Perfect zero-one matrices

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ABSTRACT A zero-one matrix is called perfect if the polytope of the associated set packing problem has integral vertices only. By this definition, all totally unimodular zero-one matrices are perfect. In this paper we give a characterization of perfect zero-one matrices in terms of forbidden submatrices. The notion of a perfect zero-one matrix is closely related to that one of a perfect graph as well as that one of a "balanced" matrix as introduced by BERGE. Furthermore, the results obtained here bear on an unsolved problem in graph theory, the strong perfect graph conjecture due to C. BERGE.

RESUME Nous disons que la matrice A aux éléments zéro-un est parfaite si le polytope $\{x \in \mathbb{R}^n \mid Ax \leq e, x \geq 0\}$ n'a que des sommets entiers, où le vecteur e a toutes ses composantes égales à 1. Toutes les matrices totalement unimodulaires sont donc parfaites. Dans cet article nous donnons une caractérisation des matrices parfaites en termes de sous-matrices interdites. Le concept de matrice parfaite est relié au concept de graphe parfait et aussi à celui de BERGE d'une matrice équilibrée. En effet, les matrices parfaites sont fournies par les matrices d'incidence d'un hypergraphe normal. Les résultats obtenus ici sont reliés à la conjecture forte de BERGE concernant les graphes parfaits.

1. INTRODUCTION

In this paper, we consider the polytope defined by the constraints of the following set packing problem :

$$
(P) \qquad \begin{aligned} & \max\ c\,x \\ & A\,x \le e \\ & x_j = 0 \text{ or } 1 \quad \forall\, j \in N = \{1, \ldots, n\} \end{aligned}
$$

where A is a mxn matric of zeroes and ones having no zero columns, $e^T = (1, \ldots, 1)$ is the vector having all m components equal to one, and c is an arbiratry vector of reals. This problem has recently obtained much attention, see e.g. [1], [2], [6], [14], [18]. By (LP) we denote the linear programming problem obtained from (P) by dropping the integrality requirement on x.

If the matrix A involved in problem (P) is totally unimodular [10], then all basic feasible solutions to (LP) are integral, i.e. for <u>any</u> vector c the integer programming problem (P) can be solved as an <u>ordinary</u> linear programming problem. Generally, the matrix A encountered in (P) is not totally unimodular. Nevertheless, for certain matrices A the property that all basic feasible solutions to (LP) are integral, remains true (see Section 2 for relevant examples , also [5]). We call such matrices <u>perfect</u> zero-one matrices. Using some results from graph theory, we give a complete characterization of perfect matrices A in terms of <u>forbidden submatrices</u>. We give examples that show that -like in the case of balanced matrices, see [5]- it is not possible to characterize perfect zero-one matrices by means of forbidden determinantal values (as is the case for totally unimodular matrices). Indeed, given any natural number k, there exists a perfect zero-one matrix A such that has a minor with determinant k.

This paper in an abbreviated version of the original paper [15] ; see also [16] where the main result of this paper is derived using a different proof techniques.

2. PERFECT ZERO-ONE MATRICES

Let A be any mxn matrix of zeroes and ones having no zero column, and define the polytopes P and P_I as follows :

(2.1) $$P = \{x \in R^n \mid Ax \le e, x_j \ge 0, j = 1, \dots, n\}$$

$$P_I = \text{conv} \{x \in R^n \mid Ax \le e, x_j = 0 \text{ or } 1, j = 1, \dots, n\}$$

where $e^T = (1, \dots, 1)$ has m components all equal to one. The matrix A is called perfect if $P = P_I$, i.e. if the polytope P defined in (2.1) has only integral vertices. Note that $\dim P = \dim P_I = n$ holds. Denote G the (intersection) graph associated with the matrix A, i.e. the nodes of G correspond to the columns of A and two nodes of G are linked by an edge if the associated columns a^i and a^j of A have at least one + 1 entry in common, se e.g. [14]. A clique in G is a maximal complete subgraph of G. Let C denote the node set of any clique in G. Then by Theorem 2.1. of [14], the inequality

