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POINT CONTROL WITH STATE CONSTRAINTS

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CONTROLE PONCTUEL AVEC CONTRAINTES SUR L'ETAT

par

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RESUME :

Des résultats de controllabilité sont présentés pour des problèmes pseudo-paraboliques dans un domaine de $\mathbb{R}^p$, $p = 1$, 2 ou 3, ainsi que des conditions de compatibilité pour les contraintes sur l'état pour des problèmes non contrôlables.

Un problème concret est envisagé et un résultat de régularité est donné pour le contrôle optimal.

Point Control with State Constraints

by

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Controllability results are presented for pseudoparabolic problems with spacial domains in R^p with $p = 1$, 2 , or 3 , as well as compatibility conditions for state constraints of noncontrollable problems. A concrete problem is considered and a regularity result is given for the optimal control.

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Point Control with State Constraints

by

L. W. White

1. Introduction.

In this note we consider the control of a pseudoparabolic (or parabolic) equation at a point with constraints imposed upon the state function. Indeed, let Ω be an open parallelopiped in R^p with sides parallel to the coordinate axes with $a \in \Omega$, $Q = \Omega \times (0,T)$, and $\Sigma = \partial\Omega \times (0,T)$. The underlying equations governing the control problem are

(1) $$(1 - \Delta)y_t - \Delta y = v(t)\delta(x - a) \quad \text{in} \quad Q$$

(2) $$y(x,0) = 0 \quad \text{in} \quad \Omega$$

(3) $$y(x,t) = 0 \quad \text{on} \quad \Sigma$$

with the minimization problem

$$\begin{aligned} &\text{minimize} \quad \|v\|^2_{L^2(0,T)} \\ &\text{subject to} \quad v \in L^2(0,T) \\ &\qquad\qquad\quad \|y(T;v) - z\|_{L^2(\Omega)} \le \rho \end{aligned}$$

where $z \in L^2(\Omega)$ and $\rho > 0$.

In order for problem (4) to have a solution the set

$$U = \{v \in L^2(0,T) : \|y(T;v) - z\|_{L^2(\Omega)} \le \rho\}$$

must be nonempty. Hence, we consider the controllability of problem (1)-(3). That is, we seek to determine whether the set given by

$$Y(T) = \{y(\cdot,T;v) : v \in L^2(0,T)\} \tag{5}$$

is dense in $L^2(\Omega)$. Here we establish conditions for controllability of (1)-(3) for $\Omega \subset R^p$ with $p = 1, 2,$ or 3. This is in contrast with the parabolic case in which controllability holds only in the case that $p = 1$. If (1)-(3) is controllable then U is nonempty for any $\rho > 0$. In the case that (1)-(3) is not controllable, we give here a compatibility condition depending on z in order for $U \neq \phi$.

After having given criteria for the formulation of (4), we determine regularity of the optimal control u_0. For ease we devote our attention to the case $\Omega = (0,1) \subset R^1$, but we point out extensions for $\Omega \subset R^p$. Furthermore, we indicate results for the parabolic case and for more general equations than (1).

2. Controllability of (1)-(3).

We determine conditions under which $\overline{Y(T)} = L^2(\Omega)$. Hence, let $\xi \in L^2(\Omega)$ have the property that for every $v \in L^2(0,T)$

$$(\xi, y(T;v))_{L^2(\Omega)} = 0 . \tag{6}$$

Hence, we seek conditions under which (6) implies $\xi = 0$ in $L^2(\Omega)$. As is usual [1, 2, 4], we introduce the adjoint problem

$$-(1-\Delta)p_t - \Delta p = 0 \quad \text{in} \quad Q \tag{7}$$

$$p(\cdot,T) = (1-\Delta)^{-1}\xi(\cdot) \quad \text{in} \quad \Omega \tag{8}$$

$$p(x,t) = 0 \quad \text{on} \quad \Sigma . \tag{9}$$

Multiplying (1) by p and integrating by parts, we see that equation (6) implies that for all $v \in L^2(0,T)$

$$(p(a,\cdot),v)_{L^2(0,T)} = 0 . \tag{10}$$

Hence, it follows that

$$p(a,t) = 0 \tag{11}$$

for almost all t in $(0,T)$.

