# Un résultat de type ”Bang-bang” pour un controle ponctuel

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Rapports de Recherche

N°65

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Avril 1981

# UN RESULTAT DE TYPE "BANG-BANG" POUR UN CONTROLE PONCTUEL

par

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RESUME :

On considère le contrôle d'un problème parabolique ou pseudo-parabolique en un point $a$ dans $\Omega \subset R^p$, p=1, 2 ou 3, soumis à des contraintes.

On obtient une propriété "bang-bang" pour un contrôle optimal qui dépend de l'irrationalité des composantes de $a$.

On montre également que ce résultat n'est pas vrai pour le cas parabolique avec $p = 2$ ou 3.

# A Bang-Bang Result for Point Control

by

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## Abstract

In this note we consider the control of a pseudo-parabolic or parabolic problem at a point $a$ in $\Omega \subset R^P$ , $P = 1, 2,$ or $3$ subject to constraints. We obtain a bang-bang property for an optimal control that depends on whether the components of $a$ are irrational. We also show that such a result does not hold for the parabolic case $p = 2$ or $3$ .

*Supported in part by National Science Foundation Grant No. MCS-7902037, a University of Oklahoma Energy Resources Center Grant, and the Institute National de Recherche d'Informatique et d'Automatique.

# A Bang-Bang Result for Point Controls

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## 1. Introduction.

In this paper we consider a sample problem involving a pseudo-parabolic equation on an interval $\Omega = (0,1)$ with control at a point $a \in (0,1)$. The underlying equation is given by

(1) $$(1 - \Delta)y_t - \Delta y = v(t)\delta(x - a) \quad \text{in} \quad Q$$

(2) $$y(x,0) = 0 \quad \text{in} \quad \Omega$$

(3) $$y(0,t) = y(1,t) = 0 \quad \text{in} \quad (0,T)$$

where $Q = \Omega \times (0,T)$. In addition, we consider the minimization problem

(4) $$\begin{aligned} &\text{minimize} \quad J(v) = \|y(\cdot,T;v) - z(\cdot)\|_0^2 = \int_0^1 (y(x,T;v) - z(x))^2 dx \\ &\text{subject to} \quad v \in U \end{aligned}$$

where the set $U$ is defined by

(5) $$U = \{v \in L^\infty(0,T) : |v(t)| \le 1 \quad \text{a.e.} \quad [0,T]\} .$$

We study here the necessary conditions for $u$ to be a solution of (1)-(5) and determine certain bang-bang properties that are dependent on whether the number $a$ is rational or irrational. In section 3 we give some examples illustrating our results. Finally, in the last section we indicate extensions of these results to higher dimensions as well as which results remain valid for parabolic equations.

## 2. Necessary conditions for a solution.

Let $\{v_n\}_{n=1}^{\infty}$ be a minimizing sequence for (4). Then there is a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ that converges weak star in $L^{\infty}(0,T)$ , and thus weakly in $L^2(0,T)$ , to an element $u$ in $U$ .

<u>Lemma</u> 1. If $w_n \to w$ weakly in $L^2(0,T)$ , then $y(\cdot,T;w_n) \to y(\cdot,T;w)$ weakly in $L^2(\Omega)$ .

Proof. Introduce the adjoint problem

$$-(1 - \Delta)p_t - \Delta p = 0 \quad \text{in} \quad Q \tag{6}$$

$$p(x,T) = (1 - \Delta)^{-1}\theta(x) \quad \text{in} \quad \Omega \tag{7}$$

$$p(0,t) = p(1,t) = 0 \quad \text{in} \quad (0,T) \tag{8}$$

with $\theta \in L^2(\Omega)$ . If we multiply (1) by $p$ and integrate by parts, we obtain

$$(y(\cdot,T;w_n),\theta)_{L^2(\Omega)} = \int_0^T w_n(t)p(a,t)dt \ . \tag{9}$$

Since $p \in H^1(0,T;H^2(\Omega))$ , it follows that $p(a,t) \in L^2(0,T)$ , [2]. Hence, we see that

$$\int_0^T w_n(t)p(a,t)dt \to \int_0^T w(t)p(a,t)dt \ ,$$

and the result follows.

From Lemma 1 and the weak lower semicontinuity of the $L^2(\Omega)$ norm, we have the following.

