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Rapports de Recherche

N° 12

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Mars 1980

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CONTROL AND STABILIZATION FOR THE WAVE EQUATION IN AN EXPANDING DOMAIN

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Résumé : Nous utilisons des invariantes d'énergie pour étudier des estimations de croissance et de décroissance pour les solutions de l'équation des ondes dans un domaine en expansion. Nous donnons des conditions suffisantes pour obtenir une contrôlabilité exacte (avec paramètre distribué) de l'équation des ondes.

Abstract : We use energy invariants to study the growth and decay estimates for solutions of the wave equation in an expanding domain. Sufficient conditions are formulated which ensure the exact (distributed-parameter) controllability of the wave equation.

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§ 0. INTRODUCTION

In three of the earlier papers [1], [2], [3], we have studied controllability, stabilizability and observability theory for the wave equation in a bounded domain $\Omega \subseteq R^N$. Ω was a fixed domain with the passage of time. In practical situations, many processes evolve in domains whose boundary has moving parts. A simple model, e.g., is a heat process in a combustion chamber where a piston is attached. Part of the boundary moves with the motion of the piston. Partial differential equations in domains with moving boundary have been studied by [4], [5], [6], [7], etc... ; see also the references therein.

In this paper,we will be concerned with the controllability and stabilizability problems for the wave equation in a domain with an expanding boundary. As for as we known, the present paper is the first attempt to resolve the above questions. We prove that, under certain conditions, (1) the wave equation is satbilizable with the introduction of viscous damping and compensation and (2) the wave equation is exactly controllable with distributed parameter controllers.

We first note that the energy of the wave equation increases as the domain expands and decreases as the domain contracts. Therefore no backward stabilizability [10] is obtainable in an expanding domain as one reverses the sense of time. This causes some complication to the study of controllability when one tries to apply Russell's complete stabilizability method [10]. Here we devise a scheme with high compensation so that, when combined with the energy method of Morawetz, it can provide us with an upper bound for the energy growth during time reversal.

Basically, our assumptions are as follows. (A) the space dimension $N \neq 2$. (B) the domain rests still until $t = T_0$. (C) after T_0, the domain expands, but every point on the domain lies within the distance θt, $o<\theta < 1$, of the origin at $t \geq T_0$. (D) a geometrical condition on the boundary which in the stationnary case reduces to the star-shapedness condition. Condition (A) is very restrictive. Conditions (B) and (D) ensure that the energy of the wave equation will not change too drastically in the process of boundary deformation.

In § 1, we start with some basic notations and the statement of the controllability problem. The existence, uniqueness and continuity of solutions of wave equations is stated without proof.

In § 2, we derive the growth and decay estimates for the wave equation (without control) in an expanding domain. It is important in itself as well as preparatory for the material in §3.

In § 3, the distributed-parameter exact controllability theorem is given.

We give an example of an expanding sphere in § 4.

The authors acknowledge the influence of the work of Strauss [11] and Tartar [12] on this paper.

§ 1. NOTATIONS - STATEMENT OF THE PROBLEM

Points in space R^N are denoted as $x = (x_1, x_2, \ldots, x_N)$. Also

$\gamma^2 = |x|^2 = \Sigma\, x_i^2$, $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$, $\nabla_{x,t} = (\nabla, \partial/\partial t)$ and $\Delta = \Sigma\, \partial^2/\partial x_i^2$.

For $t \geq o$ we postulate bounded open sets $\Omega(t)$. Let

$$Q(t_1, t_2) \equiv \bigcup_{t=t_1}^{t_2} \Omega(t) \times \{t\}, \quad \Sigma(t_1, t_2) = \bigcup_{t=t_1}^{t_2} \partial\Omega(t) \times \{t\}$$

denote the space-time domain and the lateral surface from t_1 to t_2. In case $t_1 = o$, we simply denote $Q(o, t_2)$ and $\Sigma(o, t_2)$ as $Q(t_2)$ and $\Sigma(t_2)$, respectively. We assume that $\Sigma(t)$ is precewise smooth for all $t > o$. Let $\nu = (\nu_1, \ldots, \nu_N, \nu_t) = (\nu_x, \nu_t)$ be the unit outward normal at (x, t) on Σ. Throughout this paper, we assume

(H_o) $\quad |\nu_t| \leq |\nu_x|$ on $\Sigma(T)$, for any $T > o$

From 6 , we understand taht (H_o) holds if and only if each point on the boundary moves in the normal direction at a speed less than or equal to one.

If u is a smooth function satisfying $u = o$ on Σ. Then all the tangential derivatives of u also vanishing on Σ. So

$$\nabla_{x,t}\, u = \frac{\partial u}{\partial \nu}\, \nu$$

$$(1.1) \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \nu}\, \nu_x, \quad \frac{\partial u}{\partial r} = \frac{\partial u}{\partial \nu}\, \nu_r \quad (\text{here } \nu_r = \nu_x \cdot \frac{x}{r})$$

The above remains valid in the sense of distributions if u has a well-defined trace.

For each $t \geq o$, $H_o^1(\Omega(t)) \oplus H^o(\Omega(t))$ denotes the Hilbert space equipped with the norm

$$||(w,v)||^2 = \int_{\Omega(t)} [|\nabla w|^2 + v^2]dx \equiv E_1(w,v) = \text{the energy of the state } (w,v)$$

or an equivalent norm :

$$||(w,v)||_\gamma^2 = \int_{\Omega(t)} [|\nabla w|^2 + \gamma w^2 + v^2]dx, \quad \gamma \geq o$$

for any $(w,v) \in H_0^1(\Omega(t))\ \ H^0(\Omega(t))$.

We are now in a position to pose the
Exact Controllability Problem (ECP). Let

$$
(CS)\quad \begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} - \Delta w(x,t) = f(x,t), & (x,t) \in Q(T) \\[2ex] \begin{bmatrix} w(x,o) \\ \frac{\partial w}{\partial t}(x,o) \end{bmatrix} = \begin{bmatrix} w_0(x) \\ v_0(x) \end{bmatrix} \in H_0^1(\Omega(o)) \oplus H^0(\Omega(o)) \\[2ex] w|_{\Sigma(T)} = o \end{cases}
$$

be a given distributed parameter control system governed by the wave equation. For any initial state $(w_0, v_0) \in H_0^1(\Omega(o)\ \ H^0(\Omega(o))$ and any preassigned state $(w_T, v_T) \in H_0^1(\Omega(T)) \oplus H^0(\Omega(T))$, find an admissible control $f \in L^2(Q(T))$ such that the solution $w(x,t)$ satisfies the preassigned terminal state (w_T, v_T) at $t = T$.

The theorem of existence, uniqueness and continuity of solutions is given below.