(2.2) $$\sum_{j=1}^{n} a_j^c x_j \le 1, \quad a_j^c = \begin{cases} 1 & \text{if } j \in C \\ 0 & \text{if not} \end{cases}$$

yields a facet of P_I, i.e. a face of dimension n-1 of P_I. Clearly, every facet of the polytope P_I is essential in defining P_I. Hence it is a necessary condition for A to be perfect that A contain the incidence (row-) vectors of all cliques of the associated graph G. In order to characterize perfect matrices we can thus restrict ourselves to considering "clique"-matrices, i.e. matrices A which contain the incidence vectors of all cliques of the associated graph G.

Let A be any clique-matric of size mxn and let G be the associated graph. Let $\bar{G}$ denote the complement of G and denote B a clique matrix of $\bar{G}$. Similarly to (2.1) define the polytopes Q and Q_I, respectively as follows :

(2.3) $$Q = \{x \in R^n \mid Bx \le \hat{e}, x_j \ge 0, j = 1, \dots, n\}$$

$$Q_I = \text{conv} \{x \in R^n \mid Bx \le \hat{e}, x_j = 0 \text{ or } 1, j = 1, \dots, n\}$$

where $\hat{e}^T = (1, \ldots, 1)$ has all components equal to one and is dimensioned compatibly with B. Note that B has no zero column and that $\dim Q = \dim Q_I = n$ holds. The vectices of Q_I correspond to <u>complete</u> subgraphs of G and vice versa. Furthermore, every <u>maximal</u> independent node set in G defines a clique of $\bar{G}$ (and vice versa). Consequently, there exists a (incomplete) "duality" relation between the vertices of P_I and the facets of Q (and hence, between the vertices of Q_I and the facets of P), see e.g. [11]. In the terminology of Fulkerson [7], Q(P, resp.) is the anti-blocker of $P_I(Q_I$, resp.).

Let A be any clique-matrix of size mxn. By Theorem 1 of [6], A is perfect if and only if the associated graph G is perfect. Consequently, A is imperfect if and only if the graph G contains an induced subgraph of G' which is <u>almost perfect</u>, i.e. G' is imperfect, but every proper induced subgraph of G' is perfect, see [13], [15], [17]. Since induced subgraphs of G having k nodes correspond uniquely to mxk submatrices of A (and vice versa) we can make, without loss of generality, the assumption that G is an almost perfect graph. Denote $\alpha(G)$ the maximum cardinality of a stable (independent) node set in G and define $\alpha(\bar{G})$ likewise for $\bar{G}$.

<u>Lemma 1</u> : <u>Let A be a clique-matrix of an almost perfect graph G, $\alpha = \alpha(G)$ and $\bar{\alpha} = \alpha(\bar{G})$. Then $\sum_{j=1}^{n} x_j \le \bar{\alpha}$ provides a facet of Q_I and $x = (1/\bar{\alpha})e$ is a fractional vertex of P</u>.

<u>Proof</u> : Denote by e^j the row vector with n-1 components equal to +1 and having a zero in the j-th component. Since G is almost perfect it follows Theorem 1 of [13] that :

$$\max_{x \in P} e^j x = \alpha \text{ for } j = 1, \ldots, n.$$

Define H_j to be the halfspace given by

$$H_j = \{x \in R^n \mid e^j x \le \alpha\} \text{ for } j = 1, \ldots, n .$$

Consequently $P \subseteq H_j$ for $j = 1, \ldots, n$ or equivalently,

$$P \subseteq H = \bigcap_{j=1}^{n} H_j$$

By definition of e^j, $j = 1, \ldots, n$, H is a pointed polyhedral convex cone with its apex at $\bar{x} = \frac{\alpha}{n-1} e$. By Theorem 1 of [13], we have that $\alpha(G)\,\alpha(\bar{G}) = n - 1$ holds, since G is almost perfect. Hence it follows that :

$$A\bar{x} \leq \frac{\bar{\alpha}\,\alpha}{n-1} e = e.$$

Consequently, since H is pointed, $\bar{x} = \frac{\alpha}{n-1} e = \frac{1}{\bar{\alpha}} e$ is a vertex of P, satisfying $0 < \bar{x} < e$. Consequently, the inequality $\sum_{j=1}^{n} x_j \leq \bar{\alpha}$ provides a facet of Q_I. Q.E.D.