We consider the implications of (11). Changing variables, for convenience, $\tau = T - t$, with $q(\cdot,\tau) = p(\cdot,t)$ equations (7)-(9) become

$$(1-\Delta)q_t - \Delta q = 0 \quad \text{in} \quad Q \tag{12}$$

$$q(\cdot,0) = (1-\Delta)^{-1}\xi(\cdot) \quad \text{in} \quad \Omega \tag{13}$$

$$q(x,\tau) = 0 \quad \text{on} \quad \Sigma . \tag{14}$$

Now for $\Omega = (0,1)$ with $\Delta = \frac{\partial^2}{\partial x^2}$ and zero Dirichlet conditions at 0 and 1 , we consider equations (12)-(14) in terms of the Fourier sine series. Set $\xi = \sum_{k=1}^{\infty} \xi_k \sin(k\Pi x)$. The solution of (12)-(14) may be given by

$$q(x,t) = \sum_{k=1}^{\infty} \frac{\xi_k}{1+k^2\Pi^2} e^{-\mu_k t} \sin(k\Pi x) \tag{15}$$

where $\mu_k = \frac{k^2\Pi^2}{1+k^2\Pi^2}$. We note that q is the uniform limit of functions continuous in x and τ . Hence, we may evaluate $q(a,t)$ to have

(16) $$q(a,t) = \sum_{k=1}^{\infty} \frac{\xi_k}{1 + k^2\Pi^2} e^{-\mu_k t} \sin(k\Pi a)$$

for all $t > 0$. From equation (11), we have $q(a,t) = 0$ for all $t \in (0,T)$. In fact, $q(a,t)$ is analytic for $t > 0$ since from (16) it is the uniform limit of analytic functions. Hence, in fact $q(a,t) = 0$ for all $t > 0$.

Since convergence is uniform in (16), we may take the Laplace transform to obtain

(17) $$\hat{q}(a,s) = \sum_{k=1}^{\infty} \frac{\xi_k}{1 + k^2\Pi^2} \sin(k\Pi a) \frac{1}{s + \mu_k}$$

with $\hat{q}(a,s) = 0$ for all $s > 0$. Define the funciton

$$f(z) = \sum_{k=1}^{\infty} \frac{\xi_k}{1 + k^2\Pi^2} \sin(k\Pi a) \frac{1}{z + \mu_k} .$$

Now $f(z)$ is a meromorphic function with poles at $-\mu_k$. Furthermore, $f(z) = 0$ for z real and positive. We conclude then that $f(z) \equiv 0$ and has zero residues. Thus, we have

(18) $$\xi_k \sin(k\Pi a) = 0$$

for all k. It is an immediate consequence of (18) that if $a \in (0,1)$ is an irrational number then $\xi_k = 0$ for every $k \in N$. Hence, ξ is zero in $L^2(0,1)$, and $\overline{Y(T)} = L^2(0,1)$.

Theorem 1. If $a \in (0,1)$ is irrational, then (1)-(3) is controllable.

In the case a is rational say $\frac{n}{m}$ then (18) implies $\xi_k = 0$ only for $k \neq \ell m$ for any ℓ in N. Otherwise, $\sin(\ell m\Pi \frac{n}{m}) = \sin(\ell n\Pi) = 0$. Hence, if $\xi = \sum_{\ell=1}^{\infty} \xi_\ell \sin(\ell m\Pi x)$, then $\xi \perp Y(T)$ and $\xi \not\equiv 0$. Thus, we may describe the orthogonal complement of $Y(T)$ and gain information about $Y(T)$ itself.

Theorem 2. If $a = \frac{n}{m}$, then $Y(T)^{\perp}$ is the span of the functions $\{\sin(\ell m\Pi x)\}_{\ell=1}^{\infty}$. Hence, $\overline{Y(T)}$ is the closure of the space of finite linear combinations of the functions $\{\sin k\Pi x\}_{k\neq \ell m}$.

Now in the case of R^p, with $p = 2$ or 3, an analogous proof remains valid with say Ω a rectangle or a parallelopiped with sides parallel to the coordinate axes. Hence, we have the following.

