

# Propriétés de différentiabilité pour des problèmes pseudoparaboliques de controle pontuel

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**DIFFERENTIABILITY PROPERTIES  
OF PSEUDOPARABOLIC  
POINT CONTROL SYSTEMS**

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de Recherche  
en Informatique  
et en Automatique

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PROPRIETES DE DIFFERENTIABILITE POUR DES  
PROBLEMES PSEUDOPARABOLIQUES DE CONTROLE PONCTUEL

par

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RESUME :

On étudie la différentiabilité d'une solution  $u_a$  au problème de contrôle optimal :

$$\left\{ \begin{array}{l} My_t + Ly = v(t)\varphi(x-a) \text{ dans } \Omega \times (0,T) \\ y(0) = 0 \text{ dans } \Omega \\ y|_{\Sigma} = 0 \\ j(a) = \text{minimum} \int_0^T \|v\|^2_{L^2(0,T)} + \|y(T;v)-z\|^2_{L^2(\Omega)} \\ \text{soumis à } v \in L^2(0,T) \end{array} \right.$$

par rapport au point  $a$ .

Pour le cas où  $\varphi$  est une "identité approchée" indéfiniment différentiable, on trouve que  $j$  est indéfiniment différentiable.

Lorsque  $\varphi$  est la masse de Dirac en  $a$ ,  $\delta(x-a)$ , on montre que  $j(a)$  est différentiable si  $\Omega \subset \mathbb{R}^2$  et  $z \in H^{\frac{1}{2}}(\Omega)$ .

Differentiability Properties of  
Pseudoparabolic Point Control Problems\*

by

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Abstract

We study the differentiability of the solution  $u_a$  to the optimal  
control problem

$$\begin{aligned} My_t + Ly &= v(t)\varphi(x - a) \quad \text{in } \Omega \times (0, T) \\ y(0) &= 0 \quad \text{in } \Omega \\ y|_{\Sigma} &= 0 \\ j(a) &= \text{minimum } \|v\|_{L^2(0, T)}^2 + \|y(T; v) - z\|_{L^2(\Omega)}^2 \\ &\text{subject to } v \in L^2(0, T) \end{aligned}$$

with respect to the point  $a$ . For the case  $\varphi$  an infinitely differentiable  
"approximate identity", we find that  $j$  is infinitely differentiable. For  
 $\varphi$  the Dirac measure at  $a$ ,  $\delta(x - a)$ , we show that  $j(a)$  is differentiable  
if  $\Omega \subset \mathbb{R}^2$  and  $z \in H^{\frac{1}{2}}(\Omega)$ .

AMS (MOS) Subject Classification (1970). Primary 49A20, 49B25.

Key Words and Phrases. Pseudo-parabolic equation, point control.

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Differentiability Properties of  
Pseudoparabolic Point Control Problems

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L. W. White

1. Introduction.

In this paper we study the following problem. Let  $\Omega$  be a nonempty bounded open subset of  $\mathbb{R}^D$ ,  $p = 2$  or  $3$ , with a smooth boundary  $\Gamma$ , and let  $Q = \Gamma \times (0, T)$ ,  $\Sigma = \Gamma \times (0, T)$ , and  $a \in \Omega$ . Consider the pseudo-parabolic problem

$$(1) \quad \begin{aligned} M y_t + Ly &= v(t)\varphi(x - a) \quad \text{in } Q \\ y(x, 0; v) &= 0 \quad \text{in } \Omega \\ y(x, t; v) &= 0 \quad \text{on } \Sigma \end{aligned}$$

where  $M = M(x)$  and  $L = L(x)$  are second order symmetric uniformly elliptic operators. The function  $\varphi$  may be an "approximate identity" with the properties:

$$(2) \quad \begin{aligned} \varphi &\in C_0^\infty(\mathbb{R}^D), \quad \text{supp } \varphi(x - a) \subset B(a, \varepsilon) \subset \Omega \quad \text{where } B(a, \varepsilon) \\ &= \{y \in \mathbb{R}^D : \|y - a\| \leq \varepsilon\}, \quad \varphi \geq 0, \quad \int_\Omega \varphi(x - a) dx = 1 \quad \text{or may be the Dirac} \\ &\text{measure at } a, \quad \delta(x - a). \quad \text{Together with the equation (1), we study} \\ &\text{the optimization problem} \end{aligned}$$

$$(3) \quad \begin{aligned} \text{minimize } J(v) &= \int_0^T v^2(t) dt + \int_\Omega (y(x, T; v) - z(x))^2 dx \\ \text{subject to } v &\in L^2(0, T) \end{aligned}$$

with  $z \in L^2(\Omega)$ .

