A polynomial time algorithm for determining the isomorphism of graphs of fixed genus

I.S. Filotti, J.N. Mayer

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105 78150 Le Chesnay
France
Tél.954 90 20

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Ion S. FILOTTI

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(working paper)

Ion S. FILOTTI[†] and Jack N. MAYER

Department of Computer Science   Department of Mathematics

Columbia University, New York, N.Y. 10027

Résumé

Nous décrivons un algorithme en temps $p_\gamma(\alpha_0)$ pour déterminer l'isomorphie de deux graphes de genre γ à α_0 sommets, où p_γ est un polynôme de degré linéaire en γ. La méthode repose sur : (1) un théorème de rigidité des graphes généralisant un théorème de Whitney et (2) l'algorithme de Filotti et Miller pour plonger un graphe dans une surface compacte de genre γ.

Abstract

We describe an algorithm running in time $p_\gamma(\alpha_0)$ for determining the isomorphism of two graphs of genus γ on α_0 vertices, where p_γ is a polynomial of degree linear in γ. The method is based on : (1) a rigidity theorem for graphs generalizing a theorem of Whitney and (2) the algorithm of Filotti and Miller for embedding a graph in a compact surface of genus γ.

[†] On visit at INRIA, 78150 LE CHESNAY, FRANCE.

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I. S. Filotti* Jack N. Mayer

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1. Introduction

The isomorphism problem for graphs has been in recent years the object of a much research (see e.g. [Col 78] or [Re-Cor 77]). Its complexity is still unknown. It is not known whether the problem is NP-complete, although it is NP, of course. It is not known whether there exists a polynomial-time algorithm for it. Recently, Babai [Ba 79] has discussed probabilistic algorithms. For additional information see also [Mi 77]. The problem has also some practical applications. Of the known algorithms let us only quote the work of Weinberg [We 66] and of Hopcroft and Tarjan [Ho-Ta 72]. Weinberg's algorithm rums in quadratic time (in α_o, the number of vertices of the graphs). Hopcroft and Tarjan's runs in time $\mathcal{O}(\alpha_o \log \alpha_o)$ and uses their powerful technique of depth-first search. Both these algorithms apply only to planar (Weinberg's only to 3-connected planar) graphs. They rely on a well-known rigidity theorem of Withney [Withney 32].

It was natural to try to extend these algorithms to classes of non-planar graphs. This was not possible in the absence of better algorithms for embedding graphs in surfaces. Previous work by Filotti [Fi 78] and Filotti and Miller [Fi-Mi 79] was a necessary stepping stone to this end. This enabled us to give the algorithm we shall present here for determining the isomorphism of graphs of genus γ. The algorithm runs in time $\mathcal{O}(\alpha_o^{m\gamma+n})$ for some positive constants m and n. This enlarges considerably the class of graphs for which there exists a polynomial algorithm for isomorphism.

Weinberg's method relies on the fact that a planar 3-connected graph has only one embedding in the plane (actually two embeddings if we count an embedding and its mirror image as different). This theorem is due to Withney [Withney 32]. Weinberg shows how to canonically associate a code to a planar embedding. From Withney's theorem it follows that for planar 3-connected graphs the code depends only on the graph. To etablish the isomorphism of planar 3-connected graphs it suffices to compare the corresponding codes. This algorithm runs in time bounded by a quadratic polynomial in α_o. Weinberg does not generalize his algorithm to arbitrary planar graphs. This is done by Hopcroft and Tarjan [Ho-Ta 72] who show how to construct a code for a graph from the codes of its 2-connected components and how to construct a code for a 2-connected graph from those associated to its 3-connected components. The latter depends on a decomposition due to Tutte [Tu 66]. This decomposition, and hence this part of the algorithm of Hopcroft and Tarjan, holds for arbitrary graphs. Hopcroft and Tarjan show how this decomposition can be achieved within time bounded by a polynomial in α_o (actually, within $\alpha_o \log\alpha_o$ steps). We can therefore confine ourselves to 3-connected graphs.

For 3-connected graphs of higher genus the immediate generalization of Withney's theorem is false. Let S_γ denote the compact surface of genus γ. A graph with only one embedding (up to mirror image) in S_γ will be called γ-*rigid*. A graph G is called *rigid* if it is $\gamma(G)$-rigid, where $\gamma(G)$ is the genus of G. Thus Withney's theorem asserts that planar 3-connected graphs are (o)-rigid. In contrast, it is easy to construct for every γ>o an infinity of 3-connected graphs that are not γ-rigid.

Our algorithm relies on a generalization of Withney's rigidity theorem. Let H be a subgraph of G and let H^2 be an embedding of H in S_γ. The pair (H^2, G) will be called *rigid* if there is just one (up to mirror image) embedding G^2 of G in S_γ that extends H^2. We shall also say that the extension problem (H^2, G) is rigid. Theorem 7 below, a generalization of Withney's theorem, will yield an important class of rigid extension problems. We then use techniques inspired from the embedding algorithms of Filotti and Miller ([Fi 78], [Fi-Mi 79]) to construct our algorithm.

*) On visit at INRIA, 78150 Le Chesnay, FRANCE.

2. Notation, terminology and some basic results

Notation and terminology is more or less as in Tutte [Tu 66] and White [White 73] with the following additions. More details will be found in [Fi-Mi 79].

