Un résultat de type ”Bang-bang” pour un controle ponctuel

L.W. White

IRIA

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:954 90 20

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Luther W. WHITE

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UN RESULTAT DE TYPE "BANG-BANG" POUR UN CONTROLE PONCTUEL

par

Luther W. WHITE

Université de l'Oklahoma
Département de Mathématiques et Centre
de Ressources Energétiques
NORMAN OKLAHOMA 73019 (USA)

RESUME :

On considère le contrôle d'un problème parabolique ou pseudo-parabolique en un point a dans $\Omega \subset R^p$, p=1, 2 ou 3, soumis à des contraintes.

On obtient une propriété "bang-bang" pour un contrôle optimal qui dépend de l'irrationalité des composantes de a.

On montre également que ce résultat n'est pas vrai pour le cas parabolique avec $p = 2$ ou 3.

A Bang-Bang Result for Point Control

by

L. W. White*

Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73019

Abstract

In this note we consider the control of a pseudo-parabolic or parabolic problem at a point a in $\Omega \subset R^P$, $P = 1, 2,$ or 3 subject to constraints. We obtain a bang-bang property for an optimal control that depends on whether the components of a are irrational. We also show that such a result does not hold for the parabolic case $p = 2$ or 3.

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A Bang-Bang Result for Point Controls

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1. Introduction.

In this paper we consider a sample problem involving a pseudo-parabolic equation on an interval $\Omega = (0,1)$ with control at a point $a \in (0,1)$. The underlying equation is given by

$$(1) \qquad (1 - \Delta)y_t - \Delta y = v(t)\delta(x - a) \quad \text{in} \quad Q$$

$$(2) \qquad y(x,0) = 0 \quad \text{in} \quad \Omega$$

$$(3) \qquad y(0,t) = y(1,t) = 0 \quad \text{in} \quad (0,T)$$

where $Q = \Omega \times (0,T)$. In addition, we consider the minimization problem

$$(4) \qquad \begin{aligned} &\text{minimize} \quad J(v) = \|y(\cdot,T;v) - z(\cdot)\|_0^2 = \int_0^1 (y(x,T;v) - z(x))^2 dx \\ &\text{subject to} \quad v \in U \end{aligned}$$

where the set U is defined by

$$(5) \qquad U = \{v \in L^\infty(0,T) : |v(t)| \leq 1 \quad \text{a.e.} \quad [0,T]\} .$$

We study here the necessary conditions for u to be a solution of (1)-(5) and determine certain bang-bang properties that are dependent on whether the number a is rational or irrational. In section 3 we give some examples illustrating our results. Finally, in the last section we indicate extensions of these results to higher dimensions as well as which results remain valid for parabolic equations.

2. Necessary conditions for a solution.

Let $\{v_n\}_{n=1}^{\infty}$ be a minimizing sequence for (4). Then there is a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ that converges weak star in $L^{\infty}(0,T)$, and thus weakly in $L^2(0,T)$, to an element u in U .

<u>Lemma</u> 1. If $w_n \to w$ weakly in $L^2(0,T)$, then $y(\cdot,T;w_n) \to y(\cdot,T;w)$ weakly in $L^2(\Omega)$.

Proof. Introduce the adjoint problem

$$-(1-\Delta)p_t - \Delta p = 0 \quad \text{in} \quad Q \tag{6}$$

$$p(x,T) = (1-\Delta)^{-1}\theta(x) \quad \text{in} \quad \Omega \tag{7}$$

$$p(0,t) = p(1,t) = 0 \quad \text{in} \quad (0,T) \tag{8}$$

with $\theta \in L^2(\Omega)$. If we multiply (1) by p and integrate by parts, we obtain

$$(y(\cdot,T;w_n),\theta)_{L^2(\Omega)} = \int_0^T w_n(t)p(a,t)dt \ . \tag{9}$$

Since $p \in H^1(0,T;H^2(\Omega))$, it follows that $p(a,t) \in L^2(0,T)$, [2]. Hence, we see that

$$\int_0^T w_n(t)p(a,t)dt \to \int_0^T w(t)p(a,t)dt \ ,$$

and the result follows.

From Lemma 1 and the weak lower semicontinuity of the $L^2(\Omega)$ norm, we have the following.