The differential equation (1) arises in the modelling of various physical systems such as flow of fluid in fissured strata [2] and the flow of second order fluids [6]. We refer to the work of Carroll and Showalter [3] for an extensive bibliography concerning these equations.

The control problem embodied in (1) and (3) is studied in [7, 8]. There the existence of a unique solution  $u_a$  is established. Furthermore, it is shown that the function from  $\Omega$  into  $R$  defined by  $a \rightarrow j(a) = J(u_a)$  is continuous from  $\Omega$  to  $R$ . Here we determine differentiability properties of this function. More specifically for the case of the Dirac measure we show that for  $\Omega \subset R^2$  and  $z \in H^{\frac{1}{2}}(\Omega)$ , the function  $a \rightarrow j(a)$  is differentiable. In the approximate identity case the differentiability properties are independent of the space dimension and the smoothness of  $z$ . Section 2 considers the case for an approximate identity and section 3 treats the Dirac function case.

## 2. The case for an approximate identity.

We begin with the equations that characterize the solution of the control problem (1) and (3), c.f. [7, 8].

Proposition 1. The control problem (1) and (3) where  $\varphi$  satisfies (2) has a unique solution characterized by the system

$$\begin{aligned}
 (4) \quad & My_t + Ly = u_a(t)\varphi(x - a) \quad \text{in } Q \\
 & y(\cdot, 0; u_0) = 0 \quad \text{in } \Omega \\
 & y(x, t; u_a) = 0 \quad \text{on } \Sigma
 \end{aligned}$$

$$\begin{aligned}
 & -Mq_t + Lq = 0 \quad \text{in } Q \\
 (5) \quad & q(\cdot, T; u_a) = M^{-1}(y(\cdot, T; u_a) - z(\cdot)) \quad \text{in } \Omega \\
 & q(x, t; u_a) = 0 \quad \text{on } \Sigma
 \end{aligned}$$

$$(6) \quad u_a(t) + \int_{\Omega} q(x, t; u_a) \varphi(x - a) dx = 0 \quad \text{a.e. in } (0, T).$$

As a function of  $a \in \Omega$ , we have

$$(7) \quad j(a) = J(u_a) = \|u_a\|_{L^2(0, T)}^2 + \|y(\cdot, T; u_a) - z(\cdot)\|_{L^2(\Omega)}^2.$$

We calculate the gradient of  $j$  to obtain

$$(8) \quad \nabla j(a) = (j_{a_1}(a), j_{a_2}(a), j_{a_3}(a))$$

where

$$(9) \quad j_{a_i}(a) = 2(u_a, \delta_{a_i} u_a)_{L^2(0, T)} + 2(y(T; u_a) - z, \delta_{a_i} y(T; u_a))_{L^2(\Omega)}$$

for  $i = 1, 2, 3$ , and show that equation (9) makes sense.

We consider  $j_{a_1}$ , the other derivatives being similar. Set  $\eta_1 = \delta_{a_1} y$ ,  $\zeta_1 = \delta_{a_1} q$ ,  $\varphi_1 = \frac{\partial \varphi}{\partial a_1}$ , and  $w_1 = \delta_{a_1} u_a$ . Taking the variation of equations (4)-(6), we have

$$\begin{aligned}
 (10) \quad & M \frac{\partial \eta_1}{\partial t} + L\eta_1 = w_1 \varphi(x - a) - u_a \varphi_1(x - a) \quad \text{in } Q \\
 & \eta_1(0) = 0 \quad \text{in } \Omega
 \end{aligned}$$

$$\begin{aligned}
 & \eta_1|_{\Sigma} = 0 \\
 (11) \quad & -M \frac{\partial \zeta_1}{\partial t} + L\eta_1 = 0 \quad \text{in } Q \\
 & \zeta_1(T) = M^{-1} \eta_1(T) \quad \text{in } \Omega \\
 & \zeta_1|_{\Sigma} = 0
 \end{aligned}$$

$$(12) \quad w_1(t) + \int_{\Omega} \zeta_1(x, t) \varphi(x - a) dx - \int_{\Omega} q(x, t) \varphi_1(x - a) dx = 0$$