Theorem 1.1 Consider the equation

$$
\begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} - \Delta w(x,t) = f(x,t) & (x,t) \in Q(T)\ ,\quad f \in L^2(Q(T)) \\[2ex] w(x,o) = w_0(x) \in H_0^1(\Omega(o)) \\ \frac{\partial w}{\partial t}(x,o) = v_0(x) \in H^0(\Omega(o)) \\[2ex] w|_{\Sigma(T)} = o \end{cases}
$$

in $Q(T)$. Under the assumption (H_0), the solution $w(x,t)$ exists and is unique such that

$$
(w, \tfrac{\partial w}{\partial t}) \in C^0([o,T]\ ;\ H_0^1\ (\Omega(t)) \oplus H^0(\Omega(t)))
$$

The proof can be done by a change of coordinates [4] and then use theorems in [8] Chapter 3 to prove continuity.

§ 2. GROUWTH AND DECAY ESTIMATES FOR THE WAVE EQUATION IN AN EXPANDING DOMAIN

In the sequel, we will need the following assumptions from time to time.

(H_1) The space is not two dimensional, i.e., $N \neq 2$.
(H_2) The domain $\Omega(o)$ rests still until $t = T_o$ for some $T_o > o$, i.e.,

$$\Omega(t) \equiv \Omega(o) \text{ for } o \leq t \leq T_o.$$

(H_3) For $t \geq T_o$, the domain expands, i.e., $\Omega(t_1) \subseteq \Omega(t_2)$ for $T_o \leq t_1 \leq t_2$ and for any $x \in \Omega(t)$,

$$\gamma = |x| \leq \theta t \quad , \quad o < \theta < 1 \qquad \text{for some } \theta$$

Our first theorem, which is independent of the preceding hypothesis, indicates that the energy of the wave grows and decays as the boundary shrinks and expands.

Theorem 2.1. Let

$$(WE)\quad \begin{cases} \dfrac{\partial^2 w}{\partial t^2} - \Delta w = o & \text{in } Q(T) \\ w(x,o) = w_o(x) \in H_o^1(\Omega(o)) \\ \dfrac{\partial w}{\partial t}(x,o) = v_o(x) \in H^o(\Omega(o)) \\ w|_{\Sigma(T)} = o \end{cases}$$

be a wave equation in $Q(T)$. Then the energy $E_1(t)$ is non-increasing if $\Omega(t)$ is expanding, and $E_1(t)$ is non-decreasing if $\Omega(t)$ is contracting.

Proof : Let the motion of a point on $\partial\Omega(t)$ be given by $x = x(t)$. Then $(dx/dt,1)$ is tangent to $\Sigma(T)$ at $(x(t),t)$. Thus

$$(2.1)\qquad \nu_x \cdot \frac{dx}{dt} + \nu_t = o$$

As the boundary $\partial\Omega(t)$ of $\Omega(t)$ is expanding outward, the component of dx/dt in the direction of ν_x must be non-negative, thus $\nu_x\, dx/dt$ is non-negative.

Hence ν_t is non-positive.

Now writing

$$(2.2)\qquad o = \frac{\partial w}{\partial t}\left(\frac{\partial w}{\partial t} - \Delta w\right) = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)\right] - \operatorname{div}\left(\frac{\partial w}{\partial t}\nabla_x w\right)$$

and integrating over $Q(t)$, $o \le t \le T$, we obtain

$$(2.3)\qquad \frac{1}{2}\int_{\Omega(t)}\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)d_x = \frac{1}{2}\int_{\Omega(o)}\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx -$$

$$-\frac{1}{2}\int_{\Sigma(t)}\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)\nu_t\, d\sigma + \int_{\Sigma(t)}\frac{\partial w}{\partial t}\nabla w.\,\nu_x\, d\sigma$$

$$(2.4)\qquad = \frac{1}{2}\int_{\Omega(o)}\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx + \frac{1}{2}\int_{\Sigma(t)}\frac{\partial w}{\partial t}^2\,\nu_t\left(|\nu_x|^2 - \nu_t^2\right)d\sigma$$

The second term is always non-positive because ν_t is non-positive and (H_o).

Therefore $E_1(t)$ is non-increasing.

On the other hand, if $\partial\Omega(t)$ is contracting inward, ν_t is non-positive. The second term of (2.4) becomes non-negative. Thus $E_1(t)$ is non-decreasing.

Q.E.D.

In what follwos, we will use C.S. Morawetz's energy invariants [9] to study the growth and decay estimates for the wave equation. They are

$$E_2(t) \equiv \int_{\Omega(t)}\left[t\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right) + 2\gamma\frac{\partial w}{\partial\gamma}\frac{\partial w}{\partial t} + (N-1)w\frac{\partial w}{\partial t}\right]dx$$

$$E_3(t) \equiv \int_{\Omega(t)}\left[(\gamma^2+t^2)\left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right) + 4t\gamma\frac{\partial w}{\partial t}\frac{\partial w}{\partial\gamma} + 2(N-1)tw\frac{\partial w}{\partial t} - (N-1)w^2\right]$$

<u>Lemma 2.2</u> For any domain $\Omega\subseteq R^N$, $N \neq 2$, we have

$$\int_\Omega\left[\Sigma(N-1)\frac{x_i}{\gamma^2}\frac{\partial w}{\partial x_i}w + \frac{(N-1)^2}{4}\frac{w^2}{\gamma^2}\right] = -\frac{(N-1)(N-3)}{4}\int_\Omega\frac{w^2}{\gamma^2}dx$$

$$\int_\Omega (r^2 + t^2)\ [\Sigma (N-1) \frac{x_i}{\gamma^2} \frac{\partial w}{\partial x_i}\ w + \frac{(N-1)^2}{4}\ \frac{w^2}{\gamma^2}\ dx] = -\ (N-1) \int_\Omega w^2 dx -$$

$$\frac{(N-1)(N-3)}{4} \int_\Omega \frac{(\gamma^2 + t^2)}{\gamma^2}\ w^2 dx$$

provided that $w = o$ on $\partial\Omega$

Proof : They are straightforward integrations by parts. One need only note that the singularity at $\gamma = o$ does not make any contribution because $N \geq 2$

Theorem 2.3. Assume $(H_1 - H_3)$. Let $w(x,t)$ be the solution of (WE) in $Q(T)$, $T \geq T_o$

(i) If $t\nu_t + \gamma\nu_\gamma = o$ on $\partial\Omega(t)$ for $T_o \leq t \leq T$, then $E_2(t)$ is conserved during $[T_o, T]$. We have the energy decay

(2.5) $$E_1(t) \leq \frac{(1+\theta)T_o}{1-\theta}\ \frac{1}{t}\ E_1(o), \quad t > o$$

(ii) If (H_4) $t\nu_t + \gamma\nu_\gamma \leq o$ on $\partial\Omega(t)$, $t \geq T_o$ holds then $E_2(t)$ is non decreasing, and (2.5) also holds.