Proposition 2. There exists a solution $u$ of (1)-(5).

By taking the first variation of the functional in (4), we may obtain the necessary condition

$$(10) \qquad (y(T;u) - z \ , \ y(T;v) - y(T;u))_{L^2(\Omega)} \geq 0$$

for all $v$ in $U$. Now, setting $\theta = y(T;u) - z$ in equation (7), the condition in (10) implies the following.

Proposition 3. A solution of problem (1)-(5) satisfies the inequality

$$(11) \qquad \int_0^T p(a,t)(v(t) - u(t))dt \geq 0$$

for all $v \in U$.

We now study the consequences of the condition in (11). Suppose $|u(t)| < 1$ for $t$ in a set $E \subset [0,1]$ with meas $E > 0$. If $\theta = y(T;u) - z$ in equation (7), the solution $p$ of (6)-(8) has the property that $p(a,t)$ is a continuous function of $t$ in $(-\infty,\infty)$. Hence, there is a number $M > 0$ such that $|p(a,t)| \leq M$ on $[0,T]$. Now set

$$E_n = \{t \in E : 1 - |u(t)| \geq \frac{1}{n}\} .$$

By assumtpion there exists $k$ such that meas $E_k \neq 0$. Define the function on $[0,T]$

$$v(t) = \begin{cases} u(t) \pm \delta_k p(a,t) & \text{on } E_k \\ u(t) & \text{on } [0,T] - E_k \end{cases}$$

with the number $\delta_k \in (0, \frac{1}{kM})$ such that

$$\begin{aligned} |v(t)| &= |u(t) \pm \delta_k p(a,t)| \\ &\le |u(t) + \delta_k M| \\ &\le 1 - \frac{1}{k} + \delta_k M . \end{aligned}$$

$$|v(t)| \le 1 \quad \text{on} \quad E_k .$$

Thus, we see that $v \in U$. From the inequality (11), it follows that

(12) $$\int_{E_k} p^2(a,t)dt = 0 ,$$

and $p(a,t) = 0$ a.e. in $E_k$. Since $E$ is the countable union of countably many such sets, we conclude

(13) $$p(a,t) = 0 \quad \text{a.e. in } E .$$

Hence, we have shown the following

Theorem 4. If $u$ is a solution of (1)-(5) and if $p$ is the solution of (6)-(8) with $0 = y(T;u) - z$, then

(14) $$(|u(t)| - 1)p(a,t) = 0$$

holds a.e. in $[0,T]$.

Now we consider the implications of equation (14). Note that if $z \in L^2(0,T)$ and if we set $y(T;u) - z = \sum_{n=1}^{\infty} h_n \sin n\Pi x$, then the solution of (6)-(8) with $\theta = y(T;u) - z$ is given by

$$p(x,t) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-(t-t)\mu_n} \sin(n\Pi x) \tag{15}$$

where $_n = \frac{n^2\Pi^2}{1 + n^2\Pi^2}$ . For ease set $\tau = T - t$ and set

$$q(x,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-\tau\mu_n} \sin(n\Pi x) . \tag{16}$$

Clearly, the series in (16) is uniformity convergent on $[0,1] \times [0,\infty)$ . Thus, we have

$$q(a,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-\tau\mu_n} \sin(n\Pi a) \tag{17}$$

in $[0,\infty)$ , and taking the Laplace transform, we obtain

$$\hat{q}(a,s) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} \frac{1}{s + \mu_n} \sin(n\Pi a) . \tag{18}$$

Define the meromorphic function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{\zeta_n \sin(n\Pi a)}{1 + n^2\Pi^2}\right) \frac{1}{z + \mu_n} . \tag{19}$$

Now $q(a,\tau)$ is an analytic function of $\tau$ . If $|u(t)| < 1$ for $t \in E$ where meas $E > 0$ , then $q(a,\tau) = 0$ for $\tau \in \{T\} - E$ . But the set $\{T\} - E$ has a cluster point since it has positive measure. Thus, $q(a,\tau) \equiv 0$ in $[0,+\infty)$ from analyticity. Accordingly, we see that $\hat{q}(a,s) = 0$ for all $s > 0$ so that $f(z) = 0$ for all $z$ with $\operatorname{Re} z > 0$ . Hence, we have $f(z) \equiv 0$ . However, this implies that the residues of $f$ are zero, and we see that

$$\zeta_n \sin(n\Pi a) = 0 \tag{20}$$

for each $n$ . It is from equation (20) that we may deduce properties of the sequence $\{\zeta_n\}_{n=1}^{\infty}$ .