Multiplying (10) by  $q$  and integrating, we have

$$(y(T; u_a) - z, \eta_1(T))_{L^2(\Omega)} = \int_0^T (w_1(t) \int_{\Omega} q(x,t) \varphi(x-a) dx - u_a(t) \int_{\Omega} q(x,t) \varphi_1(x-a) dx) dt .$$

Thus, we may rewrite equation (9) for  $i = 1$  as

$$(13) \quad j_{a_1}(a) = 4 \int_0^T w_1(t) u_a(t) dt - 2 \int_0^T u_a(t) \int_{\Omega} q(x,t) \varphi_1(x-a) dx dt .$$

Lemma 2. Equation (13) defines  $j_{a_1}(a)$  if the system (10)-(12) has a unique solution.

We approach the problem of proving the existence and uniqueness of a solution of (10)-(12) by considering the following quadratic control problem.

$$(14) \quad \begin{aligned} M \frac{\partial \eta_1(v)}{\partial t} + L \eta_1(v) &= v(t) \varphi(x-a) - u_a(t) \varphi_1(x-a) \quad \text{in } Q \\ \eta_1(0, v) &= 0 \quad \text{in } \Omega \\ \eta_1(v)|_{\Sigma} &= 0 \end{aligned}$$

$$(15) \quad \begin{aligned} \text{minimize} \quad & \|v\|_{L^2(0,T)}^2 + \|\eta_1(T; v)\|_{L^2(\Omega)}^2 \\ & - 2(v, \int_{\Omega} q(x,t) \varphi_1(x-a) dx)_{L^2(0,T)} \end{aligned}$$

subject to  $v \in L^2(0, T)$  .

Remark 3. Note that since  $\varphi$  is smooth, the solution of (14) has trace at time  $T$  in  $L^2(\Omega)$  for any  $v \in L^2(0, T)$  . That is,  $\eta_1(\cdot, T; v) \in L^2(\Omega)$  for any  $v \in L^2(0, T)$  .

The functional in (15) makes sense, and the following is a standard result.

Lemma 4. There exists a unique solution  $w_1$  to problem (15).

By taking the variation of (15) at  $w_1$  and introducing equation (11), we obtain equation (12). Hence, we have proved the following.



Lemma 5. There exists a unique solution to the system (10)-(12).

It follows then from Lemmas 2 and 5.

Theorem 6. The partial derivative  $j_{a_1}(a)$  is given by equation (13).

Remark 7. Note if  $\varphi$  satisfies (2) then the set  $\Omega$  may be taken to be in  $\mathbb{R}^p$ . By inspecting the previous arguments, we can see that further differentiability is possible depending on the differentiability of  $\varphi$ . In particular, if  $\varphi$  is infinitely differentiable, then so is  $j$ .

### 3. The delta function case.

We now study the problem for  $\varphi = \delta$ .

$$(16) \quad \begin{aligned} My_t + Ly &= v(t)\delta(x - a) \text{ in } Q \\ y(0) &= 0 \text{ in } \Omega \\ y|_{\Sigma} &= 0 \end{aligned}$$

where  $\Omega$  is in  $\mathbb{R}^2$  and  $\Gamma$  is smooth.

Remark 8. Since for  $\Omega \subset \mathbb{R}^p$ , it follows  $H^n(\Omega) \subset C^0(\bar{\Omega})$  if  $n > \frac{p}{2}$ , [1]. For  $p = 2$ , we see that  $H^{3/2}(\Omega) \subset C^0(\bar{\Omega})$  and  $\delta \in (H^{3/2}(\Omega))^*$ . Further, by interpolation it follows that  $y \in H^1(0, T; H^{1/2}(\Omega))$  so that the trace  $y(\cdot, T; v) \in H^{1/2}(\Omega)$  for each  $v$  in  $L^2(0, T)$ .