Graphs (i.e. CW-complexes of dimension 1) are written with superscript 1 (as in H^1, G^1, etc), embeddings (i.e. CW-complexes of dimension 2) are written with superscript 2. The superscript will be omitted when it is clear from the context. G^1 will in general denote the 1-skeleton of G^2. V(G), E(G) and F(G) denote respectively the set of vertices, edges and faces of G. Their cardinalities are respectively $\alpha_0(G)$, $\alpha_1(G)$ and $\alpha_2(G)$. The Euler characteristic of G^1 is $\chi(G^1) = \alpha_0(G^1) - \alpha_1(G^1)$. The Euler characteristic of G^2 is $\chi(G^2) = \alpha_0(G^2) - \alpha_1(G^2) + \alpha_2(G^2)$. To any complex one can associate homology groups $H_i(G^j)$ ($i = 0,1,\ldots,j$; $j = 1,2$). These are free abelian groups. The rank of $H_i(G^j)$ is denoted by $\beta_i(G^j)$. By the "alternating sum theorem", $\sum_{i=0}^{j}(-1)^i\alpha_i(G^j) = \sum_{i=0}^{j}(-1)^i\beta_i(G^j)$. Further, it can be shown that $\beta_0(G^2) = \beta_2(G^2)$. $\beta_0(G^j)$ equals the number of connected components of the underlying graph. For a graph, $\beta_1(G^1)$ equals its cyclomatic number, i.e. the number of independent cycles. From the alternating sum theorem, it follows that $\beta_1(G^1) = \alpha_1(G^1) - \alpha_0(G^1) + \beta_0(G^1)$. It can be shown that $\chi(G^2)$ is always even so that $\beta_1(G^2) = 2\gamma(G^2)$ and $\chi(G^2) = 2(\gamma(G^2) - \beta_0(G^2))$ (Euler-Poincaré formula). $\gamma(G^2)$ is called the *genus* of G^2. One says that G^2 is an embedding of G^1 in $S\gamma$, the surface of genus γ. For more details see e.g. [Fi-Mi 79] or standard textbooks on combinatorial topology. The *genus of a graph* G^1 is $\gamma(G^1) = \min\{\gamma(G^2)/G^2 = G^1\}$.

An embedding G^2 can be described by a method due to, among others, Hefter and Edmonds (see e.g. [White 73]). Briefly, G^2 can be completely described by giving at each vertex of G^1 a cyclical ordering of the edges incident to it. To every embedding of G one can associate a graph G_* (depending on the embedding) as follows : associate a vertex to every face of the embedding and connect two vertices of G_* if the corresponding faces are adjacent(i.e. share an edge). G_* is called a (geometric) dual of G. There is a natural bijection between the edges of G and those of G_*. Edmonds ([Ed 65]) has shown that an embedding is completely described by a geometric dual. In general the dual is not unique. This notion of dual must not be confused with that of *algebraic dual* introduced by Withney, although the two are intimately connected. Further, the dual has a canonical embedding G_*^2 of the same genus as G^2.

To a graph G^1 one associates a directed graph by associating to every edge of G^1 two arcs with opposite orientations. If the edge was e the two arcs will be denoted arbitrarily by e and e^{-1} and will be called the *sides* of the edge. A face of G^2 is a circuit in the digraph associated to G^1. If a face never traverses an edge or a vertex more than once it is called *simply connected*. An embedding all of whose faces are simply connected is called *quasi-planar* or *simplicial*. The first term was used in [Fi 78] and [Fi-Mi 79], but we prefer now the second. An edge or a vertex that is traversed more than once by the circuit associated to a face shall be called *internal* or *repeated*.

All our notions are combinatorical, but it will be quite often useful to use a geometrical language. This is motivated by the fact that to every embedding there is associated a topological space obtained as follows. To every face associate a copy of the unit disk of the complex plane. Map the oriented boundary of the disk onto the oriented circuit corresponding to the face. When composed with the morphism from the directed graph to the undirected graph that identifies opposite sides of the same edge and various copies of the same vertex, this results in a disk with a certain number of identifications on the boundary which we call the *closed faced*. The *open face* is simply the open disk before any identification. The closed faces are not necessarily simply connected and this is the source of many difficulties.

If H^1 is a *subgraph* of K^1 we write $H^1 \subset K^1$. K^2 is an *extension* of H^2 if (a) $H^1 \subset K^1$ and (b) the cyclical orientation at a vertex of H^1 is the restriction of the cyclical orientation in K^2. An extension $K^2 \supset H^2$ is *conservative* if $\gamma(K^2) = \gamma(H^2)$.

Other notations are as follows :

$I\gamma(G)$ denotes the class of embeddings of genus γ of G.

$I(G) = \bigsqcup_{\gamma\geq 0} I\gamma(G)$, $\iota_\gamma(G) = |I\gamma(G)|$, $\iota(G) = |I(G)|$.

$K\gamma(G)$ denotes the class of all embeddings of subgraphs of G.

$K(G) = \bigsqcup_{\gamma\geq 0} K\gamma(G)$, $\kappa_\gamma(G) = |K\gamma(G)|$, $\kappa(G) = |K(G)|$.

$F\gamma(G)$ denotes the class of all embeddings of G that are frames (see section 3).

$F(G) = \bigsqcup_{\gamma\geq 0} F\gamma(G)$, $\phi_\gamma(G) = |F\gamma(G)|$, $\phi(G) = |F(G)|$.

3. The Rigidity Theorem

Let G^2 be an embedding. A set of faces is said to contain a simplex if that simplex is contained in some subdivision of the faces. A non-contractible cycle on a surface is also called *essential*. A *bracelet* is a pair (f_1, f_2) of simply connected faces whose union contains an essential cycle. It is easy to see (using Jordan's theorem) that a bracelet must contain a pair (s_1, s_2) of disjoint simplexes such that the cycle c consisting of two chains c_1 and c_2 joining s_1 to s_2 in f_1 and f_2 is essential. Such a pair is called a *strap* of the bracelet. A bracelet may have more than one strap. A *frame* is a bracelet-free quasi-planar embedding.

Our goal in this section is to prove the Rigidity Theorem, Theorem 7 below.

The details of this proof will appear elsewhere [Fi-May 80]. We shall only give a series of lemmas from which our theorem will follow.