(iii) If $t\nu_t + \gamma\nu_\gamma \geq o$ on $\partial\Omega(t)$ for $t \geq T_o$, then $E_2(t)$ is non-decreasing and

(2.6) $$E_1(t) \geq \frac{(1-\theta)T_o}{(1-\theta)}\ \frac{1}{t}\ E_1(o), \quad t \geq T_o$$

Proof : We know that

$$\begin{aligned} 0 &= 2\left(\frac{\partial^2 w}{\partial t^2} - \Delta w\right)\left(t\,\frac{\partial w}{\partial t} + \gamma\frac{\partial w}{\partial \gamma} + \frac{N-1}{2}\,w\right) \\ &= \frac{\partial}{\partial t}\ \left[t\left(\frac{\partial w}{\partial t}^2 + |\nabla_x w|^2\right) + 2\gamma\,\frac{\partial w}{\partial \gamma}\,\frac{\partial w}{\partial t} + (N-1)w\,\frac{\partial w}{\partial t}\right] \\ &\quad - \operatorname{div}\left[2t\,\frac{\partial w}{\partial t}\,\nabla w + 2(x.\nabla w)\,\nabla w + \frac{\partial w}{\partial t}^2 x + (N-1)w\nabla w - |\nabla w|^2 x\right] \end{aligned}$$

Integrating the above over $Q(T_o;t)$, we obtain

$$(2.7)\quad E_2(t) = E_2(T_o) + \int_{\Sigma(T_o,t)} \{-[t(\frac{\partial w}{\partial t}^2 + |\nabla w|^2) + 2\gamma \frac{\partial w}{\partial \gamma} \frac{\partial w}{\partial t} + (N-1)w\frac{\partial w}{\partial t}]\nu_t +$$

$$+ [2t\frac{\partial w}{\partial t} \nabla w + 2(x.\nabla w)\nabla w + \frac{\partial w}{\partial t}^2 x + (N-1)w\nabla w - |\nabla w|^2 x].\nu_x\} \, d\sigma$$

Using the relations (1.1), we can simplify the above boundary integral to

$$(2.8)\quad \int_{\Sigma(T_o,t)} \frac{\partial w}{\partial \nu}^2 (|\nu_x|^2 - \nu_t^2)\,[t\nu_t + r\nu_r] \, d\sigma$$

From here one easily sees that the conservation of $E_2(t)$ is proved. Now, define

$$(2.9)\quad \lambda_i \equiv \frac{\partial w}{\partial x_i} + \frac{N-1}{2} \frac{x_i}{\gamma^2} w$$

Then $E_2(t) = \int_{\Omega(t)} \{t[\frac{\partial w}{\partial t}^2 + \sum_i(\lambda_i - \frac{N-1}{2} \frac{x_i}{\gamma^2} w)^2] + 2\gamma \frac{\partial w}{\partial t} \frac{\partial}{\partial \gamma} + (N-1)w\frac{\partial w}{\partial t}]dx$

(integration by parts once and simplification)

$$= \int_{\Omega(t)} [t(\frac{\partial w}{\partial t}^2 + \Sigma \lambda_i^2) + 2 \frac{\partial w}{\partial t} \Sigma x_i \lambda_i] \, dx + \frac{t(N-1)(N-3)}{4} \int_{\Omega(t)} \frac{w^2}{\gamma^2} \, dx$$

By (H_3), $\gamma \le \theta t$ on $\Omega(t)$, we have

$$|2 \frac{\partial w}{\partial t} \Sigma x_i \lambda_i| \le 2|\frac{\partial w}{\partial t}| \gamma (\Sigma \lambda_i^2)^{1/2} \le \gamma(\frac{\partial w}{\partial t}^2 + \Sigma \lambda_i^2) \le \theta t(\frac{\partial w}{\partial t}^2 + \Sigma\lambda_i^2)$$

Therefore

$$(2.10)\quad (1+\theta)t \int_{\Omega(t)} (\frac{\partial w}{\partial t}^2 + \Sigma \lambda_i^2) \, dx + \frac{t(N-1)(N-3)}{4} \int_{\Omega(t)} \frac{w^2}{\gamma^2} \, dx \ge E_2(t) \ge$$

$$\ge (1-\theta)t \int_{\Omega(t)} (\frac{\partial w}{\partial t}^2 + \Sigma\lambda_i^2) \, dx + \frac{t(N-1)(N-3)}{4} \int_{\Omega(t)} \frac{w^2}{\gamma^2} \, dx$$

Substituting (2.9) back, using Lemma 2.2 and simplifying, we get

$$(2.11)\quad (1+\theta)t \int_{\Omega(t)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx - \frac{(N-1)(N-3)}{4}\theta t \int_{\Omega(t)} \frac{w^2}{\gamma^2}\,dx \geq E_2(t) \geq$$

$$\geq (1-\theta)t \int_{\Omega(t)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx + \frac{(N-1)(N-3)}{4}\theta t \int_{\Omega(t)} \frac{w^2}{\gamma^2}\,dx$$

Combining the above inequalities with (2.7), (2.8), we conclude that for $t \geq T_o$, if $t\nu_t + \gamma\nu_\gamma \leq o$ on $\Sigma(T_o,t)$, then

$$(1+\theta)T_o \int_{\Omega(T_o)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx \geq (1-\theta)t \int_{\Omega(t)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx$$

Hence (2.5) is proved because $E_1(T_o) = E_1(o)$

If $t\nu_t + \gamma\nu_\gamma \geq o$ on $\Sigma(T_o,t)$, then

$$(1+\theta)T_o \int_{\Omega(T_o)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx \leq (1-\theta)T \int_{\Omega(t)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right)dx$$

so (2.6) is proved.

Remark : The assumption (H_4) is similar to the "pulse illumination" condition of Cooper and Strauss [6] . For example, after T_o if $x \in \partial\Omega(t)$ moves according to the law $|x| = \theta t$, $o < \theta < 1$, then (H_4) reduces to $\nu_t + \theta\nu_\gamma \leq o$. An example illustrating various situations in Theorem 2.3. can be found in §4.

As for $E_3(t)$, we have the following theorem.

