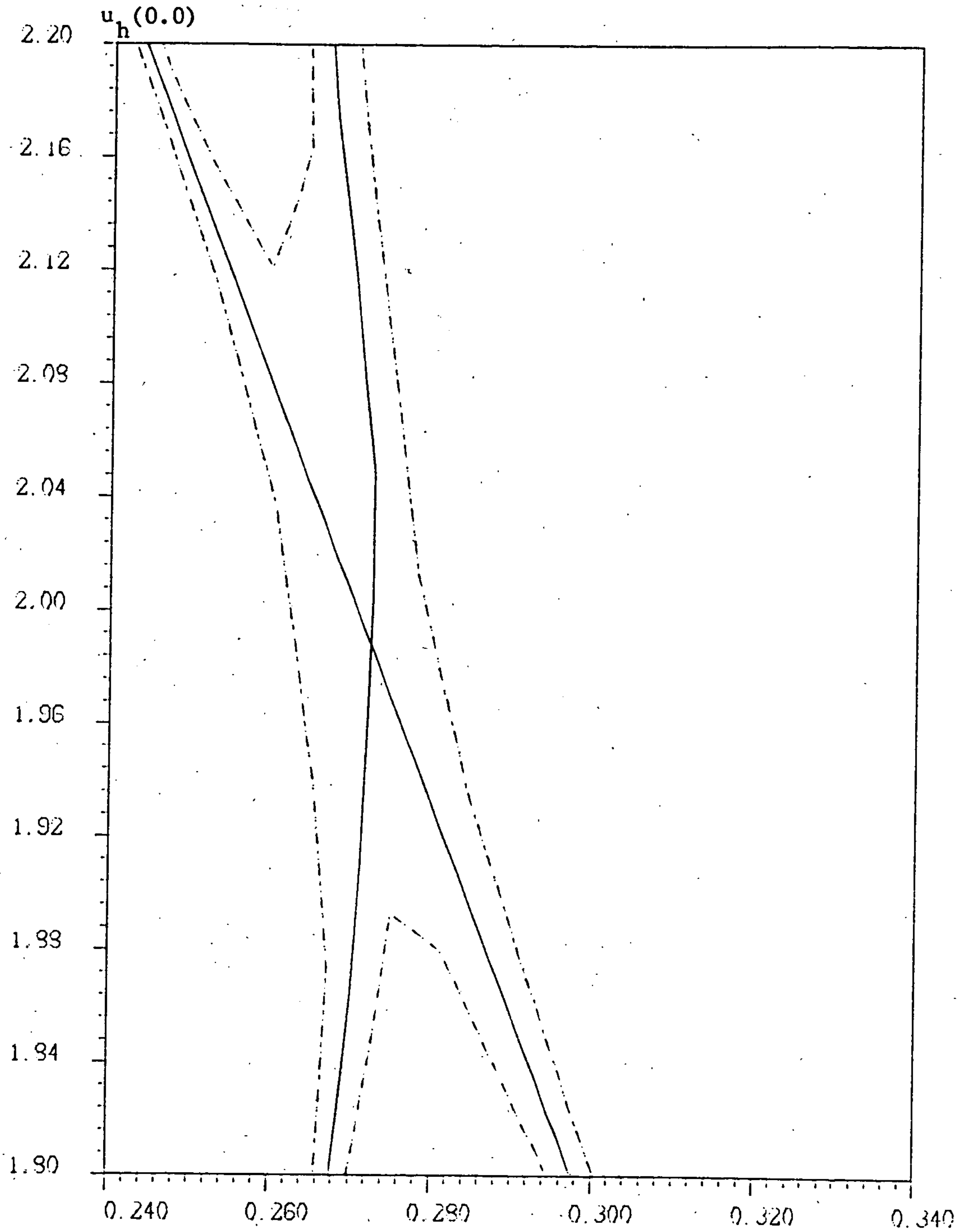


Figure 4.19

$$\begin{cases} -\dfrac{u''}{\pi^2} + u = \lambda e^u \\ u'(0) = -u'(1) = \varepsilon \end{cases}$$

--- $\varepsilon = 0.01$

—— $\varepsilon = 0$

-·- $\varepsilon = -0.01$

5. - APPLICATION TO THE NAVIER-STOKES EQUATIONS FOR IMCOMPRESSIBLE VISCOUS FLUIDS.

5.1. Formulation of the Navier-Stokes equations.

Let Ω be a domain of $\mathbb{R}^N$ (N=2,3 in practice) and Γ be its boundary. The *steady* flows of an incompressible and viscous newtonian fluid, in Ω, are modelled by the following *Navier-Stokes equations*

$$-\nu\Delta\underset{\sim}{u} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} + \underset{\sim}{\nabla} p = \underset{\sim}{f} \textit{ in } \Omega, \tag{5.1}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \textit{ in } \Omega \textit{ (incompressibility condition)}. \tag{5.2}$$

In (5.1),(5.2) :

$\underset{\sim}{u} = \{u_i\}_{i=1}^N$ is the *flow velocity*,

p is the *pressure*,

ν is a *viscosity* parameter,

$\underset{\sim}{f}$ is a *density of external forces*,

$(\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u}$ is a symbolic notation for the vector-function $\{u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^N$.

Typical *boundary conditions* associated to (5.1),(5.2) are

$$\underset{\sim}{u} = \underset{\sim}{u}_\beta \textit{ on } \Gamma, \tag{5.3}$$

where $\underset{\sim}{u}_\beta$ is a given function defined over Γ and satisfying (from the incompressibility condition (5.2))

$$\int_\Gamma \underset{\sim}{u}_\beta\cdot\underset{\sim}{n}\, d\Gamma = 0 \tag{5.4}$$

where $\underset{\sim}{n}$ is the outward normal unit vector at Γ.

The Navier-Stokes equations for incompressible viscous fluids have motivated a countless number of papers, reports, books, ... from both the theoretical and numerical points of view. Concentrating on books only, we shall mention, among others, LIONS [7], LADYZENSKAYA [20], TEMAM [21], GIRAULT-RAVIART [22], RAUTMANN [23], THOMASSET [24] ; we refer also to the numerous references contained in these books.

It follows in particular from [7], [20], [21] that if $\underset{\sim}{f}$ and $\underset{\sim}{u}_\beta$ are sufficiently smooth, then problem (5.1),(5.2),(5.3) has a solution $\{\underset{\sim}{u},p\}$ belonging to $(H^1(\Omega))^N \times (L^2(\Omega)/\mathbb{R})$ (the pressure p is clearly determinated only to within an arbitrary constant). If we suppose in addition that ν is sufficiently large (or equivalently - if ν is given - that $\underset{\sim}{f}$ and $\underset{\sim}{u}_\beta$ are sufficiently small), then problem (5.1)-(5.3) has unique solution in $(H^1(\Omega))^N \times (L^2(\Omega)/\mathbb{R})$.

5.2. Stream function-vorticity formulation of the Navier-Stokes equations.

We suppose from now on that Ω is a bounded domain of $\mathbb{R}^2$. We also assume for simplicity that Ω is *simply connected* (see e.g. GLOWINSKI-PIRONNEAU [25] for the case where Ω is q (≥ 1) connected, i.e. contains q holes). With Γ the boundary of Ω, let $\underset{\sim}{n}$, $\underset{\sim}{s}$ be respectively the unit vector of the outward normal at Ω on Γ and the unit vector of the corresponding *oriented tangent*.

There exists from (5.2) a stream function ψ (determinated only to whithin an arbitrary constant) such that

(5.5) $$u_1 = \frac{\partial\psi}{\partial x_2}, \quad u_2 = -\frac{\partial\psi}{\partial x_1},$$

and it follows from (5.1),(5.5) that ψ satisfies the following (well-known) *nonlinear biharmonic equation*

(5.6) $$\nu\Delta^2\psi + \frac{\partial\psi}{\partial x_1}\frac{\partial}{\partial x_2}\Delta\psi - \frac{\partial\psi}{\partial x_2}\frac{\partial}{\partial x_1}\Delta\psi = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \textit{ in } \Omega .$$

Concerning the *boundary conditions* we have

(5.7) $$\frac{\partial\psi}{\partial s} = \underset{\sim}{u}_\beta\cdot\underset{\sim}{n} \textit{ on } \Gamma \ ;$$

since $\int_\Gamma \underset{\sim}{u}_\beta\cdot\underset{\sim}{n}\, d\Gamma = 0$, (5.7) implies that

(5.8) $$\psi(M) = \int_{\widehat{M_oM}} \underset{\sim}{u}_\beta\cdot\underset{\sim}{n}\, d\Gamma \quad \forall M \in \Gamma ,$$

where $M_o \in \Gamma$ (M_o can be arbitrarily chosen and we have prescribed $\psi(M_o) = 0$). We also have

(5.9) $$\frac{\partial\psi}{\partial n} = -\underset{\sim}{s}\cdot\underset{\sim}{u}_\beta \textit{ on } \Gamma.$$

Actually (5.6),(5.8),(5.9) is a particular case of the more general family of nonlinear biharmonic problems

(5.10) $$\nu\Delta^2\psi + \frac{\partial\psi}{\partial x_1}\frac{\partial}{\partial x_2}\Delta\psi - \frac{\partial\psi}{\partial x_2}\frac{\partial}{\partial x_1}\Delta\psi = f \text{ in } \Omega,$$

(5.11) $$\psi = g_1 \text{ on } \Gamma,$$

(5.12) $$\frac{\partial\psi}{\partial n} = g_2 \text{ on } \Gamma.$$

An equivalent formulation of (5.10)-(5.12) as a nonlinear system of coupled *second order* elliptic equations is

(5.13) $$-\nu\Delta\omega + \frac{\partial\omega}{\partial x_1}\frac{\partial\psi}{\partial x_2} - \frac{\partial\omega}{\partial x_2}\frac{\partial\psi}{\partial x_1} = f \text{ in } \Omega,$$

(5.14) $$-\Delta\psi = \omega \text{ in } \Omega,$$

with the boundary conditions (5.11),(5.12). In (5.13),(5.14), ω is the *vorticity* function.

5.3. Variational formulations.

We suppose that $g = \{g_1, g_2\}$ is sufficiently smooth (see [25] for the precise requirement), so that there exists ψ_0 such that, $\psi_0|_\Gamma = g_1$, $(\partial\psi_0/\partial n)|_\Gamma = g_2$. Let us define V_g by

(5.15) $$V_g = \{\phi \mid \phi \in H^2(\Omega),\ \phi|_\Gamma = g_1,\ \frac{\partial\psi}{\partial n}|_\Gamma = g_2\},$$

then V_g is a *non-empty, closed, affine* subspace of $H^2(\Omega)$, where

$$H^2(\Omega) = \{\phi \mid \phi, \frac{\partial\phi}{\partial x_i}, \frac{\partial^2\phi}{\partial x_i \partial x_j} \in L^2(\Omega),\ \forall\ i,j\}.$$

In particular $V_0 = \{\phi \mid \phi \in H^2(\Omega),\ \phi|_\Gamma = (\partial\phi/\partial n)|_\Gamma = 0\}$ $(= H_0^2(\Omega))$ is a *closed* subspace of $H^2(\Omega)$ (we recall - Ω being bounded - that $\phi \to (\int_\Omega |\Delta\phi|^2\,dx)^{1/2}$ is a norm on V_0 equivalent to the H^2-norm).

A *variational formulation* of (5.10)-(5.12) is then

(5.16) $$\begin{cases} \textit{Find } \psi \in V_g \textit{ such that } \forall\phi \in V_0 \\ \nu\int_\Omega \Delta\psi\,\Delta\phi\,dx + \int_\Omega \Delta\psi\left(\frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2}\right)dx = \int_\Omega f\phi\,dx. \end{cases}$$

To obtain a variational formulation of (5.11)-(5.14) seems to be more complicated ; in fact, introducing $\theta \in L^2(\Omega)$ such that $\theta = -\Delta\phi$ and using (5.14), it follows from (5.16) that the pair $\{\omega,\psi\}$ satisfies

(5.17)
$$\begin{cases} \{\omega,\psi\} \in W_g, \textit{ and } \forall\ \{\theta,\phi\} \in W_o \textit{ we have} \\ \nu\int_\Omega \omega\theta\ dx + \int_\Omega \omega\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx = \int_\Omega f\phi\ dx, \end{cases}$$

where

(5.18)
$$W_o = \{\{\theta,\phi\} \mid \theta \in L^2(\Omega),\ \phi \in V_o,\ -\Delta\phi = \theta \textit{ in } \Omega\},$$

(5.19)
$$W_g = \{\{\theta,\phi\} \mid \theta \in L^2(\Omega),\ \phi \in V_g,\ -\Delta\phi = \theta \textit{ in } \Omega\}.$$

Conversely if a pair $\{\omega,\psi\}$ satisfies (5.17), then $\{\omega,\psi\}$ is also a solution of the nonlinear boundary value problem (5.11)-(5.14) (and ψ a solution of (5.10)-(5.12)).