Tutte in [Tutte 66,p.112] has introduced an operation which we shall call the *expansion of a vertex*. It consists of replacing a vertex v of a graph by two vertices v_1 and v_2 connected by an edge. The edges incident to v will be partitioned into two sets incident to v_1 and v_2 respectively. The remaining vertices and edges will stay unchanged. The expansion of a vertex will be called *proper* if each member of the partition contains at least two members. A graph K^1 is called an *expansion* of H^1 if it is obtained by a sequence of expansions of vertices (notation : $H^1 \leq K^1$). The expansion is proper if each new vertex has valence ≥ 3. An embedding K^2 of K^1 induces a canonical embedding H^2 of H^1. Finally, $H^2 \subset K^2$ if and only if K^1_* is an expansion of H^1_*.

LEMMA 1. If F^2 is a frame and $F^2 \subset G^2$ then G^2 is a frame.

LEMMA 2. Let $G^2 \subset H^2$ be such that H^2_* is a proper expansion of G^2_*.

(a) If G^1_* is 3-connected then so is H^2_*.

(b) if G^1 is simplicial (i.e. without loops or multiple edges) then so is H^1.

LEMMA 3. Let G^2 be a frame of a 3-connected simplicial graph G^1. Then G^1_* is simplicial and 3-connected.

For any graph G we denote by C(G) the matroid of its cycles.

LEMMA 4 (Withney). Let G^2_1 and G^2_2 be two planar embeddings of the same graph G. Then the natural edge bijection between G^1_{1*} and G^1_{2*} induces an isomorphism of $C(G^1_{1*})$ and $C(G^1_{2*})$.

The following lemma is our generalization of the previous one.

LEMMA 5. Let F^2 be a frame and let $F^1 \subset G^1$. Let G^2_1 and G^2_2 be two embeddings of G^1 that are conservative extensions of F^2. Then the natural bijection between the edges of G^1_{1*} and of G^1_{2*} induces an isomorphism of $C(G^1_{1*})$ and $C(G^1_{2*})$.

LEMMA 6 (Withney). Let G and H be 3-connected graphs. Let f be a bijection between the edges of G and those of H that induces an isomorphism of C(G) and C(H). Then f induces an isomorphism of G and H.

This leads us to the following important theorem.

THEOREM 7 (The Rigidity Theorem). Let G^1 be a 3-connected simplicial graph. Let F^2 be a frame and let $\gamma = \gamma(F^2)$. Let G^2_1 and G^2_2 be two conservative extensions of F^2 with G^1 as 1-skeleton. Let G^1_{1*} and G^1_{2*} be the corresponding duals. Then the canonical bijection between the edges of G^1_{1*} and G^1_{2*} induces an isomorphism of G^1_{1*} and G^1_{2*}. Hence G^2_1 and G^2_2 are identical. In particular, the extension (F^2, G^1) is rigid.

Proof. G^1_{1*} and G^1_{2*} are proper expansions of F^1_*. By Lemma 1, G^2_1 and G^2_2 are frames. By Lemma 3, G^1_{1*} and G^1_{2*} are simplicial and 3-connected. By Lemma 5, $C(G^1_{1*}) \simeq C(G^1_{2*})$ under the natural bijection. Finally, $G^1_{1*} \simeq G^1_{2*}$ under the natural bijection. Hence the two embeddings are the same by Edmond's characterization.

Remark. There is an easy algorithm that will construct the unique extension of F^2 to G^1. Namely, select a chain of G^1-F^1 that rests on F^1. We are sure by the Theorem that this chain can be embedded in a single way. Let G^2_2 be the resulting embedding. Proceed in the same manner with G^2_2. In this way a sequence of embeddings is obtained terminating with G^2. This algorithm clearly runs in time bounded by a polynomial in $\alpha_0(G^1)$. A closer analysis would result in a linear polynomial.

4. Separation

4.1. The proof of the Main Theorem in the following section depends on the notion of *separation* or *removal*.

Let $H^2 \subset G^2$ be a conservative extension and let e be a chain internal to a face f of H^2. If no edge of e is internal in G^2 we shall say that the internal chain e has been *removed* or has been *separated* in G^2. Similary, if v is an internal vertex of H^2 that is no longer internal in G^2 we shall say that the internal vertex v has been *removed*. Finally, if (s_1, s_2) is a strap in H^2 but is no longer one in G^2, we shall say that it has been *removed* or *separated*.

LEMMA 8. Let $H^2 \subset G^2$ be a conservative extension.

(a) If a chain e of G^2 is internal, then it is also internal in H^2.

(b) If a vertex v of G^2 is internal, then it is also internal in H^2.

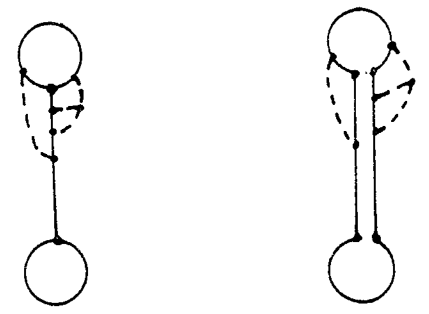
(c) If (s_1, s_2) is a strap in G^2 it is also one in H^2.

Proof. Immediate.
Thus an internal vertex that has been removed in some extensions cannot become internal in some further extension. Similar remarks hold for internal edges and straps.

4.2. The following version of Menger's theorem will be very useful. Let S and T be two disjoints subsets of $V(G^1)$. A set $X \subset V(G^1)$ is said to *separate* S and T if after its deletion no component contains both a vertex of S and one of T (cf. e.g. [Bo-Mu 75]). In particular, if S is a singleton, removal of the vertex trivially separates S from T.

LEMMA 9 (Menger). Let S and T be two disjoint sets of at least k+1 vertices. Let $c_1, c_2, \ldots, c_k$ $(k \geq 1)$ be vertex-disjoint chains connecting S to T. Then S and T are connected by k+1 vertex-disjoint chains if and only if no k-tuple $(v_1, v_2, \ldots, v_k)$ of vertices of $c_1, c_2, \ldots, c_k$ respectively separates S from T.

