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PROPRIETES DE DIFFERENTIABILITE POUR DES PROBLEMES PSEUDOPARABOLIQUES DE CONTROLE PONCTUEL

par

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RESUME :

On étudie la différentiabilité d'une solution u_a au problème de contrôle optimal :

$$\left|\begin{array}{l} My_t + Ly = v(t)\varphi(x-a) \text{ dans } \Omega\times(0,T) \\ y(0) = 0 \text{ dans } \Omega \\ y|_\Sigma = 0 \\ j(a) = \text{minimum } \|v\|^2_{L^2(0,T)} + \|y(T;v)-z\|^2_{L^2(\Omega)} \\ \qquad \text{soumis à } v\in L^2(0,T) \end{array}\right.$$

par rapport au point a.

Pour le cas où φ est une "identité approchée" indéfiniment différentiable, on trouve que j est indéfiniment différentiable.

Lorsque φ est la masse de Dirac en a, $\delta(x-a)$, on montre que $j(a)$ est différentiable si $\Omega\subset R^2$ et $z\in H^{\frac{1}{2}}(\Omega)$.

Differentiability Properties of

Pseudoparabolic Point Control Problems*

by

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Abstract

We study the differentiability of the solution u_a to the optimal control problem

$$My_t + Ly = v(t)\varphi(x - a) \quad \text{in} \quad \Omega \times (0,T)$$
$$y(0) = 0 \quad \text{in} \quad \Omega$$
$$y|_\Sigma = 0$$
$$j(a) = \text{minimum} \quad \|v\|^2_{L^2(0,T)} + \|y(T;v) - z\|^2_{L^2(\Omega)}$$
$$\text{subject to} \quad v \in L^2(0,T)$$

with respect to the point a. For the case φ an infinitely differentiable "approximate identity", we find that j is infinitely differentiable. For φ the Dirac measure at a, $\delta(x - a)$, we show that $j(a)$ is differentiable if $\Omega \subset R^2$ and $z \in H^{\frac{1}{2}}(\Omega)$.

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Differentiability Properties of

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1. Introduction.

In this paper we study the following problem. Let Ω be a nonempty bounded open subset of R^p , $p = 2$ or 3 , with a smooth boundary Γ , and let $Q = \Gamma \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, and $a \in \Omega$. Consider the pseudo-parabolic problem

$$
(1) \qquad \begin{aligned}
& My_t + Ly = v(t)\varphi(x - a) \quad \text{in } Q \\
& y(x,0;v) = 0 \quad \text{in } \Omega \\
& y(x,t;v) = 0 \quad \text{on } \Sigma
\end{aligned}
$$

where $M = M(x)$ and $L = L(x)$ are second order symmetric uniformly elliptic operators. The function φ may be an "approximate identity" with the properties:

(2) $\varphi \in C_0^\infty(R^p)$, $\operatorname{supp} \varphi(x - a) \subset B(a,\varepsilon) \subset \Omega$ where $B(a,\varepsilon)$ $= \{y \in R^p : \|y - a\| \le \varepsilon\}$, $\varphi \ge 0$, $\int_\Omega \varphi(x - a)dx = 1$ or may be the Dirac measure at a , $\delta(x - a)$. Together with the equation (1), we study the optimization problem

$$
(3) \qquad \begin{aligned}
& \text{minimize} \quad J(v) = \int_0^T v^2(t)dt + \int_\Omega (y(x,T;v) - z(x))^2 dx \\
& \text{subject to} \quad v \in L^2(0,T)
\end{aligned}
$$

with $z \in L^2(\Omega)$.

The differential equation (1) arises in the modelling of various physical systems such as flow of fluid in fissured strata [2] and the flow of second order fluids [6]. We refer to the work of Carroll and Showalter [3] for an extensive bibliography concerning these equations.

The control problem embodied in (1) and (3) is studied in [7, 8]. There the existence of a unique solution u_a is established. Furthermore, it is shown that the function from Ω into R defined by $a \to j(a) = J(u_a)$ is continuous from Ω to R . Here we determine differentiability properties of this function. More specifically for the case of the Dirac measure we show that for $\Omega \subset R^2$ and $z \in H^{\frac{1}{2}}(\Omega)$, the function $a \to j(a)$ is differentiable. In the approximate identity case the differentiability properties are independent of the space dimension and the smoothness of z . Section 2 considers the case for an approximate identity and section 3 treats the Dirac function case.

2. The case for an approximate identity.

We begin with the equations that characterize the solution of the control problem (1) and (3), c.f. [7, 8].