The variational formulation (5.17) of problem (5.11)-(5.14) contains *second order* derivatives in the definition of W_o and W_g ; having in view the approximation of (5.11)-(5.14) by simple finite element methods it is of great interest to have a variational formulation of (5.11)-(5.14) containing first order derivatives, only. Such a goal is easily achieved since W_o and W_g have the alternative definitions

(5.20)
$$W_o = \{\{\theta,\phi\} \in L^2(\Omega) \times H_o^1(\Omega),\ \int_\Omega \nabla\phi\cdot\nabla q\ dx = \int_\Omega \theta q\ dx \quad \forall q \in H^1(\Omega)\},$$

(5.21)
$$\begin{cases} W_g = \{\{\theta,\phi\} \in L^2(\Omega) \times H^1(\Omega),\ \phi = g_1 \textit{ on } \Gamma,\ \int_\Omega \nabla\phi\cdot\nabla q\ dx = \int_\Omega \theta q\ dx\ + \\ \quad + \int_\Gamma g_2 q\ d\Gamma \quad \forall q \in H^1(\Omega)\}\ , \end{cases}$$

respectively. The equivalence between (5.18),(5.19) and (5.20),(5.21) follows easily from the *Green's formula*

$$\int_\Gamma \frac{\partial\phi}{\partial n} q\ d\Gamma = \int_\Omega \Delta\phi q\ dx + \int_\Omega \nabla\phi\cdot\nabla q\ dx \quad \forall q \in H^1(\Omega),\ \forall\ \phi \in H^2(\Omega),$$

and supposes that Γ (= $\partial\Omega$) is *sufficiently smooth* (or Ω *convex*).

A variational formulation such as (5.17),(5.20),(5.21) is usually known as a *mixed variational formulation.*

5.4. Continuation solution of problem (5.10)-(5.12).

5.4.1. Synopsis.

We apply now the solution methods of Sec. 2 to the nonlinear boundary value problem (5.10) - (5.12). As parameter λ we choose $\lambda = \frac{1}{\nu}$; λ is directly proportional to the *Reynold's number* if we fix the boundary conditions as λ varies.

We consider first (in sec. 5.4.2) the solution of (5.10) - (5.12) via the variational formulation (5.16) ; the solution of (5.10) - (5.12) via (5.17) will be dicussed in sec. 5.4.3.

A mixed finite element implementation will be discussed in sec. 5.5.

5.4.2. Solution of (5.10) - (5.12) via the variational formulation (5.16).

The space V_o $(= H_o^2(\Omega))$ which plays a fundamental role in the sequel is equipped with the inner product

$$\{v,w\} \to \int_\Omega \Delta v \Delta w \, dx$$

and the corresponding norm $v \to (\int_\Omega |\Delta v|^2 \, dx)^{1/2}$.

Taking λ as parameter the problem to be solved is

$$(5.22) \qquad \Delta^2\psi = \lambda(\frac{\partial\psi}{\partial x_2} \frac{\partial}{\partial x_1} \Delta\psi - \frac{\partial\psi}{\partial x_1} \frac{\partial}{\partial x_2} \Delta\psi) + \lambda f \quad in \ \Omega,$$

$$(5.23) \qquad \psi = g_1 \quad on \quad \Gamma ,$$

$$(5.24) \qquad \frac{\partial\psi}{\partial n} = g_2 \quad on \quad \Gamma.$$

A variational formulation of (5.22) - (5.24) is given by

$$(5.25) \qquad \begin{cases} \text{Find } \psi \in V_g \text{ such that } \forall\phi \in V_o \\ \int_\Omega \Delta\psi\Delta\phi \, dx = \lambda\int_\Omega \Delta\psi(\frac{\partial\psi}{\partial x_1} \frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2} \frac{\partial\phi}{\partial x_1}) \, dx + \lambda\int_\Omega f\phi \, dx. \end{cases}$$

Description of the continuation procedure :

In the particular case of problem (5.22)-(5.24) the continuation techniques of Secs. 2.3, 2.4 lead to the following algorithm :

(a) Initialization

(5.26) $\quad$ *Take* $\lambda^o = 0$;

the corresponding ψ^o is the *unique* solution of the following *linear* variational problem.

(5.27)
$$\begin{cases} \textit{Find } \psi^o \in V_g \textit{ such that} \\ \int_\Omega \Delta\psi^o \Delta\phi \, dx = 0 \quad \forall \phi \in V_o. \end{cases}$$

Problem (5.27) is in fact equivalent to the linear biharmonic problem

(5.28)
$$\begin{cases} \Delta^2 \psi^o = 0 \textit{ in } \Omega, \\ \psi^o = g_1 \textit{ on } \Gamma, \ \frac{\partial \psi^o}{\partial n} = g_2 \textit{ on } \Gamma. \end{cases}$$

We take $\{\psi^o,0\}$ as the origin of the arc of solutions passing through it, and define the arc length s by

(5.29)
$$(\delta s)^2 = \int_\Omega |\Delta\delta\psi|^2 \, dx + (\delta\lambda)^2.$$

Denote $\frac{dX}{ds}$ by $\dot{X}$; by differentiation of (5.25) with respect to s, we obtain at s=0

(5.30)
$$\begin{cases} \int_\Omega \Delta\dot{\psi}(0)\,\Delta\phi \, dx = \dot{\lambda}(0) \int_\Omega \Delta\psi^o \left(\frac{\partial \psi^o}{\partial x_1} \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi^o}{\partial x_2} \frac{\partial \phi}{\partial x_1}\right) dx \\ + \dot{\lambda}(0) \int_\Omega f\phi \, dx \quad \forall \ \phi \in V_o \ ; \ \dot{\psi}(0) \in V_o. \end{cases}$$

We have also by definition of s

(5.31)
$$\int_\Omega |\Delta\dot{\psi}(0)|^2 \, dx + \dot{\lambda}^2(0) = 1.$$

Define $\hat{\psi}$ as the solution of the following problem

$$(5.32)\qquad \begin{cases} \hat{\psi} \in V_o, \\ \int_\Omega \Delta\hat{\psi}\ \Delta\phi\, dx = \int_\Omega \Delta\psi^o \left(\frac{\partial\psi^o}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi^o}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx + \int_\Omega f\phi\ dx \quad \forall\phi \in V_o\ ; \end{cases}$$

we clearly have from (5.30)-(5.32), that

$$(5.33)\qquad \dot{\psi}(0) = \dot{\lambda}(0)\hat{\psi},$$

$$(5.34)\qquad \dot{\lambda}(0) = \left(1 + \int_\Omega |\Delta\hat{\psi}|^2\ dx\right)^{-1/2}.$$

b) Continuation

With Δs (> 0) an elementary arc length, we define for $n \geq 0$ an approximation $\{\psi^{n+1},\lambda^{n+1}\}$ ($\in V_g \times \mathbb{R}$) of $\{\psi((n+1)\Delta s),\ \lambda((n+1)\Delta s)\}$ as the solution of the following nonlinear variational system :

Find $\{\psi^{n+1},\lambda^{n+1}\} \in V_g \times \mathbb{R}$ *such that*

$$(5.35)_1\qquad \begin{cases} \int_\Omega \Delta\psi^{n+1}\Delta\phi\ dx = \lambda^{n+1}\int_\Omega \Delta\psi^{n+1}\left(\frac{\partial\psi^{n+1}}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi^{n+1}}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx + \\ \qquad\qquad + \lambda^{n+1}\int_\Omega f\phi\ dx \quad \forall\phi \in V_o, \end{cases}$$

$$(5.35)_2\qquad \int_\Omega \Delta(\psi^1-\psi^o)\Delta\dot{\psi}(0)dx + (\lambda^1-\lambda^o)\dot{\lambda}(0) = \Delta s \ \textit{if}\ n=0,$$

$$(5.35)_3\qquad \int_\Omega \Delta(\psi^{n+1}-\psi^n)\Delta\left(\frac{\psi^n-\psi^{n-1}}{\Delta s}\right)dx + (\lambda^{n+1}-\lambda^n)\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) = \Delta s \ \textit{if}\ n \geq 1. \ \blacksquare$$

With V_o equipped with the inner product

$$\{v,w\} \to \int_\Omega \Delta v\ \Delta w\ dx$$

a convenient nonlinear least squares formulation of (5.35) is then

$$(5.36)\qquad \begin{cases} \textit{Find}\ \{\psi^{n+1},\lambda^{n+1}\} \in V_g \times \mathbb{R}\ \textit{such that} \\ J_{n+1}(\psi^{n+1},\lambda^{n+1}) \leq J_{n+1}(\chi,\mu) \quad \forall\ \{\chi,\mu\} \in V_g \times \mathbb{R}\ , \end{cases}$$

where in (5.36) the functional $J_{n+1}(\cdot,\cdot)$ is defined by

$$(5.37)\qquad J_{n+1}(\chi,\mu) = \frac{1}{2}\int_\Omega |\Delta\tilde{\chi}|^2\ dx + \frac{1}{2}\,|\tilde{\mu}|^2 \quad ;$$

in (5.37), $\tilde{\chi}$ and $\tilde{\mu}$ are nonlinear functions of $\{\chi,\mu\}$, obtained as the solutions of the *linear* problems

$$(5.38)\qquad \begin{cases} \tilde{\chi}\in V_o \ ; \ \forall\phi\in V_o \ \textit{we have} \\ \int_\Omega \Delta\tilde{\chi}\,\Delta\phi\,dx = \int_\Omega \Delta\chi\Delta\phi\,dx - \mu\int_\Omega \Delta\chi\left(\frac{\partial\chi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\chi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx - \mu\int_\Omega f\phi\,dx, \end{cases}$$

$$(5.39)\qquad \tilde{\mu} = \int_\Omega \Delta(\chi-\psi^n)\Delta\left(\frac{\psi^n-\psi^{n-1}}{\Delta s}\right)dx + (\mu-\lambda^n)\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) - \Delta s,$$

respectively ; problem (5.38) is a biharmonic problem. In that particular situation the conjugate gradient algorithm (2.37)-(2.46) reduces to :

Step 0 : Initialization

$$(5.40)\qquad \{\psi^o,\lambda^o\}\in V_g\times\mathbb{R}\ \textit{is given.}$$

Compute then $\{g^o_\psi,g^o_\lambda\}\in V_o\times\mathbb{R}$ *as the solution of*

$$(5.41)\qquad \begin{cases} \int_\Omega \Delta g^o_\psi\,\Delta\phi\,dx = \left\langle\frac{\partial J_{n+1}}{\partial\psi}(\psi^o,\lambda^o),\phi\right\rangle \quad \forall\phi\in V_o, \\ g^o_\psi\in V_o, \end{cases}$$

$$(5.42)\qquad g^o_\lambda = \frac{\partial J_{n+1}}{\partial\lambda}(\psi^o,\lambda^o),$$

and set

$$(5.43)\qquad \{z^o_\psi,z^o_\lambda\} = \{g^o_\psi,g^o_\lambda\}\ .\ \blacksquare$$

Then for $m\geq 0$, *assuming that* $\{\psi^m,\lambda^m\}$, $\{g^m_\psi,g^m_\lambda\}$, $\{z^m_\psi,z^m_\lambda\}$ *are known, compute* $\{\psi^{m+1},\lambda^{m+1}\}$, $\{g^{m+1}_\psi,g^{m+1}_\lambda\}$, $\{z^{m+1}_\psi,z^{m+1}_\lambda\}$ *by*

Step 1 : Descent

$$(5.44)\qquad \begin{cases} \textit{Find } \rho_m\in\mathbb{R} \textit{ such that, } \forall\rho\in\mathbb{R}, \\ J_{n+1}(\psi^m-\rho_m z^m_\psi,\lambda^m-\rho_m z^m_\lambda) \leq J_{n+1}(\psi^m-\rho z^m_\psi,\lambda^m-\rho z^m_\lambda), \end{cases}$$

and set

$$(5.45)\qquad \psi^{m+1} = \psi^m - \rho_m z_\psi^m \,,\quad \lambda^{m+1} = \lambda^m - \rho_m z_\lambda^m \,.$$

