



# A note on the knapsack problem with special ordered sets

E.L. Johnson, M.W. Padberg

► **To cite this version:**

| E.L. Johnson, M.W. Padberg. A note on the knapsack problem with special ordered sets.  
| [Research Report] RR-0045, INRIA. 1980. <inria-00076516>

**HAL Id: inria-00076516**

**<https://hal.inria.fr/inria-00076516>**

Submitted on 24 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**IRIA**

Institut National  
de Recherche  
en Informatique  
et en Automatique

Domaine de Voluceau  
Rocquencourt  
B.P. 105 78150 Le Chesnay  
France  
Tél. 954 90 20

Rapports de Recherche

N° 45

**A NOTE ON  
THE KNAPSACK PROBLEM  
WITH  
SPECIAL ORDERED SETS**

**Ellis L. JOHNSON  
Manfred W. PADBERG**

Décembre 1980

A NOTE ON THE KNAPSACK PROBLEM

WITH SPECIAL ORDERED SETS

Ellis L. JOHNSON  
IBM Thomas J. Watson Research Center

&

Manfred W. PADBERG  
New York University and INRIA

RESUME :

Nous montrons que le problème du sac à dos avec des bornes supérieures généralisées et des coefficients de signe arbitraire est équivalent à un problème de ce type qui n'a que des coefficients positifs ; ce dernier problème est dit standard. Nous démontrons deux propositions qui donnent un algorithme pour le programme linéaire (relaxé) associé au problème standard. Cet algorithme est une généralisation naturelle de l'algorithme de Dantzig pour le problème du sac à dos sans bornes supérieures généralisées. Nous dérivons aussi plusieurs propriétés de l'enveloppe convexe des solutions zéro-un pour cette classe de problèmes.

ABSTRACT :

The knapsack problem with special ordered set and arbitrarily signed coefficients is shown to be equivalent to a standard problem of the same type but having all coefficients positive. Two propositions are proven which define an algorithm for the linear programming relaxation of the standard problem that is a natural generalization of the Dantzig solution to the problem without special ordered sets. Several properties of the convex hull of the associated zero-one polytope are derived.

Key words : knapsack problem, special ordered sets, GUB, algorithms, facets.

## INTRODUCTION

Consider the following problem

$$\max \sum_{i \in I} \sum_{j \in K_i} c_j x_j$$

subject to

$$\sum_{i \in I} \sum_{j \in K_i} a_j x_j \leq a_0$$

(KPSOS)

$$\sum_{j \in K_i} x_j \leq 1 \quad \text{for all } i \in I$$

$$x_j \geq 0 \quad \text{for all } j \in K_i, i \in I$$

where the  $K_i$  satisfy  $|K_i| \geq 1$  for all  $i \in I$ ,  $K_i \cap K_j = \emptyset$  holds for all  $i \neq j$  and the data  $c_j$  and  $a_j$  are arbitrarily signed real numbers. This problem has received much attention in recent literature, see e.g. Zemel [6] for a survey of the literature. In this note we first show that (KPSOS) can always be brought into a "standard" form with positive data and prove two propositions which define an algorithm for (KPSOS) that is a natural extension of Dantzig's solution [2] to (KPSOS) when  $|K_i| = 1$  for all  $i \in I$ . Note that arbitrary upper bounds  $u_i$  on the special ordered sets can be dealt with by scaling. We then consider the zero-one-variables version of (KPSOS) and derive several properties of the facets of the associated convex hull of zero-one solutions to (KPSOS).

### 1. The Algorithm

Note that any special ordered set constraint of the form  $\sum_{j \in K_i} x_j = 1$  can be brought to the inequality form by eliminating one of the variables in such set. Next we show that (KPSOS) can always be brought to a standard form with all data positive. Define

$$(1.1) \quad a_{j_i} = \min \{a_j \mid j \in K_i\} \quad \text{for } i \in I$$

and let  $H = \{i \in I \mid a_{j_i} < 0\}$ . For all  $i \in H$  we set

$$a_j' = \begin{cases} a_j - a_{j_i} & \text{for all } j \in K_i - \{j_i\} \\ -a_{j_i} & \text{for } j = j_i \end{cases}$$

$$(1.2) \quad c_j' = \begin{cases} c_j - c_{j_i} & \text{for all } j \in K_i - \{j_i\} \\ -c_{j_i} & \text{for } j = j_i \end{cases}$$

$$a_o' = a_o - \sum_{i \in H} a_{j_i}$$

Let  $K = \bigcup_{i \in H} K_i$  and define  $a_j' = a_j'$ ,  $c_j' = c_j$  for all  $j \notin K$ . Consider the problem