From the above remark, it is clear that the minimization problem (3) makes sense. In [7] it is shown that there exists a unique solution  $u_a$  in  $L^2(0, T)$ , in fact in  $C^\infty(0, T)$ .

Proposition 9. There exists a unique solution  $u_a$  for the problem given by

(16) and (3) that is characterized by the system

$$\begin{aligned}
 (17) \quad & My_t + Ly = u_a(t)\delta(x - a) \text{ in } Q \\
 & y(0) = 0 \text{ in } \Omega \\
 & y|_{\Sigma} = 0
 \end{aligned}$$

$$\begin{aligned}
 (18) \quad & -Mq_t + Lq = 0 \text{ in } Q \\
 & q(T) = M^{-1}(y(T; u_a) - z) \text{ in } \Omega \\
 & q|_{\Sigma} = 0
 \end{aligned}$$

$$(19) \quad u_a(t) + q(a, t; u_a) = 0 \text{ in } (0, T).$$

As in the previous section we (formally) calculate  $j_{a_1}(a)$  and the variation of equations (17)-(19) to obtain the system of equations

$$\begin{aligned}
 (20) \quad & M \frac{\partial \eta_1}{\partial t} + L\eta_1 = w_1(t)\delta(x - a) - u_a(t)\delta_1(x - a) \text{ in } Q \\
 & \eta_1(0) = 0 \text{ in } \Omega \\
 & \eta_1|_{\Sigma} = 0
 \end{aligned}$$

$$\begin{aligned}
 (21) \quad & -M \frac{\partial \zeta_1}{\partial t} + L\zeta_1 = 0 \text{ in } Q \\
 & \zeta_1(T) = M^{-1}\eta_1(T) \text{ in } \Omega \\
 & \zeta_1|_{\Sigma} = 0
 \end{aligned}$$

$$(22) \quad w_1(t) + \zeta_1(a, t) = q_{x_1}(a, t) = 0 \text{ in } (0, T),$$

and

$$(23) \quad j_{a_1}(a) = -2 \int_0^T u_a(t) \zeta_1(a, t) dt + 4 \int_0^T w_1(t) q(a, t) dt.$$

We seek to provide the proper setting for these equations. Because of the irregularity involved, we prove existence of a solution of the system (20)-(22) by transposition [4, 5].

We begin with some observations concerning the regularity of the solution of (17)-(19) that follow from interpolation and results in [5].

Lemma 10. The solution  $y(u_a)$  of equation (17) belongs to  $H^1(0,T;H^{\frac{1}{2}}(\Omega))$ . The solution  $q$  of equation (18) belongs to  $H^k(0,T;H_0^1(\Omega) \cap H^{5/2}(\Omega))$  for  $k \geq 0$  if  $z \in H^{\frac{1}{2}}(\Omega)$ .

Remark 11. The map  $t \rightarrow q(\cdot, t)$  is an infinitely differentiable map of  $(0, T)$  into  $H_0^1(\Omega) \cap H^{5/2}(\Omega)$ . Hence,  $t \rightarrow q(a, t)$  is continuous and in  $L^2(0, T)$ . Further, with  $q_{x_1}(\cdot, t) \in H^{3/2}(\Omega)$  for each  $t$ , we see that  $t \rightarrow q_{x_1}(a, t)$  is continuous and in  $L^2(0, T)$ .

For equations (20)-(22) with the variation  $w_1$  in  $L^2(0, T)$  and with  $\delta_1$  belonging to  $H^{-5/2}(\Omega)$ , the right side of equation (20) is in  $L^2(0, T; H^{-5/2}(\Omega))$ . Thus, we seek a solution  $\eta_1$  in  $H^1(0, T; H^{-\frac{1}{2}}(\Omega))$ .

Remark 12. In this case we have only  $\eta_1(T)$  in  $H^{-\frac{1}{2}}(\Omega)$ . Hence, the method of demonstrating the existence of a solution to the variational equations that is used in section is not applicable here.

