Some results on feedback stabilization of a one-link flexible arm

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Quelques résultats sur la stabilisation d'un bras robot flexible

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Abstract : The aim of this report is to exhibit a class of stabilizing feedback laws for a structurally undamped one–link flexible arm with control applied at one extremity. A finite–dimensional model of the arm, corresponding to a discretization in space, is first considered. The arm is then modeled by a classical partial differential equation, no finite dimensional approximation being made in the control analysis. In both cases, stability of the closed–loop system is proven, using a Lyapunov method and applying Lasalle's principle. The practical significance of the proposed results is commented upon and illustrated by simulation experiments.

Key words: modeling and control of flexible structures, partial differential equations, feedback stabilization.

Résumé : On étudie dans ce rapport une classe de feedbacks stabilisants pour un bras robot flexible sans amortissement structurel commandé à l'une de ses extrémités. On considère tout d'abord un modèle de dimension finie du bras, correspondant à une discrétisation spatiale, puis une équation au dérivées partielles classique, exploitée sans approximation dans l'analyse de la commande. Dans les deux cas, on prouve la stabilité du système en boucle fermée, en utilisant la méthode de Lyapunov et le principe d'invariance de Lasalle. L'intêret pratique des résultats proposés est commenté et illustré par des simulations numériques.

Mots clés : modélisation et commande de structures flexibles, équations aux dérivées partielles, stabilisation par feedback.

Contents

List of Figures

1 Introduction

It is known that rigid mechanisms such as robot manipulators can be stabilized, either to a point or along a given trajectory, by using simple feedback laws the computation of which does not require a precise modeling of the system. For example, it has been shown that decentralized proportional-derivative (P.D.) control laws with negative control gains unconditionally stabilize any rigid manipulator [AM] [SLE]. In order to prove this type of result, inherent properties of mechanical structures, such as positivity of the system's inertia matrix or energy passivity associated with Hamiltonian systems, are explicitly used in the stability analysis. The practical interest of such feedback controls is that i) they are simple and thus easily implementable, and ii) they are robust with respect to modeling errors.
The background motivation for the present study is to explore whether similar stability and robustness results can be derived for mechanical systems with distributed flexibility. To our understanding, this question is closely related to the issue of determining the minimal amount of "knowledge" one has to have about the system in order to be able to design (efficient) stabilizing controls. The fact that Hamilton's principle is general and applies to flexible systems as well already constitute an incentive for investigating in this direction, as illustrated by many studies dedicated to active vibrational damping in the case of systems with collocated sensors and actuators. In the robotics literature, robust control of flexible arms has also motivated various studies, and we have been particularly intrigued by the work performed in [LKK] and [LAM] where a 'PDS' control, i.e. a conventional PD control complemented by a feedback of strain deformation detected at the arm's root, is considered and tested experimentally.
In order to approach these issues, point out some possibilities and provide a basis for future studies, we have chosen to work on the problem of stabilizing a simple one degree-of-freedom arm subject to torsional flexibility. When discretized in space (by a method of finite elements, for example), this system can be modeled by a linear finite dimensional ordinary differential equation the dimension of which is given by the number of retained vibration modes. Alternatively, it can also be continuously modeled by a one-dimensional partial differential wave equation (P.D.E.) with non-constant coefficients and two Neumann boundary conditions. Both modeling possibilities are considered here in order to show that the proposed class of boundary feedback controls (which involve boundary *and* distributed observations) and associated stability conditions can be derived and expressed in quite similar ways in the two cases. Controls of the 'PDS' type are obtained as particular elements of this class.

The paper is organized as follows. In section 2, we present both discrete and continuous models of our flexible torque arm. We set up some notations and present the class of feedback laws which is studied in the remaining of the report. We state the main results. Section 3 is devoted to the study of the discrete modeled arm. It is shown, by exhibiting a Lyapunov's function and applying Lasalle's theorem that the system is stable. Some simulations illustrates our purpose. The P.D.E. modeling case is developped in section 4. We give some useful properties of the eigenfunctions of the autonomous system and deduce by the same method as before (Lyapunov and Lasalle) that the closed-loop system is stable. Moreover, we show that for a boundary feedback the decay of the energy is exponential and we give an upper bound for the energy decay rate.

2 Models, notations and problem statement

A way of deriving a finite dimensional model of a shaft which is flexible in torsion (or in compression) consists of considering a succession of $n+1$ rigid bodies with mass m_l $(l = 1, ..., n+1)$ sequentially connected by n springs (see **figure 1**). Application of a torque (or force) control τ to the first body then yields the following model equation:

$$M_{n+1} \begin{bmatrix} \ddot{\alpha} \\ \ddot{q}_n \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_n \end{bmatrix} \begin{bmatrix} \alpha \\ q_n \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, \tag{2.1}$$

where:

- K_n is a $(n \times n)$ diagonal matrix composed of the spring stiffnesses k_i $(i = 1, ..., n)$,
- α is the absolute displacement of the first body,
- q_n is the vector in R^n of (relative) springs' deformations,
- $M_{n+1}(i,j) \; = \; \sum_{l=1}^{sup(i,j)} \; m_l$.

Notice that the inertia matrix may recursively be defined as follows:

$$M_{n+1} = \begin{bmatrix} M_n(1,1) + m_{n+1} & M_n(1)^T \\ M_n(1) & M_n \end{bmatrix},$$

where $M_n(1)$ is the first column of M_n. By eliminating the absolute position variable α, one deduces from (2.1) the following equation which characterizes the internal deformations of the system:

$$N_n \ddot{q}_n + K_n q_n = -\frac{M_n(1)}{M_{n+1}(1,1)} \tau, \tag{2.2}$$

with:

$$N_n = M_n - \frac{M_n(1) M_n(1)^T}{M_{n+1}(1,1)}.$$

N_n is obviously a symmetrical matrix. It is also positive, as established further.
Let $\{\Phi_n(i)\}_{i=1,...,n}$ denote a basis of eigenvectors of the matrix $K_n^{-1} N_n$, and $D_n^{-1} = diag\{d_i^{-1}\}$ the positive diagonal matrix similar to $K_n^{-1} N_n$, it is simple to verify from (2.1) (and well known) that the deformations of the "free" system (when $\tau = 0$) are oscillatory and given by :

$$q_n(t) \; = \; \sum_{i=1}^{n} \; \Phi_n(i) z_n(i)(t),$$

with:

$$\ddot{z}_n(i) \; + \; d_i z_n(i) \; = \; 0 \quad (i = 1, ..., n).$$

According to a classical terminology, the frequencies $\frac{d_i^{1/2}}{2\pi}$ are called natural "eigenfrequencies" of the system, and the associated vectors $\Phi_n(i)$ are the system's "eigenforms".

The purpose of the first part of this report is to show that any feedback control of the form :

$$\tau \; = \; -k_p(\alpha - G_n^T q_n) \; - \; k_v(\dot{\alpha} - G_n^T \dot{q}_n),$$

with $k_p > 0$, $k_v > 0$, and :

$$(G_n^T \Phi_n(i))\ \Phi_n(i,1)\ \geq\ 0\,,$$

exponentially stabilizes the system (2.1).

When the arm's flexibility is distributed, it is also possible to use a partial differential equation (P.D.E.) approach in order to model the system. More precisely, a torsionally flexible unit-lenght arm, with torsional rigidity $k(x)$, mass density $m(x)$, and a boundary control τ applied to one of its extremities can be modeled by the one-dimensional wave equation:

$$m(x)y_{tt}(x,t) - (k(x)y_x)_x(x,t) \ = \ 0 \quad , \quad x \in]0,1[\,, \tag{2.3}$$

with boundary conditions:

$$\begin{cases} (k\, y_x)(0,t) = -\tau(t)\,, \\ (k\, y_x)(1,t) = 0\,, \end{cases} \tag{2.4}$$

where $y(x,t)$ represents the arm's angle at coordinate x and time t, $y_x = \frac{\partial y}{\partial x}$, and $y_t = \frac{\partial y}{\partial t}$. With $\{\Psi_i(x)\}_{i\in N}$ denoting, as in the finite dimensional case, a set of natural eigenfunctions of the system, it will be shown in the second part of the article that the following feedback control :

$$\tau(t) \ = \ -k_p(y(0,t) - \int_0^1 g(x)y_x(x,t)dx) \ - \ k_v(y_t(0,t) - \int_0^1 g(x)y_{xt}(x,t)dx)\,, \tag{2.5}$$

with $k_p > 0$, $k_v > 0$ and a smooth enough function $g(x)$ such that $g(1) = 0$ and :

$$(\int_0^1 g(x)\Psi'_i(x)dx)\ \Psi_i(0)\ \leq -g(0)\,\Psi_i^2(0) \quad , \quad \forall i \in N\,,$$

asymptotically stabilizes the system. In comparison with the finite dimensional case, exponential convergence will be proven only when $g(x) = 0$.

3 Finite dimensional case

3.1 Preliminary results

Before proving the main result, let us establish some useful properties.