$$\max \sum_{i \in I} \sum_{j \in K_i} c_j' z_j$$

subject to

$$\sum_{i \in I} \sum_{j \in K_i} a_j' z_j \leq a_o'$$

(KPSOS\*)

$$\sum_{j \in K_i} z_j \leq 1 \quad \text{for all } i \in I$$

$$z_j \geq 0 \quad \text{for all } j \in K_i, i \in I.$$

Problem (KPSOS\*) satisfies  $a_j' \geq 0$  for all  $j$  and the variable substitution

$$(1.3) \quad \begin{array}{ll} x_j = z_j & \text{for } j \notin K \\ x_j = z_j & \text{for } j \in K_i - \{j_i\} \\ x_j = 1 - \sum_{j \in K_i} z_j & \text{for } j = j_i \end{array} \left\{ \begin{array}{l} \\ \\ \end{array} \right. \text{for all } i \in H$$

provides a one-to-one mapping between feasible solutions to (KPSOS) and (KPSOS\*), respectively, which preserves - up to the constant term  $\sum_{i \in H} c_{j_i}$  - the value of the objective function.

If  $c'_j \leq 0$  and  $a'_j \geq 0$  in (KPSOS\*) we set  $z_j = 0$ ; if  $c'_j > 0$  and  $a'_j = 0$  for some  $j \in K_i$  and some  $i \in I$ , it follows that every optimum solution to (KPSOS\*) satisfies  $\sum_{j \in K_i} z_j = 1$  and letting

$$(1.4) \quad c'_{j_i} = \max \{c'_j \mid a'_j = 0, j \in K_i\}$$

we can eliminate variable  $z_j$  from the problem. The eliminated variable is recorded and its value determined at the end of the calculation. After this reduction (possibly used repeatedly) we thus obtain the linear knapsack problem with special ordered sets in the following standard

form:

$$\max \sum_{i \in L} \sum_{j \in S_i} d_j z_j$$

subject to

(KPSOSX)

$$\sum_{i \in L} \sum_{j \in S_i} g_j z_j \leq g_0$$

$$\sum_{j \in S_i} z_j \leq 1 \quad \text{for all } i \in L$$

$$z_j \geq 0 \quad \text{for all } j \in S_i, i \in L$$

where  $d_j > 0$ ,  $g_j > 0$ ,  $g_0 > 0$  for all  $j$  and  $|S_i| \geq 1$  for all  $i \in L$ . Let for  $i \in L$

$$(1.5) \quad \frac{d_{j_i}}{g_{j_i}} = \max \left\{ \frac{d_j}{g_j} \mid j \in S_i \right\}$$

and suppose that the special ordered sets have been ordered such that

$$(1.6) \quad \frac{d_{j_1}}{g_{j_1}} \geq \frac{d_{j_2}}{g_{j_2}} \geq \dots \geq \frac{d_{j_h}}{g_{j_h}}$$

where  $h = |L|$ .

**Proposition 1.1:** There exists an optimal solution  $z^*$  to (KPSOSX) such that  $\sum_{k \in S_1} z^*_k = 1$  or an optimal solution is given by  $z^*_{j_1} = g_0/g_{j_1}$ ,  $z^*_j = 0$  for all  $j \neq j_1$ .

**Proof:** Suppose that  $\sum_{k \in S_1} z_k < 1$  holds in every optimal solution to (KPSOSX). Let  $z$  be any optimal solution and  $z_j > 0$  for some  $j \neq j_1$ . Define  $z^*$  by  $z^*_{j_1} = z_{j_1} + \epsilon$ ,  $z^*_j = z_j - \epsilon g_{j_1}/g_j$ ,

$z^*_k = z_k$  for all  $k \neq j_1$ ,  $j$  where

$$(1.7) \quad \epsilon = \min \left\{ 1 - \sum_{k \in S_1} z_k, z_j g_j / g_{j_1} \right\}.$$

Clearly  $\epsilon > 0$  and  $z^*$  is a feasible solution to (KPSOSX). We then get

$$(1.8) \quad \sum_{\text{all } k} d_k z^*_k - \sum_{\text{all } k} d_k z_k = d_{j_1} \epsilon - d_j \epsilon g_{j_1} / g_j = \left( \frac{d_{j_1}}{g_{j_1}} - \frac{d_j}{g_j} \right) g_{j_1} \epsilon \geq 0$$

and thus  $z^*$  is optimal as well. If  $\epsilon = 1 - \sum_{k \in S_1} z_k$  holds, then  $\sum_{k \in S_1} z_k = 1$ . Else we can iterate unless  $z^*_{j_1}$  is the only positive variable and  $z^*_{j_1} = g_0 / g_{j_1}$  holds since  $\sum_{k \in S_1} z_k < 1$  holds in every optimal solution. The proposition follows.