However, we note that if  $\eta_1(\cdot, T)$  is in  $H^{-\frac{1}{2}}(\Omega)$ , the solution  $\zeta_1$  of equation (21) belongs to  $H^p(0, 1; H_0^1(\Omega) \cap H^{3/2}(\Omega))$ . Accordingly, for each  $a \in \Omega$ ,  $\zeta_1(a, t)$  is defined and is a continuous function of  $t$  in  $[0, T]$ .

Lemma 13. If there exists a solution to the system of equation (20)-(22) with  $\zeta_1(a, t)$  in  $L^2(0, T)$ , then formula (23) has meaning.

We prove the existence of a solution to (20)-(22) by transposition. To this end, we consider the following system.

$$(24) \quad \begin{aligned} -M\psi_t + L\psi &= \theta \quad \text{in } Q \\ \psi(T) &= M^{-1}\alpha(T) \quad \text{in } \Omega \end{aligned}$$

$$\psi|_{\Sigma} = 0$$

$$\begin{aligned}
(25) \quad & M\alpha_t + L\alpha = \beta - \psi(a,t)\delta(x-a) \quad \text{in } Q \\
& \alpha(0) = 0 \quad \text{in } \Omega \\
& \alpha|_{\Sigma} = 0
\end{aligned}$$

where  $\theta \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$  and  $\beta \in L^2(0,T;H^{-3/2}(\Omega))$ .

Multiplying equation (20) by  $\psi$  and using equation (22), we integrate to obtain

$$\begin{aligned}
(26) \quad & \int_{\Omega} \eta_1(x,T)\alpha(x,T)dx + \int_0^T \int_{\Omega} \eta_1(x,t)\theta(x,t)dxdt \\
& = \int_0^T (q_{x_1}(a,t) - \zeta_1(a,t))\psi(a,t)dt \\
& - \int_0^T u_a(t)\psi_{x_1}(a,t)dt.
\end{aligned}$$

Similarly, multiplying equation (21) by  $\alpha$  and integrating, we find that

$$(27) \quad \int_{\Omega} \eta_1(x,T)\alpha(x,T)dx = \int_0^T \int_{\Omega} \zeta_1(x,t)\beta(x,t)dxdt - \int_0^T \psi(a,t)\zeta_1(a,t)dt.$$

Combining equations (26) and (27), we have

$$\begin{aligned}
(28) \quad & \int_0^T \int_{\Omega} \zeta_1(x,t)\beta(x,t)dt + \int_0^T \int_{\Omega} \eta_1(x,t)\theta(x,t)dt \\
& = \int_0^T q_x(a,t)\psi(a,t)dt - \int_0^T u_a(t)\psi_{x_1}(a,t)dt
\end{aligned}$$

Lemma 14. If for every pair  $(\theta, \beta)$  in  $L^2(0,T;H^{\frac{1}{2}}(\Omega)) \times L^2(0,T;H^{-3/2}(\Omega))$  there exists a unique solution of (24) and (25), then the solution  $(\zeta_1, \eta_1)$  in  $L^2(0,T;H^{3/2}(\Omega)) \times L^2(0,T;H^{-1/2}(\Omega))$  of (20)-(22) is defined by equation (28).

We now show that the system of equations (24) and (25) has a unique solution. Thus, we consider the problem

$$\begin{aligned}
(29) \quad & M\alpha_t(v) + L\alpha(v) = \beta - v(t)\delta(x-a) \quad \text{in } Q \\
& \alpha(0) = 0 \quad \text{in } \Omega \\
& \alpha|_{\Sigma} = 0.
\end{aligned}$$

With  $\beta \in L^2(0,T;H^{-3/2}(\Omega))$  given, the equation (29) defines  $\alpha \in H^1(0,T;H^{1/2}(\Omega))$ , c.f. [7], by interpolation [5]. Hence, it follows that the trace  $\alpha(\cdot,T)$  belongs to  $H^{1/2}(\Omega)$ , [5], and, as in the previous section, we introduce the minimization problem

$$(30) \quad \begin{aligned} & \text{minimize} \quad \|v\|_{L^2(0,T)}^2 + \|\alpha(T;v)\|_{L^2(\Omega)}^2 + 2(\theta, \alpha(v))_{L^2(Q)} \\ & \text{subject to} \quad v \in L^2(0,T) . \end{aligned}$$