Proposition 1. The control problem (1) and (3) where φ satisfies (2) has a unique solution characterized by the system

$$(4) \qquad \begin{aligned} My_t + Ly &= u_a(t)\varphi(x - a) \quad \text{in } Q \\ y(\cdot,0;u_0) &= 0 \quad \text{in } \Omega \\ y(x,t;u_a) &= 0 \quad \text{on } \Sigma \end{aligned}$$

$$(5)\qquad \begin{aligned} -Mq_t + Lq &= 0 \quad \text{in } Q \\ q(\cdot,T;u_a) &= M^{-1}(y(\cdot,T;u_a) - z(\cdot)) \quad \text{in } \Omega \\ q(x,t;u_a) &= 0 \quad \text{on } \Sigma \end{aligned}$$

$$(6)\qquad u_a(t) + \int_\Omega q(x,t;u_a)\varphi(x - a)dx = 0 \quad \text{a.e. in } (0,T) .$$

As a function of $a \in \Omega$, we have

$$(7)\qquad j(a) = J(u_a) = \|u_a\|^2_{L^2(0,T)} + \|y(\cdot,T;u_a) - z(\cdot)\|^2_{L^2(\Omega)} .$$

We calculate the gradient of j to obtain

$$(8)\qquad \nabla j(a) = (j_{a_1}(a), j_{a_2}(a), j_{a_3}(a))$$

where

$$(9)\qquad j_{a_i}(a) = 2(u_a, \delta_{a_i} u_a)_{L^2(0,T)} + 2(y(T;u_a) - z, \delta_{a_i} y(T;u_a))_{L^2(\Omega)}$$

for $i = 1,2,3$, and show that equation (9) makes sense.

We consider j_{a_1} , the other direvatives being similar. Set $\eta_1 = \delta_{a_1} y$, $\zeta_1 = \delta_{a_1} q$, $\varphi_1 = \dfrac{\partial\varphi}{\partial a_1}$, and $w_1 = \delta_{a_1} u_a$. Taking the variation of equations (4)-(6), we have

$$(10)\qquad \begin{aligned} M\frac{\partial\eta_1}{\partial t} + L\eta_1 &= w_1\varphi(x - a) - u_a\varphi_1(x - a) \quad \text{in } Q \\ \eta_1(0) &= 0 \quad \text{in } \Omega \\ \eta_1|_\Sigma &= 0 \end{aligned}$$

$$(11)\qquad \begin{aligned} -M\frac{\partial\zeta_1}{\partial t} + L\eta_1 &= 0 \quad \text{in } Q \\ \zeta_1(T) &= M^{-1}\eta_1(T) \quad \text{in } \Omega \\ \zeta_1|_\Sigma &= 0 \end{aligned}$$

$$(12)\qquad w_1(t) + \int_\Omega \zeta_1(x,t)\varphi(x - a)dx - \int_\Omega q(x,t)\varphi_1(x - a)dx = 0$$

Multiplying (10) by q and integrating, we have

$$(y(T;u_a) - z\ ,\ \eta_1(T))_{L^2(\Omega)} = \int_0^T (w_1(t)\int_\Omega q(x,t)\varphi(x-a)dx - u_a(t)\int_\Omega q(x,t)\varphi_1(x-a)dx)dt\ .$$

Thus, we may rewrite equation (9) for $i = 1$ as

$$j_{a_1}(a) = 4\int_0^T w_1(t)u_a(t)dt - 2\int_0^T u_a(t)\int_\Omega q(x,t)\varphi_1(x-a)dxdt\ . \tag{13}$$

Lemma 2. Equation (13) defines $j_{a_1}(a)$ if the system (10)-(12) has a unique solution.

We approach the problem of proving the existence and uniqueness of a solution of (10)-(12) by considering the following quadratic control problem.

$$\begin{aligned} M\frac{\partial\eta_1(v)}{\partial t} + L\eta_1(v) &= v(t)\varphi(x-a) - u_a(t)\varphi_1(x-a) \quad \text{in } Q \\ \eta_1(0,v) &= 0 \quad \text{in } \Omega \\ \eta_1(v)|_\Sigma &= 0 \end{aligned} \tag{14}$$

$$\text{minimize} \quad \|v\|^2_{L^2(0,T)} + \|\eta_1(T;v)\|^2_{L^2(\Omega)} - 2(v,\int_\Omega q(x,t)\varphi_1(x-a)dx)_{L^2(0,T)} \tag{15}$$

subject to $v \in L^2(0,T)$.

Remark 3. Note that since φ is smooth, the solution of (14) has trace at time T in $L^2(\Omega)$ for any $v \in L^2(0,T)$. That is, $\eta_1(\cdot,T;v) \in L^2(\Omega)$ for any $v \in L^2(0,T)$.