Proposition 1 i) *M_n is a symmetric positive definite matrix,*
ii) *N_n is a symmetric positive definite matrix,*
iii) *$K_n^{-1}N_n$ is similar to a positive diagonal matrix denoted as D_n^{-1},*
iv) *Let Φ_n denote a $(n \times n)$ matrix the columns of which form a basis of eigenvectors of $K_n^{-1}N_n$, then: $\Phi_n(i,1) \neq 0$, for $i = 1, ..., n$,*
v) *The eigenvalues of $K_n^{-1}N_n$ are all distinct.*

Proof :
i) It is simple to verify i) by induction on n.

ii) Setting:

$$y \ = \ \begin{bmatrix} -\frac{M_n(1)^T}{M_{n+1}(1,1)}x \\ x \end{bmatrix}\,,$$

we have, from the definition of N_n :

$$x^T N_n x = y^T M_{n+1} y \,,$$

and the positivity of N_n thus simply results from the positivity of M_{n+1}.

iii) Since $K_n^{-1/2} N_n K_n^{-1/2}$ is symmetric positive definite, it is similar to a positive diagonal matrix D_n^{-1} and there exists a unitary matrix P_n such that:

$$K_n^{-1/2} N_n K_n^{-1/2} \;=\; P_n D_n^{-1} P_n^T \,.$$

Therefore:

$$\Phi_n{}^{-1} K_n^{-1} \Phi_n \;=\; D_n^{-1} \,, \text{ with } \Phi_n \;=\; K_n^{-1/2} P_n \,.$$

iv) Let us prove this point by induction and contradiction. The property is obviously true for $n = 1$. Let us assume that it is true up to n, but not for $(n+1)$. Let:

$$\Phi_{n+1}(i) \;=\; \begin{bmatrix} 0 \\ \Psi_n(i) \end{bmatrix}$$

denote an eigenvector of $K_{n+1}^{-1} N_{n+1}$, and $\lambda_{n+1}(i)$ the associated positive eigenvalue. By using the definitions of M_n and N_n, we verify that :

$$N_{n+1} \Phi_{n+1}(i) \;=\; \begin{bmatrix} (1 - \dfrac{M_{n+1}(1,1)}{M_{n+2}(1,1)}) M_n(1)^T \Psi_n(i) \\ \\ (M_n - \dfrac{M_n(1) M_n(1)^T}{M_{n+2}(1,1)}) \Psi_n(i) \end{bmatrix} .$$

Thus :

$$K_{n+1}^{-1} N_{n+1} \Phi_{n+1}(i) \;=\; \begin{bmatrix} (1/k_1)(1 - \dfrac{M_{n+1}(1,1)}{M_{n+2}(1,1)}) M_n(1)^T \Psi_n(i) \\ \\ K_n^{-1}(M_n - \dfrac{M_n(1) M_n(1)^T}{M_{n+2}(1,1)}) \Psi_n(i) \end{bmatrix} . \tag{3.1}$$

On the other hand, since $\Phi_{n+1}(i)$ is an eigenvector :

$$K_{n+1}^{-1} N_{n+1} \Phi_{n+1}(i) \;=\; \lambda_{n+1}(i)\; \Phi_{n+1}(i) \;=\; \begin{bmatrix} 0 \\ \lambda_{n+1}(i)\; \Psi_n(i) \end{bmatrix} . \tag{3.2}$$

By comparing (3.1) and (3.2) :

$$M_n(1)^T \Psi_n(i) \;=\; 0 \,, \tag{3.3}$$

and :

$$K_n^{-1} M_n \Psi_n(i) \;=\; \lambda_{n+1}(i)\; \Psi_n(i) \,. \tag{3.4}$$

Therefore, by premultiplying the members of (3.4) by $[1, 0, ..., 0]$, and using (3.3) :

$$\Psi_n(i, 1) \;=\; 0 \,. \tag{3.5}$$

We also have, by using the definition of N_n :

$$K_n^{-1} N_n \Psi_n(i) \;=\; K_n^{-1} M_n \Psi_n(i) - K_n^{-1} \tfrac{M_n(1) M_n(1)^T}{M_{n+1}(1,1)} \Psi_n(i) \;.$$

Thus, $\Psi_n(i)$ is an eigenvector of $K_n^{-1} N_n$, and relation (3.5) then contradicts our initial assumption.

v) Let us assume that there exists a double eigenvalue λ associated with two independent eigenvectors $\Phi_n(i)$ and $\Phi_n(j)$ of $K_n^{-1} N_n$. Due to iv), one may normalize these vectors so that :
$\Phi_n(i,1) \;=\; \Phi_n(j,1)$. Then $\Phi_n(i) - \Phi_n(j)$ is also an eigenvector of $K_n^{-1} N_n$, and the first component of this vector is zero. This contradicts iv). ■

Let us then define a vector z_n as follows:

$$z_n \;=\; {\Phi_n}^{-1}\, q_n \,, \tag{3.6}$$

with:

$$\begin{array}{l} \Phi_n \;=\; K_n^{-1/2} P_n \,, \\ K_n^{-1/2} N_n K_n^{-1/2} \;=\; P_n D_n^{-1} {P_n}^T \,, \\ D_n \;:\; \textit{positive diagonal}\,, \\ P_n \;:\; \textit{unitary matrix}\,. \end{array} \tag{3.7}$$

Φ_n is a matrix the columns of which form a basis of eigenvectors of $K_n^{-1} N_n$ (since ${\Phi_n}^{-1} K_n^{-1} N_n \Phi_n = D_n^{-1}$), and the components of the diagonal matrix D_n^{-1} are the eigenvalues of $K_n^{-1} N_n$.
We then have the following result :

Proposition 2 *z_n satisfies the equation :*

$$\ddot{z}_n \;+\; D_n z_n \;=\; -\frac{k_1}{m_{n+1}} \begin{bmatrix} \Phi_n(1,1) \\ \cdot \\ \cdot \\ \Phi_n(n,1) \end{bmatrix} \tau \,. \tag{3.8}$$

Proof :
– From 2.2 and the definition of z_n, one easily verifies that:

$$\ddot{z}_n \;+\; D_n z_n \;=\; -\frac{1}{M_{n+1}(1,1)}\; D_n {\Phi_n}^T M_n(1)\; \tau \,. \tag{3.9}$$

By definition of N_n:

$$N_n(1)^T = \left(1 - \frac{M_n(1,1)}{M_{n+1}(1,1)}\right) M_n(1)^T = \frac{m_{n+1}}{M_{n+1}(1,1)} M_n(1)^T \,. \tag{3.10}$$

and, since $N_n = K_n \Phi_n D_n^{-1}$:

$$N_n(1)^T \Phi_n \;=\; k_1\, [\Phi_n(1,1) ... \Phi_n(n,1)]\; D_n^{-1} \,. \tag{3.11}$$

– From (3.10) and (3.11) :

$$\frac{1}{M_{n+1}(1,1)} M_n(1)^T \Phi_n \;=\; \frac{k_1}{m_{n+1}} D_n^{-1} \begin{bmatrix} \Phi_n(1,1) \\ \cdot \\ \cdot \\ \Phi_n(n,1) \end{bmatrix}, \tag{3.12}$$

and (3.8) is obtained by using (3.12) in (3.9). ■

3.2 Main result

Theorem 1 *Let $\{\Phi_n(i)\}$ denote an eigenbasis of $K_n^{-1} N_n$. Let k_p and k_v be two positive real numbers. Let G_n be a vector in R^n such that :*

$$(G_n^T \Phi_n(i)) \Phi_n(i,1) \geq 0\,, \text{ for } i = 1, ..., n\,.$$

Then, the feedback control:

$$\tau \;=\; -k_p(\alpha - G_n^T q_n) \;-\; k_v(\dot{\alpha} - G_n^T \dot{q}_n) \tag{3.13}$$

exponentially stabilizes the system 2.1.

Proof : We will use for the proof the particular eigenbasis Φ_n defined in (3.7).
By using the definition of z_n, it is simple to verify that the control (3.13) may also be written:

$$\tau \;=\; -k_p(\alpha - \sum_{i=1}^{n} \delta_i \Phi_n(i,1) z_n(i)) \;-k_v(\dot{\alpha} - \sum_{i=1}^{n} \delta_i \Phi_n(i,1) \dot{z}_n(i))\,, \tag{3.14}$$

with $\delta_i \;=\; \frac{1}{\Phi_n(i,1)}(G_n^T \Phi_n(i))$ (≥ 0, by assumption).
Let us now introduce some positive functions that will be used to form a Lyapunov-like function for the system.

$$V_0 \;=\; 1/2\;[\begin{bmatrix} \dot{\alpha} \\ \dot{q}_n \end{bmatrix}^T M_{n+1} \begin{bmatrix} \dot{\alpha} \\ \dot{q}_n \end{bmatrix} \;+\; q_n^T K_n q_n]\,,$$

V_0 is the total energy of the system,

$$V_1 \;=\; 1/2\;[\dot{z}_n^T \Delta_n \dot{z}_n \;+\; z_n^T \Delta_n D_n z_n] \text{ with } \Delta_n \;=\; \frac{m_{n+1}}{k_1}\; diag\{\delta_i\}_{i=1,...,n}\,,$$

and

$$V_2 \;=\; 1/2\; k_p(\alpha \;-\; \sum_{i=1}^{n} \delta_i \Phi_n(i,1) z_n(i))^2\,.$$

– From (2.1), we have :

$$\frac{d}{dt} V_0 \;=\; \dot{\alpha}\tau\,.$$

– From (3.8) :

$$\frac{d}{dt} V_1 \;=\; -(\sum_{i=1}^{n} \delta_i \Phi_n(i,1) \dot{z}_n(i))\tau\,.$$

Setting :

$$V \;=\; V_0 \;+\; V_1 \;+\; V_2\,,$$

we have :

$$\frac{d}{dt}V \;=\; (\dot\alpha - \sum_{i=1}^{n}\delta_i\Phi_n(i,1)\dot z_n(i))\;[\tau + k_p(\alpha - \sum_{i=1}^{n}\delta_i\Phi_n(i,1)z_n(i))]\,,$$

and, by using the control expression (3.14) :

$$\frac{d}{dt}V \;=\; -k_v(\dot\alpha - \sum_{i=1}^{n}\delta_i\Phi_n(i,1)\dot z_n(i))^2 \;\;(\le 0)\,.$$

Whatever the initial conditions at time $t=0$, the function V is thus decreasing. Its time-derivative being bounded (since the boundedness of V yields the boundedness of α and q_n, and, in turn, the boundedness of all derivatives of α and z_n), $\frac{d}{dt}V$ tends to zero.
To finish the proof, we only need to show, by application of Lasalle's Principle [LL], that, under the constraint $\frac{d}{dt}V=0$, the system (2.1) has only one solution: $(\alpha=0, q_n=0)$.