Step 2 : Construction of the new descent direction

Define $\{g_\psi^{m+1}, g_\lambda^{m+1}\}$ *as the solution of*

$$(5.46)\qquad \begin{cases} \int_\Omega \Delta g_\psi^{m+1}\, \Delta\phi\, dx = <\frac{\partial J_{n+1}}{\partial\psi}(\psi^{m+1},\lambda^{m+1}),\phi> \quad \forall\phi\in V_o, \\ g_\psi^{m+1} \in V_o, \end{cases}$$

$$(5.47)\qquad g_\lambda^{m+1} = \frac{\partial J_{n+1}}{\partial\lambda}(\psi^{m+1},\lambda^{m+1}),$$

compute

$$(5.48)\qquad \gamma_m = \frac{\int_\Omega \Delta(g_\psi^{m+1}-g_\psi^m)\Delta g_\psi^{m+1}\, dx + (g_\lambda^{m+1}-g_\lambda^m) g_\lambda^{m+1}}{\int_\Omega |\Delta g_\psi^m|^2\, dx + |g_\lambda^m|^2}$$

and set

$$(5.49)\qquad z_\psi^{m+1} = g_\psi^{m+1} + \gamma_m z_\psi^m \,,\quad z_\lambda^{m+1} = g_\lambda^{m+1} + \gamma_m z_\lambda^m.$$

Do then m = m+1 *and go to* (5.44). ■

According to Sec. 2.4 we should use $\{2\psi^n-\psi^{n-1}, 2\lambda^n-\lambda^{n-1}\}$ as initializer in (5.40) to compute $\{\psi^{n+1},\lambda^{n+1}\}$ by the conjugate gradient algorithm (5.40)-(5.49). About the partial derivatives $\frac{\partial J_{n+1}}{\partial\psi}$, $\frac{\partial J_{n+1}}{\partial\lambda}$ we should prove that at $\{\chi,\mu\}\in V_g\times\mathbb{R}$ we have

$$(5.50)\qquad \begin{cases} <\frac{\partial J_{n+1}}{\partial\psi}(\chi,\mu),\phi> = \int_\Omega \Delta\tilde\chi\,\Delta\phi\, dx - \mu\int_\Omega \left(\frac{\partial\chi}{\partial x_1}\frac{\partial\tilde\chi}{\partial x_2} - \frac{\partial\chi}{\partial x_2}\frac{\partial\tilde\chi}{\partial x_1}\right)\Delta\phi\, dx \\ -\mu\int_\Omega \Delta\chi\left(\frac{\partial\phi}{\partial x_1}\frac{\partial\tilde\chi}{\partial x_2} - \frac{\partial\phi}{\partial x_2}\frac{\partial\tilde\chi}{\partial x_1}\right)dx + \tilde\mu\int_\Omega \Delta\left(\frac{\psi^n-\psi^{n-1}}{\Delta s}\right)\Delta\phi\, dx \\ \forall\phi\in V_o, \end{cases}$$

$$(5.51)\qquad \frac{\partial J_{n+1}}{\partial\lambda}(\chi,\mu) = \tilde\mu\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) - \int_\Omega \Delta\chi\left(\frac{\partial\chi}{\partial x_1}\frac{\partial\tilde\chi}{\partial x_2} - \frac{\partial\chi}{\partial x_2}\frac{\partial\tilde\chi}{\partial x_1}\right) dx - \int_\Omega f\tilde\chi\, dx,$$

where, in (5.50),(5.51), $\{\tilde\chi,\tilde\mu\}$ is obtained from $\{\chi,\mu\}$ through the solution of (5.38),(5.39).

Remark 5.1 : Despite its apparent complexity algorithm (5.40)-(5.49) is quite easy to use in practice ; what it requires is essentially a good solver for linear biharmonic problems of the following type

$$(5.52)\qquad \begin{cases} \Delta^2 w = f \text{ in } \Omega, \\ w = g_1 \text{ on } \Gamma, \ \dfrac{\partial w}{\partial n} = g_2 \text{ on } \Gamma. \end{cases}$$

Finite element solvers for (5.52) will be discussed (briefly) in Sec. 5.5 ; they are founded on the mixed variational formulation (5.17).

5.4.3. Solution of (5.10)-(5.12) via the mixed variational formulation (5.17).

Using $\lambda = 1/\nu$ as parameter, the nonlinear mixed variational problem (5.17) becomes

$$(5.53)\qquad \begin{cases} \text{Find } \{\omega,\psi\} \in W_g \text{ such that } \forall\ \{\theta,\phi\} \in W_o \text{ we have} \\ \displaystyle\int_\Omega \omega\theta\ dx + \lambda\int_\Omega \omega\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\phi}{\partial x_1}\right)dx = \lambda\int_\Omega f\phi\ dx, \end{cases}$$

with W_o and W_g still defined by (5.20),(5.21), respectively.

Description of the continuation procedure :

We clearly have from Sec. 5.4.2

(a) Initialization

$$(5.54)\qquad \text{Take } \lambda^o = 0 \ ;$$

then $\{w^o,\psi^o\}$ is the unique solution of the following linear mixed variational problem (equivalent to (5.27))

$$(5.55)\qquad \begin{cases} \text{Find } \{\omega^o,\psi^o\} \in W_g \text{ such that} \\ \displaystyle\int_\Omega \omega^o\theta\ \ dx = 0 \quad \forall\ \{\theta,\phi\} \in W_o. \end{cases}$$

We take then $\{\{\omega^o,\psi^o\},0\}$ as the origin of the arc of solutions passing through it and define the arc length s by

$$(5.56)\qquad (\delta s)^2 = \int_\Omega (\delta\omega)^2 dx + (\delta\lambda)^2.$$

By differentiation of (5.53) with respect to s, we obtain at s=0

$$(5.57)\quad \begin{cases} \int_\Omega \dot{\omega}(0)\theta\ dx = \dot{\lambda}(0)\int_\Omega \omega^o\left(\frac{\partial\psi^o}{\partial x_2}\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi^o}{\partial x_1}\frac{\partial\phi}{\partial x_2}\right)dx + \dot{\lambda}(0)\int_\Omega f\phi\ dx \\ \forall\ \{\theta,\phi\}\in W_o\ ;\ \{\dot{\omega}(0),\dot{\psi}(0)\}\in W_o. \end{cases}$$

Since we have

$$\int_\Omega |\dot{\omega}(0)|^2\ dx + \dot{\lambda}^2(0) = 1,$$

we obtain from (5.57) that

$$\dot{\lambda}(0) = \left(1+\int_\Omega |\hat{\omega}|^2\ dx\right)^{-1/2},$$

$$\{\dot{\omega}(0),\dot{\psi}(0)\} = \dot{\lambda}(0)\{\hat{\omega},\hat{\psi}\},$$

where $\{\hat{\omega},\hat{\psi}\}$ is the solution of the linear mixed variational problem

$$(5.58)\quad \begin{cases} \{\hat{\omega},\hat{\psi}\}\in W_o, \\ \int_\Omega \hat{\omega}\theta\ dx = \int_\Omega \omega^o\left(\frac{\partial\psi^o}{\partial x_2}\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi^o}{\partial x_1}\frac{\partial\phi}{\partial x_2}\right)dx + \int_\Omega f\phi\ dx\quad \forall\{\theta,\phi\}\in W_o. \end{cases}$$

(b) Continuation

With Δs (> 0) an elementary arc length we define for $n \geq 0$ an approximation $\{\{\omega^{n+1},\psi^{n+1}\},\lambda^{n+1}\}\in W_g\times\mathbb{R}$ of $\{\{\omega((n+1)\Delta s),\psi((n+1)\Delta s)\},\lambda((n+1)\Delta s)\}$ as the solution of the following nonlinear mixed variational system :

Find $\{\{\omega^{n+1},\psi^{n+1}\},\lambda^{n+1}\}\in W_g\times\mathbb{R}$ *such that*

$$(5.59)_1\quad \int_\Omega \omega^{n+1}\theta\ dx = \lambda^{n+1}\int_\Omega \omega^{n+1}\left(\frac{\partial\psi^{n+1}}{\partial x_2}\frac{\partial\phi}{\partial x_1} - \frac{\partial\psi^{n+1}}{\partial x_1}\frac{\partial\phi}{\partial x_2}\right)dx + \lambda^{n+1}\int_\Omega f\phi\ dx,\quad \forall\ \{\theta,\phi\}\in W_o,$$

$$(5.59)_2\quad \int_\Omega (\omega^1-\omega^o)\dot{\omega}(0)dx + (\lambda^1-\lambda^o)\dot{\lambda}(0) = \Delta s\ \text{if}\ n=0,$$

$$(5.59)_3\quad \int_\Omega (\omega^{n+1}-\omega^n)\left(\frac{\omega^n-\omega^{n-1}}{\Delta s}\right)dx + (\lambda^{n+1}-\lambda^n)\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) = \Delta s\ \text{if}\ n\geq 1.\ \blacksquare$$

The space W_o can be equipped with

$$\{\{\theta_1,\phi_1\},\{\theta_2,\phi_2\}\}\to\int_\Omega \theta_1\theta_2\ dx$$

as inner product. A convenient least squares formulation of (5.59) is then

$$(5.60)\qquad \begin{cases} \textit{Find } \{\{\omega^{n+1},\psi^{n+1}\},\lambda^{n+1}\} \in W_g \times \mathbb{R} \textit{ such that} \\ j_{n+1}(\{\omega^{n+1},\psi^{n+1}\},\lambda^{n+1}) \le j_{n+1}(\{\eta,\chi\},\mu) \quad \forall\{\{\eta,\chi\},\mu\} \in W_g \times \mathbb{R} , \end{cases}$$

where in (5.60) the functional $j_{n+1}(\cdot,\cdot)$ is defined by

$$(5.61)\qquad j_{n+1}(\{\eta,\chi\},\mu) = \frac{1}{2}\int_\Omega |\tilde{\eta}|^2\, dx + \frac{1}{2}\, |\tilde{\mu}|^2 ;$$

in (5.61), $\{\tilde{\eta},\tilde{\chi}\}$ and $\tilde{\mu}$ are nonlinear function of η,χ,μ obtained as the solutions of the *linear* problems

$$(5.62)\qquad \begin{cases} \{\tilde{\eta},\tilde{\chi}\} \in W_o \ ; \ \forall\{\theta,\phi\} \in W_o \textit{ we have} \\ \displaystyle\int_\Omega \tilde{\eta}\theta\, dx = \int_\Omega \eta\theta\, dx - \mu\int_\Omega \eta\Big(\frac{\partial\chi}{\partial x_2}\frac{\partial\phi}{\partial x_1} - \frac{\partial\chi}{\partial x_1}\frac{\partial\phi}{\partial x_2}\Big)dx - \mu\int_\Omega f\phi\, dx, \end{cases}$$

$$(5.63)\qquad \tilde{\mu} = \int_\Omega (\eta-\omega^n)\Big(\frac{\omega^n-\omega^{n-1}}{\Delta s}\Big)dx + (\mu-\lambda^n)\Big(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\Big) - \Delta s,$$

respectively ; problem (5.62) is equivalent to the biharmonic problem (5.38).