Theorem 3. If $\Omega \subseteq R^p$ with $p = 2$ or 3 and Ω is the square $(0,1) \times (0,1)$ or is the cube $(0,1) \times (0,1) \times (0,1)$ and if $a \in \Omega$ is a p-tuple with irrational components, then the initial value problem (1)-(3) is controllable.

Remark 4. These results are in contrast to those for parabolic equations which are controllable only for the case $\Omega \subset R$, [2]. The proof here is true for higher dimensions because the factors $\frac{1}{1 + \lambda_k^2}$ in (15) (where $\lambda_k^2 = k^2\Pi^2$) imply uniform convergence for $\xi \in L^2(\Omega)$. This, of course, is due to the presence of the term $1 - \Delta$ in equation (1).

3. A control problem.

In this section we consider control of (1)-(3) by means of the following minimization problem with state constraints

$$\begin{aligned} &\text{minimize} \quad \|v\|^2_{L^2(0,T)} \\ (19) \qquad &\text{subject to} \quad v \in L^2(0,T) \\ &\qquad \|y(T;v) - z\|_{L^2(\Omega)} \leq \rho \end{aligned}$$

where $z \in L^2(\Omega)$. Hence, we have the set of admissible controls

$$U(\ ) = \{v \in L^2(0,T) : \|y(T;v) - z\|_{L^2(\Omega)} \leq \rho\} .$$

In order to have a meaningful problem, it is necessary that U be nonempty. An immediate consequence of Theorems 1 and 3 is the following

<u>Corollary</u> 5. If $a \in (0,1)$ is irrational, then $U(\rho)$ is nonempty for any $\rho > 0$.

<u>Corollary</u> 6. If Ω is $(0,1) \times (0,1)$ or $(0,1) \times (0,1) \times (0,1)$, then $U(\rho)$ is nonempty for any $\rho > 0$ whenever all the components of a are irrational.

Suppose now that for (1)-(3) the number a is rational. A similar argument holds for a being a p-tuple with one or more rational components. In this case it has been shown in Theorem 2 that $\overline{Y(T)}$ is a proper subspace of $L^2(\Omega)$. Now certainly in order for $U(\rho)$ to be nonempty, the condition

$$\rho > d = \underset{y \in \overline{Y(T)}}{\text{minimum}} \|y - z\|_{L^2(\Omega)} \tag{20}$$

must be satisfied.

To fix ideas let $z \in L^2(0,1)$ and $a = \frac{n}{m}$. Then we may express z in terms of a Fourier series $z = \sum_{k=1}^{\infty} \zeta_k \sin(k\Pi x)$. Furthermore, we may write

$$z = \sum_{k \neq \ell m} \zeta_k \sin(k\Pi x) + \sum_{\ell=1}^{\infty} \zeta_{\ell m} \sin(\ell m \Pi x) .$$

Thus, it follows that

$$\begin{aligned} d &= \|z - \sum_{k \neq \ell m}^{\infty} \zeta_k \sin(k\Pi x)\|_{L^2(0,1)} \\ &= \|\sum_{\ell=1}^{\infty} \zeta_{\ell m} \sin(\ell m\Pi x)\|_{L^2(0,1)} \end{aligned}$$

$$d = \left(\sum_{\ell=1}^{\infty} \zeta_{\ell m}^2 \right)^{\frac{1}{2}} , \tag{21}$$

and we have

$$d = \frac{1}{2}\left[\sum_{\ell=1}^{\infty} (z,\sin(\ell m\Pi x))^2_{L^2(0,1)} \right]^{\frac{1}{2}} \tag{22}$$

Hence, we have the following

Theorem 7. Let $z \in L^2(\Omega)$. If ρ satisfies $\rho > d$ where d is given by (22), then $U(\rho)$ is nonempty. That is, there exists $v \in L^2(0,T)$ such that $\|y(T;v) - z\|_{L^2(0,1)} < \rho$.

Having determined conditions to guarantee that $U(\rho)$ is nonempty, we now consider problem (19).

Lemma 8. The admissible set $U(\rho)$ is a nonempty closed convex set in $L^2(0,T)$.

Proof. That $U(\rho)$ is nonempty is a consequence of Theorem 7. Certainly $U(\rho)$ is convex from the linearlity of the map $v \to y(T;v)$ and the triangle inequality.