– From the control expression (3.14) and (3.2), $\frac{d}{dt}V=0$ yields :

$$\tau \;=\; -k_p(\alpha - \sum_{i=1}^{n}\delta_i\Phi_n(i,1)z_n(i)) \;=\; constant\,, \tag{3.15}$$

and :

$$\ddot\alpha \;=\; \sum_{i=1}^{n}\delta_i\ddot z_n(i)\,.$$

On the other hand, from (2.1) :

$$M_{n+1}(1,1)\ddot\alpha \;+\; M_n(1)^T\ddot q_n \;=\; \tau\,. \tag{3.16}$$

Thus, by using the definition 3.6 of z_n and relation 3.12 :

$$\ddot\alpha \;+\; \frac{k_1}{m_{n+1}}\sum_{i=1}^{n}d_i^{-1}\Phi_n(i,1)\ddot z_n(i) \;=\; \frac{\tau}{M_{n+1}(1,1)}\,, \tag{3.17}$$

where d_i is the ith component of the diagonal matrix D_n. By eliminating $\ddot\alpha$ between (3.16) and (3.17) :

$$\sum_{i=1}^{n}(\delta_i + \frac{k_1}{m_{n+1}}d_i^{-1})\Phi_n(i,1)\ddot z_n(i) \;=\; \frac{\tau}{M_{n+1}(1,1)}\,, \tag{3.18}$$

and, by combining this relation with relation (3.8) :

$$\sum_{i=1}^{n}(\delta_i d_i + \frac{k_1}{m_{n+1}})\Phi_n(i,1)z_n(i) = [\frac{k_1}{m_{n+1}}\sum_{i=1}^{n}(\delta_i + \frac{k_1}{m_{n+1}}d_i^{-1})\Phi_n(i,1)^2 + \frac{1}{M_{n+1}(1,1)}]\tau\,.$$

Differentiating both members of the previous relation twice with respect to time, we obtain :

$$C_n^T\ddot z_n \;=\; 0\,, \tag{3.19}$$

with :

$$C_n^T \;=\; [(\delta_1 d_1 + \frac{k_1}{m_{n+1}})\Phi_n(1,1)...(\delta_n d_n + \frac{k_1}{m_{n+1}})\Phi_n(n,1)]\,.$$

Notice that C_n cannot be the null vector, since $\Phi_n(i,1) \neq 0$ ($\forall i$).
Now, when τ is constant, the solutions of equation (3.8) are of the form :

$$z_n(i) \;=\; -\frac{k_1}{d_i m_{n+1}}\Phi_n(i,1)\tau \;+\; b_i sin(\sqrt{d_i}t + \theta_i)\,. \tag{3.20}$$

Thus :

$$\ddot{z}_n \;=\; \begin{bmatrix} d_1 b_1 sin(\sqrt{d_1}t + \theta_1) \\ . \\ . \\ d_n b_n sin(\sqrt{d_n}t + \theta_n) \end{bmatrix}. \tag{3.21}$$

Since the d_i's are all distinct (property *v)* of Proposition 1), relations (3.19) and (3.21) are compatible only if $b_i = 0$ ($\forall i$).
Therefore, $\ddot{z}_n = 0$, $\tau = 0$ (from (3.18)), $z_n = 0$ (from (3.20)), and $\alpha = 0$ (from (3.15)). ∎

3.3 A few comments and simulation results

What is the practical interest of the control (3.13)?
With $G_n = 0$ one falls back upon a rather classical result according to which a simple Proportional-Derivative (P.D.) feedback using only the position and velocity of the arm's extremity where the control force is applied tends to stabilize the system. In the robotics literature, it has been proven for example that such feedbacks can theoretically stabilize complex (nonlinear) structures such as multi-linked robot manipulators with [T], or without [AM], transmission flexibilities (modeled by one spring per link). The advantages of this type of control are their extreme simplicity and the fact that they can be implemented with no knowledge whatsoever of the system's mechanical characteristics. In this sense, this control appears to be particularly robust with respect to a large class of structured modeling errors. However, the stability margin associated with it may be small in the case of flexible systems and the vibrational damping effect may prove to be insufficient in practice.
This motivates the study of more sophisticated, and hopefully more efficient, state-feedback control laws. Such a control is obtained by choosing G_n different from zero. We have shown here, in the case of a one-link torsionally flexible arm, that a sufficient condition for stability is $(G_n^T\Phi_n(i))\Phi_n(i,1) \geq 0$, $(i = 1,...,n)$ where $\{\Phi_n(i)\}$ are the natural eigenforms of the system. In order to implement this control, some knowledge of the system's natural eigenforms is thus needed. From a practical point of view, this condition is less constraining than requiring the knowledge of all the system's parameters (masses and springs' stiffnesses). Moreover, since the condition is an inequality, it may still be satisfied in the case where the system's eigenforms are not precisely known. This can be interpreted as a sign of robustness with respect to modeling imprecisions. In particular, this condition is automatically satisfied when choosing: $G_n^T = \beta[1,0,...,0]$ with $\beta \geq 0$. This corresponds to only using the arm's deflection q_1 (and its time-derivative $\dot{q}_1$), which can be measured next to the actuator's hub, in the control expression. As in the case where $G_n = 0$, the obtained control theoretically stabilizes the system whatever its elasticity and mass repartition. However, it is more efficient in terms of convergence rate (vibrations are damped more rapidly), as can be observed from our simulation results.

For example, **figure 2** shows the evolution of the system's total energy for a system with $n = 3$ flexible modes, unitary mass, and constant stiffness $k = 50$, and for different values of β (0,1, and 2). Initial conditions are all zero except for the first deflection $q_1(0) = 0.05$. The PD coefficients k_p and k_v have been set to 8 and 2 respectively. This simulation tends to indicate that energy's dissipation increases with β.
Figure 3 shows the corresponding evolution of the arm's extremity position α. Since choosing β different from zero does not improve the convergence of α to zero, it is clear that the overall improvement comes from a better dampening of the elastic modes. A smoothening effect on α is also noticeable.
The effect of modifying the velocity coefficient k_v can be observed from **figure 4**.
Non constant stiffness has been simulated in **figure 5**, with $k_1 = 50$, $k_2 = 70$, and $k_3 = 25$. Parameters are otherwise the same as in the case of **figure 2**.
All these facts seem to be in accordance with the experimental results given in [LAM] and [LKK] for the "PDS" (PD-Strain) control considered by the authors (who also studied the case of bending flexibility).

4 Continuous model

Consider now the P.D.E. (2.3) subject to boundary conditions (2.4) and assume that the stiffness and the mass functions k and m are of class C^1 and such that $k(x)\,,\, m(x) \geq \rho > 0$, $x \in [0,1]$. (This is for the sake of simplicity. For less restrictive assumptions under which the following holds, see [CLM]).

4.1 Preliminary results

We first establish some properties of the deformation operator involved in the autonomous system (4.1) associated to (2.3) and (2.4), with $\tau = 0$:

$$\begin{cases} m(x)\, y_{tt}(x,t) - \frac{1}{m}\,(k\, y_x)_x(x,t) = 0 \;,\; x \in]0,1[\,, \\ (k\, y_x)(0,t) = 0\,, \\ (k\, y_x)(1,t) = 0\,. \end{cases} \tag{4.1}$$

Introduce the Hilbert spaces $H = L^2(0,1)$ and $V = H^1(0,1)$ (the Sobolev space of L^2 functions wich admit L^2 derivatives) respectively equipped with the scalar products :

$$(u,v)_V = \textstyle\int_0^1 k\, u_x\, v_x\, dx + k_p\, u(0)\, v(0) \quad \text{and} \quad (u,v)_H = \int_0^1 m\, u\, v\, dx\,,$$

and with the associated norms $\|\ \|_V$ and $\|\ \|_H$, for some positive constant k_p. Let V' be the dual space of V and $<,>_{V'V}$ denote the duality product between V' and V. Define then the linear operator $A \in \mathcal{L}(V, V')$ by :

$$< Av, w >_{V'V} = \int_0^1 k\, v_x\, w_x\, dx\,, \tag{4.2}$$

for every $v, w \in V$. We denote also by A the associated unbounded operator of $\mathcal{L}(H)$ with domain :

$$\mathcal{D}(A) = \{u \in V \text{ s.t. } k\, u_x \in V \;,\; (k\, u_x)(1) = (k\, u_x)(0) = 0\}\;.$$

Note that $\mathcal{D}(A)$ is dense in V and that, if $u \in \mathcal{D}(A)$, then :

$$A\,u = -\frac{1}{m}(k\,u_x)_x \in H\,.$$

And the autonomous system (4.1) rewrites :

$$y_{tt} + A\,y = 0 \text{ in } H\ ,\ y \in \mathcal{D}(A)\,.$$

Lemma 1
(i) A is monotonous and self–adjoint.
(ii) $(I + A)^{-1}$ is a compact operator of $\mathcal{L}(H)$.
(iii) H admits an orthonormal basis of eigenfunctions $\{\Psi_i\}$ of A such that $\Psi_i(0) \neq 0$.