<u>Remark 1</u> : By Theorem 1 of [12] it follows that G is almost perfect if and only if $\bar{G}$ is almost perfect. Consequently, we have that $\bar{\bar{x}} = \frac{1}{\alpha} e$ is an fractional vertex of Q and $\sum_{j=1}^{n} x_j \leq \alpha$ provides a facet for P_I.

<u>Remark 2</u> : Since $\bar{x} = \frac{1}{\bar{\bar{\alpha}}} e$ is a vertex of P, we can reorder the rows of the matrix A as follows :

$$A = \begin{pmatrix} \tilde{A}_1 \\ \tilde{A}_2 \end{pmatrix}$$

where the row sums of $\tilde{A}_1$ all equal $\bar{\alpha}$ and the row sums of $\tilde{A}_2$ are all (strictly) less than $\bar{\alpha}$. Furthermore, $\tilde{A}_1$ is of size $\hat{m}$xn with $m \geq \hat{m} \geq n$ and $\tilde{A}_1$ contains a (at least one) nonsingular submatrix of size nxn. By Remark 1, we have a similar partitioning of B into $\tilde{B}_1$ and $\tilde{B}_2$, where $\tilde{B}_1$ has $\tilde{m}$ rows, $\tilde{m} > n$, having row sums equal to α and $\tilde{B}_1$ contains a nxn nonsingular submatrix.

<u>Remark 3</u> : In graphical notation, Remark 2 implies that every almost perfect graph G contains at least $|G|$ maximum cliques of the cardinality $\bar{\alpha} = \alpha(\bar{G})$. This has been observed earlier by Sachs and can be found, though without proof, in [17].

For $j = 1, \ldots, n$ define :

(2.4) $$P_j = P \cap \{x \in \mathbb{R}^n \mid x_j = 0\}$$

and let Q_j be defined analogously with P replaced by Q. Intersecting P with $x_j = 0$ corresponds to the operation of deleting from the graph G node j and all edges incident to it. Hence we have

Remark 4 : If A is a clique-matrix of an almost perfect graph G, then $P_j \subseteq P_I$ ($Q_j \subseteq Q_I$, resp.) holds for $j = 1, \ldots, n$.

Lemma 2 : Let A be a mxn clique-matrix of an almost perfect graph G, then A contains a nxn nonsingular submatrix A_1 whose column and row sums are all equal to $\bar{\alpha} = \alpha(\bar{G})$. Furthermore, any row of A which is not in A_1 is either componentwise identical to some row of A_1, or has a row sum strictly less that $\bar{\alpha}$.

Proof : By Remark 1, the point $\bar{\bar{x}} = (1/\alpha)e$ is a fractional vertex of Q. The point $x^j = (1/\alpha)e^j$ satisfies $x^j \in Q_j$ for $j = 1, \ldots, n$ where e^j is the vector defined in the proof of Lemma 1. Since $Q_j \subseteq Q_I$ holds for $j = 1, \ldots, n$ it follows by convexity that $\tilde{x} \in Q_I$ where

$$\tilde{x} = (1/n) \sum_{j=1}^{n} x^j = (\bar{\alpha}/n)e \tag{2.5}$$

Since $\sum_{j=1}^{n} \tilde{x}_j = \bar{\alpha}$ holds it follows from Carathéodory's theorem [11] that there exists a nxn nonsingular submatrix A_1 of the matrix $\tilde{A}_1$ defined in Remark 2 such that :

$$\tilde{x}^T = \gamma^T A_1, \ \gamma_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} \gamma_i = 1 \tag{2.6}$$

holds. Let B_1 be any nxn nonsingular submatrix of the matrix $\tilde{B}_1$ defined in Remark 2. Then we obtain from (2.6) :

$$\gamma^T A_1 B_1^T = \tilde{x}^T B_1^T = \frac{\bar{\alpha}}{n} e^T B_1^T = \frac{n-1}{n} e^T$$