The functional in (15) makes sense, and the following is a standard result.

Lemma 4. There exists a unique solution w_1 to problem (15).

By taking the variation of (15) at w_1 and introducing equation (11), we obtain equation (12). Hence, we have proved the following.

Lemma 5. There exists a unique solution to the system (10)-(12).

It follows then from Lemmas 2 and 5.

Theorem 6. The partial derivative $j_{a_1}(a)$ is given by equation (13).

Remark 7. Note if φ satisfies (2) then the set Ω may be taken to be in R^p. By inspecting the previous arguments, we can see that further differentiability is possible depending on the differentiability of φ. In particular, if φ is infinitely differentiable, then so is j.

3. The delta function case.

We now study the problem for $\varphi = \delta$.

$$
\begin{aligned}
My_t + Ly &= v(t)\delta(x - a) \quad \text{in } Q \\
y(0) &= 0 \quad \text{in } \Omega \\
y|_\Sigma &= 0
\end{aligned}
\tag{16}
$$

where Ω is in R^2 and Γ is smooth.

Remark 8. Since for $\Omega \subset R^p$, it follows $H^n(\Omega) \subset C^0(\bar{\Omega})$ if $n > \frac{p}{2}$, [1]. For $p = 2$, we see that $H^{3/2}(\Omega) \subset C^0(\bar{\Omega})$ and $\delta \in (H^{3/2}(\Omega))^*$. Further, by interpolation it follows that $y \in H^1(0,T;H^{\frac{1}{2}}(\Omega))$ so that the trace $y(\cdot,T;v) \in H^{\frac{1}{2}}(\Omega)$ for each v in $L^2(0,T)$.

From the above remark, it is clear that the minimization problem (3) makes sense. In [7] it is shown that there exists a unique solution u_a in $L^2(0,T)$, in fact in $C^\infty(0,T)$.

Proposition 9. There exists a unique solution u_a for the problem given by

(16) and (3) that is characterized by the system

$$\text{(17)} \qquad \begin{aligned} My_t + Ly &= u_a(t)\delta(x - a) \quad \text{in} \quad Q \\ y(0) &= 0 \quad \text{in} \quad \Omega \\ y|_\Sigma &= 0 \end{aligned}$$

$$\text{(18)} \qquad \begin{aligned} -Mq_t + Lq &= 0 \quad \text{in} \quad Q \\ q(T) &= M^{-1}(y(T;u_a) - z) \quad \text{in} \quad \Omega \\ q|_\Sigma &= 0 \end{aligned}$$

$$\text{(19)} \qquad u_a(t) + q(a,t;u_a) = 0 \quad \text{in} \quad (0,T) \ .$$

As in the previous section we (formally) calculate $j_{a_1}(a)$ and the variation of equations (17)-(19) to obtain the system of equations

$$\text{(20)} \qquad \begin{aligned} M\frac{\partial\eta_1}{\partial t} + L\eta_1 &= w_1(t)\delta(x - a) - u_a(t)\delta_1(x - a) \quad \text{in} \quad Q \\ \eta_1(0) &= 0 \quad \text{in} \quad \Omega \\ \eta_1|_\Sigma &= 0 \end{aligned}$$

$$\text{(21)} \qquad \begin{aligned} -M\frac{\partial\eta_1}{\partial t} + L\zeta_1 &= 0 \quad \text{in} \quad Q \\ \zeta_1(T) &= M^{-1}\eta_1(T) \quad \text{in} \quad \Omega \\ \zeta_1|_\Sigma &= 0 \end{aligned}$$

$$\text{(22)} \qquad w_1(t) + \zeta_1(a,t) = q_{x_1}(a,t) = 0 \quad \text{in} \quad (0,T) \ ,$$

and

$$\text{(23)} \qquad j_{a_1}(a) = -2\int_0^T u_a(t)\zeta_1(a,t)dt + 4\int_0^T w_1(t)q(a,t)dt \ .$$

We seek to provide the proper setting for these equations. Because of the irregularity involved, we prove existence of a solution of the system (20)-(22) by transposition [4,5].

We begin with some observations concerning the regularity of the solution of (17)-(19) that follow from interpolation and results in [5].

Lemma 10. The solution $y(u_a)$ of equation (17) belongs to $H^1(0,T;H^{\frac{1}{2}}(\Omega))$. The solution q of equation (18) belongs to $H^k(0,T:H_0^1(\Omega) \cap H^{5/2}(\Omega))$ for $k \geq 0$ if $z \in H^{\frac{1}{2}}(\Omega)$.