Proof : The assertion (i) follows easily from the definition (4.2) of A and the fact that, for every $u, v \in \mathcal{D}(A)$,

$$< A\,u, v >_{V'V} = (A\,u, v)_H\,.$$

That $I + A$ is invertible and has a compact inverse ($(I + A)^{-1}$ is an integral operator on H) may be found in [KAT, III,2,3] for example. This is (ii).
Now, (iii) follows from (ii) and [B, thm.VI.11]. Of course, the Ψ_i may be chosen to be normalized in H :

$$\|\Psi_i\|_H^2 = \int_0^1 m\,\Psi_i^2\,dx = 1\,.$$

That $\Psi_i(0) \neq 0$ may be shown as in [CP]. ■

Let $\lambda_i = \omega_i^2 \geq 0$ be the eigenvalues associated with Ψ_i. Defining then the positive bilinear form $\tilde{a}$ on V by

$$\tilde{a}(v, w) = < Av, w >_{V'V} \quad \forall v, w \in V$$

one can set the following lemma :

Lemma 2 *The family $\{\Psi_i\}$ is complete in V with respect to $\|\ \|_V$ and :*

$$\tilde{a}(\Psi_i, \Psi_i) = \int_0^1 k\,\Psi_{i_x}^2\,dx = \omega_i^2\,.$$

Proof : The equality above is obvious, integrating by parts. Now for every $\{f_i\}$, $1 \leq i \leq N$, one has :

$$\begin{aligned}
\|\textstyle\sum_{p=1}^N f_i\,\Psi_i - f\|_V{}^2 &= \int_0^1 k\,(\sum f_i\,\Psi_{ix} - f_x)^2\,dx + k_p\,(\sum f_i\,\Psi_i(0) - f(0))^2 \\
&\leq \frac{1}{\rho}\int_0^1 (\sum f_i\,k\Psi_{ix} - k f_x)^2\,dx + k_p\,(\sum f_i\,\Psi_i(0) - f(0))^2 \\
&\leq (k_p + \frac{1}{\rho})\int_0^1 (\sum f_i\,(k\Psi_{ix})_x - (k f_x)_x)^2\,dx \text{ by Poincaré's inequalities} \\
&\leq (k_p + \frac{1}{\rho})\int_0^1 (\sum f_i\,\omega_i^2\,m\,\Psi_i - (k f_x)_x)^2\,dx \\
&\leq (k_p + \frac{1}{\rho})\|\textstyle\sum f_i\,\omega_i^2\,\Psi_i - \frac{1}{m}(k f_x)_x\|_H{}^2\,.
\end{aligned}$$

Moreover, for every $f \in \mathcal{D}(A)$, $A f = \frac{1}{m}(kf_x)_x \in H$ and since $\{\Psi_i\}$ is complete in H, for every $\epsilon' > 0$, there exists $\{g_i\}$, $1 \leq i \leq N$, such that,

$$\| \sum_{p=1}^{N} g_i \, \Psi_i - \frac{1}{m}(kf_x)_x \|_H^{\,2} < \epsilon' .$$

Hence, by the inequality above, for every $\epsilon = (k_p + \frac{1}{\rho})\epsilon'$, there exists $\{f_i\}$, $f_i = \frac{g_i}{\omega_i^2}$, such that :

$$\| \textstyle\sum_{p=1}^{N} f_i \, \Psi_i - f \|_V^{\,2} \leq (k_p + \dfrac{1}{\rho}) \, \| \textstyle\sum f_i \, \omega_i^2 \, \Psi_i - \frac{1}{m}(kf_x)_x \|_H^{\,2} < \epsilon .$$

Since $\mathcal{D}(A)$ is dense in V, this ends the proof. ∎

4.2 Main result

Consider now the control system (2.3), (2.4) :

$$\begin{cases} m \, y_{tt} - (k \, y_x)_x = 0 \, , \\ (ky_x)(0,t) = -\tau(t) \, , \\ (ky_x)(1,t) = 0 \, , \end{cases} \tag{4.3}$$

and the energy of the solutions of (4.7) given by :

$$E(t) = \frac{1}{2} \left[\int_0^1 m \, y_t^2 \, dx + \int_0^1 k \, y_x^2 \, dx + k_p \, y^2(0) \right] .$$

We have :

Theorem 2 *Let $\{\Psi_i\}$ be a family of eigenfunctions of the autonomous system (4.1) which provides a basis of H. Let k_p, k_v, be positive, and let $g \in V$ such that $g(1) = 0$ and :*

$$\left(\int_0^1 g(x) \Psi'_i(x) dx \right) \; \Psi_i(0) \; \leq -g(0) \, \Psi_i^2(0) \qquad \forall i \geq 0 \, . \tag{4.4}$$

Then : (i) The control law (2.5) :

$$\tau(t) \;\; = \;\; -k_p (y(0,t) - \int_0^1 g(x) y_x(x,t) dx) \;\; - \;\; k_v (y_t(0,t) - \int_0^1 g(x) y_{xt}(x,t) dx)$$

stabilizes the closed–loop system (4.3).
(ii) If $g \equiv 0$ (τ being a boundary feedback), system (4.3) is uniformly exponentially stable.

Remark that the compatibility condition (4.4) between g and $\{\Psi_i\}$ of theorem 2 is the continuous equivalent of the assumption of theorem 1 in the discrete case. With the notation $G = g_x \in H$, condition (4.4) can equivalently be written as :

$$\left(\int_0^1 G(x) \, \Psi_i(x) \, dx \right) \; \Psi_i(0) = \left(\tfrac{1}{m} \, G, \Psi_i \right)_H \Psi_i(0) \geq 0 \;\; , \; \forall i \geq 0 \, , \tag{4.5}$$

Introducing $\kappa_0 = 1 + g(0)$, the closed-loop control law (2.5) may also be written as :

$$\tau(t) = -k_p \, \left[\kappa_0 \, y(0,t) + \int_0^1 G(x) \, y(x,t) \, dx \right] - k_v \, \left[\kappa_0 \, y_t(0,t) + \int_0^1 G(x) \, y_t(x,t) \, dx \right] ,$$

or equivalently,

$$\tau(t) = -k_p \left[\kappa_0 \, y(0,t) + \left(\tfrac{1}{m}G, y\right)_H\right] - k_v \left[\kappa_0 \, y_t(0,t) + \left(\tfrac{1}{m}G, y_t\right)_H\right] . \tag{4.6}$$

The closed–loop system given by (4.3) and (4.6) is then :

$$\begin{cases} m \, y_{tt} - (k \, y_x)_x = 0 \, , \\ (ky_x)(0,t) = k_p \left[\kappa_0 \, y(0,t) + \left(\tfrac{1}{m}G, y(t)\right)_H\right] + k_v \left[\kappa_0 \, y_t(0,t) + \left(\tfrac{1}{m}G, y_t(t)\right)_H\right] , \\ (ky_x)(1,t) = 0 \, . \end{cases} \tag{4.7}$$

In the following, we assume (4.7) (or (4.3)) to be well–posed : for every initial condition $(y_0, y_1) \in V \times H$, there exists a unique solution $(y(t), y_t(t)) \in V \times H$ of (4.7) such that $(y(0), y_t(0)) = (y_0, y_1)$. When $G \equiv 0$ (or $g \equiv 0$), this assumption is proved to be true in appendix A. When $G \neq 0$, the well–posedness of (4.7) in $V \times H$ remains to be justified. To the authors'knowledge, it has only been recently established in the case of a feedback τ involving a single distributed term in position (F. Conrad, private communication).

The following of this section is devoted to the proof of theorem 2. We use the notations of equations (4.7). To establish part (i), we first exhibit a Lyapunov's function for the system (4.7) in section 4.2.1 and apply Lasalle's invariance principle in section 4.2.2. Part (ii) then follows from a boundedness property of $E(t)$ (section 4.2.4).

4.2.1 The Lyapunov function

Given $(y_0, y_1) \in V \times H$, let $(y(t), y_t(t)) \in V \times H$ be the solution of (4.7), for $t \geq 0$. Let us then introduce the following positive functions of t.

$$\begin{aligned} V_0(t) &= \frac{1}{2} \, (1, y_t(t))_H^2 \, , \\ V_i(t) &= \frac{1}{2} \, (\Psi_i, y_t(t))_H^2 + \frac{1}{2} \frac{1}{\omega_i^2} \, \tilde{a}(\Psi_i, y(t)) \; , \quad i \geq 1 \, . \end{aligned} \tag{4.8}$$

One has :

$$\begin{aligned} \frac{dV_0}{dt} &= (1, y_t)_H \, (1, y_{tt})_H = (1, y_t)_H \left(1, \tfrac{1}{m}(ky_x)_x\right)_H = -(1, y_t)_H \, \tau(t) \, , \\ \frac{dV_i}{dt} &= (\Psi_i, y_t)_H \, (\Psi_i, y_{tt})_H + \tfrac{1}{\omega_i^2} \, \tilde{a}(\Psi_i, y) \, \tilde{a}(\Psi_i, y_t) \; , \; i \geq 1 \, . \end{aligned}$$

But,

$$\tilde{a}(\Psi_i, y) = [\Psi_i \, k \, y_x]_0^1 - \left(\Psi_i, \tfrac{1}{m}(ky_x)_x\right)_H = \Psi_i(0) \, \tau(t) - (\Psi_i, y_{tt})_H \, ,$$

and

$$\tilde{a}(\Psi_i, y_t) = [k \, \Psi_{ix} \, y_t]_0^1 - \int_0^1 (k \, \Psi_{ix})_x \, y_t \, dx = \omega_i^2 \, (\Psi_i, y_t)_H \, .$$

So,

$$\frac{dV_i}{dt} = (\Psi_i, y_t)_H \, \Psi_i(0) \, \tau(t)$$

Thanks to these functions, we define now a Lyapunov function composed of three components.

a) Let

$$\xi_i = \frac{1}{\Psi_i(0)} \left(\tfrac{1}{m}G, \Psi_i\right)_H, \; p \geq 0\,. \tag{4.9}$$