In the case that the number $a$ is irrational, equation (20) implies

$\zeta_n = 0$ for each $n$. Hence, we conclude that $p(x,t) \equiv 0$ so that $y(T;u) = z$ in $L^2(\Omega)$.

<u>Theorem</u> 5. If $a \in (0,1)$ is an irrational number, then either

$$y(\ ,T;u) = z(\cdot) \quad \text{in} \quad L^2(\Omega) \tag{21}$$

or

$$|u(t)| = 1 \quad \text{a.e. in} \quad [0,T] . \tag{22}$$

3. Examples.

We consider the problem with $T = 1$.

$$(1 - \frac{\partial^2}{\partial x^2})y_t - \frac{\partial^2}{\partial x^2} y = v(t)\delta(x - a) \quad \text{in} \quad (0,1) \times (0,1) \tag{23}$$

$$y(x,0) = 0 \quad \text{in} \quad (0,1) \tag{24}$$

$$y(0,t) = y(1,t) = 0 \quad \text{in} \quad (0,1) . \tag{25}$$

First, we note that in terms of a Fourier series the Green's function of the operator $M = 1 - \frac{\partial^2}{\partial x^2}$ on $(0,1)$ with the Dirichlet boundary conditions is given by

$$g(x|\xi) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2\Pi^2} \sin(n\Pi\xi)\sin(n\Pi x) .$$

Hence, the solution of (23)-(25) may be given in terms of a Fourier series by

$$y(x,t;v) = \sum_{n=1}^{\infty} \left(\int_0^t e^{-\mu_n(t-s)} v(s)ds\right)\frac{\sin(n\Pi a)}{1 + n^2\Pi^2} \sin(n\Pi x) \tag{26}$$

where $\mu_n = \frac{n^2\Pi^2}{1 + n^2\Pi^2}$ . Setting $z = \sum_{z=1}^{\infty} c_n \sin(n\Pi x)$ we see that

(27) $$J(v) = \frac{1}{2} \sum_{n=1}^{\infty} [(\int_0^1 e^{-\mu_n(1-t)} v(t)dt)\frac{\sin(n\Pi a)}{1 + n^2\Pi^2} - c_n]^2 .$$

Example 1. Let $a$ be irrational and $z$ be defined by a Fourier series

$$z(x) = \sum_{n=1}^{\infty} \frac{\sin(n\Pi a)}{n^2\Pi^2}(1 - e^{-\frac{\mu_n}{2}})\sin(n\Pi x) .$$

Take $u(t) \equiv 0$ on $[0,\frac{1}{2})$ and $u(t) \equiv 1$ on $[\frac{1}{2},1]$ , then

$$\begin{aligned} \int_0^1 e^{-\mu_n(1-t)} u(t)dt &= \int_{\frac{1}{2}}^1 e^{-\mu_n(1-t)} dt \\ &= \frac{1}{\mu_n}(1 - e^{-\frac{\mu_n}{2}}) , \end{aligned}$$

and we see that $J(u) = 0$ . In this case we see have $y(T;u) = z$ in $L^2(0,1)$ and $|u(t)| = 0 < 1$ for $t \in [0,\frac{1}{2})$ .

Example 2. Again let $a$ be irrational and choose $a \in (0,\frac{1}{4})$ so that $\sin(2\Pi a) > 0$ . Let $z$ be given in terms of a Fourier series

$$z(x) = c_2 \sin 2\Pi x + \sum_{n\neq 2}^{\infty} \frac{\sin(n\Pi a)}{n^2\Pi^2}(1 - e^{-\mu_n})\sin(n\Pi x) .$$

Note that for $u(t) \equiv 1$ on $[0,1]$ ,

(28) $$J(u) = \frac{1}{2}](\int_0^1 e^{\frac{-4\Pi^2}{1+4\Pi^2}} u(t)dt)\frac{\sin(2\Pi a)}{1 + 4\Pi^2} - c_2]^2 .$$