Theorem 2.4. Assume (H_0)-(H_3). Let $w(x,t)$ be the solution of (WE) in $Q(T)$, $T \geq T_o$. If the condition

$$(H_5)\quad (\gamma^2 + t^2)\,\nu_t + 2t\gamma\nu_\gamma \geq o, \qquad x \in \partial\Omega(t),\ t \geq T_o$$

is satisfied, then $E_3(t)$ is increasing $(t \geq T_o)$ and

$$(2.12)\quad E_1(t) \geq \frac{1}{1+\theta^2}\left(\frac{1-\theta}{1+\theta}\right)^2 \frac{T_o^2}{t^2}\; E_1(o) \qquad t > T_o$$

Remark : (i) (H_5) is satisfied provided that $t\nu_t+\gamma\nu_\gamma \geq o$ is satisfied, because

$$(\gamma^2+ t^2)\ \nu_t + 2t\gamma\nu_\gamma \geq (1+\theta^2)t^2\nu_t + 2t\gamma\nu_\gamma \geq 2t^2\nu_t + 2t\gamma\nu_\gamma =$$

$$= 2t(t\nu_t + \gamma\nu_\gamma) \geq o$$

(ii) The estimate (2.12) says that under (H_5), $E_1(t)$ cannot decay with a rate faster than $1/t^2$. This estimate does not seem to be too useful from the energy decay point of view, because we already known that $E_1(t) \leq E_1(0)$. However, such an estimate is very important in the controllability study in §3. Compare (3.5).

(iii) For a sphere expanding with a uniform speed $\theta < 1$, it is impossible to have

$$(\gamma^2+ t^2)\nu_t + 2\ t\gamma\nu_\gamma \leq o \text{ on } \partial\Omega_t$$

for all $t \geq T_0$. Therefore one in general cannot expect a result like (2.5) from E_3.

Proof : We follow along the same line of argument as in the proof of the preceeding theorem.

It is known that

$$0 = 2\left(\frac{\partial^2 w}{\partial t^2} - \Delta w\right) \left[(\gamma^2+ t^2)\ \frac{\partial w}{\partial t} + 2t\gamma\frac{\partial w}{\partial \gamma} + (N-1)tw\ \right]$$

$$= \frac{\partial}{\partial t}\left[(\gamma^2+t^2)\left(\frac{\partial w}{\partial t}^2 + |\Delta\ w|^2\right) + 4t\gamma\ \frac{\partial w}{\partial t}\ \frac{\partial w}{\partial \gamma} + 2(N-1)tw\ \frac{\partial w}{\partial t} - (N-1)w^2\right]$$

$$- 2\ \mathrm{div}\left[(\gamma^2+t^2)\ \frac{\partial w}{\partial t}\ \ w + t\ \frac{\partial w}{\partial t}^2\ x + 2t(x.\nabla w)\ \nabla w - t|\nabla w|^2\ x + 2(N-1)tw\nabla w\right]$$

Integrating over $Q(T_0,t)$, we obtain

$$(2.13)\quad E_3(t) = E_3(T_0) + \int_{\Sigma(T_0,t)} \{[(\gamma^2+t^2)\frac{\partial w}{\partial t}\nabla w + t\frac{\partial w^2}{\partial t} x + 2t(x.\nabla w)\nabla w - t|\nabla w|^2 x$$

$$+ 2(N-1)tw\nabla w].\nu_x - [(\gamma^2+t^2)(\frac{\partial w^2}{\partial t} + |\nabla w|^2) + 4t\gamma\frac{\partial w}{\partial t}\frac{\partial w}{\partial \gamma} +$$

$$+ 2(N-1)tw\frac{\partial w}{\partial t} - (N-1)w^2]\nu_t\}\, d\sigma$$

Using (2.3), we again simplify the above boundary integral and get

$$(2.14)\quad E_3(t) = E_3(T_0) + \int_{\Sigma(T_0,t)} \frac{\partial w^2}{\partial \nu}(|\nu_x|^2 - \nu_t^2)[(\gamma^2+t^2)\,\nu_t + 2tr\nu_r]\, d\sigma$$

Now we use (2.9) and integrate by parts, we get

$$E_3(t) = \int_{\Omega(t)} \{(\gamma^2+t^2)[(\frac{\partial w^2}{\partial t} + \Sigma\,\lambda_i^2] + 4t\frac{\partial w}{\partial t}\Sigma x_i\, \lambda_i\}dx + \frac{(N-1)(N-3)}{4}\int_{\Omega(t)} \frac{\gamma^2+t^2}{\gamma^2} w^2 dx$$

For $t \geq T_0$, $\gamma \leq \theta t$ on $\Omega(t)$, so $2t\gamma \leq \frac{2\theta}{1+\theta^2}(\gamma^2+t^2)$

Thus

$$4t\frac{\partial w}{\partial t}\Sigma x_i\lambda_i \leq 4t\gamma\frac{\partial w}{\partial t}(\Sigma\lambda_i^2)^{1/2} \leq 2t\gamma(\frac{\partial w^2}{\partial t} + \Sigma\lambda_i^2) \leq \frac{2\theta}{1+\theta^2}(\gamma^2+t^2)(\frac{\partial w^2}{\partial t} + \Sigma\lambda_i^2)$$

Therefore

$$(1+\frac{2\theta}{1+\theta^2})\int_{\Omega(t)}(\gamma^2+t^2)(\frac{\partial w^2}{\partial t} + \Sigma\lambda_i^2)dx + \frac{(N-1)(N-3)}{4}\int_{\Omega(t)}\frac{\gamma^2+t^2}{\gamma^2}w^2dx \geq E_3(t) \geq$$

$$\geq (1-\frac{2\theta}{1+\theta^2}).\int_{\Omega(t)}(\gamma^2+t^2)(\frac{\partial w^2}{\partial t} + \Sigma\lambda_i^2)dx + \frac{(N-1)(N-3)}{4}\int_{\Omega(t)}\frac{\gamma^2+t^2}{\gamma^2}w^2dx$$

$$\geq \frac{(1-\theta)^2}{1+\theta^2}t^2\{\int_{\Omega(t)}(\frac{\partial w^2}{\partial t} + \Sigma\lambda_i^2)dx + \frac{(N-1)(N-3)}{4}\int_{\Omega(t)}\frac{w^2}{\gamma^2}dx\}$$

Substituting (2.9) back and using Lemma 2.2, we obtain

$$(2.15)\quad \frac{(1+\theta)^2}{1+\theta^2}\int_{\Omega(t)}(\gamma^2+t^2)(\frac{\partial w^2}{\partial t} + |\nabla w|^2)dx - (N-1).\frac{(1+\theta)^2}{1+\theta^2}\int_{\Omega(t)}w^2dx -$$

$$-\frac{2\theta}{1+\theta^2}\,\frac{(N-1)(N-3)}{4}\int_{\Omega(t)}\frac{\gamma^2+t^2}{\gamma^2}w^2dx \geq E_3(t) \geq \frac{(1-\theta)^2}{1+\theta^2}t^2\int_{\Omega(t)}(\frac{\partial w^2}{\partial t} + |\nabla w|^2)\,dx$$

Now

$$(2.16) \qquad \gamma^2+t^2 \le (1+\theta^2)t^2$$

combining (H_5), (2.14), (2.15), (2.16), we obtain

$$(1+\theta)\, t^2 \int_{\Omega(t)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right) dx \ge E_3(t) \ge E_3(T_0) \ge$$

$$\ge \frac{(1-\theta)^2}{1+\theta^2}\, T_0^{\;2} \int_{\Omega(T_0)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right) dx$$

$$= \frac{(1-\theta)^2}{1+\theta^2}\, T_0^{\;2} \int_{\Omega(T_0)} \left(\frac{\partial w}{\partial t}^2 + |\nabla w|^2\right) dx$$

so (2.12) is proved. Q.E.D.