According to hypothesis (4.5), $\xi_i \geq 0$ for every $i \geq 0$ and the function

$$V_{dis}(t) = \sum_{i\geq 0} \xi_i \, V_i(t) \tag{4.10}$$

is positive for all $t \geq 0$. Moreover, $\{\Psi_i\}$ being an orthonormal basis of H, we have :

$$\frac{1}{m}G = \sum_{i\geq 0} \left(\tfrac{1}{m}G, \Psi_i\right)_H \Psi_i\,,$$

and :

$$\frac{dV_{dis}}{dt} = \sum_{i\geq 0} \xi_i \frac{dV_i}{dt} = \tau(t) \sum_{i\geq 0} \left(\tfrac{1}{m}G, \Psi_i\right)_H (\Psi_i, y_t)_H = \tau(t) \left(\tfrac{1}{m}G, y_t\right)_H.$$

b) Define now the positive function ($\kappa_0 \geq 0$) :

$$V_{bo} = \kappa_0 \sum_{i\geq 0} V_i\,. \tag{4.11}$$

It verifies :

$$\frac{dV_{bo}}{dt} = \kappa_0 \sum_{i\geq 0} \frac{dV_i}{dt} = \kappa_0 \, \tau(t) \sum_{i\geq 0} \Psi_i(0)\, (\Psi_i, y_t)_H = \kappa_0\, \tau(t)\, y_t(0)\,.$$

c) Let us finally introduce the third positive function ($k_p \geq 0$) :

$$V_{pos} = \frac{k_p}{2} \left[\kappa_0\, y(0) + \left(\tfrac{1}{m}G, y\right)_H\right]^2. \tag{4.12}$$

We have :

$$\frac{dV_{pos}}{dt} = k_p \left[\kappa_0\, y(0) + \left(\tfrac{1}{m}G, y\right)_H\right] \left[\kappa_0\, y_t(0) + \left(\tfrac{1}{m}G, y_t\right)_H\right].$$

Define the following Lyapunov function :

$$V = V_{dis} + V_{bo} + V_{pos} \tag{4.13}$$

One has $V(t) \geq 0$ for all $t \geq 0$. Moreover, by using the expression (4.6) of the feedback τ,

$$\begin{aligned} \frac{dV}{dt} &= \left[\kappa_0\, y_t(0) + \left(\tfrac{1}{m}G, y_t\right)_H\right]\left[\tau(t) + k_p \left(\kappa_0\, y(0) + \left(\tfrac{1}{m}G, y\right)_H\right)\right] \\ &= -k_v \left[\kappa_0\, y_t(0) + \left(\tfrac{1}{m}G, y_t\right)_H\right]^2 \leq 0\,. \end{aligned}$$

4.2.2 Lasalle's invariance principle

In view of the boundary conditions, the variational equation associated with (4.7) is :

$$y_{tt} + Ay + C^*\tau = 0\,. \tag{4.14}$$

with the operator A defined in (4.2) and $C \in \mathcal{L}(V, R)$ defined by :

$$Cv = \frac{1}{M} v(0)\,,$$

for all $v \in V$, $M = \int_0^1 m\,dx = m$ denoting the whole mass of the arm, and C^* denoting the adjoint operator of C.
Let us show that the constraint $\frac{dV}{dt} = 0$ implies $y(x,t) = y_t(x,t) = 0$ for $t \geq 0$ and $x \in [0,1]$.
We first remark that $\frac{dV}{dt} = 0$ implies

$$\kappa_0\, y_t(0) + \left(\tfrac{1}{b} G, y_t\right)_H = 0\,,\ \forall t \geq 0\,. \tag{4.15}$$

In this case, the control law τ is constant :

$$\tau(t) = \tau(0) = k_p\left[\kappa_0\, y_0(0) + (G, y_0)_{L^2}\right].$$

As $\{\Psi_i\}$ is a orthonormal basis of H (lemma 1), one can set :

$$y(x,t) = \sum_{i\geq 0} \theta_i(t)\,\Psi_i(x)\,,\ \text{with } \theta_i(t) = (y, \Psi_i)_H\,.$$

The inner product of (4.14) with Ψ_q yields :

$$\theta_{q_{tt}} + \lambda_q \theta_q + \tau(t)\Psi_q(0) = 0\,, \tag{4.16}$$

where $\lambda_q = \omega_q^2$ is the eigenvalue associated with Ψ_q. The solution of the above equation is :

$$\theta_q(t) = \theta_q(0)\cos\,\omega_q t + \frac{1}{\omega_q}\theta_{q_t}(0)\sin\,\omega_q t + \int_0^1 \frac{1}{\omega_q}\tau(t)\Psi_q(0)\sin\,\omega_q(s-t)ds\,.$$

The control law τ being constant, one has :

$$\int_0^1 \frac{1}{\omega_q}\,\tau(t)\Psi_q(0)\sin\omega_q(s-t)ds = -\tau(0)\,\Psi_i(0)\,\frac{1-\cos\omega_q t}{\omega_q^2}$$

Then, every solution of (4.14) satisfying (4.15) is given by :

$$y(x,t) = \sum_{i\geq 0}\left[\theta_i(0)\cos\omega_i t + \theta_{it}(0)\frac{\sin\omega_i t}{\omega_i} - \tau(0)\,\Psi_i(0)\,\frac{1-\cos\omega_i t}{\omega_i^2}.\right]\Psi_i(x)\,,$$

and

$$y_t(x,t) = \sum_{i\geq 0}\left[-(\theta_i(0)\,\omega_i + \frac{\tau(0)\,\Psi_i(0)}{\omega_i})\,\sin\omega_i t + \theta_{it}(0)\,\cos\omega_i t\right]\Psi_i(x)\,.$$

Now, the constraint (4.15) leads to :

$$\sum_{i\geq 0}\left[-(\theta_i(0)\,\omega_i + \frac{\tau(0)\,\Psi_i(0)}{\omega_i})\,\cos\omega_i t + \theta_{it}(0)\,\cos\omega_i t\right] \times (\kappa_0\,\Psi_i(0) + \left(\tfrac{1}{m}G, \Psi_i\right)_H) = 0\,. \tag{4.17}$$

By (4.5), since $\kappa_0 \geq 0$ and $\Psi_i(0) \neq 0$, one has :

$$\kappa_0 \, \Psi_i(0) + \left(\tfrac{1}{m}G, \Psi_i\right)_H \neq 0 \quad , \ \forall i \geq 0 \, ,$$

and, as $\{\sin \omega_i t, \cos \omega_i t\}$ is complete in $L^2(0, \infty; \mathbb{R})$, (4.17) implies that, $\forall i \geq 0$,

$$\left\{ \begin{array}{l} \theta_i(0) \, \omega_i + \frac{1}{\omega_i} \, \tau(0) \, \Psi_i(0) = 0 \, , \\ \theta_{it}(0) = 0 \, . \end{array} \right. \tag{4.18}$$

Let

$$\rho_i = \frac{\theta_i(0)}{\Psi_i(0)} = -\frac{\tau(0)}{\omega_i^2} \, , \ i \geq 0 \, .$$

According to the expressions of τ and y, one has from (4.18) :

$$\theta_i(0) \, \omega_i^2 + k_p \, \Psi_i(0) \sum_{j \geq 0} \theta_j \left(\left(\tfrac{1}{m}G, \Psi_j\right)_H + \kappa_0 \, \Psi_j(0) \right) = 0 \, ,$$

or equivalently :

$$\rho_i \, \omega_i^2 + k_p \sum_{j \geq 0} \rho_j \left(\left(\tfrac{1}{m}G, \Psi_j\right)_H \Psi_j(0) + \kappa_0 \, \Psi_j^2(0) \right) = 0 \quad , \ \forall i \geq 0 \, . \tag{4.19}$$

By assumption (4.5) and since the ρ_j all have the same sign for $j \geq 0$, (4.19) implies that $\rho_i = 0$, for all $i \geq 0$, hence $\theta_i(0) = 0$, for all $i \geq 0$, and $\tau = 0$. We deduce from the expressions of y and y_t above that $y(x,t) = y_t(x,t) = 0$: the unique solution of the system under the constraint $\frac{dV}{dt} = 0$ is zero. Applying Lasalle's invariance principle [LL], this means that $V(t)$ decreases to zero.

4.2.3 Comparison of V and E

Let us now compare $V(t)$ and the energy $E(t)$, and thus finally show that the proposed control law stabilizes the system. Recall that by definition (4.11),

$$V_{bo}(t) = \frac{\kappa_0}{2} \sum_{i \geq 0} (\Psi_i, y_t)_H^2 + \frac{\kappa_0}{2} \sum_{i \geq 0} \frac{\tilde{a}(\Psi_i, y)^2}{\omega_i^2} \, .$$

Since $\{\Psi_i\}$ is an orthonormal basis of H,

$$\sum_{i \geq 0} (\Psi_i, y_t)_H^2 = \|y_t\|_H^2 \, ,$$

and according to lemma 2 :

$$\sum_{i \geq 0} \frac{\tilde{a}(\Psi_i, y)}{\omega_i^2} = \tilde{a}(y, y) \, .$$

Now, since :

$$E(t) = \frac{1}{2}\|y_t\|_H^2 + \frac{1}{2}\tilde{a}(y,y) + \frac{1}{2}k_p y^2(0) \, ,$$

we have :

$$E(t) \leq \frac{1}{\kappa_0} \, V_{bo}(t) + \frac{1}{\kappa_0^2} \, V_{pos}(t) \leq \frac{1}{\min(\kappa_0, \kappa_0^2)} \, V(t) \, .$$

Hence, $E(t) \to 0$ since $V(t)$ does. The closed-loop system is thus stable and this proves part (i) of theorem 2.

4.2.4 Exponential stability

We prove the part (ii) of theorem 2 in the case of constant mass and stiffness distributions. For a general proof of (ii) with non–constant functions m and k in suitable spaces, we refer to [CLM].

Thus, assume that $m \equiv 1$ and $k \equiv 1$, and consider the boundary control law $\tau(t) = -k_p\, y(0,t) - k_v\, y_t(0,t)$. System (4.7) becomes :

$$\begin{cases} y_{tt} - y_{xx} = 0\,, \\ y_x(0,t) = k_p\, y(0,t) + k_v\, y_t(0,t)\,, \\ y_x(1,t) = 0\,. \end{cases} \tag{4.20}$$

and the energy of its solutions :

$$E(t) = \frac{1}{2}\left[\int_0^1 y_t^2\, dx + \int_0^1 y_x^2\, dx + k_p\, y^2(0)\right]\,. \tag{4.21}$$

The following lemma is proved in appendix B, using multiplier techniques.