Define $D = A_1 B_1^T$. Then D is a nxn nonsingular matrix of zeroes and ones since the columns of B_1^T are (a subset of) vertices of P_I (satisfying $e^T x \leq \alpha$ with equality). Furthermore, D cannot contain a row consisting of +1 entries only. For, if there were such a row, then necessarily B_1 would have to be singular since the row sums of B_1 all equal α. Then we have that :

$$(2.7) \qquad \gamma^T D = \frac{n-1}{n} e^T$$

where $\gamma_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \gamma_i = 1$. From (2.7) we obtain :

$$\gamma^T De = n - 1$$

Consequently, $\gamma_i > 0$ implies that the row sum of row i of D equals n-1, since $\sum_{i=1}^{n} \gamma_i = 1$. Suppose now that $\gamma_1 \leq \ldots \leq \gamma_k$ with $k < n$ satisfies $\gamma_1 > 0$, $\gamma_{k+1} = \ldots = \gamma_n = 0$. Since D is nonsingular we can rearrange the rows and columns of D such that D has the form

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

with

$$D_1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \qquad D_2 = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

where D_1 is of size kxk and has zeroes only in the main diagonal, D_2 is of size (n-k) x (n-k) and consists entirely of ones. If $k < n$, obviously, (2.7) cannot have a solution satisfying $\gamma_i \geq 0$ and $\sum_{i=1}^{n} \gamma_i = 1$. On the other hand $\sum_{i=1}^{n} \gamma_i = 1$ implies $k \geq 1$. Consequently, $k = n$ and D has the general form E - R, where E is the nxn matrix consisting entirely of ones and R is a nxn permutation matrix. Consequently, from (2.7) we have that

$$\gamma^T = \frac{n-1}{n} e^T D^{-1} = \frac{1}{n} e^T.$$

To complete the proof of Lemma 2, we note that the relation $A_1 B_1^T = E - R$ implies that :

$$A_1^{-1} = \frac{\alpha}{n-1} E - B_1^T R^T.$$

Suppose now that the matrix A contains a row a^T satisfying $a^T e = \bar{\alpha}$ which is not contained in the submatrix A_1. Then we have :

$$(2.8) \qquad 0 \le a^T A_1^{-1} = e^T - a^T B_1^T R^T = Z \le e^T.$$

From $Z e = 1$ and the integrality of Z it follows that such row a^T is componentwise identical to some row in A_1 and thus Lemma 2 follows. Q.E.D.

Remark 5 : In graphical notation, Lemma 2 implies that every almost perfect graph G has exactly $|G|$ maximum cliques of the cardinality $\bar{\alpha} = \alpha(\bar{G})$ and $|G|$ maximal independent node sets of the cardinality $\alpha = \alpha(G)$.

Lemma 2 suggests the following definition :

Definition : Let A be a zero-one matrix of size mxn, $m \geq n$. A is said to have property $\pi_{\beta,n}$ if the following conditions are met :

(i) A contains a nxn nonsingular submatrix A_1 whose row and column sums are all equal to β.

(ii) Each row of A which is not a row of A_1 either is componentwise equal to some row of A_1 or has a row sum strictly less than β.

Remark 6 : Let A be a mxn matrix of zeroes and ones and G its associated intersection graph. A is a clique-matrix, i.e. A contains as rows vectors the incidence vectors of all cliques in G, if and only if A does not contain any mxk submatrix A' having the property $\pi_{\beta,k}$ with $\beta = k-1$ and $\beta \geq 2$.