For the cas N=2, the function w^2/γ^2 is not integrable in general due to the singularity at x=o. In order to be able to derive some energy estimate, we must assume taht each $\Omega(t)$ is of annular shape.

Theorem 2.5. Assume that for each $t \ge o$, $\Omega(t)$ is an annular region and its boundary $\partial\Omega(t)$ consists of two parts $\Gamma_0 x\{t\}$ and Γ_t. The interior part Γ_0 is star-shaped with respect to the origin. Γ_0 is fixed, but the exterior part Γ_t may be moving. Assume (H_0), $(H_2$, (H_3), (H_4). Then $E_2(t)$ is non-increasing $(t \ge T_0)$ and

$$(2.17) \quad E_1(t) - \frac{1}{4}\frac{\theta}{1-\theta} \int_{\Omega(t)} \frac{w^2}{\gamma^2}\, dx \le \frac{T_0}{t} \left\{\frac{1+\theta}{1-\theta} E_1(T_0) + \frac{1}{4}\frac{\theta}{1-\theta} \int_{\Omega(T_0)} \frac{w^2}{\gamma^2} \right\} dx$$

for $t \ge T_0$.

Proof : It is the same as that of theorem 2.3.except some modifications. We replace (2.7) and (2.8) by

$E_2(t) = E_2(T_0)$ + boundary integral over the exterior lateral surface of $Q(T_0, t) + 2(t - T_0) \int_{\Gamma_0} (x.\, \nu_x) \frac{\partial w^2}{\partial \gamma}\, d\sigma_x$

The last additional term is always non-positive. So $E_2(t)$ is non-increasing Now the proof of (2.17) follows immediately from (2.11). Q.E.D.

§ 3. STABILIZABILITY AND EXACT CONTROLLABILITY

Our first theorem is a stabilizability theorem. Here we obtain "forward" decay estimates by using high damping compensation. It is an analogue of [2, Theorem 4.1.].

Theorem 3.1. Let $w(x,t)$ be the solution of

$$(E_t) = \begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} + \gamma_1 \dfrac{\partial w(x,t)}{\partial t} - \Delta w(x,t) + \gamma_2 w(x,t) = o & (x,t) \in Q(T) \\ w(x,o) = w_o(x) \in H_o^1(\Omega(o)) \\ \dfrac{\partial w}{\partial t}(x,o) = v_o(x) \in L^2(\Omega(o)) \\ w(x,t)|_{\Sigma(t)} = o \end{cases}$$

Assume that the domains are expanding, i.e., $\Omega(t_1) \equiv \Omega(t_2)$ for $o \le t_1 \le t_2 \le T$ For any $\varepsilon > o$, if $\gamma_1 \equiv \lambda \equiv 1/2\varepsilon$, $\gamma_2 \equiv 1/\varepsilon^3$, then

$$\int_{\Omega(t)} [\frac{\partial w}{\partial t}^2 + |\nabla w|^2 + \gamma_2 w^2]dx \le \frac{1+\varepsilon}{1-\varepsilon+2\lambda T} \int_{\Omega(o)} [v_o^2 + |\nabla w_o|^2 + \gamma_2 w_o^2]dx$$

Proof : We use $\partial w/\partial t$ as multiplier and obtain

$$(3.1) \quad 0 = \frac{\partial w}{\partial t}(\frac{\partial^2 w}{\partial t^2} - \Delta w + 2\gamma_1 \frac{\partial w}{\partial t} + \gamma_2 w) = \frac{\partial}{\partial t}[\frac{1}{2}(\frac{\partial w}{\partial t}^2 + |\nabla w|^2 + \gamma_2 w^2)] - \operatorname{div}(\frac{\partial w}{\partial t}\nabla w) + $$
$$+ 2\gamma_1 \frac{\partial w}{\partial t}^2 .$$

Integrating over $Q(t)$, we obtain

$$\frac{1}{2}\int_{\Omega(t)} (\frac{\partial w}{\partial t}^2 + |\nabla w|^2 + \gamma_2 w^2)dx = \frac{1}{2}\int_{\Omega(o)} (v^2 + |\nabla w_o|^2 + \gamma_2 w_o^2)dx + \int_{\Sigma(t)} [(\frac{\partial w}{\partial t}\nabla w).\nu_x -$$
$$- \frac{1}{2}(\frac{\partial w}{\partial t}^2) + |\nabla w|^2 + \gamma_2 w^2)\nu_t] d\sigma - 2\gamma_1 \iint_{Q(t)} \frac{\partial w}{\partial t}^2 dx\, dt$$

The boundary integral is the same as those occured in (2.2), therefore it is non-positive. The third term on the right is always non-positive. Therefore we conclude that

$$(3.2)\quad \int_{\Omega(t)} \left(\frac{\partial w^2}{\partial t} + |\nabla w|^2 + \gamma_2 w^2\right)dx$$

is non-increasing as a function of t.

Now using λw as multiplier, we get

$$(3.3)\quad o = \lambda w\left(\frac{\partial^2 w}{\partial t} - \Delta w + 2\gamma_1 \frac{\partial w}{\partial t} + \gamma_2 w\right) = \lambda\left\{ \frac{\partial}{\partial t}\left[w \frac{\partial w}{\partial t} + \gamma_2 w^2\right] - \operatorname{div}(w\nabla w) + \right.$$

$$\left. |\nabla w|^2 - \frac{\partial w^2}{\partial t} + \gamma_2 w^2 \right\} \qquad [(3.2) + (3.3) \text{ and integration over } Q(T)] \Longrightarrow$$

$$\int_{\Omega(T)} \left[\frac{1}{2}\left(\frac{\partial w^2}{\partial t} + |\nabla w|^2\right) + \lambda w \frac{\partial w}{\partial t} + \left(\frac{\gamma_2}{2} + \lambda\gamma_1\right)w^2\right]dx = \int_{\Omega(o)} \left[\frac{1}{2}\left(\frac{\partial w^2}{\partial t} + |\nabla w|^2\right) + \right.$$

$$\left. + \lambda w \frac{\partial w}{\partial t} + \left(\frac{\gamma_2}{2} + \lambda\gamma_1\right)w^2\right] dx + \int_{\Sigma(T)} (\text{negative term})d\sigma - \iint_{Q(T)} \left[\lambda |\nabla w|^2 + \right.$$

$$\left. + (2\gamma_1 - \lambda) \frac{\partial w^2}{\partial t} + \lambda\gamma_2 w^2 \right] dxdt$$