Lemma 3 *There exists a constant $C > 0$ such that*

$$\int_0^T E(t)\, dt \leq C\, E(0)\,,\ \forall T > 0\,.$$

We can then deduce an upperbound for the decay rate of the energy. Indeed, as E is decreasing, lemma 3 implies :

$$T\, E(T) \leq C\, E(0)\,.$$

By choosing, $T \geq 2\,C$:

$$E(T) \leq \frac{1}{2}\, E(0)\,,$$

i.e. for every integer $p \geq 0$,

$$E(p\,T) \leq \frac{1}{2^p}\, E(0)\,.$$

But, for all $t \geq 0$, there exists a positive integer p such that $p\,T \leq t < (p+1)\,T$, and then $E(t) \leq E(p\,T)$. Let

$$\mu = \frac{1}{2T}\, \log 2\ > 0\,,$$

and remark that if $p \geq 1$, one has :

$$\exp(-p \log 2) \leq \exp(-\frac{p+1}{2} \log 2)\,.$$

Therefore :

$$E(t) \leq \frac{1}{2^p}\, E(0) \leq e^{-(p+1)\mu T}\, E(0) \leq e^{-\mu\, t}\, E(0)\,.$$

This proves part (ii) of theorem 2. Moreover, we can give an upper bound for the decay rate. Actually, the assumption $T \geq 2\,C$, implies :

$$\mu \leq \frac{\log 2}{4C}\,,$$

with the constant C given by (B.9).

5 Conclusion

We studied here the stabilizing properties of a feedback control law for a flexible torque arm. This feedback law requires only a reduced knowledge of the system, and the measurement of an output which may be an arbitrarily small part of the whole state. This is a guarantee of the robustness of the proposed control scheme with respect to uncertainties concerning the system. It is also necessary from an implementation point of view.
This control law has been applied to both a discrete model of the arm, an ordinary differential system, and a continuous model consisting of a P.D.E. For these two models, a sufficient condition for stability consists of a compatibility constraint between the gain involved in the distributed part of the control and the eigenfunctions of the autonomous system : it requires that this gain acts "positively" on the eigenfunctions, so as to dissipate enough energy. Simulations illustrate some possible choices for this gain and the superior performence of 'PDS' type control over simple PD boundary feedback.
To complete this work, it remains to establish the exponential stability of the continuous model, with non–constant mass and density functions, under the effect of the whole feedback law (including the effect of the distributed part). At this stage, a study of the energy decay rate with respect to the various gain coefficients will be of interest.
The results presented here apply also to an arm which is flexible in compression, since the equations take the same form. Their extension to bending flexibility remains to be done.
Another perspective of this work is its generalization to the stabilization of multi–linked robots, although this will induce non–linearities in the models.
It will also be important to study robustness aspects with respect to various pertubations (mass at one extremity, delay at observation...).

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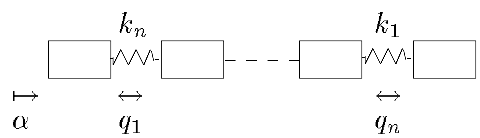

Figure 1: The discrete system

A Well-posedness in case of boundary observation

Consider the feedback

$$\tau(t) = -k_p\, y(0,t) - k_v\, y_t(0,t)\,,$$

for some $k_p, k_v > 0$. The closed-loop system is then the following :

$$\left\{ \begin{array}{l} m\, y_{tt}(x,t) - (ky_x)_x = 0\,, \\ ky_x(0,t) = k_p\, y(0,t) + k_v\, y_t(0,t)\,, \\ ky_x(1,t) = 0\,. \end{array} \right. \tag{A.1}$$

and its variational form :

$$y_{tt} + Ay + C^* k_p C y + C^* k_v C y_t = 0\,,$$

with the operators A and C defined previously. At the first order, it rewrites :

$$\begin{pmatrix} y \\ y_t \end{pmatrix}_t + \mathcal{A} \begin{pmatrix} y \\ y_t \end{pmatrix} = 0\,,$$

where $\mathcal{A}$ is the unbounded operator of $\mathcal{L}(V \times H)$ defined by :

$$\mathcal{A} = \begin{pmatrix} 0 & -I \\ A + C^* k_p C & C^* k_v C \end{pmatrix}\,,$$

with domain :

$$\mathcal{D}(\mathcal{A}) = \left\{ (u,v) \in H^2(0,1) \times H^1(0,1) \text{ such that } k u_x \in H^1(0,1) \text{ and } \begin{array}{l} k u_x(1) = 0 \\ k u_x(0) = k_p u(0) + k_v v(0) \end{array} \right\}.$$

Note that if $(u,v) \in \mathcal{D}(\mathcal{A})$, $\mathcal{A}\,(u,v) = \begin{pmatrix} -v \\ -\frac{1}{m}(k u_x)_x \end{pmatrix}$.
Let the energy space $V \times H$ be equipped with the inner product

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \mu \\ \nu \end{pmatrix} \right)_{V \times H} = \int_0^1 k\, u_x\, \mu_x\, dx + \int_0^1 m\, v\, \nu\, dx + k_p u(0)\mu(0) = (u,\mu)_V + (v,\nu)_H .$$

Let $V_0 = \{v \in H^1(0,1) \, \text{s.t. } v(1) = 0\}$ be equipped with the norm :

$$\|v\|_{V_0} = \left[\int_0^1 k\, v_x^2\, dx \right]^{\frac{1}{2}} .$$

Let us introduce, the linear form φ defined on V_0 by :

$$\varphi : v \mapsto \int_0^1 m\, h\, v\, dx + k_v f(0) v(0) ,$$

and the bilinear form :

$$a(u,v) = \int_0^1 m\, u\, v\, dx + \int_0^1 k\, u_x\, v_x\, dx + k_s u(0) v(0) .$$

We will need later the following result :

Lemma 4 .
i) φ *is a continuous form on* V_0.
ii) a *is a continuous* V_0*-elliptic bilinear form.*

Proof : For $v \in V_0$, one has the following Poincaré's inequalities :

$$|v(0)| = \left| \int_0^1 v_x\, dx \right| \le \|v_x\|_{L^2} \le \rho^{-\frac{1}{2}} \|v\|_{V_0} ,$$

so that :

$$\|v\|_{L^2}^2 \le 2 \int_0^1 v_x^2\, dx + 2\, v^2(0) = 2\|v_x\|_{L^2}^2 + 2\, v^2(0) \le \frac{4}{\rho} \|v\|_{V_0}^2 ,$$

and $\|v\|_{L^2} \le 2\rho^{-\frac{1}{2}} \|v\|_{V_0}$. Therefore, by definition of φ, one has for all $v \in V_0$:

$$\begin{array}{rcl} |\varphi(v)| & \le & |\int_0^1 m\, h\, v\, dx| + |k_v f(0)|\, |v(0)| \\ & \le & \|mh\|_{L^2}\, \|v\|_{L^2} + \rho^{-\frac{1}{2}} |k_v f(0)|\, \|v\|_{V_0} \\ & \le & \frac{1}{\rho} \|h\|_{L^2}\, \|v\|_{L^2} + \rho^{-\frac{1}{2}} |k_v f(0)|\, \|v\|_{V_0} \\ & \le & \rho^{-\frac{1}{2}} (\frac{2}{\rho} \|h\|_{L^2} + |k_v f(0)|)\, \|v\|_{V_0} \quad , \end{array}$$

and then, φ is continuous. Moreover, for all u, $v \in V_0$,

$$\begin{array}{rcl} |a(u,v)| & \le & |\int_0^1 m\, u\, v\, dx| + |\int_0^1 k\, u_x\, v_x\, dx| + k_s |u(0)|\, |v(0)| \\ & \le & \frac{1}{\rho} \|u\|_{L^2}\, \|v\|_{L^2} + \frac{1}{\rho} \|u_x\|_{L^2}\, \|v_x\|_{L^2} + k_s |u(0)|\, |v(0)| \\ & \le & \frac{1}{\rho}\frac{4}{\rho} \|u\|_{V_0}\, \|v\|_{V_0} + \frac{1}{\rho}\frac{1}{\rho} \|u\|_{V_0}\, \|v\|_{V_0} + \frac{k_s}{\rho} \|u\|_{V_0}\, \|v\|_{V_0} \\ & \le & \frac{1}{\rho} (\frac{5}{\rho} + k_s)\, \|u\|_{V_0}\, \|v\|_{V_0} \quad , \end{array}$$

so that a is continuous, and

$$|a(v,v)| = \int_0^1 m\, v^2\, dx + \int_0^1 k\, v_x^2\, dx + k_s v^2(0) \geq \int_0^1 k\, v_x^2\, dx \geq \|v\|_{V_0}^2\,,$$

so that a is V_0-elliptic. ∎

We are now ready to prove the :

Lemma 5 *$\mathcal{A}$ is maximal monotone on $V \times H$.*

Proof : First, $\mathcal{A}$ is monotone. Indeed, if $(u,v) \in \mathcal{D}(\mathcal{A})$, we have

$$\begin{aligned}
\left(\mathcal{A}(u,v), \begin{pmatrix} u \\ v \end{pmatrix}\right)_{V\times H} &= \int_0^1 -k u_x\, v_x\, dx + \int_0^1 -(k u_x)_x\, v\, dx - k_p u(0) v(0) \\
&= k u_x(0)\, v(0) - k_p u(0) v(0) \\
&= k_v v^2(0) \;>\; 0\,.
\end{aligned}$$