4.3 Let $H^2 \subset G^2$ be a conservative extension. We shall associate to the extension a planar embedding called its *planar representation*, $P(H^2,G^2)$. To every face $f \in F(H^2)$ associate an undirected cycle in the plane, different faces receiving disjoint disks. The pieces of G^1-H^1 can then be embedded in these disks in exactly the same manner as in G^2. Figure 1 shows an example of such a representation.



H^2 is in solid lines
G^2-H^2 is in dotted lines

$P(H^2,G^2)$

Figure 1

By an abuse of language we denote the cycle that corresponds to f in $P(H^2,G^2)$ by the same letter and the component corresponding to face f by $f(G^2)$. To every internal edge of H^2 there will correspond two edges in $f(G^2)$ called its *sides*. To every internal vertex of H^2 there will correspond a number of vertices on $f(G^2)$ called its *corners*. We call $P(H^2,H^2)$ the planar representation of H^2. If H^2 is quasi-planar, it consists simply of disjoint disks in the plane. Each edge of the boundary of such a disk corresponds to an edge of H^2 and the two sides of an edge always occur in different disks.

4.4. Let again $H^2 \subset G^2$ be a conservative extension, and let $f \in F(H^2)$ have an internal edge e. f is then of the form $eae^{-1}b$ for some directed chains a and b of the directed graph associated to H^1. We shall call a and b the components of the *rim* of f.

LEMMA 10. (a) e is no longer internal in G^2 if and only if $f(G^2)$ has no articulation points in the interior of e.

(b) if G^2 is quasi-planar, then $f(G^2)$ has no articulation points on e.

(c) if e is no longer internal in G^2, then $f(G^2)$ contains two independant chains connecting the opposite sides of the rim.

(d) if G^2 is quasi-planar, then $f(G^2)$ contains two vertex-disjoint chains connecting the rims.

Proof. (a) and (b) follow from the fact that a planar embedding has an internal vertex if and only if the vertex is an articulation point. (c) and (d) follow from Lemma 9.

A pair of chains as in (c) or (d) is said to *separate* the chain e in f.

We now turn to internal vertices.

LEMMA 11. Let $f \in F(H^2)$ and v' and v" be two corners of the same vertex v. If G^2 has no repeated vertices then there exists a chain in $f(G^2)$ that separates v' and v".

4.5 Extending slightly a notion of Tutte [Tu 66], let us call a pair (s_1,s_2) of non-adjacent simplexes of H^1 a *hinge* if its removal disconnects the graph. It can be shown that in a planar graph a pair of simplexes is a hinge if and only if it is common to two faces of any embedding of the graph.

We now modify slightly the notion of strap. First, we shall allow the s_i's to be chains of edges, s_2 and s_2 being non-adjacent. Further, we shall insists that a strap be maximal, i.e. that no pair of non-adjacent chains (s_1',s_2') be a strap if $s_i' \supset s_i$.

Consider now an extension $H^2 \subset G^2$ with H^2 quasi-planar. Let (f_1,f_2) be a bracelet of H^2. In $P(H^2)$, the embedding induced on a bracelet is, after identifying the common chains, of one of the following three types (Figure 2) :

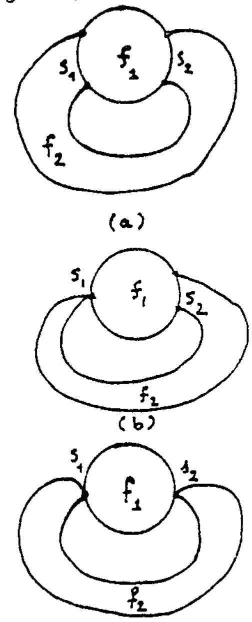


Figure 2

Define the *rim* of (f_1,f_2) with respect to the strap as consisting of the vertices of (the boundary) f_1 and f_2 that are not interior to s_1 or to s_2 (a vertex being considered interior to itself). In cases (a) and (b) the rim has two components. It has four in case (c).

We shall say that the strap (s_1,s_2) has been *removed* from (f_1,f_2) if G^2 has no pair (s_1',s_2') of non-adjacent chains and no pair (f_1',f_2') of faces such that $(s_1',s_2')\subset(s_1,s_2)$, $(f_1',f_2')\subset(f_1,f_2)$ and that (s_1',s_2') is a strap of (f_1',f_2').

A chain of G^2-H^2 all of whose edges are embedded in f_1 or f_2 will be said to be embedded in $f_1\cup f_2$.

LEMMA 12. Let H^2 be quasi-planar and let $H^2\subset G^2$ be a conservative extension. Assume that H^2 possesses a bracelet (f_1,f_2) with strap (s_1,s_2) that has been removed from (f_1,f_2) in G^2. Then

(a) if both s_1 and s_2 are chains of at least one edge, then G^2 possesses three vertex-disjoint chains connecting the two components of the rim and embedded in $f_1\cup f_2$.

(b) if one only of the two components of (s_1,s_2) is a chain of at least one edge, then G^2 possesses two vertex-disjoint chains connecting the two components of the rim and embedded in $f_1\cup f_2$.

(c) if s_1 and s_2 are both vertices, then they are separated by a chain embedded in either f_1 or f_2.

Proof. The proof follows from Lemma 9 and the remark concerning the hinges of planar graphs.

4.6. The set $K(G^1)$ of embeddings of subgraphs of G is partially ordered by $\subset$. This induces a partial ordering on any subset of $K(G^1)$. An *antichain* is a set of mutually incomparable embeddings under this ordering. It is maximal if no superset is an antichain.

Let X, $Y\subset K(G)$. We shall say that Y dominates X (notation : $X\leq Y$) if every embedding of Y extends some embedding of X. A *base* for X is an antichain dominated by X.

5. The Main Theorem

5.1 The Main Theorem that we shall prove in this section shows that a 3-connected graph cannot have too many frames of genus γ. Moreover, they can be listed in polynomial time.