Hence for every $\varepsilon > o$, we have

$$\int_o^T \int_{\Omega(t)} \left[|\nabla_x w|^2 + (2\gamma_1 - \lambda) \frac{\partial w^2}{\partial t} + \lambda\gamma_2 w^2\right] dxdt + \int_{\Omega(T)} \left\{ \frac{1}{2}\left[(1-\varepsilon) \frac{\partial w^2}{\partial t} + |\nabla w|^2\right] + \left[\frac{\gamma_2}{2} + \right.\right.$$

$$\left.\left. \left(\lambda - \frac{1}{2\varepsilon}\right)\right] w^2 \right\}dx \le \int_{\Omega(o)} \left\{ \frac{1}{2} \left[(1+\varepsilon) \frac{\partial w^2}{\partial t} + |\nabla w|^2\right] + \left[\frac{\gamma_2}{2\varepsilon} + \lambda\left(\gamma_1 + \frac{1}{2\varepsilon}\right)\right]w^2\right\} dx$$

We choose $\lambda = \gamma_1 = 1/2\varepsilon$, $\gamma_2 = 1/\varepsilon^3$ and use the decreasing property of (3.2) to get

$$(3.4)\quad \left(\lambda T + \frac{1-\varepsilon}{2}\right) \int_{\Omega(t)} \left[\frac{\partial w^2}{\partial t} + |\nabla w|^2 + \gamma_2 w^2\right]dx \le \frac{1+\varepsilon}{2} \int_{\Omega(o)} \left[\frac{\partial w^2}{\partial t} + |\nabla w^2 + \gamma_2 w^2\right]dx$$

Q.E.D.

So the proof is complete.

When we reverse the sense of time, an expanding domain becomes a contracting one. Therefore by theorem 2.1., we can not expect backward decay as we had in [2], [3], [10]. Instead, we use a scheme of high compensation in keeping with the compensation we made in theorem 3.1. Under those assumptions in § 2, we desive the following backward growth estimate.

Lemma 3.2. Assume that (H_0),-(H_3) and (H_5) hold. Let $w(x,t)$ be the solution of the backward equation

$$(E-)\quad \begin{cases} \dfrac{\partial^2 w(x,t)}{\partial t^2} - \Delta w(x,t) + \gamma\, w(x,t) = 0, & (x,t) \in Q(T),\quad \gamma \geq 0 \\ w(x,T) = w_T(x) \in H_0^1\,(\Omega(T)) & \\ \dfrac{w}{t}\,(x,T) = v_T(x) \in H^0(\Omega(T)) & \end{cases}$$

Then the following inequality

$$(3.5)\qquad \int_{\Omega(T)} \Big[\, \frac{\partial w^2}{\partial t}\,(x,T) + |\nabla w(x,T)|^2 + \gamma\, w^2(x,T)\Big]\, dx \geq \frac{1}{1+\theta^2}\Big(\frac{1-\theta}{1+\theta}\Big)^2 \frac{T_0^{\,2}}{T^2} \int_{\Omega(o)} \Big[\frac{\partial w^2}{\partial t}(x,o) + |\nabla\, w(x,\ )| + \gamma\, w^2(x,o)\Big]\, dx$$

holds.

Proof : It is straightforward to verity that

$$0 = 2\Big(\frac{\partial^2 w}{\partial t^2} - \Delta w + \gamma w\Big)\, \Big[(r^2+t^2)\frac{\partial w}{\partial t} + 2tr\frac{\partial w}{\partial r} + 2(N-1)tw\Big] = \frac{\partial}{\partial t}\, \Big[(r^2+t^2)\Big(\frac{\partial w^2}{\partial t} + |\nabla\, w|^2 + \gamma w^2\Big) +$$

$$+ 4tr\frac{\partial w}{\partial t}\,\frac{\partial w}{\partial r} + 2(N-1)tw\frac{\partial w}{\partial t} - (N-1)w^2\Big] - 2\mathrm{div}\, \Big[(r^2+t^2)\, \frac{\partial w}{\partial t}\, \nabla w + t\, \frac{\partial w^2}{\partial t}\, x + 2t(x.\nabla w)\, \nabla w -$$

$$- t|\nabla\, w|^2 x + (N-1)tw\nabla\, w - \gamma t w^2 x\, \Big] - 4\gamma\, tw^2$$

Integrating over $Q(T_0,T)$, we obtain

$$(3.6)\qquad \int_{\Omega(T)} \Big[(r^2+T^2)\Big(\frac{\partial w^2}{\partial t} + |\,\nabla w|^2 + \gamma w^2\Big) + 4Tr\, \frac{\partial w}{\partial t}\, \frac{\partial w}{\partial r} + 2(N-1)T\, w\, \frac{\partial w}{\partial t} - (N-1)w^2\Big]\ \ dx =$$

$$= \int_{\Omega(T_0)} \Big[(r^2+T^2)\Big(\frac{\partial w^2}{\partial t} + |\nabla w|^2 + \gamma w^2\Big) + 4T_0 r\, \frac{\partial w}{\partial t}\ \ \frac{\partial w}{\partial r} + 2(N-1)T_0 w\, \frac{\partial w}{\partial t} - (N-1)w^2\ \Big] dx +$$

+ integral over the boundary $\Sigma\,(T_o,T)$ - $\iint_{Q(T_o,T)} 4\gamma\, tw^2 dxdt$

The above boundary integral is the same as the one in (2.13) with t=T, so it is non-positive by (H_5). The very last integral over $Q(T_o,T)$ is always non-positive.

Again using λ_i as defined in (2.9), with (H_3), we obtain

$$(1+\frac{2\theta}{1+\theta^2})\int_{\Omega(T)}(r^2+T^2)(\frac{\partial w^2}{\partial t}+\Sigma\lambda_i^2+\gamma w^2)dx+\frac{(N-1)(N-3)}{4}\int_{\Omega(T)}\frac{r^2+T^2}{r^2}w^2dx \geq$$

$$\geq (1-\frac{2\theta}{1+\theta^2})\int_{\Omega(T_o)}(r^2+T_o^2)(\frac{\partial w^2}{\partial t}+\Sigma\lambda_i^2+\gamma w^2)dx+\frac{(N-1)(N-3)}{4}\int_{\Omega(T_o)}\frac{r^2+T^2}{r^2}w^2dx \geq$$

$$\geq \frac{(1-\theta)^2}{1+\theta^2}T_o^2\,[\int_{\Omega(T_o)}(\frac{\partial w^2}{\partial t}+\Sigma\lambda_i^2+\gamma w^2)dx+\frac{(N-1)(N-3)}{4}\int_{\Omega(T_o)}\frac{w^2}{r^2}dx]$$

The rest of theproof follows from the same type of argument as in Theorem 2.4. and will not be reproduced here. Q.E.D.