Clearly, there exists a unique solution  $u$  to problem (30), see [4, 7]. Again, a characterization may be obtained by taking the variation at  $u$  of the functional in (30). We have

$$(31) \quad (u,v)_{L^2(0,T)} + (\alpha(T;u), (\delta\alpha)(T))_{L^2(\Omega)} + (\theta, (\delta\alpha))_{L^2(Q)} = 0$$

where the variations satisfy

$$(32) \quad \begin{aligned} M(\delta\alpha)_t + L(\delta\alpha) &= -v(t)\delta(x-a) \quad \text{in } Q \\ (\delta\alpha)(0) &= 0 \quad \text{in } \Omega \\ (\delta\alpha)|_{\Sigma} &= 0 . \end{aligned}$$

We introduce the adjoint equation

$$(24) \quad \begin{aligned} -M\psi_t + L\psi &= \theta \quad \text{in } Q \\ \psi(T) &= M^{-1}\alpha(T;u) \quad \text{in } \Omega \\ \psi|_{\Sigma} &= 0 , \end{aligned}$$

and we note that, with  $\theta \in L^2(0,T;H^{1/2}(\Omega))$ , the solution  $\psi$  of (24) belongs to  $H^1(0,T;H_0^1(\Omega) \cap H^{5/2}(\Omega))$ . Multiplying (32) by  $\psi$  and integrating, we see that

$$\int_0^T \int_{\Omega} \psi (M(\delta\alpha)_t + L(\delta\alpha)) dx dt = - \int_0^T v(t) \psi(a,t) dt$$

so that

$$(\alpha(T;u), (\delta\alpha)(T))_{L^2(\Omega)} + (\theta, (\delta\alpha))_{L^2(Q)} = -\int_0^T v(t)\psi(a,t)dt .$$

Hence, we see that

$$(u - \psi(a, \cdot), v)_{L^2(0,T)} = 0$$

for all  $v \in L^2(0,T)$ , and we have

$$(33) \quad u(t) = \psi(a,t)$$

almost everywhere in  $[0,T]$ . The characterizing equations then are given by

$$(25) \quad \begin{aligned} M\alpha_t + L\alpha &= \beta - \psi(a,t)\delta(x-a) \quad \text{in } Q \\ \alpha(0) &= 0 \quad \text{in } \Omega \\ \alpha|_{\Sigma} &= 0 \end{aligned}$$

$$(24) \quad \begin{aligned} -M\psi_t + L\psi &= \theta \quad \text{in } Q \\ \psi(T) &= M^{-1}\alpha(T) \quad \text{in } \Omega \\ \psi|_{\Sigma} &= 0 , \end{aligned}$$

and we have shown that the system of equations (24) and (25) has a solution.

If  $\theta = 0$  and  $\beta = 0$ , we have by multiplying (25) by  $\omega$  and integrating that

$$\|\alpha(T)\|_{L^2(\Omega)}^2 + \|\psi(a, \cdot)\|_{L^2(0,T)}^2 = 0$$

so that  $\psi = 0$  and  $\alpha = 0$ .

Proposition 15. If  $\beta \in L^2(0,T;H^{-3/2}(\Omega))$  and  $\theta \in L^2(0,T;H^{1/2}(\Omega))$ , there

exists a unique solution  $(\alpha, \psi)$  of (24) and (25) with  $\psi \in H^1(0, T; H_0^1(\Omega) \cap H^{5/2}(\Omega))$  and  $\alpha \in H^1(0, T; H^{1/2}(\Omega))$ .

From Proposition 15 and Lemma 14, we deduce the following.

Corollary 16. There exists a solution  $\zeta_1$  such that  $\zeta_1(a, \cdot)$  belongs to  $L^2(0, T)$ , in fact, in  $C(0, T)$ .

Thus, from Lemma 13 we conclude the following.

Theorem 17. Let  $\Omega \subset \mathbb{R}^2$  and  $z \in H^{1/2}(\Omega)$ . Then  $j_{a_1}(a)$  is well-defined and is given by equation (23).

Remark 18. An analogous argument holds for  $j_{a_2}(a)$ , and thus,  $\nabla j(a)$  is defined for each  $a \in \Omega$ .

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