**Proposition 1.2:** If  $g_{j_1} \geq g_0$ , then  $z_{j_1} = g_0 / g_{j_1}$ ,  $z_k = 0$  for all  $k \neq j_1$  is an optimal solution to (KPSOSX).

**Proof:** If  $g_{j_1} \geq g_0$  holds, the solution of proposition 1.2 is basic and a feasible basis is obtained by making all slack variables of the special ordered sets basic. Denoting  $s_0$  the slack of the knapsack constraint we compute

$$\sum_{i \in L} \sum_{j \in S_1} d_j z_j = d_{j_1} \frac{g_0}{g_{j_1}} + \sum_{j \neq j_1} \left( d_j - g_j \frac{d_{j_1}}{g_{j_1}} \right) z_j - \frac{d_{j_1}}{g_{j_1}} s_0$$

and thus the reduced cost of all nonbasic variables are nonnegative, establishing optimality of the proposed solution. The proposition follows.

Propositions 1.1 and 1.2 define an algorithm for the linear knapsack problem with special ordered sets which is a natural generalization of Dantzig's method for the linear knapsack problem with variables having upper bounds. Let  $j_1, j_2, \dots, j_k$  be defined as in (1.6). If  $g_{j_1} \geq g_0$  holds, we are done by proposition 1.2. Else  $g_{j_1} < g_0$  holds, thus we know by proposition 1.1 that



$$(1.9) \quad \sum_{j \in S_1} z_j = 1$$

holds at the optimum. If  $|S_1| = 1$  holds, we fix the corresponding variable, record the change of  $g_0$  and iterate. Else we eliminate from  $S_1$  the variable  $j_*$  defined by

$$(1.10) \quad d_{j_*} = \max \{d_j \mid g_j = g^*\} \text{ where } g^* = \min \{g_j \mid j \in S_1\}.$$

We replace  $S_1$  by the set

$$(1.11) \quad S_1^* = \{j \in S_1 \mid d_j > d_{j_*}\},$$

replace  $d_j$  by  $d_j - d_{j_*}$ ,  $g_j$  by  $g_j - g_{j_*}$  for all  $j \in S_1^*$  and replace  $g_0$  by  $g_0 - g_{j_*}$ . If  $S_1^* = \phi$  holds, we fix  $z_{j_*}$  at value 1 and iterate. If  $S_1^* \neq \phi$  holds, we record the eliminated variable and compute

$$(1.2) \quad \frac{d_{j_+}}{g_{j_+}} = \max \left\{ \frac{d_j}{g_j} \mid j \in S_1^* \right\}.$$

We merge the special ordered set  $S_1^*$  into its proper place according to the value of this ratio among the remaining (decreasingly ordered) ratios to be worked at and iterate.

In the following iterations it may happen that the eliminated variable is set to zero if  $S_1^*$  is used again for elimination or that the eliminated variable is set to equal to  $1 - g_0/g_{j_i}$  when the special ordered set corresponding to the current index  $i$  (i.e. in the redefined ordering) at which the algorithm stops corresponds to the original set  $S_1$ . In any case, it may be that a previously eliminated variable may have to be reset at such fractional value if the current index  $i$  corresponds to some other special ordered set which was used previously for elimination and the algorithm stops with at most two variables at fractional values.

## 2. PROPERTIES OF FACETS FOR THE ZERO-ONE POLYTOPE

We consider again the general knapsack problem with special ordered sets and require that all variables be zero-one variables. As we are interested in the convex hull of zero-one solution we dispense with the objective function and consider the constraint set

$$(C1) \quad \sum_{i \in I} \sum_{j \in K_i} a_j x_j \leq a_0$$

$$\sum_{j \in K_i} x_j \leq 1 \quad \text{for all } i \in I$$

$$x_j = 0 \text{ or } 1 \quad \text{for all } j \in K_i, i \in I.$$

Using the variable substitution (1.3) we bring (C1) into the equivalent form

$$\sum_{i \in I} \sum_{j \in K_i} \alpha_j z_j \leq \alpha_0$$

(C2)

$$\sum_{j \in K_i} z_j \leq 1 \quad \text{for all } i \in I$$

$$z_j = 0 \text{ or } 1 \quad \text{for all } j \in K_i, i \in I.$$

where the  $\alpha_j$  are non-negative and correspond to the  $a'_j$  of (1.2).