The conjugate gradient algorithm (2.37)-(2.46) becomes in that context

Step 0 : Initialization

$$(5.64)\qquad \{\{\omega^o,\psi^o\},\lambda^o\} \in W_g \times \mathbb{R} \textit{ is given .}$$

Compute then $\{\{g_\omega^o,g_\psi^o\},g_\lambda^o\} \in W_o \times \mathbb{R}$ *as the solution of*

$$(5.65)\qquad \begin{cases} \displaystyle\int_\Omega g_\omega^o\theta\, dx = \Big\langle\frac{\partial j_{n+1}}{\partial(\omega,\psi)}(\{\omega^o,\psi^o\},\lambda^o),\{\theta,\phi\}\Big\rangle \quad \forall\{\theta,\phi\} \in W_o, \\ \{g_\omega^o,g_\psi^o\} \in W_o, \end{cases}$$

$$(5.66)\qquad g_\lambda^o = \frac{\partial j_{n+1}}{\partial\lambda}(\{\omega^o,\psi^o\},\lambda^o),$$

and set

$$(5.67)\qquad \{z_\omega^o,z_\psi^o,z_\lambda^o\} = \{g_\omega^o,g_\psi^o,g_\lambda^o\}. \blacksquare$$

Then for $m \geq 0$, *assuming that* $\{\omega^m, \psi^m, \lambda^m\}$, $\{g_\omega^m, g_\psi^m, g_\lambda^m\}$, $\{z_\omega^m, z_\psi^m, z_\lambda^m\}$ *are known, compute* $\{\omega^{m+1}, \psi^{m+1}, \lambda^{m+1}\}$, $\{g_\omega^{m+1}, g_\psi^{m+1}, g_\lambda^{m+1}\}$, $\{z_\omega^{m+1}, z_\psi^{m+1}, z_\lambda^{m+1}\}$ by

Step 1 : Descent

(5.68) $$\begin{cases} \text{Find } \rho_m \in \mathbb{R} \text{ such that, } \forall \rho \in \mathbb{R}, \\ j_{n+1}(\{\omega^m - \rho_m z_\omega^m, \psi^m - \rho_m z_\psi^m\}, \lambda^m - \rho_m z_\lambda^m) \leq j_{n+1}(\{\omega^m - \rho z_\omega^m, \psi^m - \rho z_\psi^m\}, \lambda^m - \rho z_\lambda^m), \end{cases}$$

and set

(5.69) $$\omega^{m+1} = \omega^m - \rho_m z_\omega^m, \quad \psi^{m+1} = \psi^m - \rho_m z_\psi^m, \quad \lambda^{m+1} = \lambda^m - \rho_m z_\lambda^m .$$

Step 2 : Construction of the new descent direction

Define $\{g_\omega^{m+1}, g_\psi^{m+1}\}$, g_λ^{m+1} *as the solutions of*

(5.70) $$\begin{cases} \int_\Omega g_\omega^{m+1}\, \theta \, dx = \left\langle \frac{\partial j_{n+1}}{\partial(\omega,\psi)}(\{\omega^{m+1}, \psi^{m+1}\}, \lambda^{m+1}), \{\theta, \phi\} \right\rangle \ \forall \{\theta, \phi\} \in W_o, \\ \{g_\omega^{m+1}, g_\psi^{m+1}\} \in W_o, \end{cases}$$

(5.71) $$g_\lambda^{m+1} = \frac{\partial j_{n+1}}{\partial \lambda}(\{\omega^{m+1}, \psi^{m+1}\}, \lambda^{m+1}),$$

respectively. Compute then

(5.72) $$\gamma_m = \frac{\int_\Omega (g_\omega^{m+1} - g_\omega^m) g_\omega^{m+1}\, dx + (g_\lambda^{m+1} - g_\lambda^m) g_\lambda^{m+1}}{\int_\Omega |g_\omega^m|^2\, dx + |g_\lambda^m|^2}$$

and set

(5.73) $$z_\omega^{m+1} = g_\omega^{m+1} + \gamma_m z_\omega^m, \quad z_\psi^{m+1} = g_\psi^{m+1} + \gamma_m z_\psi^m, \quad z_\lambda^{m+1} = g_\lambda^{m+1} + \gamma_m z_\lambda^m.$$

Do then $m = m+1$ *and go to* (5.68). ■

According again to Sec. 2.4 we should use $\{\{2\omega^n - \omega^{n-1}, 2\psi^n - \psi^{n-1}\}, 2\lambda^n - \lambda^{n-1}\}$ as initializer in (5.64) to compute $\omega^{n+1}, \psi^{n+1}, \lambda^{n+1}$ by the conjugate gradient algorithm (5.64)-(5.73).

Concerning the partial derivatives $\frac{\partial j_{n+1}}{\partial(\omega,\psi)}$, $\frac{\partial j_{n+1}}{\partial\lambda}$ we should prove that at $\{\{\eta\ ,\chi\},\mu\} \in W_g \times \mathbb{R}$ we have

$$(5.74)\qquad \begin{cases} \left\langle\frac{\partial j_{n+1}}{\partial(\omega,\psi)}(\{\eta,\chi\},\mu)\ ,\ \{\theta,\phi\}\right\rangle = \int_\Omega \tilde{\eta}\theta\, dx \\ -\mu\int_\Omega\left(\frac{\partial\chi}{\partial x_2}\frac{\partial\tilde{\chi}}{\partial x_1} - \frac{\partial\chi}{\partial x_1}\frac{\partial\tilde{\chi}}{\partial x_2}\right)\theta\, dx - \mu\int_\Omega \eta\left(\frac{\partial\tilde{\chi}}{\partial x_1}\frac{\partial\phi}{\partial x_2} - \frac{\partial\tilde{\chi}}{\partial x_2}\frac{\partial\chi}{\partial x_1}\right)dx \\ +\tilde{\mu}\int_\Omega\left(\frac{\omega^n-\omega^{n-1}}{\Delta s}\right)\theta\, dx \quad \forall\{\theta,\phi\}\in W_o, \end{cases}$$

$$(5.75)\qquad \frac{\partial j_{n+1}}{\partial\lambda}(\{\eta,\chi\},\mu) = \tilde{\mu}\left(\frac{\lambda^n-\lambda^{n-1}}{\Delta s}\right) - \int_\Omega\eta\left(\frac{\partial\tilde{\chi}}{\partial x_1}\frac{\partial\chi}{\partial x_2} - \frac{\partial\tilde{\chi}}{\partial x_2}\frac{\partial\chi}{\partial x_1}\right)dx - \int_\Omega f\tilde{\chi}\, dx,$$

where in (5.74),(5.75), $\tilde{\eta},\tilde{\chi},\tilde{\mu}$ are obtained from η,χ,μ through the solution of (5.62),(5.63).

Remark 5.2 : The main motivation of the mixed variational formulation discussed in Secs. 5.3 and 5.4.3 is that it provides a convenient framework to the approximation of linear and nonlinear biharmonic problems, by very simple finite element methods like those discussed in the following Sec. 5.5.

5.5. Finite element approximation.

5.5.1. Triangulation of Ω. Fundamental discrete spaces.

We suppose for simplicity that Ω is a polygonal domain of $\mathbb{R}^2$. With $\mathcal{T}_h$ a triangulation of Ω obeying to the conditions given in Sec. 3.2.1 we define the following finite dimensional functional spaces

$$(5.76)\qquad H_h^1 = \{v_h | v_h \in C^o(\bar{\Omega}),\ v_{h|T}\in P_k \quad \forall T\in\mathcal{T}_h\},$$

$$(5.77)\qquad H_{oh}^1 = H_h^1 \cap H_o^1(\Omega)\ \ (= \{v_h | v_h\in H_h^1,\ v_{h|\Gamma} = 0)$$

with P_k = space of polynomials in x_1,x_2 of degree $\le k$; H_h^1 and H_{oh}^1 approximate $H^1(\Omega)$ and $H_o^1(\Omega)$, respectively.

We approximate then the spaces W_o and W_g (defined by (5.20) and (5.21), respectively) by

$$(5.78)\qquad W_{oh} = \{\{\theta_h,\phi_h\}\in H_h^1\times H_{oh}^1,\ \int_\Omega\nabla\phi_h\cdot\nabla q_h\, dx = \int_\Omega\theta_h q_h\, dx \quad \forall q_h\in H_h^1\},$$

$$(5.79)\quad \begin{cases} W_{gh} = \{\{\theta_h,\phi_h\}\in H_h^1\times H_h^1,\ \phi_h = g_{1h} \text{ on } \Gamma,\ \int_\Omega \nabla\phi_h\cdot\nabla q_h\, dx = \\ \int_\Omega \theta_h q_h\, dx + \int_\Gamma g_{2h} q_h\, d\Gamma\}, \end{cases}$$

where, in (5.79), g_{1h} and g_{2h} are convenient approximations of g_1 and g_2, respectively. We observe that $W_{oh} \not\subset W_o$; similarly $W_{gh} \not\subset W_g$, even in the simple case where $g_{1h} = g_1$, $g_{2h} = g_2$.

5.5.2. Approximation of the Navier-Stokes equations via the $\{\omega,\psi\}$ formulation.

Using $\lambda = 1/\nu$ as parameter, a mixed variational formulation of the Navier-Stokes equations was given in Sec. 5.4.3 by (5.53). We approximate then (5.53) by

$$(5.80)\quad \begin{cases} \textit{Find } \{\omega_h,\psi_h\}\in W_{gh} \textit{ such that } \forall\ \{\theta_h,\phi_h\}\in W_{oh} \textit{ we have} \\ \int_\Omega \omega_h\theta_h\, dx + \lambda\int_\Omega \omega_h\left(\frac{\partial\psi_h}{\partial x_1}\frac{\partial\phi_h}{\partial x_2} - \frac{\partial\psi_h}{\partial x_2}\frac{\partial\phi_h}{\partial x_1}\right)dx = \lambda\int_\Omega f_h\phi_h\, dx, \end{cases}$$

with f_h a convenient approximation of f.

We refer to GIRAULT-RAVIART [22], for the convergence properties of $\{\omega_h,\psi_h\}$ as $h \to 0$.

Concentrating on the numerical solution of problem (5.80) by *continuation least-squares methods* we should adapt easily the algorithms of Sec. 5.4.3 (which are concerned with the continuous problem (5.53)) to the solution of the approximate problem (5.80) (see REINHART [3] for more details on the solution of (5.80) by the methods of the present paper).

In fact applying, to the solution of (5.80), the discrete analogues of the methods described in Sec. 5.4.3 requires an efficient solver for the various *discrete linear biharmonic problems* coming from the mixed finite element approximation ; such a solver is particularly required by the conjugate gradient algorithm solving the least squares problem encountered at each step of the continuation process.

5.5.3. On the solution of the discrete linear biharmonic problems.

5.5.3.1. Generalities. Synopsis.

A careful examination of the algorithms discussed in Sec. 5.4.3 shows that the *discrete linear biharmonic problems* to be solved are in fact mixed finite element approximations of biharmonic problems of the following class

$$(5.81)\qquad \begin{cases} \Delta^2\psi = f_o - \dfrac{\partial f_1}{\partial x_1} - \dfrac{\partial f_2}{\partial x_2} - \Delta f_3 \text{ in } \Omega, \\ \psi|_\Gamma = g_1, \ \dfrac{\partial\psi}{\partial n}|_\Gamma = g_2 \ ; \end{cases}$$

in (5.81), $f_i \in L^2(\Omega)$, $\forall i = 0,1,2,3$, and the derivatives occuring there have to be understood in the sense of distributions. Assuming that g_1, g_2 are sufficiently smooth, problem (5.81) has a unique solution in V_g (see Sec. 5.3 for the definition of V_g and V_o) ; this solution ψ is also the unique solution of the following variational problem

$$(5.82)\qquad \begin{cases} \textit{Find } \psi \in V_g \textit{ such that } \forall\phi \in V_o \\ \displaystyle\int_\Omega \Delta\psi\, \Delta\phi\, dx = \int_\Omega (f_o\phi + f_1 \frac{\partial\phi}{\partial x_1} + f_2 \frac{\partial\phi}{\partial x_2} - f_3\Delta\phi)dx. \end{cases}$$

An equivalent *mixed variational formulation* of (5.82) is given by

$$(5.83)\qquad \begin{cases} \textit{Find } \{\omega,\psi\} \in W_g \textit{ such that } \forall\{\theta,\phi\} \in W_o \\ \displaystyle\int_\Omega \omega\theta\, dx = \int_\Omega (f_o\phi + f_1 \frac{\partial\phi}{\partial x_1} + f_2 \frac{\partial\phi}{\partial x_2} + f_3\theta)dx, \end{cases}$$

where W_o and W_g are defined by (5.20) and (5.21), respectively.

Starting from the mixed formulation (5.83) we shall discuss in the following sections the finite element approximation of (5.83) and solution methods for the approximate problems.

5.5.3.2. Finite element approximation of (5.83).

Following CIARLET-RAVIART [26] and GLOWINSKI-PIRONNEAU [25] we should approximate (5.83) by

$$(5.84)\qquad \begin{cases} \textit{Find } \{\omega_h,\psi_h\} \in W_{gh} \textit{ such that } \forall\{\theta_h,\phi_h\} \in W_{oh} \\ \displaystyle\int_\Omega \omega_h\theta_h\, dx = \int_\Omega (f_{oh}\phi_h + f_{1h} \frac{\partial\phi_h}{\partial x_1} + f_{2h} \frac{\partial\phi_h}{\partial x_2} + f_{3h}\theta_h)dx, \end{cases}$$

where W_{oh} and W_{gh} are still defined by (5.78) and (5.79), respectively, and where f_{ih} is, for i=0,1,2,3, a convenient approximation of f_i.