Let us prove now that for any $(f,g) \in V \times H$, the equation :

$$(u,v) + \mathcal{A}(u,v) = (f,g)$$

admits a unique solution $(u,v) \in \mathcal{D}(\mathcal{A})$ (see [B]).
If $u \in V$, then $v = u - f \in V$. Therefore, by setting $h = f + g$ and $k_s = k_p + k_v$, we have to solve the following equivalent problem : for any $(f,h) \in V \times H$, find u such that $k u_x \in H^1(0,1)$ and

$$\left\{ \begin{array}{l} m u - (k u_x)_x = m h \\ k u_x(1) = 0 \\ k u_x(0) = k_s\, u(0) - k_v f(0)\,. \end{array} \right. \tag{A.2}$$

Multiply and integrate the first equation of (A.2) by any test function $v \in V_0$ to obtain :

$$\int_0^1 m\, u\, v\, dx + \int_0^1 k\, u_x\, v_x\, dx + k_s u(0) v(0) = \int_0^1 m\, h\, v\, dx + k_v f(0) v(0)\,,$$

according to boundary conditions, or :

$$a(u,v) = \phi(v)\,, \tag{A.3}$$

for any $v \in V_0$. From lemma 4 and the Lax-Milgram theorem [B, cor.V.8] there exists of a unique $u \in V_0$ satisfying (A.3) for all $v \in V_0$. That u belongs in fact to $H^2(0,1)$ and is the unique solution of (A.2), is easy to see, integrating by part and taking a test function satisfying $v(0) = 0$. Hence $\mathcal{A}$ is maximal monotone. ∎

By the Hille–Yosida theorem [B, thm.VII.4], for any initial condition $(y_0, y_1) \in \mathcal{D}(\mathcal{A})$, system (A.1) admits a unique solution $y \in C^1([0,+\infty[; L^2(0,1)) \cap C([0,+\infty[; \mathcal{D}(\mathcal{A}))$. And since $\mathcal{D}(\mathcal{A})$ is dense in $V \times H$ (see also [B]), this implies that (A.1) is well–posed in $H^1(0,1) \times L^2(0,1)$.

When the control law τ contains distributed terms, the well–posedness of the closed–loop system seems to be much more technical. It is not studied here.

B Proof of lemma 3

Let $T > 0$, and consider system (4.20) for $t \in [0, T]$. Multiplying the evolution equation by the classical $(1-x)\,y_x$ leads to : $y_x(0,t) = k_p\,y(0,t) + k_v\,y_t(0,t)$ and $y_x(1,t) = 0$ one gets :

$$\begin{aligned}&\frac{1}{2}\int_0^1\int_0^T y_t^2\,dx\,dt + \frac{1}{2}\int_0^1\int_0^T y_x^2\,dx\,dt - \frac{k_p^2}{2}\int_0^T y^2(0,t)\,dt \\ &= \int_0^1 [y_t\,(1-x)\,y_x]_0^T\,dx + \frac{1+k_v^2}{2}\int_0^T y_t^2(0,t)\,dt + k_p\,k_v\int_0^T y(0,t)\,y_t(0,t)\,dt\,.\end{aligned} \tag{B.4}$$

Multiplying now by y gives :

$$\begin{aligned}&\int_0^1\int_0^T y_t^2\,dx\,dt - \int_0^1\int_0^T y_x^2\,dx\,dt - k_p\int_0^T y^2(0,t)\,dt \\ &= \int_0^1 [y_t\,y]_0^T\,dx + k_v\int_0^T y(0,t)\,y_t(0,t)\,dt\,.\end{aligned} \tag{B.5}$$

Let $A > 0$. Multiply (B.4) by A and substract (B.5) to obtain :

$$\begin{aligned}&(A-1)\int_0^1\int_0^T y_t^2\,dx\,dt + (A+1)\int_0^1\int_0^T y_x^2\,dx\,dt + (1 - A\,k_p)\,k_p\int_0^T y^2(0,t)\,dt \\ &= 2\,A\int_0^1 [y_t\,(1-x)\,y_x]_0^T\,dx - \int_0^1 [y_t\,y]_0^T\,dx \\ &+A\,(1+k_v^2)\int_0^T y_t^2(0,t)\,dt + (A\,k_p - 1)\,k_v\int_0^T y(0,t)\,y_t(0,t)\,dt\end{aligned} \tag{B.6}$$

If $A > 1$, by applying Young's inequality to the last term of the second member of (B.6), and defining B as follows, one gets :

$$\begin{aligned}B =&\ (A-1)\int_0^1\int_0^T y_t^2\,dx\,dt + (A-1)\int_0^1\int_0^T y_x^2\,dx\,dt + (1 - A\,k_p)\,(k_p + \frac{k_v}{2})\int_0^T y^2(0,t)\,dt \\ \leq&\ 2\,A\int_0^1 [y_t\,(1-x)\,y_x]_0^T\,dx - \int_0^1 [y_t\,y]_0^T\,dx + [(A\,k_p - 1)\,\frac{k_v}{2} + A\,(1+k_v^2)]\int_0^T y_t^2(0,t)\,dt\end{aligned} \tag{B.7}$$

We first bound B from above. For this, set :

$$\left|2\,A\int_0^1 [y_t\,(1-x)\,y_x]_0^T\,dx - \int_0^1 [y_t\,y]_0^T\,dx\right| \leq 4\,(A + 1 + \frac{1}{k_p})\,E(0) \tag{B.8}$$

Thanks to Young's inequality, one has :

$$\begin{aligned}&2\left|\int_0^1 y_t(x,t)\,(1-x)\,y_x(x,t)\,dx\right| \leq \int_0^1 y_t^2\,dx + \int_0^1 (1-x)^2\,y_x^2\,dx \\ &\leq \int_0^1 y_t^2\,dx + \int_0^1 y_x^2\,dx \leq 2\,E(t)\,.\end{aligned}$$

So,

$$\left|2\,A\int_0^1 [y_t\,(1-x)\,y_x]_0^T\,dx\right| \leq 2\,A\,(E(T)+E(0)) \leq 4\,A\,E(0)\,,$$

because $E(t)$ is a decreasing function.
On another hand, using Poincaré's inequality

$$\begin{aligned}
\left|\int_0^1 y_t(x,t)\,y(x,t)\,dx\right| &\leq \tfrac{1}{2}\int_0^1 y_t^2\,dx + \tfrac{1}{2}\int_0^1 y^2\,dx \\
&\leq \int_0^1 y_t^2\,dx + \int_0^1 y_x^2\,dx + y^2(0,t) \\
&\leq 2\,(1+\frac{1}{k_p})\,E(t) \quad \leq \quad 4\,(1+\frac{1}{k_p})\,E(0)\,.
\end{aligned}$$

So,

$$\left|\int_0^1 [y_t\,y]_0^T\,dx\right| \leq 4\,(1+\frac{1}{k_p})\,E(0)$$

Moreover, as $\frac{d\,E}{d\,t} = -k_v\,y_t^2(0,t)$

$$\int_0^T y_t^2(0,t)\,dt = -\frac{1}{k_v}\,(E(T)-E(0)) \leq \frac{1}{k_v}\,E(0)$$

So, according to (B.7) one has :

$$B \leq \left(A\,\left(\frac{k_p}{2}+k_v+4+\frac{1}{k_v}\right)+\frac{4}{k_p}+\frac{7}{2})\right)\,E(0)\,.$$

Now, if :

$$\frac{(1-A\,k_p)\,(k_p+\frac{k_v}{2})}{A-1} \geq k_p\,,$$

which is verified by any $A>1$ if $2k_p+k_v>1$, it follows from the definition (4.21) of E that

$$\int_0^T E(t)\,dt \leq \frac{B}{A-1}\,.$$

Finally, $\int_0^T E(t)\,dt \leq C\,E(0)$, with

$$C = \frac{1}{A-1}\left\{A\,\left(\frac{k_p}{2}+k_v+4+\frac{1}{k_v}\right)+\frac{4}{k_p}+\frac{7}{2}\right\}\,. \qquad \text{(B.9)}$$