Proposition 2.1: Let  $\pi z \leq \pi_0$  be a facet (a valid inequality) of the convex hull of (C2), then

$$(2.1) \quad \sum_{i \in H} (-\pi_{j_i}) x_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\pi_j - \pi_{j_i}) x_j + \sum_{j \in K} \pi_j x_j \leq \pi_0 - \sum_{i \in H} \pi_{j_i}$$

is a facet of (a valid inequality) of the convex hull of (C1). Conversely, let  $\mu x \leq \mu_0$  be a facet (a valid inequality) of the convex hull of (C1), then

$$(2.2) \quad \sum_{i \in H} (-\mu_{j_i}) z_{j_i} + \sum_{i \in H} \sum_{j \in K_i - \{j_i\}} (\mu_j - \mu_{j_i}) z_j + \sum_{j \in K} \mu_j z_j \leq \mu_0 - \sum_{i \in H} \mu_{j_i}$$

is a facet (a valid inequality) of the convex hull of (C2), where the notation of Section 1 is

used.

**Proof:** The variable substitution (1.3) can be written as  $x = f + Az$  where  $f$  is a zero-one vector and  $A$  is a nonsingular matrix. On the other hand one verifies that  $z = f + Ax$  holds as well. Validity of the respective inequalities follows from the fact that the variable substitution provides a one-to-one mapping of feasible solutions. The facet defining property of the respective inequalities follows from the non-singularity of  $A$  and thus the proposition follows.

We assume now without restriction of generality that  $\alpha_j \leq \alpha_0$  holds for all  $j$ . It follows that the inequalities

$$(2.3) \quad \begin{aligned} \sum_{j \in K_i} z_j &\leq 1 && \text{for all } i \in I \\ -z_j &\leq 0 && \text{for all } j \in K_i, i \in I \end{aligned}$$

define facets of the convex hull of (C2). We call those facets the trivial facets. Due the non-negativity of the constraint system (C2) we know e.g. from [3] that every facet  $\pi z \leq \pi_0$  of (C2) which is not a non-negativity condition satisfies  $\pi \geq 0$  and  $\pi_0 > 0$ .

**Proposition 2.2:** If  $\alpha_j = 0$  for some  $j \in K_i, i \in I$ , then  $\pi_j = 0$  in every nontrivial facet  $\pi z \leq \pi_0$  of the convex hull of (C2).

**Proof:** Suppose the proposition is false and let  $\pi z \leq \pi_0$  be a facet which has  $\pi_j > 0$  for some  $j \in K_i, i \in I$ , with  $\alpha_j = 0$ . Since  $\pi z \leq \pi_0$  is nontrivial there exists a feasible solution  $z^*$  to (C2) such that  $\pi z^* = \pi_0$  and  $\sum_{j \in K_i} z_j^* = 0$ , since otherwise we have  $\pi z = \sum_{j \in K_i} z_j$  and  $\pi_0 = 1$ . But then  $z^{**}$  defined by  $z^{**}_k = z^*_k$  for all  $k \neq j$  and  $z^{**}_j = 1$  is a feasible solution to (C2) satisfying  $\pi z^{**} = \pi_0 + \pi_j > \pi_0$ , contradiction.

Consequently, we can purge all zero  $\alpha_j$  in (C2) and assume that  $\alpha_0 \geq \alpha_j > 0$  holds for all  $j \in K_i$  and all  $i \in I$ .

Assume next that the  $K_i$  are indexed such that

$$K_i = \{j_i, j_i + 1, \dots, j_i + t_i\}$$

(2.4)

$$\alpha_{j_i} \leq \alpha_{j_i+1} \leq \dots \leq \alpha_{j_i+t_i}$$

where  $t_i \geq 0$  holds.

Proposition 2.3: If  $\pi z \leq \pi_0$  is a nontrivial facet of the convex hull of (C2) and for some  $i \in I$  and  $j \in K_i$  we have  $\pi_j > 0$ , then

$$(2.5) \quad \pi_k \geq \pi_j \quad \text{for all } k \in K_i, k > j$$

holds .