Is it quite easy to prove that (5.84) has a unique solution ; concerning the convergence of $\{\omega_h,\psi_h\}$ to $\{-\Delta\psi,\psi\}$ as $h \to 0$, it follows from CIARLET-RAVIART [26], SCHOLZ [27] that

(5.85) $$\lim_{h\to 0} \|-\Delta\psi-\omega_h\|_{L^2(\Omega)} = 0, \quad \lim_{h\to 0} \|\psi-\psi_h\|_{H^1(\Omega)} = 0$$

for all $k \geq 1$ (in the definition of H_h^1 ; cf. (5.76)). Actually the convergence result (5.85) supposes that some mild assumptions on the angles are satisfied as $h \to 0$ (see the two above references for more details).

5.5.3.3. Decomposition properties of the approximate problem (5.84).

We follow (and complete on some points) GLOWINSKI-PIRONNEAU [25].

A direct solution of (5.84) is a non trivial task ; however taking into account the very special structure of (5.84) we shall be able, *via a decomposition principle*, to reduce its solution to the solution of a family of *discrete Poisson problems (much easier to solve)*.

The starting point of our discussion is the fact that the pair $\{\omega_h,\psi_h\}$ solution of (5.84) is characterized by the existence of p_h such that

(5.86)$_1$ $$\begin{cases} p_h \in H_h^1, \\ \displaystyle\int_\Omega \nabla p_h \cdot \nabla \phi_h \, dx = \int_\Omega \left(f_{oh}\phi_h + f_{1h}\frac{\partial \phi_h}{\partial x_1} + f_{2h}\frac{\partial \phi_h}{\partial x_2}\right) dx \quad \forall \phi_h \in H_{oh}^1, \end{cases}$$

(5.86)$_2$ $$\begin{cases} \omega_h \in H_h^1, \\ \displaystyle\int_\Omega \omega_h\theta_h \, dx = \int_\Omega (f_{3h}+p_h)\theta_h \, dx \quad \forall \theta_h \in H_h^1, \end{cases}$$

(5.86)$_3$ $$\begin{cases} \psi_h \in H_h^1, \ \psi_h = g_{1h} \text{ on } \Gamma, \\ \displaystyle\int_\Omega \nabla\psi_h \cdot \nabla q_h \, dx = \int_\Omega \omega_h q_h \, dx + \int_\Gamma g_{2h} q_h \, d\Gamma \quad \forall q_h \in H_h^1. \end{cases}$$

To prove the characterization (5.86) we should observe that (5.84) is equivalent to the minimization problem

(5.87) $$\begin{cases} \text{Find } \{\omega_h,\psi_h\} \in W_{gh} \text{ such that} \\ j_h(\omega_h,\psi_h) \leq j_h(\theta_h,\phi_h) \quad \forall \{\theta_h,\phi_h\} \in W_{gh}, \end{cases}$$

where

$$(5.88)\qquad j_h(\theta_h,\phi_h) = \frac{1}{2}\int_\Omega \theta_h^2\, dx - \int_\Omega (f_{oh}\phi_h + f_{1h}\frac{\partial\phi_h}{\partial x_1} + f_{2h}\frac{\partial\phi_h}{\partial x_2} + f_{3h}\theta_h)dx \;;$$

hence p_h appears as a *lagrange multiplier* for the *linear equality constraint* satisfied by $\{\omega_h,\psi_h\}$ in $(5.86)_3$ (and in the definition of W_{gh} ; see (5.79)).

To go further in the decomposition properties we introduce now a space $\mathcal{M}_h$ obeying the following properties

$$(5.89)\qquad \begin{cases} \mathcal{M}_h \text{ is a complementary space (not precisely defined for the moment)} \\ \text{of } H^1_{oh} \text{ in } H^1_h, \text{ i.e. } \mathcal{M}_h \subset H^1_h \text{ and } H^1_{oh} \oplus \mathcal{M}_h = H^1_h \,. \end{cases}$$

It follows from (5.89) that the bilinear form $\mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$ defined by

$$\{\lambda_h,\mu_h\} \to \int_\Gamma \lambda_h\mu_h\, d\Gamma$$

is a *scalar product* over $\mathcal{M}_h$.

The key step is in fact to introduce a bilinear form $a_h : \mathcal{M}_h \times \mathcal{M}_h \to \mathbb{R}$, defined as follows :

Let $\lambda_h \in \mathcal{M}_h$ and let p_h, respectively ψ_h, be the solutions of the following approximate problems

$$(5.90)_1\qquad \int_\Omega \nabla p_h\cdot\nabla\phi_h\, dx = 0 \quad \forall\phi_h \in H^1_{oh},\; p_h \in H^1_h,\; p_h-\lambda_h \in H^1_{oh},$$

$$(5.90)_2\qquad \int_\Omega \nabla\psi_h\cdot\nabla\phi_h\, dx = \int_\Omega p_h\phi_h\, dx \quad \forall\phi_h \in H^1_{oh},\; \psi_h \in H^1_{oh}.$$

Then we define the bilinear form $a_h(\cdot,\cdot)$ by

$$(5.90)_3\qquad a_h(\lambda_h,\mu_h) = \int_\Omega p_h\mu_h\, dx - \int_\Omega \nabla\psi_h\cdot\nabla\mu_h\, dx \quad \forall\mu_h \in \mathcal{M}_h.$$

It follows then from [25 , Sec. 3.5] that $a_h(\cdot,\cdot)$ is *symmetric* and *positive definite*.

Application to the decomposition of the approximate problem (5.84) :

Let $\{\omega_h,\psi_h\}$ be the solution of (5.84) and let λ_h be the component in $\mathcal{M}_h$ of this function p_h occuring in the characterization (5.86). Let $\bar{p}_h$, $\bar{\psi}_h$ be the solutions of

(5.91) $$\int_\Omega \nabla\bar{p}_h\cdot\nabla\phi_h \, dx = 0 \quad \forall\phi_h \in H^1_{oh},\ \bar{p}_h-\lambda_h \in H^1_{oh},$$

(5.92) $$\int_\Omega \nabla\bar{\psi}_h\cdot\nabla\phi_h \, dx = \int_\Omega \bar{p}_h\phi_h \, dx \quad \forall\phi_h \in H^1_{oh},\ \bar{\psi}_h \in H^1_{oh}.$$

Let p_{oh} and ψ_{oh} be the solutions of

(5.93) $$\begin{cases} \displaystyle\int_\Omega \nabla p_{oh}\cdot\nabla\phi_h \, dx = \int_\Omega (f_{oh}\phi_h + f_{1h}\frac{\partial\phi_h}{\partial x_1} + f_{2h}\frac{\partial\phi_h}{\partial x_2}) \, dx \quad \forall\phi_h \in H^1_{oh}, \\ p_{oh} \in H^1_{oh}, \end{cases}$$

(5.94) $$\int_\Omega \nabla\psi_{oh}\cdot\nabla\phi_h \, dx = \int_\Omega (p_{oh}+f_{3h})\phi_h \, dx \quad \forall\phi_h \in H^1_{oh},\ \psi_h \in H^1_h,\ \psi_{oh} = g_{1h} \text{ on } \Gamma \ ;$$

we clearly have $p_h = \bar{p}_h + p_{oh}$ and $\psi_h = \bar{\psi}_h + \psi_{oh}$.

We shall now show that λ_h is the solution of a variational problem in $\mathcal{M}_h$.

Theorem 5.1 : *Let* $\{\omega_h,\psi_h\}$ *be the solution of* (5.84) *and let* λ_h *be the component in* $\mathcal{M}_h$ *of* p_h *defined from* $\{\omega_h,\psi_h\}$ *by* (5.86). *Then* λ_h *is the unique solution of the linear variational problem*

(5.95) $$\begin{cases} a_h(\lambda_h,\mu_h) = \displaystyle\int_\Omega \nabla\psi_{oh}\cdot\nabla\mu_h \, dx - \int_\Omega (p_{oh}+f_{3h})\mu_h \, dx - \int_\Gamma g_{2h}\mu_h \, d\Gamma \\ \forall\mu_h \in \mathcal{M}_h,\ \lambda_h \in \mathcal{M}_h \end{cases}$$

which is equivalent to a linear system with a positive definite matrix.

Proof : We have from (5.90)-(5.92) that

$$(5.96)\quad \begin{cases} a_h(\lambda_h,\mu_h) = \int_\Omega \bar{p}_h\mu_h\,dx - \int_\Omega \nabla\bar{\psi}_h\cdot\nabla\mu_h\,dx = \\ = \int_\Omega (p_h-p_{oh})\mu_h\,dx - \int_\Omega \nabla(\psi_h-\psi_{oh})\cdot\nabla\mu_h\,dx = \\ = \int_\Omega \nabla\psi_{oh}\cdot\nabla\mu_h\,dx - \int_\Omega (p_{oh}+f_{3h})\mu_h\,dx \\ - (\int_\Omega \nabla\psi_h\cdot\nabla\mu_h\,dx - \int_\Omega (p_h+f_{3h})\mu_h\,dx) \quad \forall\mu_h\in\mathcal{M}_h. \end{cases}$$

But from $(5.86)_2$, $(5.86)_3$ we have

$$\int_\Omega \nabla\psi_h\cdot\nabla\mu_h\,dx - \int_\Omega (p_h+f_{3h})\mu_h\,dx =$$
$$= \int_\Omega \nabla\psi_h\cdot\nabla\mu_h\,dx - \int_\Omega \omega_h\mu_h\,dx =$$
$$= \int_\Gamma g_{2h}\mu_h\,d\Gamma \quad \forall\mu_h\in\mathcal{M}_h,$$

which, together with (5.96), proves (5.95). The uniqueness is obvious since $a_h(\cdot,\cdot)$ is positive definite. The equivalence with a positive definite linear system is a classical result of the approximation of linear variational problems. ■

Remark 5.3 : To compute the right hand side of (5.95) it is necessary to solve two approximate Dirichlet problems ((5.93) and (5.94)). Similarly if λ_h is known, to compute p_h,ω_h and ψ_h it is necessary to solve

$$(5.97)\quad \begin{cases} p_h\in H_h^1,\ p_h-\lambda_h\in H_{oh}^1, \\ \int_\Omega \nabla p_h\cdot\nabla\phi_h\,dx = \int_\Omega (f_{oh}\phi_h + f_{1h}\frac{\partial\phi_h}{\partial x_1} + f_{2h}\frac{\partial\phi_h}{\partial x_2})dx \quad \forall\phi_h\in H_{oh}^1, \end{cases}$$

$$(5.98)\quad \begin{cases} \omega_h\in H_h^1, \\ \int_\Omega \omega_h\theta_h\,dx = \int_\Omega (f_{3h}+p_h)\theta_h\,dx \quad \forall\theta_h\in H_h^1, \end{cases}$$

$$(5.99)\quad \begin{cases} \psi_h\in H_h^1,\ \psi_h = g_{1h}\ \textit{on}\ \Gamma, \\ \int_\Omega \nabla\psi_h\cdot\nabla\phi_h\,dx = \int_\Omega (p_h+f_{3h})\phi_h\,dx \quad \forall\phi_h\in H_{oh}^1, \end{cases}$$

i.e. two discrete Dirichlet problems ((5.97) and (5.99)), and (5.98) which is a much simpler linear problem (ω_h is in fact the L^2-projection on H_h^1 of the function p_h+f_{3h}). ■

Recapitulation : It has been shown that solving the discrete biharmonic problem (5.84) is equivalent to solve (5.93),(5.94),(5.95),(5.97),(5.98),(5.99) *sequentially*. Problems (5.93),(5.94),(5.97),(5.99) are discrete Dirichlet problems, for the operator $-\Delta$, for which very efficient direct or iterative solvers exist ; the variational problem (5.98) is even simpler to solve, since the matrix of the equivalent linear system is very sparse, has a condition number in $O(1)$ and is in fact an approximation of the identity operator. Finally the only non-standard step is the solution of the variational problem (5.95) ; the solution of problem (5.95) is discussed in the following Sec. 5.5.3.4.