Recall that $\phi_\gamma(G)$ denotes the number of embeddings of G of genus γ and that $\iota_\gamma(G)$ denotes the total number of embeddings of G of genus γ .

THEOREM 13 (Main Theorem). For every genus $\gamma\geq 0$ there exist polynomial p_γ and q_γ such that for any 3-connected graph G :

(a) $\phi_\gamma(G)\leq p_\gamma(\alpha_0(G))$.

(b) the frames of G of genus γ can be effectively listed in time bounded by $q_\gamma(\alpha_0(G))$.

Let us call a class of graphs γ-*semi-rigid* if there exists a polynomial p_γ such that $\iota_\gamma(G)\leq p_\gamma(\alpha_0(G))$ for all graphs G of the class. It follows from the Main Theorem that the class of graphs admitting only frames as embeddings in S_γ is γ-semi-rigid.

To prove the Main Theorem, we shall construct a base $\mathcal{D}(G)$ for $F(G)$. The procedure will be efficient i.e. will run in time bounded by a polynomial in $\alpha_0(G)$. Consequently $|\mathcal{D}(G)|$ is bounded by such a polynomial. By the Rigidity Theorem, it now follows that G^1 has no more than $|\mathcal{D}(G)|$ frames of genus γ. Further, by the remark following the Rigidity Theorem, an extension of F^2 to G^1 can be constructed very efficiently. Thus, we have reduced the problem to that of the construction of $\mathcal{D}(G)$.

To construct $\mathcal{D}(G)$ we shall proceed along lines similar to those used in [Fi 78] and [Fi-Mi 79]. Essentially, the method consists of starting with an embedding that is non-quasi-planar and to study the ways in which a quasi-planar embedding can extend it. We then proceed to remove the straps of the bracelets. More precisely, we shall construct a sequence of antichains of embeddings $\mathcal{A}(G)\leq\mathcal{B}(G)\leq\mathcal{C}(G)\leq\mathcal{D}(G)\leq F(G)$, each member in the sequence being a base for the following one. The embeddings in $\mathcal{A}(G)$ have only one face and are non-quasi-planar. The embeddings in $\mathcal{B}(G)$ have no internal edges. $\mathcal{B}(G)$ is obtained from $\mathcal{A}(G)$ by removing the internal edges. $\mathcal{C}(G)$ is obtained by removing the internal vertices of the embeddings of $\mathcal{B}(G)$. Consequently, the embeddings of $\mathcal{C}(G)$ are all quasi-planar. Finally, the straps are removed from the embeddings in $\mathcal{C}(G)$ resulting in a base $\mathcal{D}(G)$ of frames.

In what follows $\gamma\geq 0$ is fixed, G is a fixed 3-connected graph and T is a fixed spanning tree of G.

5.2. Construction of $\mathcal{A}(G)$

For any set S of edges of G-T, we let T(S) be the subgraph constituted of S together with all the paths of T connecting two vertices u and v that are extremities of edges in S. $\nu(v)$ denotes the valence of vertex v.

LEMMA 14 ([Fi-Mi 79]). For any graph G :

(a) $\Sigma\nu(v) = 2\alpha_1(G)$, where the sum extends over all $v\in V(G)$.

(b) $\Sigma(\nu(v) - 2) = 2(\beta_1(G) - \beta_0(G))$, where the sum extends over all $v\in V(G)$.

(c) G has no more than $2(\beta_1(G) - \beta_0(G))$ vertices of valence ≥ 3.

(d) the total number of embeddings of G does not exceed $(2\beta_1(G) - 2\beta_0(G) + 1)!$

(e) the total number of embeddings of G does not exceed the number of embeddings of a bouquet of $\beta_1(G)$ circles.

Proof. (a) The sum on the left-hand side counts every edge twice.

(b) $\Sigma(\nu(v) - 2) = \Sigma\nu(v) - 2\alpha_0(G) =$

$= 2\alpha_1(G) - 2\alpha_0(G) = 2(\beta_1(G) - \beta_0(G))$.

(c) follows from (b) immediately.

(d) G has $\Pi(\nu(v) - 1)!$. The product contains $\Sigma(\nu(v) - 1)$ factors of wich only $\Sigma(\nu(v) - 2)$ are greater than 1. The product cannot therefore exceed $(\Sigma(\nu(v) - 2) + 1)! = (2\beta_1(G) - 2\beta_0(G) + 1)!$

(e) A bouquet of $\beta_1(G)$ circles has $(2\beta_1(G) - 1)!$ embeddings. Since $\beta_0(G)\geq 1$ the results follows. A direct geometric proof is also possible.

LEMMA 15 ([Fi-Mi 79]). Let G^2 be an embedding of G with $\gamma(G^2) = \gamma$. There exists a set $S\subset E(G-T)$ of size 2γ such that

(a) G^2 induces on $H = T(S)$ an embedding H^2 with $\gamma(H^2) = \gamma$.

(b) H is minimal with property (a).

(c) $\alpha_2(H^2) = \beta_o(H^2)$.

We let $A(G) = \{H^2 \mid H^1 = T(S), S\subseteq E(G-T), \gamma(H^2) = \gamma\}$. By Lemma 15, A(G) is a base for $I_\gamma(G)$. Clearly,

$$|A(G)| \leq \binom{\alpha_1(G)}{2\gamma}.(2(2\gamma-\beta_o(H))+1)! \leq \binom{\alpha_1(G)}{2\gamma}[(4\gamma+1)!].$$

Thus $|A(G)| = \mathcal{O}\left(\alpha_o(G)^{\mathcal{O}(\gamma)}\right)$.