**Proof:** Suppose the proposition is false. Since  $\pi z \leq \pi_0$  is nontrivial, there exists a feasible solution  $z^*$  to (C2) with  $\pi z^* = \pi_0$  and  $z^*_k = 1$  since otherwise  $\pi z = -z_k$  and  $\pi_0 = 0$  holds. Since  $\alpha_j \leq \alpha_k$  it follows that  $z^{**}$  defined by  $z^{**}_\ell = z^*_\ell$  for all  $\ell \neq k, j$  and  $z^{**}_k = 0$ ,  $z^{**}_j = 1$  is feasible for (C2) and that  $\pi z^{**} = \pi_0 + \pi_j - \pi_k > \pi_0$  holds, a contradiction.

To state the next proposition we need the following auxiliary problem: For  $k \in I$  and  $\ell \in K_k$  let

$$\zeta_\ell = \max_{\substack{i \in I \\ i \neq k}} \sum_{j \in K_i}^{\alpha_j} \alpha_j z_j$$

subject to

$$\sum_{\substack{i \in I \\ i \neq k}} \sum_{j \in K_i} \alpha_j z_j \leq \alpha_0 - \alpha_\ell$$

(AUXKT)

$$\sum_{j \in K_i} z_j \leq 1 \quad \text{for all } i \in I, i \neq k$$

$$z_j = 0 \text{ or } 1 \quad \text{for all } j$$

**Proposition 2.4:** If  $\zeta_j = \zeta_\ell$  for  $\ell \neq j$ ,  $\ell \in K_k$ ,  $j \in K_k$  and some  $k \in I$ , then  $\pi_j = \pi_\ell$  for every nontrivial facet  $\pi z \leq \pi_0$  of the convex hull of (C2).

Proof: Suppose the proposition is false and let  $\pi z \leq \pi_0$  be a facet with  $\pi_j \neq \pi_\ell$ . Without restriction of generality let  $\pi_j < \pi_\ell$ . Since  $\pi z \leq \pi_0$  is a nontrivial facet of the convex hull of (C2) there exists a feasible zero-one vector  $z$  with  $z_j = 1$  and  $\pi z = z_0$ . Define  $z^*$  by  $z^*_\ell = 1$ ,  $z^*_j = 0$ ,  $z^*_h = z_h$  for all  $h \neq \ell, j$ . Then

$$\sum_{i \in K} \sum_{h \in K_i} \alpha_h z^*_h \leq \alpha_\ell + \zeta_j = \alpha_\ell + \zeta_\ell \leq \alpha_0$$

holds and thus  $z^*$  is feasible. But  $\pi z^* = \pi_0 + \pi_\ell - \pi_j > \pi_0$  holds, contradiction. Thus the proposition follows.

The assumption of proposition 2.4 is satisfied e.g. if  $\alpha_j = \alpha_\ell$  holds. It follows that we can purge all but one of the variables in any special ordered set which have identical coefficients when we are interested in finding a minimal linear constraint set for the convex hull of (C2). We initially conjectured a stronger property, namely that all facets of the convex hull of (C2) could be obtained by "lifting" facets from associated knapsack inequalities with all special ordered sets consisting of singletons. While the conjecture is true for facets derived from minimal covers, see [3], [4], and from (1,k) configurations, see [5], the following example shows that the conjecture is false in general.

**Example 2.5:** Consider the inequalities

$$5x_1 + 3x_2 + 3x_3 + x_4 + 2x_5 \leq 7$$

$$x_1 + x_2 \leq 1$$

$$0 \leq x_i \leq 1 \text{ and integer, } i = 1, \dots, 5.$$

The inequality  $2x_1 + x_2 + x_3 + x_4 + x_5 \leq 3$  defines a facet of the corresponding convex hull none of whose projections into a lower-dimensional space, however, defines a facet for the associated lower-dimensional zero-one problem.

The algorithm for the linear knapsack problem with special ordered sets as well as the properties of facets of the convex hull of (C2) are currently used in a cutting-plane-based approach to large-scale zero-one programming problems [1].

## REFERENCES:

- [1] Crowder, H.P., E.L. Johnson and M.W. Padberg: "Solving large-scale zero-one problems to optimality", IBM Research Report, forthcoming.
- [2] Dantzig, G.B.: Linear Programming and Extensions, Princeton University Press, Princeton, N.J. 1963, pp. 517-520.
- [3] Hammer, P.L., E.L. Johnson, and U. Peled: "Facets of regular 0-1 polytopes", Mathematical Programming, 8(1975), 179-206.
- [4] Padberg, M.W.: "A Note on zero-one programming", Operations Research, 23(1975), 833-837.
- [5] Padberg, M.W.: "(1,k)-Configurations and facets for packing problems", Mathematical Programming, 18(1980), 94-99.
- [6] Zemel, E.: "The Linear multiple-choice knapsack problem", to appear in Operations Research.