5.5.3.4. Solution of problem (5.95).

Several methods for the solution of (5.95) have been discussed in [25, Secs. 4 and 5]. Let us mention among them a *conjugate gradient method* which yields a solution algorithm for the discrete biharmonic problem (5.84) ; the cost per iteration is essentially the solution of *two* discrete Dirichlet problems for the operator $-\Delta$; numerical experiments show a convergence in $O(\sqrt{N_h})$ iterations, where $N_h = \dim \mathcal{M}_h$. We may find also in [25, Sec. 4] a detailed analysis of a direct method for solving (5.95) requiring the construction of the symmetric, positive definite (and full) matrix A_h of the linear system equivalent to (5.95). In fact one does not construct A_h, but (using the *Cholesky factorization* method) a lower triangular-regular matrix L_h such that $A_h = L_h L_h^t$; since the construction of L_h requires (cf. [25, Sec. 4]) the solution of $2N_h$ discrete Dirichlet problems it seems preferable to use the conjugate gradient algorithm. Actually reality is more complicated for the following reasons :

(i) Since the $2N_h$ discrete Dirichlet problems mentioned above have all the same matrix which is symmetric and positive definite, a Cholesky factorization done once and for all will result in an important saving of computational time.

(ii) If a large number of linear discrete biharmonic problems have to be solved – like in time dependent problems or during an iterative process like those discussed in this paper – the solution method of (5.84), founded on the construction of L_h offers (from our numerical experiments) a more economical strategy than the conjugate gradient algorithms discussed in [25, Sec. 5].

The above comments justify the choice of the direct solution of (5.95) for the numerical experiments described in Sec. 5.6.

However in order to complete [25, Sec. 5] we shall describe in this section a *new conjugate gradient algorithm with scaling* (i.e. preconditioning) faster (if the speed of convergence is measured in number of iterations) than those discussed in [25, Sec. 5] ; compared to the conjugate gradient algorithm described in [25, Sec. 5] (algorithm (5.76)-(5.83), pp. 197,198) the new algorithm requires the solution of *three* discrete Dirichlet problems instead of *two*, at each iteration.

Description of the new conjugate gradient algorithm :

In the sequel C denotes a positive constant.

Step 0 : Initialization

(5.100) $\lambda_h^o \in \mathcal{M}_h$ *is given ;*

then compute, sequentially, p_h^o, ψ_h^o, r_h^o, z_h^o, g_h^o *as the solutions of the following finite dimensional linear variational problems (all equivalent to linear systems whose matrices are symmetric and positive definite)*

$$(5.101)\quad \begin{cases} p_h^o \in H_h^1,\ p_h^o - \lambda_h^o \in H_{oh}^1, \\ \int_\Omega \nabla p_h^o \cdot \nabla \phi_h \, dx = \int_\Omega (f_{oh}\phi_h + f_{1h}\frac{\partial \phi_h}{\partial x_1} + f_{2h}\frac{\partial \phi_h}{\partial x_2})dx \quad \forall\ \phi_h \in H_{oh}^1, \end{cases}$$

$$(5.102)\quad \begin{cases} \psi_h^o \in H_h^1,\ \psi_h^o = g_{1h} \ \text{on}\ \Gamma, \\ \int_\Omega \nabla \psi_h^o \cdot \nabla \phi_h \, dx = \int_\Omega (p_h^o + f_{3h})\phi_h \, dx \quad \forall \phi_h \in H_{oh}^1, \end{cases}$$

$$(5.103)\quad \begin{cases} r_h^o \in \mathcal{M}_h, \\ \int_\Gamma r_h^o \mu_h \, d\Gamma = \int_\Omega (p_h^o + f_{3h})\mu_h \, dx - \int_\Omega \nabla \psi_h^o \cdot \nabla \mu_h \, dx + \int_\Gamma g_{2h}\mu_h \, d\Gamma \ \forall \mu_h \in \mathcal{M}_h, \end{cases}$$

$$(5.104)\quad \begin{cases} z_h^o \in H_h^1,\ z_h^o - r_h^o \in H_{oh}^1, \\ \int_\Omega \nabla z_h^o \cdot \nabla \phi_h \, dx = 0 \quad \forall \phi_h \in H_{oh}^1, \end{cases}$$

$$(5.105)\quad \begin{cases} g_h^o \in \mathcal{M}_h, \\ \int_\Gamma g_h^o \mu_h \, d\Gamma = C\int_\Gamma r_h^o \mu_h \, d\Gamma + \int_\Omega \nabla z_h^o \cdot \nabla \mu_h \, dx \quad \forall \mu_h \in \mathcal{M}_h. \end{cases}$$

Then set

$$w_h^o = g_h^o. \quad \blacksquare \tag{5.106}$$

For $n \geq 0$*, if we suppose that* λ_h^n, r_h^n, g_h^n, w_h^n *are known, we compute* λ_h^{n+1}, r_h^{n+1}, g_h^{n+1}, w_h^{n+1} *as follows :*

<u>Step 1</u> : <u>Descent</u>

Compute sequentially π_h^n, χ_h^n, η_h^n *as the solutions of*

$$\begin{cases} \pi_h^n \in H_h^1, \ \pi_h^n - w_h^n \in H_{oh}^1, \\ \int_\Omega \nabla\pi_h^n \cdot \nabla\phi_h \, dx = 0 \quad \forall\phi_h \in H_{oh}^1, \end{cases} \tag{5.107}$$

$$\begin{cases} \chi_h^n \in H_{oh}^1, \\ \int_\Omega \nabla\chi_h^n \cdot \nabla\phi_h \, dx = \int_\Omega \pi_h^n \phi_h \, dx \quad \forall\phi_h \in H_{oh}^1, \end{cases} \tag{5.108}$$

$$\begin{cases} \eta_h^n \in \mathcal{M}_h, \\ \int_\Gamma \eta_h^n \mu_h \, d\Gamma = \int_\Omega \pi_h^n \mu_h \, dx - \int_\Omega \nabla\chi_h^n \cdot \nabla\mu_h \, dx \quad \forall\mu_h \in \mathcal{M}_h, \end{cases} \tag{5.109}$$

and then

$$\rho_n = \frac{\int_\Gamma w_h^n r_h^n \, d\Gamma}{\int_\Gamma \eta_h^n w_h^n \, d\Gamma} \quad \left(= \frac{\int_\Gamma g_h^n r_h^n \, d\Gamma}{\int_\Gamma \eta_h^n w_h^n \, d\Gamma}\right), \tag{5.110}$$

$$\lambda_h^{n+1} = \lambda_h^n - \rho_n w_h^n \, . \tag{5.111}$$

<u>Step 2</u> : <u>New descent direction</u>

$$\begin{cases} \zeta_h^n \in H_h^1 \, , \ \zeta_h^n - \eta_h^n \in H_{oh}^1 \\ \int_\Omega \nabla\zeta_h^n \cdot \nabla\phi_h \, dx = 0 \quad \forall\phi_h \in H_{oh}^1, \end{cases} \tag{5.112}$$

(5.113) $$\begin{cases} \xi_h^n \in \mathcal{M}_h, \\ \int_\Gamma \xi_h^n \mu_h \, d\Gamma = C \int_\Gamma \eta_h^n \mu_h \, d\Gamma + \int_\Omega \nabla \zeta_h^n \cdot \nabla \mu_h \, dx \quad \forall \mu_h \in \mathcal{M}_h, \end{cases}$$

(5.114) $$r_h^{n+1} = r_h^n - \rho_n \eta_h^n ,$$

(5.115) $$g_h^{n+1} = g_h^n - \rho_n \xi_h^n ,$$

(5.116) $$\gamma_n = \frac{\int_\Gamma g_h^{n+1} \, r_h^{n+1} \, d\Gamma}{\int_\Gamma g_h^n \, r_h^n \, d\Gamma} ,$$

(5.117) $$w_h^{n+1} = g_h^{n+1} + \gamma_n w_h^n . \blacksquare$$

Do $n = n+1$ *and go to* (5.107).

Remark 5.4 : The conjugate gradient algorithm (5.100)-(5.117) may appear as a complicated method for solving the discrete biharmonic problem (5.84) via the linear variational problem (5.95). In fact what it requires is mainly the solution of the three discrete Dirichlet problems (5.107),(5.108),(5.112) at each iteration ; problems (5.109),(5.113) which are of a much smaller dimension require in comparison a small computational effort.

The justification of algorithm (5.100)-(5.117) is founded on the fact that problem (5.95) is indeed the approximation of a *boundary integral* problem and that the matrix A_h associated to the bilinear form $a_h(\cdot,\cdot)$ may be viewed as the approximation of a boundary integral operator mapping the Sobolev space $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ (see [25] for more details). A more complete discussion of (5.100)-(5.117), and related algorithms for solving the Stokes problem in the pressure-velocity formulation may be found in [28].

Remark 5.5 (*On the choice of* $\mathcal{M}_h$) : Suppose that H_h^1 is made of ordinary lagrangian finite elements of order k (k=1,2 in most applications). It follows then from [25] (for which we refer for more details) that the best choice for $\mathcal{M}_h$ is given by

(5.118) $$\mathcal{M}_h = \{\mu_h | \mu_h \in H_h^1, \ \mu_h|_T = 0 \quad \forall T \in \mathcal{C}_h \text{ such that } \partial T \cap \Gamma = \emptyset\} \quad ;$$

with such a choice the elements of $\mathcal{M}_h$ are completely determinated by the values taken at those nodes of $\mathcal{C}_h$ belonging to Γ. In that direction we should take as basis functions for $\mathcal{M}_h$ those basis functions of H_h^1 associated to the boundary nodes (see - again - [25] for more details).

5.6. Numerical experiments.

5.6.1. Formulation of the test problem.

With $\Omega =]0,1[\times]0,1[$ we consider the following Navier-Stokes test problem

$$(5.119)\qquad \begin{cases} -\nu\Delta \underset{\sim}{u} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} + \underset{\sim}{\nabla} p = 0 \ \textit{in}\ \Omega, \\ \underset{\sim}{\nabla}\cdot\underset{\sim}{u} = 0 \ \textit{in}\ \Omega\ , \\ \underset{\sim}{u}(x_1,x_2)|_\Gamma = \{1,0\}\ \textit{if}\ x_2 = 1,\ = \{0,0\}\ \textit{if}\ 0 \le x_2 < 1. \end{cases}$$

Hence problem (5.119) is the classical *driven cavity* problem. The corresponding $\{\omega,\psi\}$ formulation is

$$(5.120)\qquad \begin{cases} \nu\Delta\omega + \left(\frac{\partial\psi}{\partial x_1}\frac{\partial\omega}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\omega}{\partial x_1}\right) = 0 \ \textit{in}\ \Omega, \\ -\Delta\psi = \omega \ \textit{in}\ \Omega, \\ \psi = 0 \ \textit{on}\ \Gamma\ ;\ \frac{\partial\psi}{\partial n}(x_1,x_2)|_\Gamma = 1\ \textit{if}\ x_2 = 1,\ = 0\ \textit{if}\ 0 \le x_2 < 1. \end{cases}$$

5.6.2. Triangulation of Ω.

The triangulation $\mathcal{C}_h$ used to approximate (5.119),(5.120) by the methods of Sec. 5.5, is shown on Figure 5.1. It contains 128 triangles and since *piecewese quadratic* elements are used (i.e. k=2 in (5.76)), it corresponds to 64 boundary nodes and 225 interior nodes (vertices and midpoints). Actually the above finite element grid is too coarse for high Reynold's number calculations, but using a finer grid would have been difficult with the computer used for these numerical simulations.