For any graph its *reduction* is obtained by suppressing all vertices of valence 2 and replacing the two incident edges by a single one attached to their non-common extremity. Any embedding gives rise to exactly one of the reduction. The reductions of graphs in A(G) have relatively few edges, as is seen in

LEMMA 16 ([Fi-Mi 79]). Let $H = T(S)\epsilon A(G)$ and let K be its reduction. Then

(a) $\alpha_o(K)\leq 2(\beta_1(K) - \beta_o(K))\leq 2(2\gamma - 1)$.

(b) $\alpha_1(K)\leq 3(\beta_1(K^2) - \beta_o(K^2))\leq 3(2\gamma - 1)$.

(c) H has at most $[2(2\gamma - 1) + 1]!$ embeddings.

Proof. $\beta_1(K) = \beta_1(H) = 2\gamma$ since the cyclomatic number equals the number of edges in the complement of a maximal tree. $\beta_o(K) = \beta_o(H)\leq 1$. By Lemma 14 (c), $\alpha_o(K)\leq 2(\beta_1(K) - \beta_o(K)) = 2(2\gamma - \beta_o(K)) \leq 2(2\gamma - 1)$. To obtain a bound on $\alpha_1(K)$ use Euler's formula and the fact that $\alpha_2(K^2) = \alpha_2(H^2) = \beta_o(H^2) = \beta_o(K^2)$ (by Lemma 15 (c)). This yields $\alpha_1(K^2) = \alpha_o(K^2) - \beta_o(K^2) + \beta_1(K^2)\leq$ $\leq 3(\beta_1(K^2) - \beta_o(K^2))\leq 3(2\gamma - 1)$. Finally, (c) follows from Lemma 14 (d).

We shall now estimate the time required for the construction of A(G). First the construction of T can be done in $\mathcal{O}(\alpha_o \log\alpha_o)$ steps (because, by Euler's formula, α_1 is a linear function of α_o), using for example Kruskal's greedy algorithm ([Kr 56]). There are $\mathcal{O}(\alpha_o{}^{2\gamma})$ choices for the set S. Each graph H = T(S) admits at most $\mathcal{O}(4\gamma^{4\gamma})$ embeddings. Thus the construction of A(G) requires no more than $\mathcal{O}(\alpha_o{}^{2\gamma})$ steps (for $\gamma\geq 1$, which is anyway the only case of real interest).

5.3. Construction of $\mathcal{B}(G)$

We shall construct a chain $A(G) = \mathcal{B}_o(G) \leq \mathcal{B}_1(G) \leq \ldots\leq \mathcal{B}_r(G) = \mathcal{B}(G)$. Assume $H^2\epsilon\mathcal{B}_i(G)$ has been constructed. We shall construct from H^2 a set of extensions that will belong to $\mathcal{B}_{i+1}(G)$. If H^2 has no internal edges then we let H^2 be a member of $\mathcal{B}_{i+1}(G)$.

Assume that H^2 has an internal chain e belonging to a face f. Using the notations of section 4.2, let a and b be the rims of f. Let L^1 be the subgraph of G^1 induced by the edges in $(G^1-H^1)\cup\{e\}$. By Lemma 10, any frame extending H^2 will contain two vertex-disjoint chains connecting the rims and embedded in f. We would like to place in $\mathcal{B}_{i+1}(G)$ all possible extensions obtained by augmenting H^2 by two such chains. Unfortunately, this number is much too high. Observe however that if a chain c has been embedded in f then any other chain of G^1-H^1 that touches c at an interior vertex will be also embedded in f. Therefore it suffices to specify the edges that begin and end the separating chains and then verify that such chains exist. To summarize, H^2 will generate a family of embeddings as follows : pick in all possible ways two edges e_1 and e_2 in L^1 that have exactly one extremity on a and two edges e_1' and e_2' in L^1 that have exactly one extremity on b. Using a standard max-flow-min-cut algorithm construct two disjoint chains c_1 and c_2 in G^1-H^1 connecting $\{e_1,e_2\}$ to $\{e_1',e_2'\}$. If no such chains exist take the next choice. Let K^2 be the embedding obtained by embedding c_1 and c_2 in f and removing e. $\mathcal{B}_{i+1}(G)$ will consist of all such embeddings.

Clearly, the chain $(\mathcal{B}_i(G))_{i\geq 0}$ beconmes stationary for i greater than the number of internal edges of H^2. We let $\mathcal{B}(G)$ equal its limiting value.

The size of $\mathcal{B}(G)$ is bounded by the number of choices for $\{e_1,e_2,e_1',e_2'\}$ which is $\mathcal{O}(\alpha_o(G)^4)$. An upper bound on the running time of the max-flow-min-cut algorithm is $\alpha_o(G)^2$. It is now easy to see that this algorithm runs in time $\mathcal{O}(\alpha_o(G)^{l(\gamma)})$ where l is a linear function.

For a graph H let H_{red} be its reduction. We shall now estimate $\alpha_o(K^1_{red})$ and $\alpha_1(K^1_{red})$ for an embedding $K^2\epsilon\mathcal{B}(G)$. Let $H^2\epsilon A(G)$ be such that $H^2\subseteq K^2$. By Lemma 16, $\alpha_o(H^1_{red})\leq 2\mu$ and $\alpha_1(H^1_{red})\leq 3\mu$ where we put $\mu = 2\gamma - 1$. Every internal edge of H^1_{red} gives rise in K^1_{red} to at most two additional edges. Therefore $\alpha_1(K^1_{red})\leq 6\mu$. Every time a chain is removed at most two new vertices are being created. Hence $\alpha_o(K^1_{red})\leq 2\mu + 2.3\mu = 8\mu$.

The number of internal vertices of K^2_{red} does not exceed 8μ and the multiplicity of an internal vertex does not exceed 6μ.

5.4. Constructionof C(G)

Let $K^2\epsilon\mathcal{B}(G)$. By Lemma 11, if G^2 is a frame of G, then any two co-facial corners of the same vertex of K^2 will be separated by a chain in G^2. By Lemma 8, it suffices to remove all the existing internal vertices and no new ones will be created in the process. Suppose that a face of K^2 has r repeated vertices, each with $n_1,n_2,\ldots,n_r$ corners respectively. In G^2 any two consecutive corners of the same vertex must be separated by a chain. Therefore there can be at most Σn_i separating chains and at most $2\Sigma n_i$ different extremities.