Lemma 3.3. Under the same assumptions of Theorem 3.1. and Lemma 3.2., respectively,

(i) Let w(x,t) be the solution of (E_+) with initial state (w_o,v_o). Let $\Phi_+(t) = H_o^1(\Omega(o)) \oplus H^o(\Omega(o)) \to H_o^1(\Omega(t)) \oplus H^o(\Omega(t))$ be the linear transformation defined by

$$\Phi_+(t)\,((w_o,v_o)) \equiv (w(o,t), \frac{\partial w}{\partial t}(.,t))$$

Then $\Phi_+(t)$ is continuous for each $t \in [o,T]$.

(ii) let w(x,t) be the solution of (E_-) with terminal state(w_T, v_T). Define $\Phi\text{-}(t) = H_o^1(\Omega(T))\oplus H^o(\Omega(T)) \to H_o^1(\Omega(t))\oplus H^o(\Omega(t))$ to be the linear transformation

$$\Phi\text{-}(t)((w_T,v_T)) \equiv (w(.,t)\ \frac{\partial w}{\partial t}(.,t))$$

Then Φ-(t) is also continuous for each $t \in [o,T]$.

Proof : We need only prove (i), (ii) will follow from a similar manner. Let (w_o^1, v_o^1), $(w_o^2, v_o^2) \in H^1_o(\Omega(o)) \oplus H^o(\Omega(o))$. Let w(x,t) be the solution of (E+) with initial state $(w_o^1 - v_o^1, w_o^2 - v_o^2)$. From (3.4) with T replaced by t, we have

$$(\lambda t + \frac{1-\varepsilon}{2}) \int_{\Omega(t)} [\frac{\partial w^2}{\partial t} + |\nabla w|^2 + \gamma^2 w^2]dx \le \frac{1+\varepsilon}{2} \int_{\Omega(o)} [\frac{\partial w^2}{\partial t} + |\nabla w|^2 + \gamma_2 w^2]dx$$

Thus

$$(\lambda t + \frac{1-\varepsilon}{2}) ||\Phi + (t)(w^1o - w^2o, v^1o - v^2o)||^2_{\gamma_2} \le \frac{1+\varepsilon}{2} ||(w^1o - w^2o, v^1o - v^2o)||^2_{\gamma_2}$$

The continuity of $\Phi+(t)$ is clear. Q.E.D.

Theorem 3.4. (Exact controllability of the wave equation in an expanding domain).

Assume $(H_o$-(H_3) and (H_5). Let T > o. For any given initial state (wo,vo) $\in H^1_o(\Omega(T)) \oplus H^o(\Omega(T))$, there is a control $f \in L^2(Q(T))$ which solves the (ECP).

Proof : By [2, Theorem 4.2.], we need only consider the case $T \ge T_o$. We first consider the case of controllability to zero, namely, $(w_1, v_1) = (o,o)$. Let $\tilde{w}(x,t)$ be the solution of :

$$\begin{cases} \dfrac{\partial^2 \tilde{w}}{\partial t^2}(x,t) - \Delta \tilde{w}(x,t) = -2\gamma_1 \dfrac{\partial \tilde{w}(x,t)}{\partial t} - \gamma_2 \tilde{w}(x,t), & (x,t) \in Q(T) \\ & \gamma_1 > o, \quad \gamma_2 > o \\ (\tilde{w}(x,o), \tilde{v}(x,o)) = (P(x), f(x)) \in H^1_o(\Omega(o)) \oplus H^o(\Omega(o)) & \end{cases}$$

The terminal state of $(\tilde{w}(x,t), \tilde{v}(x,t))$ at t=T is $(\tilde{w}(x,T), \tilde{v}(x,T))$, which is in $H^1_o(\Omega(T)) \oplus H^o(\Omega(T))$. Let $\bar{w}(x,t)$ denote the solution of

$$\begin{cases} \dfrac{\partial^2 \bar{w}(x,t)}{\partial t^2} - \Delta \bar{w}(x,t) = -\gamma_2 \bar{w}(x,t), & (x,t) \in Q(T) \\ (\bar{w}(x,T), \bar{v}(x,T)) = -(\tilde{w}(x,T), \tilde{v}(x,T)) \in H^1_o(\Omega(T)) \oplus H^o(\Omega(T)) & \end{cases}$$

Define

$$\begin{cases} w(x,t) \equiv \tilde{w}(x,t) + \bar{w}(x,t), & (x,t) \in Q(T) \\ f(x,t) \equiv -2\gamma_1 \frac{\partial \tilde{w}}{\partial t} - \gamma_2[\tilde{w}(x,t) + \bar{w}(x,t)] \end{cases}$$

Then $w(x,t)$ satisfies the equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \Delta w(x,t) = f(x,t), \qquad (x,t) \in Q(T)$$

with the terminal state $(w(x,T), v(x,T)) = (o,o)$. The initial state is

$$\begin{aligned} (w(x,o), v(x,o)) &= (\tilde{w}(x,o), \tilde{v}(x,o)) + (\bar{w}(x,o), \bar{v}(x,o)) \\ &= (p(x), g(x) + \Phi_-(o)\, \Phi_+(T)\, [- (p(x), g(x))] \end{aligned}$$

$$(3.7) \qquad = [\, I - \Phi_-(o)\, \Phi_+(T)]\, (p(x), f(x))$$

Now choose $\gamma_1 = \lambda = 1/2\varepsilon$, $\gamma_2 = 1/\varepsilon^3$. By theorem 3.1. and Lemma 3.2, we deduce that

$$(3.8) \qquad ||\Phi_-(o)\, \Phi_+(T)||^2 \leq (1+\theta^2)\left(\frac{1+\theta}{1-\theta}\right)^2 \frac{T^2}{T_o^2} \quad \frac{1+\varepsilon}{1-\varepsilon+2\lambda T}$$

Here the operator norm is relative to $||\ ||_\gamma$ on $H_0^1(\Omega(o)) \oplus H^0(\Omega)$ and $H_0^1(\Omega(T)) \oplus H^0(\Omega(T))$.

We choose ε so small that the right hand size (3.6) is smaller than 1. Therefore $I-\Phi_-(o)\Phi_+(T)$ is an invertible linear transformation from $H_0^1(\Omega(o)) \oplus H^0(\Omega(o))$ into itself. We choose

$$(p.g) = [I-\Phi_-(o)\, \Phi_+(T)]^{-1}(wo,vo)$$

Then f steers the system from (wo, vo) to (o,o) at t=T.