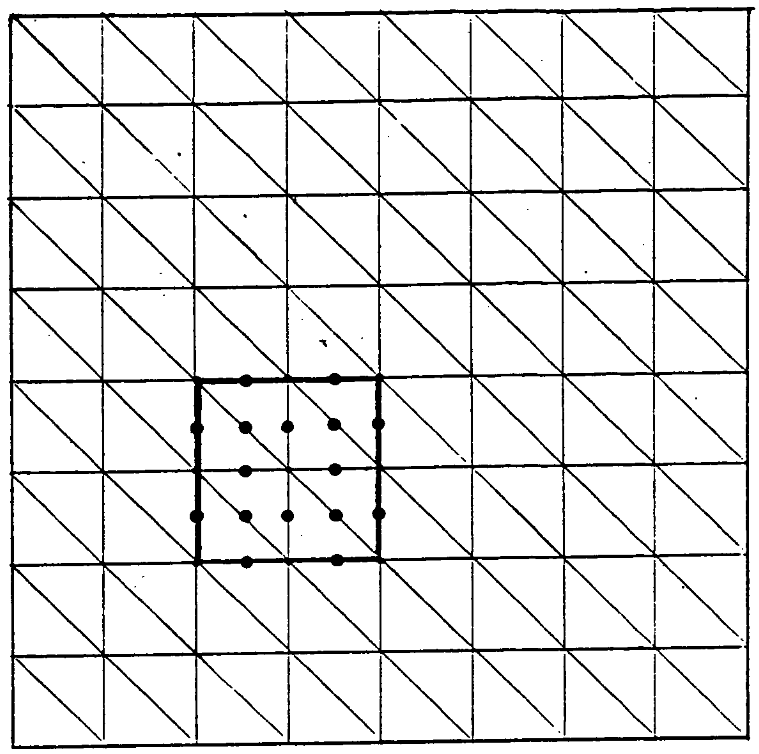


Figure 5.1

5.6.3. Numerical results. Further comments.

The numerical procedure described in Sec. 5.4.3 has been applied to the solution of the approximate problem (5.80) associated to (5.120) (with $\lambda = 1/\nu = Re$). We have used $\Delta s = 10$ for $0 \leq \lambda < 400$, $\Delta s = 25$ for $400 \leq \lambda \leq 900$, $\Delta s = 50$ for $900 \leq \lambda \leq 1200$ and $\Delta s = 100$ for $\lambda > 1200$. The conjugate gradient iterations were stopped as soon as the least squares cost functional was less than 10^{-5}. Figures 5.2, 5.3, 5.4 show the variations of the least squares cost functional, as a function of the number of conjugate gradient iterations, for $\lambda = 380, 800, 1100$, respectively ; as expected the number of iterations necessary to obtain the convergence is an increasing function of $\lambda(= Re)$. Actually restarting from time to time the conjugate gradient algorithm in the direction of the gradient (i.e. taking $\gamma_m = 0$ in (5.73)) seems to improve the convergence properties of the conjugate gradient algorithm, as shown on Figure 5.2 where this was done at iteration 7 (it was not done for the problems corresponding to Figures 5.3, 5.4) ; a systematical way of doing that restarting would be of great interest for these very large nonlinear least squares problems (for restarting procedures see POWELL [2]).

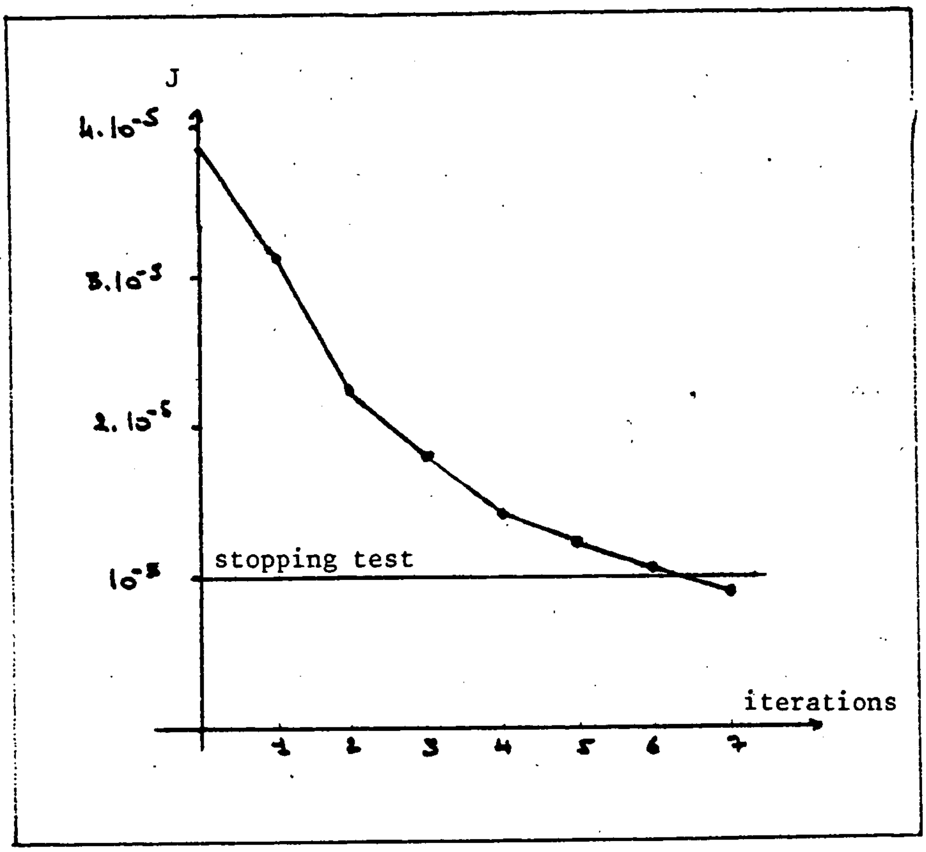


Figure 5.2

Variation of the cost function during the iterative process (Re = 800)

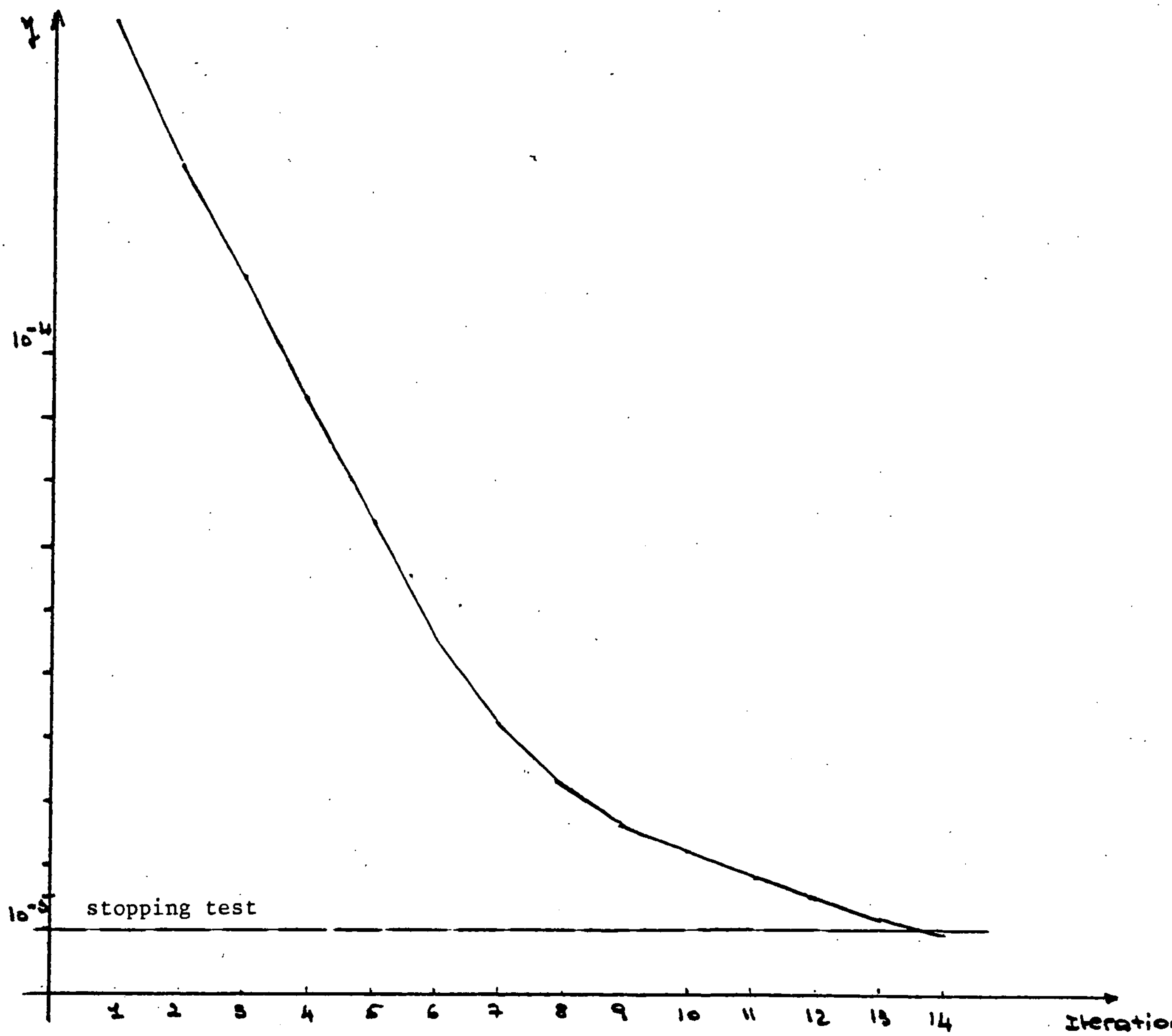


Figure 5.3

Variation of the cost function during the iterative process (Re = 1100)

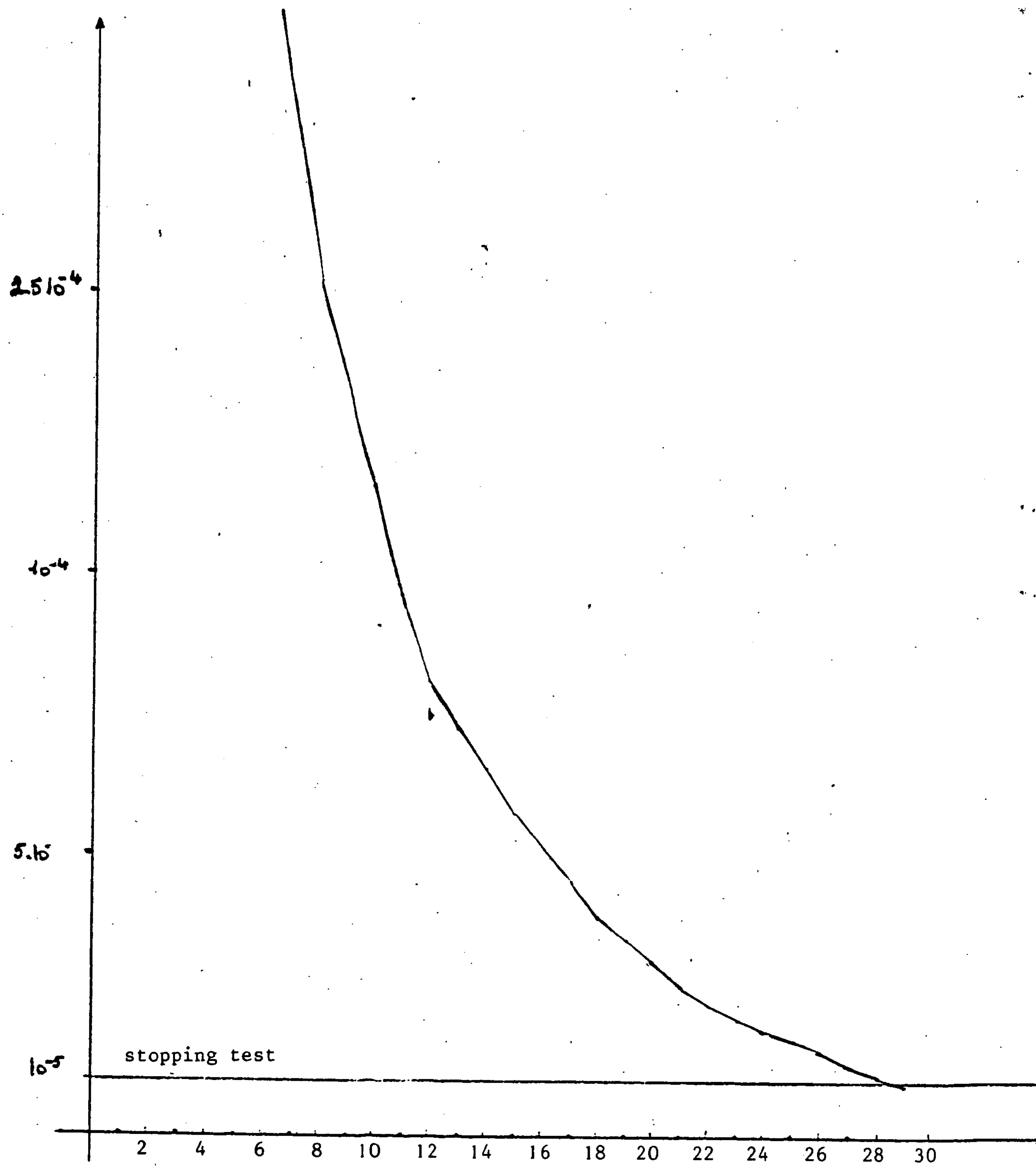


Figure 5.4

Variation of the cost function during the iterative process (Re = 1300)

The stream lines corresponding to Re = 10, 200, 400 and 1600 have been shown on Figures 5.5, 5.6, 5.7 and 5.8 respectively. It is quite clear - as mentioned previously - that the finite element grid of Figure 5.1 is not fine enough, to observe the small scale properties of the flow (boundary layers, small eddies, etc...). For more accurate calculations, for the same driven cavity problem, by other methods, see OLSON-TUANN [29].