We will sketch the proof of Remark 6 only, since the assertion of Remark 6 is probably a known result. The necessity of the condition is obvious. To prove sufficiency, let C be the node set of a clique in G such that the associated incidence vector a^T with $a_j = 1$ if $j \in C$, $a_j = 0$ otherwise, is not contained in A. Define $P' = P \cap \{x \in R^n \mid x_j = 0, \forall j \in N - C\}$ and $P'_j = P' \cap \{x \in R^n | x_j = 0\}$ for all $j \in C$. Obviously, P' must have at least one fractional vertex. We now distinguish two cases : (i) P'_j has integral vertices only for all $j \in C$ or (ii) there exists a $j \in C$ such that P'_j has a fractional vertex. In case (i), we can show by an argument completely analogous to the one used in the proof of Lemma 1 that A contains a mxk submatrix A' having property $\pi_{\beta,k}$ with $\beta = k - 1$, $\beta \geq 2$ and $k = |C|$. In case (ii), we can restrict attention to any P'_j having a fractional vertex for $j \in C$. Abusing slightly the notation, we redefine C to be $C-\{j\}$, redefine the polytopes P', P'_j with respect to the new set C and find again the two cases mentioned above. (Note that the new set C defines a complete subgraph in G, which is no longer a clique in G. This, however, does not affect the argument). Clearly, case (ii) can happen only finitely many times and finally, we obtain a set C having <u>at least three</u> elements and which is such that case (i) prevails. This completes the out-line of the proof of Remark 6.

The following theorem states a necessary and sufficient condition for an arbitrary zero-one matrix to be perfect :

<u>Theorem 1</u> : <u>Let A be any zero-one matrix of size mxn. The following two conditions are equivalent</u> :

(i) <u>A is perfect.</u>

(ii) <u>For $\beta \geq 2$ and $3 \leq k \leq n$, A does not contain any mxk submatrix A' having the property</u> $\pi_{\beta,k}$.

<u>Proof</u> : Suppose that A is perfect and that (ii) is violated. Then there exists a mxk submatrix A' of A having property $\pi_{\beta,k}$ for som $\beta \geq 2$ and $k \geq 3$. Suppose the columns of A have been ordered such that A' coincides with the k first columns. Then $\bar{x}$ defined by $\bar{x}_j = \frac{1}{\beta}$ for $j = 1, \ldots, k$, $\bar{x}_j = 0$ for $j = k+1, \ldots, n$, is a <u>fractional</u> vertex of the polytope P defined in (2.1). Hence, by definition, A cannot be perfect.

On the other hand, suppose that A is such that (ii) holds. Then A must be perfect. For if not, then by Remark 6 and Theorem 1 of [6] the intersection graph G associated with A must be imperfect.. Consequently G contains an induced subgraph G' that is almost perfect. Again by Remark 6, the clique-matrix of G' is a mxk submatrix A' of A where $k = |G'|$. Let A" denote the m'xk submatrix of A' whose rows correspond to the cliques of G'. By Lemma 2, A" has the property

$\pi_{\beta,k}$ with $\beta = \alpha(\bar{G}') \geq 2$. Since the m-m' truncated rows of A not contained in A" are dominated by some row in A", the mxk submatrix A' of A must also have property $\pi_{\beta,k}$ with $\beta = \alpha(\bar{G}')$. Thus (ii) cannot be satisfied by A. Q.E.D.

Corollary 1 : Let A be a zero-one matrix of size mxn. A is almost perfect, i.e. A is the clique-matrix of an almost perfect graph G, if and only if the following two conditions are met :

(i) A has the property $\pi_{\beta,k}$ for some β satisfying $2 \leq \beta \leq [\frac{n}{2}]$ and $k = n$.

(ii) A does not contain any mxk submatrix having property $\pi_{\beta,k}$ for $\beta \geq 2$ and $3 \leq k \leq n-1$.