For arbitrasily prescribed final state $(w_1,v_1) \in H_0^1(\Omega(T)) \oplus H^0(\Omega(T))$, we first let $w^-(x,t)$ be the solution of

$$\begin{cases} \dfrac{\partial^2 w^-(x,t)}{\partial t^2} - \Delta w^-(x,t) = o & (x,t) \in Q(T) \\ \begin{bmatrix} w^-(x,T) \\ v^-(x,T) \end{bmatrix} = \begin{bmatrix} w_1(x) \\ v_1(x) \end{bmatrix} & \text{(terminal condition)} \end{cases}$$

and next let $w^+(x,t)$ be the solution of

$$\begin{cases} \dfrac{\partial^2 w^+(x,t)}{\partial t^2} - \Delta w^+(x,t) = f(x,t) & (x,t) \in Q(T) \\ \begin{bmatrix} w^+(x,o) \\ v^+(x,o) \end{bmatrix} = \begin{bmatrix} w^-(x,o) \\ v^-(x,o) \end{bmatrix} \end{cases}$$

where f is a control which steers the system from $(w^-(x,o),\ v^-(x,o))$ to (o,o) at t=T. Then $w \equiv w^+ + w^-$ satisfies the equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} - \Delta w(x,t) = f(x,t)$$

with f steering the system from (w_o, v_o) to $(w_1,\ v_1)$. Q.E.D.

Remark : The above theorem generalizes slightly Russell's controllability via stabilizability principle [10] in the sense that we do not need to have both forward and backward decay. What is most important is to have $||\Phi_-(o)\ \Phi_+(T)||<1$, thereby ensuring the invertibility of $I - \Phi_-(o)\Phi_+(T)$.

§ 4. EXAMPLE

Let $\Omega(o)$ be a sphere with radius γ_0 in R^N, $N \neq 2$. Let $Q(T)=\bigcup_{o\le t\le T} \Omega(t)\times\{t\}$ where

$$\Omega(t) = \begin{cases} \Omega(o), & o \le t \le T_0 \\ \{z\in R^N \mid z=[|x|+\theta(t-T_0)]\, x/|x|,\ x \neq o,\ x \in \Omega(o),\ \mathbf{z=o \text{ if } x=o}\} & t \ge T_0 \end{cases}$$

$o < \theta < 1$

In other words, after time T_0, the boundary of $\Omega(t)$ expands radially outward with a velocity $\theta < 1$. Let

$$(4.1) \qquad \frac{\partial^2 w(x,t)}{\partial t^2} - \Delta_x\, w(x,t) = o$$

be the wave equation in $Q(T)$. One can, by making the follwing global change of variable

$$y \equiv \frac{1}{1+\theta(t-T_0)} \cdot x \qquad\qquad t \ge T_0$$

$$u(y,t) \equiv w(x,t) = w([1+\theta(t,T_0)]y,t)$$

derive a time-dependent hyperbolic equation

$$(4.2) \qquad \frac{\partial^2}{\partial t^2} u(y,t) - \frac{2\theta}{1+\theta(t-T_0)} \sum_{i=1}^{N} y_i \frac{\partial^2}{\partial y_i \partial t} u(y,t) + \frac{\theta^2}{[1+\theta(t-T_0)]^2}$$

$$\cdot [2 \sum_i y_i \frac{\partial}{\partial y_i} u(y,t) + \sum_{i,j} y_i y_j \frac{\partial^2}{\partial y_i \partial y_j} u(y,t)] - \frac{1}{[1+\theta(t-T_0)]^2} \Delta_y u(y,t) = o$$

on the <u>fixed</u> domain $\Omega(o)$ for $t \ge T_0$. It is easy to check that the symbol of the principal part of (4.2) which is

$$p(t,y,\xi) = \xi_0^2 - \frac{2\theta}{1+\theta(t-T_0)} \sum_i y_i \xi_0 \xi_i + \frac{\theta^2}{[1+\theta(t-T_0)]} \sum_{i,j} y_i y_j \xi_i \xi_j - \frac{1}{[1+\theta(t-T_0)]^2} \sum_i y_i^2$$

always has two distinct real roots ξ_o for $p(y,\xi)=o$ with any given $(t,y,\xi_1,\ldots,\xi_n)$

Thus the strict hyperbolicity is conserved.

On the boundary of the truncated cone $Q(T_o,T)$, let the motion of a point on $\partial\Omega(t)$ be given by (x,t). Then

$$\frac{dx(t)}{dt} = \theta \frac{x}{|x|} = \theta \frac{\nu_x}{|\nu_x|}$$

since

$$\left(\frac{dx(t)}{dt}, 1\right).(\nu_x, \nu_t) = o \qquad \text{on } \Sigma\ (T_o,T)$$

we have

$$\begin{cases} \theta\ |\nu_x| + \nu_t = o \\ |\nu_x|^2 + \nu_t^2 = 1 \end{cases} \implies |\nu_x| = \frac{1}{\sqrt{1+\theta^2}} \quad \nu_t = -\frac{\theta}{\sqrt{1+\theta^2}}$$

Returning to theorem 2.3., one easily verifies the following :

(i) if $T_o = r_o/\theta$ then $r = \theta t$ and $t\nu_t + r\nu_r = o$ on $\partial\Omega(t)$ for $t \geq T_o$. Hence after T_o, $E_2(t)$ is conserved. $E_1(t)$ decays with a rate $1/t$ and $E_3(t)$ grows with a rate t.

(ii) If $T_o > r_o/\theta$, then $r > \theta t$ and $t\nu_t + r\nu_r < o$ on $\partial\Omega(t)$ for $t \geq T_o$. Hence $E_2(t)$ is decreasing after T_o. $E_1(t)$ decays with a rate $1/t$. $E_3(t)$ is increasing after certain $T_1 > o$.

(iii) If $T_o < r_o/\theta$, then $t\nu_t + r\nu_r > o$ on $\partial\Omega(t)$ for $t \geq T_o$. Therefore $E_2(t)$ is increasing after T_o.

Suppose T_o satisfies the condition that

$$\gamma_o \leq T_o \leq \frac{\theta}{1-\sqrt{1-\theta^2}}\,\gamma o \tag{4.3}$$

then
$$\frac{1-\sqrt{1-\theta^2}}{\theta}\; t \le \theta t + (r_o - \theta T_o) = |x(t)| \le t \qquad \text{on } \partial\Omega(t),\ t \ge T_o$$

Thus
$$\frac{\nu_t}{\nu_r} + \frac{2tr}{r^2+t^2} = -\theta\frac{2tr}{r^2+t^2} \ge o \qquad \text{for all } t \ge T_o$$

so (H_5) is satisfied.

Combining (4.3) and (ii) above, we see that if

$$\gamma_o/\theta \le T_o \le \theta r_o/[1-\sqrt{1-\theta^2}]$$

then the assumptions (H_0)-(H_5) are all satisfied. By theorem 3.3, the wave equation is exactly controllable for any $T > o$.

Acknowledgement. The second author would like to thank Professor Jeffrey Rauch for some extremely helpful discussions. He is especially indebted to Professor J.L. Lions for inviting him to IRIA-LABORIA where this work was completed.

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