Each embedding $K^2\epsilon\mathcal{B}(G)$ will generate embeddings L^2 in C(G) as follows.

(a) For each face $f\epsilon F(K^2)$ containing internal vertices and for each pair (u',u") of consecutive corners of the same vertex on f, choose a pair (v',v") of corners of f skew to (u',u"). (v',v") will be candidate extremities of a chain separating u' and u" and embedded in f.

(b) Find all assignments of components of G^1-K^1 to each pair (v',v") of separating corners satisfying the following condition : if $C(v_1',v_2'')$ and

$C(v_2', v_2'')$ are the components assigned to (v_1', v_1'') and (v_2', v_2'') respectively and if (v_1', v_1'') and (v_2', v_2'') are mutually skew then $C(v_1', v_1'') = C(v_2', v_2'')$.

(c) For each assignment in (b) construct in each component $C(v', v'')$ a chain connecting v' to v''.

(d) Embed the chains constructed at (c) in the corresponding faces.

(e) For each chain embedded in (d) retain only the portion between f and the intersection point with another separating chain that is closest to the boundary of f.

L_{red} is obtained from K_{red} by adjoining at most $\sum_{v \in K^1} \nu(v)$ separating chains. By Lemma 14 (b) we have

$\Sigma\nu(v) = 2\mu + 2\alpha_0(K^1) \le 2\mu + 2.8\mu = 18\mu$, where as in 5.4, $\mu = 2\gamma - 1$. The separating chains add at most $2.18\mu = 36\mu$ vertices to those already in K_{red} by perhaps subdivising an edge of K_{red} and at most an additional $18^2\mu^2 = 324\mu^2$ possible intersection points between the separating chains. However, since at (e) we retain on each chain only at most two edges and two vertices in addition to those on f, it follows that the separating chains add at most 72μ vertices. Hence $\alpha_0(L^1_{red}) \le 72\mu + 8\mu = 80\mu$. By a similar argument, it follows that $\alpha_1(L^1_{red}) \le 96\mu$. From Euler's formula we get $\alpha_2(L^1_{red}) \le 2 - 2\gamma + \alpha_1(L^1_{red}) \le 1 + \mu + 96\mu = 97\mu + 1 \le 98\mu$.

To evaluate the number of straps and bracelets, notice that every strap of L^1_{red} must involve either an internal edge of H^2 or an internal vertex of K^2. Thus there cannot be more than 80μ straps.

The running time of this procedure is dominated by step (a). Step (a) requires the choice of no more than 36μ vertices, hence $\mathcal{O}(\alpha_0(G)^{36\mu})$ choices. Steps (b), (c), (d) and (e) require no more than $\alpha_0(G)^3$ steps.

5.4 Construction of $\mathcal{D}(G)$

This construction is based on Lemmas 8 and 12. By Lemma 8 once all straps have been removed no new ones have been created. By Lemma 12 the simplexes of a chain can be separated by one, two or three chains, depending on the nature of the strap. The number of straps of an embedding $L^2 \in C(G)$ does not exceed 80μ ($\mu = 2\gamma - 1$). To remove all straps no more than 240μ separating chains are needed. The construction is very similar to that of $\mathcal{B}(G)$. We shall omit the details.

This completely concludes the proof. We have actually shown that the degree of the polynomial p_γ and q_γ is linear in γ as was stated in the Introduction (unfortunately this statement was omitted from the statement of Theorem 13). We have the following important

COROLLARY 17. There exists an algorithm that runs in time $\mathcal{O}(\alpha_0(G)^{l(\gamma)})$ for determining the isomorphism of 3-connected graphs of genus γ that admit only frames of genus γ, where l is a linear function.

Proof. List of all frames of genus γ of both graphs. Use then any standard labeling procedure (e.g. Weinberg's) to generate codes for the embeddings and then compare the codes.

6. The isomorphism of 3-connected gragphs

Let $v \in V(G^1)$. $G\backslash v$ will denote the graph obtained by replacing v by $\nu(v)$ copies and making each adjacent to one of the $\nu(v)$ edges incident to v in G^1. We say then that v has been *exploded*. For $\mathfrak{X} \subset V(G^1)$ define $G\backslash\mathfrak{X}$ inductively on $|\mathfrak{X}|$ by $G\backslash(\mathfrak{X}\cup\{x\}) = (G\backslash\mathfrak{X})\backslash x$. An embedding G^2 of G^1 induces an embedding $G^2\backslash v$ on $G^1\backslash v$ restricting the cyclical orientations. For $\mathfrak{X} \subset E(G^1)$ we denote by $G^1\backslash\mathfrak{X}$ the subgraph induced by $E(G^1) - \mathfrak{X}$. For $e \in E(G^1)$ we write $G^1\backslash e$ instead of $G^1\backslash\{e\}$. G^2 induces an embedding $G^2\backslash\mathfrak{X}$ on $G^1\backslash\mathfrak{X}$ by restricting the rotations.

LEMMA 18 Let G^2 be an embedding of G^1.

(a) Assume G^1 is 2-connected. Then v is an internal vertex of G^2 if and only if $\gamma(G^2\backslash v) < \gamma(G^2)$.

(b) Assume G^1 is 3-connected. Then (s,t) is a strap of G^2 if and only if

(i) $\gamma(G^2\backslash s) = \gamma(G^2) = \gamma(G^2\backslash t)$,

(ii) $\gamma(G^2\backslash(s,t)) < \gamma(G^2)$.

(c) Let G^1 be 2-connected. If v is an internal vertex of some embedding of genus $\gamma(G^1)$, then $\gamma(G^1\backslash v) < \gamma(G^1)$.