Let $v_n \to v$ in $L^2(0,T)$. Again using equations (7)-(9) with $\xi \in L^2(\Omega)$ and multiplying (1) by p , we see that

$$(\xi,y(T;v_n))_{L^2(\Omega)} = (p(a,\cdot),v_n)_{L^2(0,T)} \tag{23}$$

for all v_n . In the limit we have

$$(\xi,y(\tau;v))_{L^2(\Omega)} = (p(a,\cdot),v)_{L^2(0,T)}$$

so that $y(T;v_n) \to y(T;v)$ weakly in $L^2(\Omega)$. Now there is a sequence of convex combinations $\Sigma\theta_{\eta_i} y(T;v_{\eta_i}) = y(T;\Sigma\theta_{\eta_i} v_{\eta_i})$ that converge strongly to

$y(T;v)$. Thus, it follows that

$$\|y(T;v) - z\|_{L^2(\Omega)} \le \|\Sigma\theta_{\eta_i} y(T;v_{\eta_i}) - z\|_{L^2(\Omega)}$$
$$\le \Sigma\theta_{\eta_i} \|y(T;v_{\eta_i}) - z\|_{L^2(\Omega)} .$$

Hence, we have $\|y(T;v) - z\|_{L^2(\Omega)} \le \rho$ since $v_{\eta_i} \in U(\rho)$, and $v \in U(\rho)$ so that $U(\rho)$ is closed.

We immediately have the following.

Theorem 9. There exists a unique solution u_0 of problem (19).

We now determine regularity properties of u_0 . First, we note as a consequence of the controllability and compatibility results that there exists $v \in L^2(0,T)$ such that $\|y(T;v) - z\|_{L^2(\Omega)} < \rho$. This implies, c.f. [4], the existence of a positive Lagrange multiplier, see also [3].

Theorem 10. If the conditions of Corollary 5 or 6 hold, or if the compatibility condition of Theorem 7 holds, there exists a positive number λ such that for

$$\Lambda(v) = \|v\|^2_{L^2(0,T)} + \lambda(\|y(T;v) - z\|^2_{L^2(\Omega)} - \rho^2)$$

the following holds

$$\|u_0\|^2_{L^2(0,T)} = \min_{v\in U(\rho)} \|v\|^2_{L^2(0,T)} = \min_{v\in U(\rho)} \Lambda(v) = \Lambda(u_0) . \tag{24}$$

As a corollary of Theorem 10, we have

Corollary 11. $\|y(T;u_0) - z\|_{L^2(\Omega)} = \rho$.

Calculating the variation of Λ , c.f. [4], we see that $\Lambda(u_0)(v) = 0$

for all $v \in L^2(0,T)$ so that

$$0 = (u_0, v)_{L^2(0,T)} + \lambda(y(T;u_\partial) - z,\ y(T;v))_{L^2(\Omega)} . \tag{25}$$

By introducing the adjoint problem (7)-(9) with $\xi = y(T;u_0) - z$, we obtain

$$(u_0 + \lambda p(a,\cdot), v)_{L^2(0,T)} = 0$$

for all $v \in L^2(0,T)$. Hence, we see that

$$u_0(t) = -\lambda p(a,t) \tag{26}$$

for almost all t in $(0,T)$. But $t \to p(\cdot,t)$ is an analytic map of $(0,T)$ into $H_0^1(\Omega) \cap H^2(\Omega)$. Thus, we have the following as a consequence of equation (26).

Theorem 12. The optimal control u_0 is equal almost everywhere in $(0,T)$ to an analytic function.

Remark 13. The existence of a positive Lagrange multiplier λ is dependent upon knowing that there is a v satisfying $\|y(T;v) - z\|_{L^2(\Omega)} < \rho$. Hence, compatibility and controllability conditions play an important role here as well. The result clearly holds for Ω a rectangle or a parallelopiped with sides parallel to the coordinate axes.

Remark 14. Similar arguments to those in the controllability proof remain true for Ω a more general domain and for $-\Delta$ replaced by a symmetric uniformly strongly elliptic operator $L(x)$. The ability to explicitly determine the $\overline{Y(T)}$ depends on knowing the eigenfunctions and their zeros for L on Ω.

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