Remark 7. The strong perfect graph conjecture [2], [3], if true, now is reduced to proving that the only zero-one matrix A of size mxn satisfying the conditions (i) and (ii) of Corollary 1 and $2 \leq \beta < [\frac{n}{2}]$ is the circulant of odd size having exactly two positive entries in every row and column, i.e. A is the clique-matrix of an odd cycle without chords. A characterization of almost perfect matrices in graphical terms appears advantageous if one wants to check the perfection of a zero-one matrix. For, similarly to the criterion for the total unimodularity of a matrix A, a direct check of the perfection of a zero-one matrix via the necessary and sufficient criterion of Theorem 1 is -computationally- an impossible task, whereas graphical criteria -at least in the context of total unimodularity- are relatively easily verified or known to be satisfied by the physical conditions of a problem under consideration.

Example 1 : Since a zero-one matrix A is perfect if it is the clique-matrix of a perfect graph (and vice versa), the clique-matrices of perfect graphs furnish examples of zero-one matrices satisfying the condition (ii) of Theorem 1. Among the graphs known to be perfect are the rigid circuit graphs [7] (or "triangulated" graphs [2]), the comparability graphs and the "i-triangulated" graphs, see e.g. [2]. The example of rigid circuit graphs provides examples of zero-one matrices which are perfect, but not totally unimodular. (I am indebted to D.R. Fulkerson for this example).

Example 2 : Consider the graph G of Figure 1 and its associated clique-matrix A in Figure 2. The submatrix A' made up of columns 1, 2, 3 and 4 and rows 1, 6, 11 and 16 has a determinant of 3. Consequently, A is not totally unimodular. Due to the simple structure of A, we find by inspection that A is perfect. (Checking Condition (ii) of Theorem 1) amounts to proving that G does not contain an odd cycle without chords). Furthermore, the matrix A provides us with an example where A^T, the transpose of a perfect matrix, is not perfect. This is remarkable since it is different from total unimodularity. The above example can be generalized to prove that given any natural number k there exists a perfect matrix A having a minor whose determinant in absolute values equals k. (Replace K_4 by K_{k+1} and the G_i, i = 1, ..., 4, by k+1 copies of G_1, say, each of which is connected to K_{k+1} in a similar fashion as done above). This indicates why a characterization of perfect matrices in terms of forbidden matrices is appropriate rather than a characterization in terms of forbidden subdeterminants (which is not possible here, but possible for totally unimodular matrices).