(d) Let G^1 be 3-connected and let (s,t) be a strap of an embedding of genus $\gamma(G^1)$ of G^1. Then

(i) $\gamma(G^1\backslash s) = \gamma(G^1) = \gamma(G^1\backslash t)$.

(ii) $\gamma(G^1\backslash(s,t)) < \gamma(G^1)$.

The algorithm for deciding the isomorphism of two 3-connected graphs G_1 and G_2 proceeds by bootstrapping. We assume an algorithm for genus $\gamma-1$ and lower. An algorithm for the case $\gamma = 0$ is well known [Ho-Ta 72]. For simplicity, we may assume that the graphs are not non-isomorphic in "obvious" ways (e.g. they don't have the same degree sequence). The algorithm is then the following.

(1) Check wether $\gamma(G_1) = \gamma = \gamma(G_2)$. This can be done in time $\mathcal{O}(\alpha_0(G_i)^{l(\gamma)})$ steps, where l is linear ([Fi-Mi 79]). If not the graphs are not isomorphic.

(2) Select all possible pairs $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $\gamma(G_1\backslash v_1) < \gamma(G_1)$ and $\gamma(G_2\backslash v_2) < \gamma(G_2)$ (i.e. such that v_i is an internal vertex in some embedding of G_i). For each such pair execute step 3. When step 3 has been executed for all pairs go to step 4.

(3) Check wether $G_1\backslash v_1 \sim G_2\backslash v_2$ using the isomorphism algorithm for genus $\gamma-1$. It can be shown that if $G_1\backslash v_1 \sim G_2\backslash v_2$ then $G_1 \sim G_2$. If $G_1\backslash v_1 \sim G_2\backslash v_2$ stop. Else return to step (2).

(4) Select in all possible ways two pairs (s_1, t_1) and (s_2, t_2) to play the role of straps in G_1 and G_2 respectively. For each couple, check wether $\gamma(G_1\backslash(s_1,t_1)) < \gamma(G_1)$ and $\gamma(G_2\backslash(s_2,t_2)) < \gamma(G_2)$. If this is so attach distinguishing tags to the vertices replacing s_i and t_i ($i = 1,2$). Let G_1' and

and G_2^j be the resulting graphs. For each pair (G_1^i, G_2^j) execute step (5). If all such pairs have been explored, go to step (6).

(5) Check, using the algorithm for genus $\gamma-1$, wether $G_1^i \sim G_2^j$. It can be shown that if $G_1^i \sim G_2^j$ then $G_1 \sim G_2$.

(6) At this stage, G_1 and G_2 admit only frames as embeddings of genus γ, hence G_1 and G_2 can be tested for isomorphism by the algorithm of Corollary 17.

This algorithm runs in time $p_\gamma(\alpha_0(G_i))$ for some polynomial p_γ of degree linear in γ.

7. The general case

The general case proceeds as in [Ho-Ta 72], as indicated in the Introduction.

8. References

[Ba 79] L. Babai and L. Kucera, Canonical Labelling of Graphs in Linear Average Time, Proc. of the 20-th Annual Symp. Foundations of Computer Science, Oct 29-31, 1979, San Juan, Puerto Rico,(published by the IEEE Computer Society).

[Bo-Mu 75] J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", American Elsevier Publishing Company, New York, N.Y., 1976.

[Col 78] C.J. Colbourn, A Bibliography of the Graph Isomorphism Problem, Technical Report n°123, Dept. of Computer Science, University of Toronto, 1978.

[Ed 65] J. Edmonds, On the Surface Duality of Linear Graphs, J. Res. Natl. Bur. Standards-B. Mathematics and Mathematical Physics, 69.B, 1-2 (Jan.-June 1965), 121-123.

[Fi 78] I.S. Filotti, An Algorithm for Imbedding Cubic Graphs in the Torus, J.C.S.S. 1980 (to appear).(A preliminary version of this paper was published in the Proc. Tenth Annual ACM Symp. Theory of Computing, San Diego, California, May 1980.

[Fi-Mi 79] I.S Filotti and G. Miller, Efficient Determination of the Genus of a Graph (to appear). (A preliminary version of this paper was published in the Proc. Eleventh Annual ACM Symp. Theory of Computing, Atlanta, Georgia, May 1979).

[Fi-May 80] A rigidity theorem for graph embeddings (unpublished).

[Ho-Ta 72] J.E. Hopcroft and R.E. Tarjan, Isomorphism of Planar Graphs (Working Paper) in "Complexity of Computer Computations" (Miller et al. eds.), Plenum Press, 1972.

[Kr 56] J.B. Kruskal, On the Shortest Spanning Subtree of a Graph and the Travelling Salesman Problem, Proc. Amer. Math. Soc., 7 (1956), 48-50.

[Mi 77] G.L. Miller, Graph Isomorphism, General Remarks, Proc. 9-th ACM Symposium on the Theory of Computing, 143-150 (1977).

[Re-Cor 77] R.C. Read and D.G. Corneil, The Graph Isomorphism Disease, J. Graph Theory, 1, 339-363 (1977).

[Re-Ni-De 77] Edward M. Reingold, Jurg Nievergelt and Narsingh Deo, "Combinatorial Algorithms : Theory and Practice", Prentice Hall, Englewood Cliffs, N.J.,1977.

[Tu 66] W.T. Tutte, "Connectivity in Graphs", U. of Toronto Press, Toronto 1966.

[We 66] L. Weinberg, A Simple and Efficient Algorithm for Determining Isómorphism of Planar Triply Connected Graphs, IEEE Trans. on Circuit Theory, CT-13, 2 (1966), 142-148.

[White 73] A. White, "Graphs, Groups and Surfaces", North-Holland Publishing Co.,-American Elsevier Co., New York, 1973.

[Whitney 32] H. Whitney, "Congruent Graphs and the Connectivity of Graphs", Amer. J. Math., 54 (1932),150-168.