Proposition 2. There exists a solution u of (1)-(5).

By taking the first variation of the functional in (4), we may obtain the necessary condition

$$(10) \qquad (y(T;u) - z \,,\; y(T;v) - y(T;u))_{L^2(\Omega)} \geq 0$$

for all v in U . Now, setting $\theta = y(T;u) - z$ in equation (7), the condition in (10) implies the following.

Proposition 3. A solution of problem (1)-(5) satisfies the inequality

$$(11) \qquad \int_0^T p(a,t)(v(t) - u(t))dt \geq 0$$

for all $v \in U$.

We now study the consequences of the condition in (11). Suppose $|u(t)| < 1$ for t in a set $E \subset [0,1]$ with meas $E > 0$. If $\theta = y(T;u) - z$ in equation (7), the solution p of (6)-(8) has the property that $p(a,t)$ is a continuous function of t in $(-\infty,\infty)$. Hence, there is a number $M > 0$ such that $|p(a,t)| \leq M$ on $[0,T]$. Now set

$$E_n = \{t \in E : 1 - |u(t)| \geq \frac{1}{n}\} .$$

By assumtpion there exists k such that meas $E_k \neq 0$. Define the function on $[0,T]$

$$v(t) = \begin{cases} u(t) \pm \delta_k p(a,t) & \text{on } E_k \\ u(t) & \text{on } [0,T] - E_k \end{cases}$$

with the number $\delta_k \in (0, \frac{1}{kM})$ such that

$$\begin{aligned} |v(t)| &= |u(t) \pm \delta_k p(a,t)| \\ &\leq |u(t) + \delta_k M| \\ &\leq 1 - \frac{1}{k} + \delta_k M . \end{aligned}$$

$$|v(t)| \leq 1 \quad \text{on} \quad E_k .$$

Thus, we see that $v \in U$. From the inequality (11), it follows that

(12) $$\int_{E_k} p^2(a,t)dt = 0 ,$$

and $p(a,t) = 0$ a.e. in E_k. Since E is the countable union of countably many such sets, we conclude

(13) $$p(a,t) = 0 \quad \text{a.e. in } E .$$

Hence, we have shown the following

Theorem 4. If u is a solution of (1)-(5) and if p is the solution of (6)-(8) with $0 = y(T;u) - z$, then

(14) $$(|u(t)| - 1)p(a,t) = 0$$

holds a.e. in $[0,T]$.

Now we consider the implications of equation (14). Note that if $z \in L^2(0,T)$ and if we set $y(T;u) - z = \sum_{n=1}^{\infty} h_n \sin n\Pi x$, then the solution of (6)-(8) with $\theta = y(T;u) - z$ is given by

(15) $$p(x,t) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-(t-t)\mu_n} \sin(n\Pi x)$$

where ${}_n = \dfrac{n^2\Pi^2}{1 + n^2\Pi^2}$. For ease set $\tau = T - t$ and set

(16) $$q(x,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-\tau\mu_n} \sin(n\Pi x) .$$

Clearly, the series in (16) is uniformity convergent on $[0,1] \times [0,\infty)$. Thus, we have

(17) $$q(a,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} e^{-\tau\mu_n} \sin(n\Pi a)$$

in $[0,\infty)$, and taking the Laplace transform, we obtain

(18) $$\hat{q}(a,s) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2\Pi^2} \frac{1}{s + \mu_n} \sin(n\Pi a) .$$

Define the meromorphic function

(19) $$f(z) = \sum_{n=1}^{\infty} \left(\frac{\zeta_n \sin(n\Pi a)}{1 + n^2\Pi^2}\right) \frac{1}{z + \mu_n} .$$

Now $q(a,\tau)$ is an analytic function of τ . If $|u(t)| < 1$ for $t \in E$ where meas $E > 0$, then $q(a,\tau) = 0$ for $\tau \in \{T\} - E$. But the set $\{T\} - E$ has a cluster point since it has positive measure. Thus, $q(a,\tau) \equiv 0$ in $[0,+\infty)$ from analyticity. Accordingly, we see that $\hat{q}(a,s) = 0$ for all $s > 0$ so that $f(z) = 0$ for all z with $\mathrm{Re}\ z > 0$. Hence, we have $f(z) \equiv 0$. However, this implies that the residues of f are zero, and we see that

(20) $$\zeta_n \sin(n\Pi a) = 0$$

for each n . It is from equation (20) that we may deduce properties of the sequence $\{\zeta_n\}_{n=1}^{\infty}$.