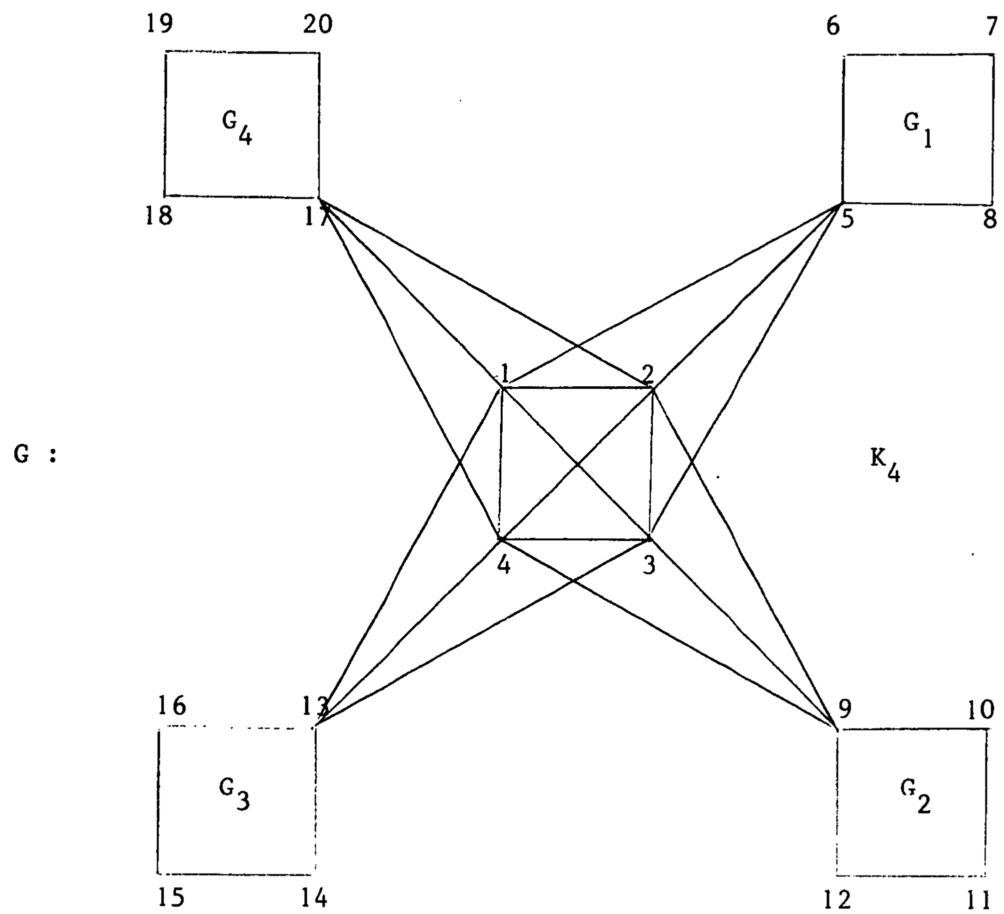

FIGURE 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0
A =	1	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 2

REFERENCES

[1] BALAS E. and PADBERG M.W.
"On the set covering problem"
Operations Reserach, Vol. 20 (1972), pp. 1152-1161.

[2] BALAS E. and PADBERG M.W.
"On the set covering problem II : an algorithm"
Operations Research, Vol 23, (1975), pp. 74-90.

[3] BERGE C.
"Graphes et hypergraphes"
Dunod, Paris 1970.

[4] BERGE C.
"Introduction à la théorie des hypergraphes"
Lectures Notes, Université de Montréal, Summer 1971.

[5] BERGE C.
"Balanced matrices"
Mathematical Programming, Vol. 2 (1972), pp. 19-31.

[6] CHVATAL V.
"On certain polytopes associated with graphs"
Journal of Combinatorial Theory (B), Vol. 18 (1975), pp. 138-154

[7] FULKERSON D.R.
"Blocking and antiblocking pairs of polyhedra"
Mathematical Programming, Vol. 1, (1971), pp. 168-194.

[8] FULKERSON D.R.
"On the perfect graph theorem"
In T.C. Hu and S.M. Robinson, eds., Mathematical Programming, Academic Press, New York, (1973), pp. 69-77.

[9] GANTMACHER
"Matrix theory"
Vol. 11, Chelsea Publishing Company, New York, N.Y. (1964) (translated by K.A. Hirsch).

[10] GARFINKEL R. and NEMHAUSER G.
"Integer programming"
John Wiley & Sons, (1972).

[11] GRÜNBAUM B.

"Convex polytopes"
John Wilez & Sons, (1966).

[12] LOVASZ L.

"Normal hypergraphs and the perfect graph conjecture"
Discrete Mathematics, Vol. 2, (1972), pp. 253-268.

[13] LOVASZ L.

"A characterization of perfect graphs"
Journal of Combinatorial Theory, (B), Bol. 13 (1972), pp. 95-98.

[14] PADBERG M.W.

"On the facial structure of set covering problem"
Mathematical Programming, Vol. 5 (1973), pp. 199-215.

[15] PADBERG M.W.

"Perfect zero-one matrices"
Mathematical Programming, Vol. 6 (1974), pp. 180-196.

[16] PADBERG M.W.

"Almost integral polyhedra related to certain combinatorial optimization problems"
Linear Algebra and its Applications, Vol. 15 (1976), pp. 69-88.

[17] SACHS H.

"On the Berge conjecture concerning perfect graphs"
Combinatorial Structures and their Applications, Gordon and Breach, N.Y. (1970), pp. 377-387.

[18] TROTTER L.

"Solution characteristics and algorithms for the vertex packing problem"
Technical Report n° 168, Ph.. D. Thesis, Cornell University, January 1973.