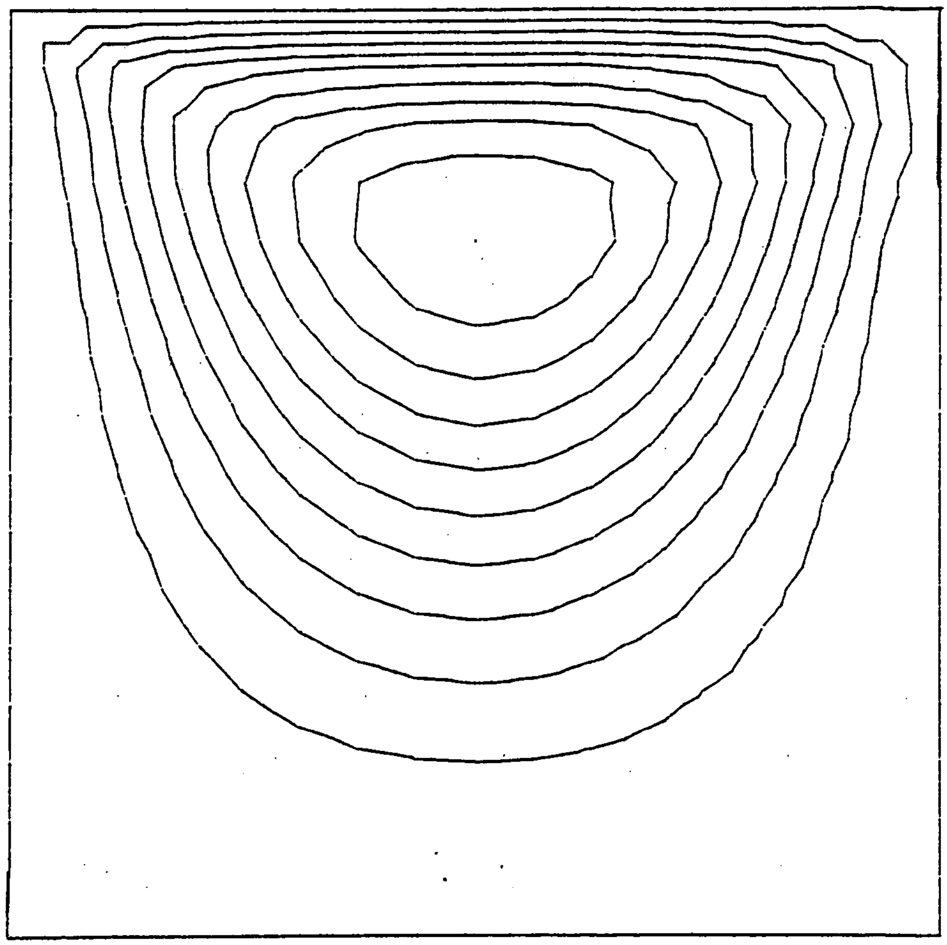


Figure 5.5

Streamlines at Re = 10

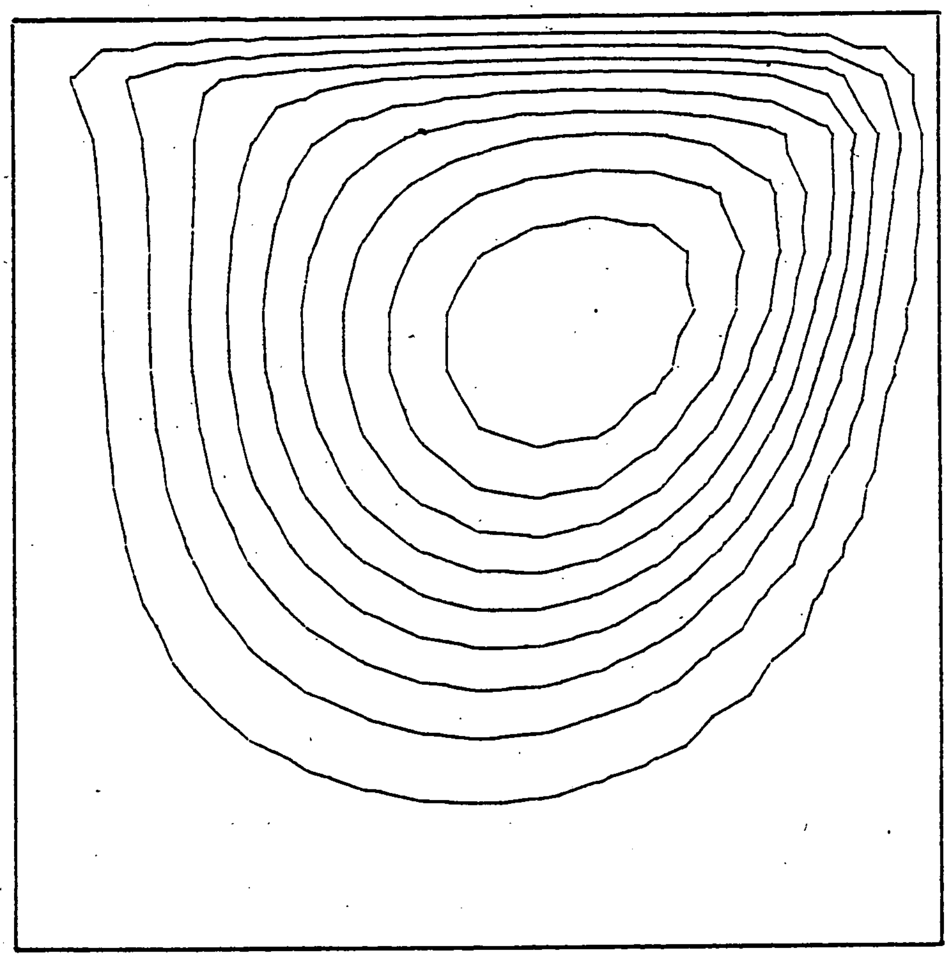


Figure 5.6

Streamlines at Re = 200

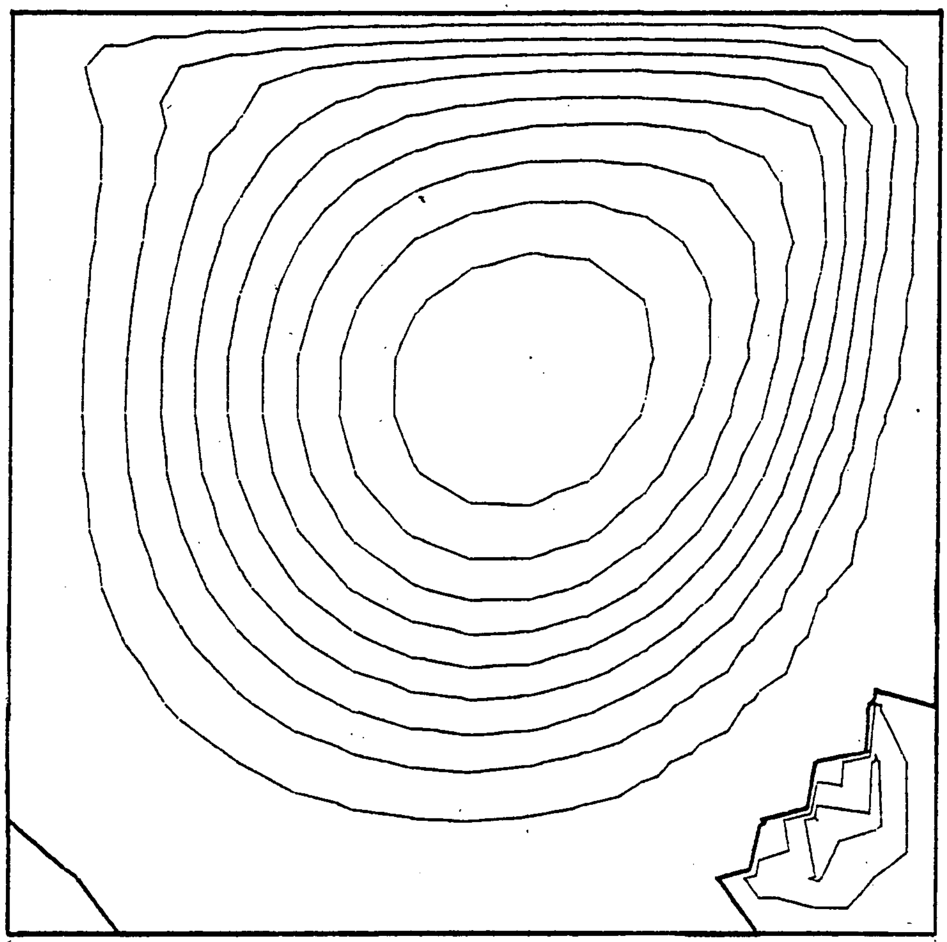


Figure 5.7

Streamlines at Re = 400

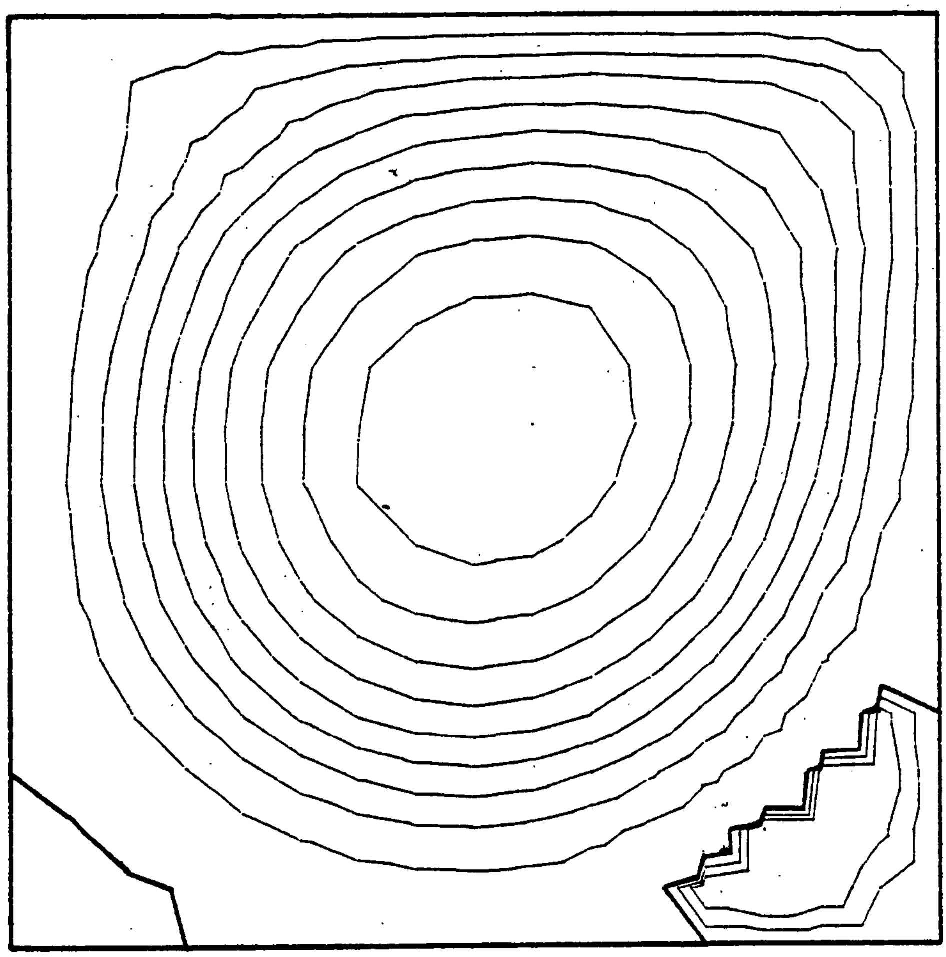


Figure 5.8

Streamlines at Re = 1600

6. - CONCLUSION.

We have discussed in this paper the solution of nonlinear boundary value problems containing a parameter by a combination of arc length continuation methods, least squares - conjugate gradient algorithms and finite element approximations. The resulting methodology is quite general and has been applied to the solution of second order and fourth order nonlinear boundary value problems whose branches of solutions may exhibit limit point, bifurcation, etc... ; actually it has been applied also to the solution of more complicated nonlinear boundary value problems than those considered in this paper, such as the *Von Karman equations* for nonlinear plates (cf. REINHART [17]) or the computation of the multiple solutions of the *full potential equation* modelling *transonic flows* for *compressible inviscid fluids*.

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