Remark 11. The map $t \to q(\cdot,t)$ is an infinitely differentiable map of $(0,T)$ into $H_0^1(\Omega) \cap H^{5/2}(\Omega)$. Hence, $t \to q(a,t)$ is continuous and in $L^2(0,T)$. Further, with $q_{x_1}(\cdot,t) \in H^{3/2}(\Omega)$ for each t, we see that $t \to q_{x_1}(a,t)$ is continuous and in $L^2(0,T)$.

For equations (20)-(22) with the variation w_1 in $L^2(0,T)$ and with δ_1 belonging to $H^{-5/2}(\Omega)$, the right side of equation (20) is in $L^2(0,T;H^{-5/2)}(\Omega))$. Thus, we seek a solution η_1 in $H^1(0,T;H^{-\frac{1}{2}}(\Omega))$.

Remark 12. In this case we have only $\eta_1(T)$ in $H^{-\frac{1}{2}}(\Omega)$. Hence, the method of demonstrating the existence of a solution to the variational equations that is used in section is not applicable here.

However, we note that if $\eta_1(\cdot,T)$ is in $H^{-\frac{1}{2}}(\Omega)$, the solution ζ_1 of equation (21) belongs to $H^p(0,1;H_0^1(\Omega) \cap H^{3/2}(\Omega))$. Accordingly, for each $a \in \Omega$, $\zeta_1(a,t)$ is defined and is a continuous function of t in $[0,T]$.

Lemma 13. If there exists a solution to the system of equation (20)-(22) with $\zeta_1(a,t)$ in $L^2(0,T)$, then formula (23) has meaning.

We prove the existence of a solution to (20)-(22) by transposition. To this end, we consider the following system.

$$
\begin{aligned}
&-M\psi_t + L\psi = \theta \quad \text{in } Q \\
(24) \qquad &\psi(T) = M^{-1}\alpha(T) \quad \text{in } \Omega \\
&\psi|_{\Sigma} = 0
\end{aligned}
$$

$$\begin{aligned} &M\alpha_t + L\alpha = \beta - \psi(a,t)\delta(x-a) \quad \text{in } Q \\ (25) \qquad &\alpha(0) = 0 \quad \text{in } \Omega \\ &\alpha|_\Sigma = 0 \end{aligned}$$

where $\theta \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and $\beta \in L^2(0,T;H^{-3/2}(\Omega))$.

Multiplying equation (20) by ψ and using equation (22), we integrate to obtain

$$\begin{aligned} (26) \qquad &\int_\Omega \eta_1(x,T)\alpha(x,T)dx + \int_0^T\int_\Omega \eta_1(x,t)\theta(x,t)dxdt \\ &= \int_0^T (q_{x_1}(a,t) - \zeta_1(a,t))\psi(a,t)dt \\ &- \int_0^T u_a(t)\psi_{x_1}(a,t)dt \ . \end{aligned}$$

Similarly, multiplying equation (21) by α and integrating, we find that

$$(27) \qquad \int_\Omega \eta_1(x,T)\alpha(x,T)dx = \int_0^T\int_\Omega \zeta_1(x,t)\beta(x,t)dxdt - \int_0^T \psi(a,t)\zeta_1(a,t)dt \ .$$

Combining equations (26) and (27), we have

$$\begin{aligned} (28) \qquad &\int_0^T\int_\Omega \zeta_1(x,t)\beta(x,t)dt + \int_0^T\int_\Omega \eta_1(x,t)\theta(x,t)dt \\ &= \int_0^T q_x(a,t)\psi(a,t)dt - \int_0^T u_a(t)\psi_{x_1}(a,t)dt \end{aligned}$$

Lemma 14. If for every pair (θ,β) in $L^2(0,T;H^{\frac{1}{2}}(\Omega)) \times L^2(0,T;H^{-3/2}(\Omega))$ there exists a unique solution of (24) and (25), then the solution (ζ_1,η_1) in $L^2(0,T;H^{3/2}(\Omega)) \times L^2(0,T;H^{-\frac{1}{2}}(\Omega))$ of (20)-(22) is defined by equation (28).