In the case that the number a is irrational, equation (20) implies

$\zeta_n = 0$ for each n. Hence, we conclude that $p(x,t) \equiv 0$ so that $y(T;u) = z$ in $L^2(\Omega)$.

Theorem 5. If $a \in (0,1)$ is an irrational number, then either

(21) $$y(\ ,T;u) = z(\cdot) \quad \text{in} \quad L^2(\Omega)$$

or

(22) $$|u(t)| = 1 \quad \text{a.e. in} \quad [0,T] .$$

3. Examples.

We consider the problem with $T = 1$.

(23) $$(1 - \frac{\partial^2}{\partial x^2})y_t - \frac{\partial^2}{\partial x^2} y = v(t)\delta(x - a) \quad \text{in} \quad (0,1) \times (0,1)$$

(24) $$y(x,0) = 0 \quad \text{in} \quad (0,1)$$

(25) $$y(0,t) = y(1,t) = 0 \quad \text{in} \quad (0,1) .$$

First, we note that in terms of a Fourier series the Green's function of the operator $M = 1 - \frac{\partial^2}{\partial x^2}$ on $(0,1)$ with the Dirichlet boundary conditions is given by

$$g(x|\xi) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2\Pi^2} \sin(n\Pi\xi)\sin(n\Pi x) .$$

Hence, the solution of (23)-(25) may be given in terms of a Fourier series by

(26) $$y(x,t;v) = \sum_{n=1}^{\infty} (\int_0^t e^{-\mu_n(t-s)} v(s)ds)\frac{\sin(n\Pi a)}{1 + n^2\Pi^2} \sin(n\Pi x)$$

where $\mu_n = \dfrac{n^2\Pi^2}{1 + n^2\Pi^2}$. Setting $z = \sum_{z=1}^{\infty} c_n \sin(n\Pi x)$ we see that

(27) $$J(v) = \frac{1}{2} \sum_{n=1}^{\infty} [(\int_0^1 e^{-\mu_n(1-t)} v(t)dt)\frac{\sin(n\Pi a)}{1 + n^2\Pi^2} - c_n]^2 .$$

Example 1. Let a be irrational and z be defined by a Fourier series

$$z(x) = \sum_{n=1}^{\infty} \frac{\sin(n\Pi a)}{n^2\Pi^2}(1 - e^{-\frac{\mu_n}{2}})\sin(n\Pi x) .$$

Take $u(t) \equiv 0$ on $[0,\frac{1}{2})$ and $u(t) \equiv 1$ on $[\frac{1}{2},1]$, then

$$\begin{aligned} \int_0^1 e^{-\mu_n(1-t)} u(t)dt &= \int_{\frac{1}{2}}^1 e^{-\mu_n(1-t)} dt \\ &= \frac{1}{\mu_n}(1 - e^{-\frac{\mu_n}{2}}) , \end{aligned}$$

and we see that $J(u) = 0$. In this case we see have $y(T;u) = z$ in $L^2(0,1)$ and $|u(t)| = 0 < 1$ for $t \in [0,\frac{1}{2})$.

Example 2. Again let a be irrational and choose $a \in (0,\frac{1}{4})$ so that $\sin(2\Pi a) > 0$. Let z be given in terms of a Fourier series

$$z(x) = c_2 \sin 2\Pi x + \sum_{n\neq 2}^{\infty} \frac{\sin(n\Pi a)}{n^2\Pi^2}(1 - e^{-\mu_n})\sin(n\Pi x) .$$

Note that for $u(t) \equiv 1$ on $[0,1]$,

(28) $$J(u) = \frac{1}{2}](\int_0^1 e^{\frac{-4\Pi^2}{1+4\Pi^2}} u(t)dt)\frac{\sin(2\Pi a)}{1 + 4\Pi^2} - c_2]^2 .$$

Let $|v(t)| \leq 1$ a.e. in $[0,1]$. Then we have

(29) $$J(v) = \int_0^1 y^2(1;v - u)dx + 2\int_0^1 y(1;v - u)(y(1;u) - z)dx + J(u) .$$