Let $|v(t)| \leq 1$ a.e. in $[0,1]$ . Then we have

(29) $$J(v) = \int_0^1 y^2(1;v - u)dx + 2\int_0^1 y(1;v - u)(y(1;u) - z)dx + J(u) .$$

From equation (26) the second term of (29) becomes

$$\int_0^1 y(1;v-u)(y(1;u)-z)dx =$$

(30) $$= [\int_0^1 e^{-\mu_2(1-t)}(v(t)-1)dt]\left(\frac{\sin(2\Pi a)}{1+4\Pi^2}\right) \times \left[\frac{\sin(2\Pi a)}{4\Pi^2}(1-e^{-\mu_2}) - c_2\right] .$$

Since $|v(t)| \le 1$ a.e., the integral is nonpositive. Hence, if $c_2 \ge \frac{1}{4\Pi^2}(1-e^{-\mu_2})\sin(2\Pi a)$ then $\int_0^1 y(1;v-u)(y(1;u)-z)dx \ge 0$ for all $v$ satisfying $|v(t)| \le 1$ a.e. in $[0,1]$. As a consequence of equation (29), we have that $J(v) \ge J(u)$ for all such $v$. Similarly, if $c_2 \le -\frac{1}{4}(1-e^{-\mu_2})\sin(2\Pi a)$, then $u(t) \equiv -1$ is a solution.

Example 3. Let $a = \frac{1}{2}$. Set $z(x) = \sum_{n=1}^{\infty} c_n \sin(n\Pi x)$ where $c_2 \ne 0$, $c_{2n} = 0$ for $n \ge 2$, and if $n$ is odd $c_n = \frac{1}{2n^2\Pi^2}(1-e^{-\mu_n})\sin(\frac{n\Pi}{2})$. Thus, from equation (27) we see that $J(\frac{1}{2}) = \frac{1}{2}c_2^2$. Clearly, $u(t) \equiv \frac{1}{2}$ for $[0,1]$ is a solution. Here, however, we have $y(x,1;\frac{1}{2}) - z(x) = c_2 \sin(2\Pi x)$. Thus, we note that

$$p(x,t) = \frac{c_2}{1+4\Pi^2} e^{-\mu_2(1-t)} \sin(2\Pi x)$$

and $p(a,t) = 0$. Hence, this example shows that if the number $a$ is rational Theorem 5 does not hold although Theorem 4 applies.

## 4. Higher dimensions and the Parabolic Case.

By inspecting the results in section 2, we see that Theorem 5 remains true for problem (1)-(3) with $(0,1)$ replaced by $\Omega$ a rectangle or parallelopiped with sides parallel to the coordinate axes. Hence, $\Omega$ may contained in $R^p$ for $p = 2$ or $3$ and convergence of the series in (16)-(19) is still uniform. This leads to the following.

Theorem 6. If $a \in \Omega \subset R^p$, $p = 2$ or $3$, with $\Omega$ a rectangle or parallelopiped with sides parallel to the coordinate axes and if the components of the p-tuple $a$ are all irrational, then either (21) or (22) holds.

For the parabolic case equation (17) becomes

$$(31) \qquad q(a,\tau) = \sum_{n=1}^{\infty} \zeta_n e^{-\tau_n^2} \sin(n\Pi a) .$$

By the results of Muntz-Szasz [1, 3] the functions $\{e^{-\tau\lambda_n}\}_{n=1}^{\infty}$ are independent if and only if $\Sigma \frac{1}{\lambda_n} < \infty$. Hence, $q(a,\tau) = 0$ for equation (31) implies that equation (20) holds for the one-dimensional case. Here, however, if $\Omega \subset R^p$ for $p = 2$ or $3$ by the Mutz-Szasz theorem $\{e^{-\tau\lambda_n}\}_{n=1}^{\infty}$ is no longer independent and (20) does not follow.

Theorem 7. For the problem

$$y_t - \Delta y = v(t)\delta(x - a) \quad \text{in } Q$$
$$y(x,0) = 0 \quad \text{in } \Omega$$
$$y(x,t)\big|_{\partial\Omega} = 0 \quad \text{in } (0,T) ,$$

the results of Theorem 5 hold if $\Omega \subset R$. If $\Omega \subset R^p$ for $p = 2$ or $3$, Theorem 5 no longer remains valid.

Remark 8. Even though Theorem 5 does not go through, Theorem 4, hence equation (14), still holds.

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