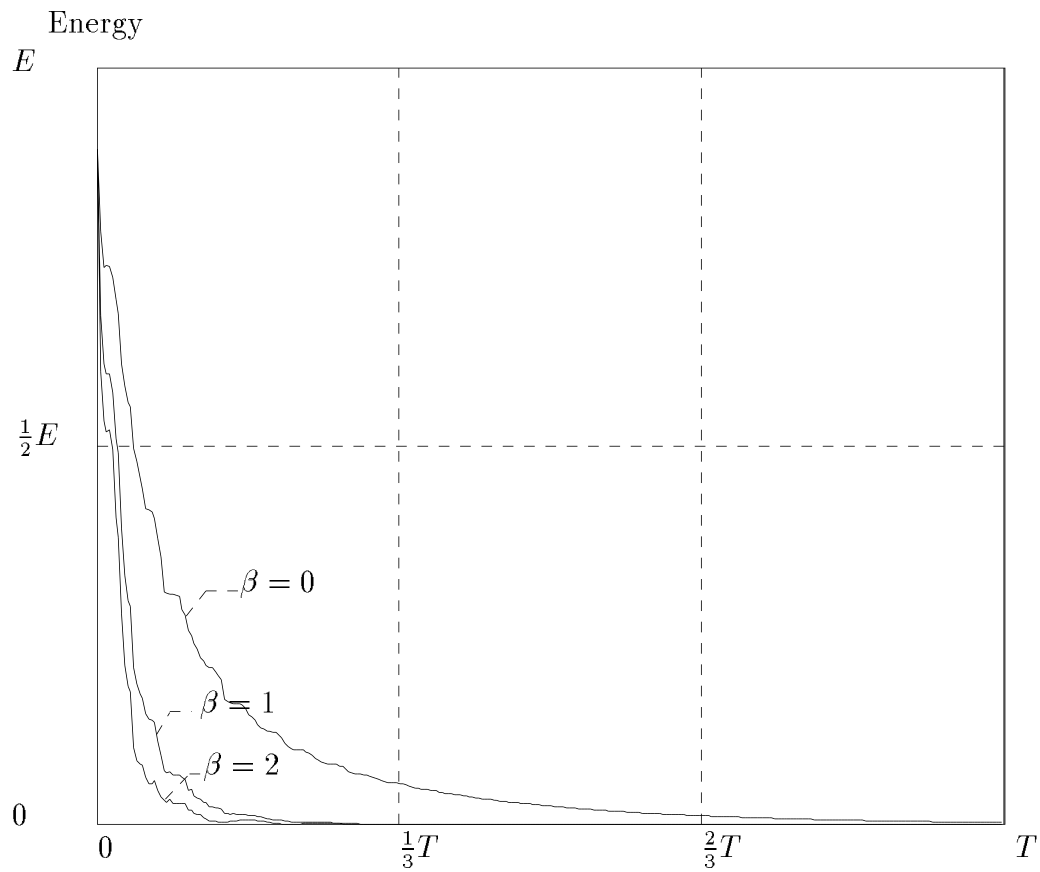

Figure 2: Energy

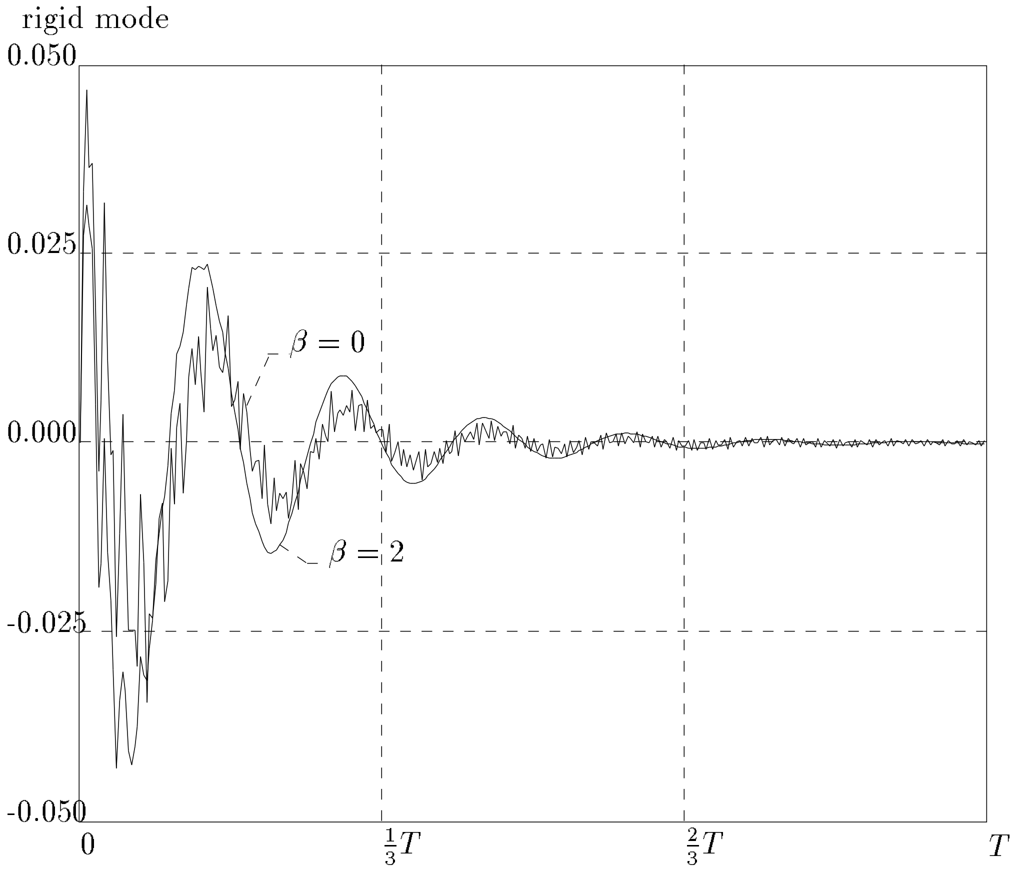

Figure 3: Rigid mode

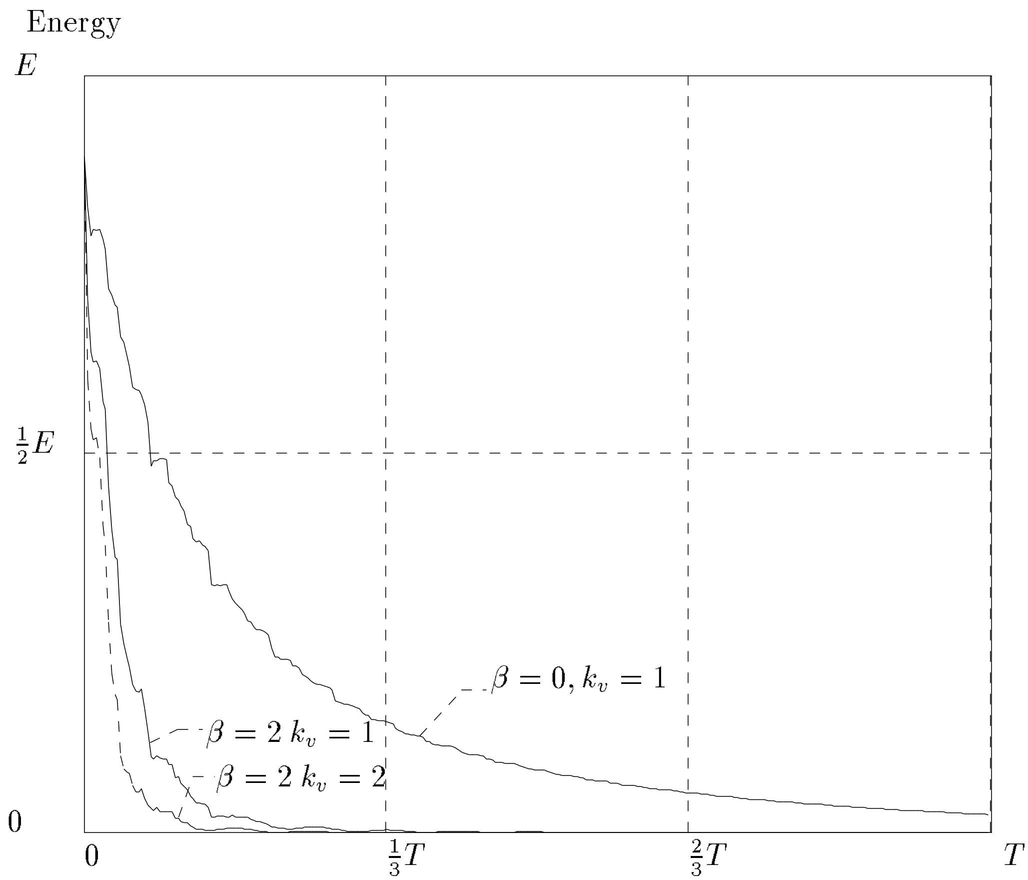

Figure 4: Energy (variations of PD coefficients)

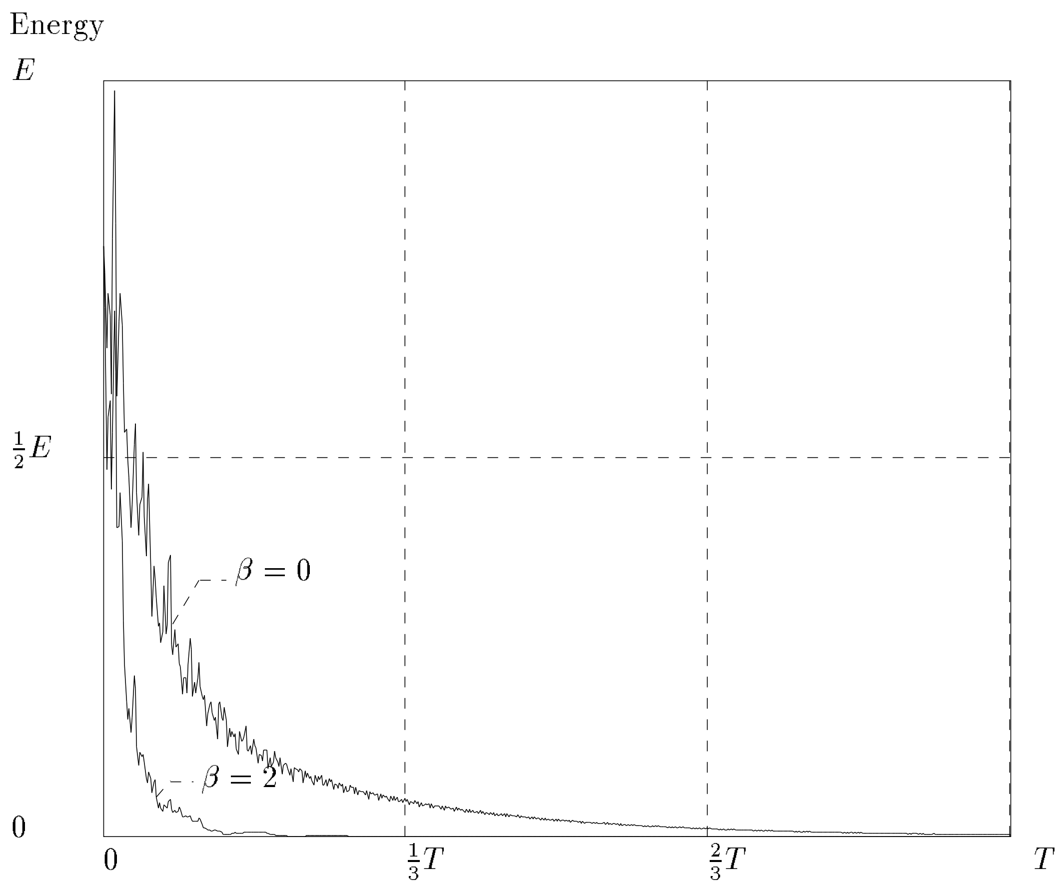

Figure 5: Energy (non-constant stiffness)