From equation (26) the second term of (29) becomes

$$\int_0^1 y(1;v-u)(y(1;u)-z)dx =$$

$$= [\int_0^1 e^{-\mu_2(1-t)}(v(t)-1)dt]\left(\frac{\sin(2\Pi a)}{1+4\Pi^2}\right) \times \tag{30}$$

$$\left[\frac{\sin(2\Pi a)}{4\Pi^2}(1-e^{-\mu_2}) - c_2\right] .$$

Since $|v(t)| \le 1$ a.e., the integral is nonpositive. Hence, if $c_2 \ge \frac{1}{4\Pi^2}(1-e^{-\mu_2})\sin(2\Pi a)$ then $\int_0^1 y(1;v-u)(y(1;u)-z)dx \ge 0$ for all v satisfying $|v(t)| \le 1$ a.e. in $[0,1]$. As a consequence of equation (29), we have that $J(v) \ge J(u)$ for all such v. Similarly, if $c_2 \le -\frac{1}{4}(1-e^{-\mu_2})\sin(2\Pi a)$, then $u(t) \equiv -1$ is a solution.

Example 3. Let $a = \frac{1}{2}$. Set $z(x) = \sum_{n=1}^{\infty} c_n \sin(n\Pi x)$ where $c_2 \ne 0$, $c_{2n} = 0$ for $n \ge 2$, and if n is odd $c_n = \frac{1}{2n^2\Pi^2}(1-e^{-\mu_n})\sin(\frac{n\Pi}{2})$. Thus, from equation (27) we see that $J(\frac{1}{2}) = \frac{1}{2}c_2^2$. Clearly, $u(t) \equiv \frac{1}{2}$ for $[0,1]$ is a solution. Here, however, we have $y(x,1;\frac{1}{2}) - z(x) = c_2 \sin(2\Pi x)$. Thus, we note that

$$p(x,t) = \frac{c_2}{1+4\Pi^2} e^{-\mu_2(1-t)} \sin(2\Pi x)$$

and $p(a,t) = 0$. Hence, this example shows that if the number a is rational Theorem 5 does not hold although Theorem 4 applies.

4. Higher dimensions and the Parabolic Case.

By inspecting the results in section 2, we see that Theorem 5 remains true for problem (1)-(3) with $(0,1)$ replaced by Ω a rectangle or parallelopiped with sides parallel to the coordinate axes. Hence, Ω may contained in R^p for $p = 2$ or 3 and convergence of the series in (16)-(19) is still uniform. This leads to the following.

Theorem 6. If $a \in \Omega \subset R^p$, $p = 2$ or 3, with Ω a rectangle or parallelopiped with sides parallel to the coordinate axes and if the components of the p-tuple a are all irrational, then either (21) or (22) holds.

For the parabolic case equation (17) becomes

$$q(a,\tau) = \sum_{n=1}^{\infty} \zeta_n e^{-\tau_n^2} \sin(n\Pi a) . \tag{31}$$

By the results of Muntz-Szasz [1, 3] the functions $\{e^{-\tau\lambda_n}\}_{n=1}^{\infty}$ are independent if and only if $\Sigma \frac{1}{\lambda_n} < \infty$. Hence, $q(a,\tau) = 0$ for equation (31) implies that equation (20) holds for the one-dimensional case. Here, however, if $\Omega \subset R^p$ for $p = 2$ or 3 by the Mutz-Szasz theorem $\{e^{-\tau\lambda_n}\}_{n=1}^{\infty}$ is no longer independent and (20) does not follow.

Theorem 7. For the problem

$$\begin{aligned} y_t - \Delta y &= v(t)\delta(x - a) \quad \text{in } Q \\ y(x,0) &= 0 \quad \text{in } \Omega \\ y(x,t)\big|_{\partial\Omega} &= 0 \quad \text{in } (0,T) , \end{aligned}$$

the results of Theorem 5 hold if $\Omega \subset R$. If $\Omega \subset R^p$ for $p = 2$ or 3, Theorem 5 no longer remains valid.

Remark 8. Even though Theorem 5 does not go through, Theorem 4, hence equation (14), still holds.

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