We now show that the system of equations (24) and (25) has a unique solution. Thus, we consider the problem

$$\begin{aligned} &M\alpha_t(v) + L\alpha(v) = \beta - v(t)\delta(x-a) \quad \text{in } Q \\ (29) \qquad &\alpha(0) = 0 \quad \text{in } \Omega \\ &\alpha|_\Sigma = 0 \ . \end{aligned}$$

With $\beta \in L^2(0,T;H^{-3/2}(\Omega))$ given, the equation (29) defines $\alpha \in H^1(0,T;H^{1/2}(\Omega))$, c.f. [7], by interpolation [5]. Hence, it follows that the trace $\alpha(\cdot,T)$ belongs to $H^{1/2}(\Omega)$, [5], and, as in the previous section, we introduce the minimization problem

$$(30)\qquad \begin{aligned} &\text{minimize} \quad \|v\|^2_{L^2(0,T)} + \|\alpha(T;v)\|^2_{L^2(\Omega)} + 2(\theta,\alpha(v))_{L^2(Q)} \\ &\text{subject to} \quad v \in L^2(0,T) . \end{aligned}$$

Clearly, there exists a unique solution u to problem (30), see [4, 7]. Again, a characterization may be obtained by taking the variation at u of the functional in (30). We have

$$(31)\qquad (u,v)_{L^2(0,T)} + (\alpha(T;u),(\delta\alpha)(T))_{L^2(\Omega)} + (\theta,(\delta\alpha))_{L^2(Q)} = 0$$

where the variations satisfy

$$(32)\qquad \begin{aligned} M(\delta\alpha)_t + L(\delta\alpha) &= -v(t)\delta(x-a) \quad \text{in } Q \\ (\delta\alpha)(0) &= 0 \quad \text{in } \Omega \\ (\delta\alpha)|_\Sigma &= 0 . \end{aligned}$$

We introduce the adjoint equation

$$(24)\qquad \begin{aligned} -M\psi_t + L\psi &= \theta \quad \text{in } Q \\ \psi(T) &= M^{-1}\alpha(T;u) \quad \text{in } \Omega \\ \psi|_\Sigma &= 0 , \end{aligned}$$

and we note that, with $\theta \in L^2(0,T;H^{1/2}(\Omega))$, the solution ψ of (24) belongs to $H^1(0,T;H^1_0(\Omega) \cap H^{5/2}(\Omega))$. Multiplying (32) by ψ and integrating, we see that

$$\int_0^T\int_\Omega \psi(M(\delta\alpha)_t + L(\delta\alpha))dxdt = -\int_0^T v(t)\psi(a,t)dt$$

so that

$$(\alpha(T;u),(\delta\alpha)(T))_{L^2(\Omega)} + (\theta,(\delta\alpha))_{L^2(Q)} = -\int_0^T v(t)\psi(a,t)dt \ .$$

Hence, we see that

$$(u - \psi(a,\cdot),v)_{L^2(0,T)} = 0$$

for all $v \in L^2(0,T)$, and we have

$$u(t) = \psi(a,t) \tag{33}$$

almost everywhere in $[0,T]$. The characterizing equations then are given by

$$\begin{aligned} M\alpha_t + L\alpha &= \beta - \psi(a,t)\delta(x - a) \quad \text{in } Q \\ \alpha(0) &= 0 \quad \text{in } \Omega \\ \alpha|_\Sigma &= 0 \end{aligned} \tag{25}$$

$$\begin{aligned} -M\psi_t + L\psi &= \theta \quad \text{in } Q \\ \psi(T) &= M^{-1}\alpha(T) \quad \text{in } \Omega \\ \psi|_\Sigma &= 0 \ , \end{aligned} \tag{24}$$

and we have shown that the system of equations (24) and (25) has a solution.

If $\theta = 0$ and $\beta = 0$, we have by multiplying (25) by ω and integrating that

$$\|\alpha(T)\|_{L^2(\Omega)} + \|\psi(a,\cdot)\|^2_{L^2(0,T)} = 0$$

so that $\psi = 0$ and $\alpha = 0$.

<u>Proposition</u> 15. If $\beta \in L^2(0,T;H^{-3/2}(\Omega))$ and $\theta \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$, there

exists a unique solution (α,ψ) of (24) and (25) with $\psi \in H^1(0,T:H_0^1(\Omega) \cap H^{5/2}(\Omega))$ and $\alpha \in H^1(0,T;H^{\frac{1}{2}}(\Omega))$.

From Proposition 15 and Lemma 14, we deduce the following.

Corollary 16. There exists a solution ζ_1 such that $\zeta_1(a,\cdot)$ belongs to $L^2(0,T)$, in fact, in $C(0,T)$.

Thus, from Lemma 13 we conclude the following.

Theorem 17. Let $\Omega \subset R^2$ and $z \in H^{\frac{1}{2}}(\Omega)$. Then $j_{a_1}(a)$ is well-defined and is given by equation (23).

Remark 18. An analogous argument holds for $j_{a_2}(a)$, and thus, $\nabla j(a)$ is defined for each $a \in \Omega$.

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