'Lion and man': upper and lower bounds

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Le problème de l'Homme et du Lion : bornes inférieures et supérieures *

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Abstract

Given a lion and a man, their initial positions, and restrictions on their ranges and speeds, how quickly can the lion get within a given distance from the man? We consider the case in which the lion and man are restricted to the interior of a circle and each is limited to the same speed.

Résumé

Imaginez un lion et un chrétien, choisissez leur position initiale et définissez le domaine dans lequel ils peuvent se mouvoir et la vitesse avec laquelle ils peuvent le faire, en combien de temps le lion peut-il rattraper le chrétien ? Nous considérerons, ici, le cas où le lion et le chrétien se déplacent à l'intérieur d'une arène circulaire à vitesse constante.

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Abstract. Given a lion and a man, their initial positions, and restrictions on their ranges and speeds, how quickly can the lion get within a given distance from the man? We consider the case in which the lion and man are restricted to the interior of a circle and each is limited to the same speed.

Key words. Games: continuous pursuit; analysis of algorithms: lower and upper bounds.

Imagine a lion pursuing a man in a circular arena and ask how long the chase can continue until the man becomes kitty chow? More formally, we consider the two-dimensional continuous pursuit problem in which we are given two point objects (the lion and the man), their initial positions, and restrictions on their ranges and speeds; we ask how quickly the lion can capture the man, that is, get within a given distance c from the man? We consider the case in which the lion and man are restricted to the interior of a circle with radius $r \gg c$ (that is, $c/r \approx 0$) and each is limited to the same speed s. We prove upper and lower bounds on the time until capture, assuming the lion and man move optimally. The bounds are not tight, so there is room for improvement.

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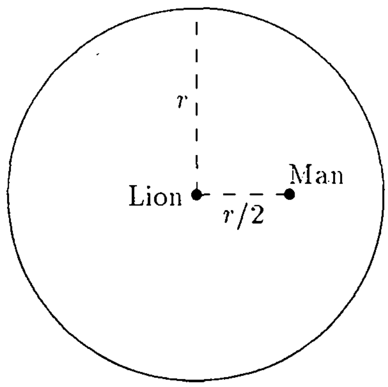


Figure 1: Initial configuration of the lion and man inside a circular region. The lion is at the center of the circle of radius r and the man is at a distance $r/2$ from the center.

The initial positions are not important with regard to the bounds, but for convenience we let the lion's initial position be the center of the circle and the man's initial position be at a distance $r/2$ from the center (see Figure 1). In any reasonable strategy, neither the lion nor the man gains anything by moving more slowly than s, so we assume that both move at speed s until capture.

Our lower bound is derived by describing a strategy for the man to elude capture as long as possible—his strategy is, in essence, to spiral outward from his initial position toward the edge of the bounding circle. The lion, on the other hand, can always pursue the man by moving toward him most of the time; this pursuit strategy leads to our upper bound.

The case $c = 0$ was posed by Rado [10, pages 114–117] where Littlewood used a result of A. S. Besicovitch to show that the man can move so as to avoid capture indefinitely. Croft [3] discussed the case $c = 0$ with the further restriction that the curvature of the paths of the lion and man are bounded and showed that the lion *can* capture the man. Other variations have been considered by Flynn [4], [5], [6], Gale [7], and Lewin [9]. A recent review of the subject was done by Benkoski, Monticino, and Weisinger [2].

Throughout this paper the notation $g(x) = O(f(x))$ means $g(x)$ grows no more quickly than $f(x)$ as $x \to \infty$; $g(x) = o(f(x))$ means $g(x)$ grows slower than $f(x)$ as $x \to \infty$; $g(x) = \Omega(f(x))$ means $g(x)$ grows at least as quickly as $f(x)$ as $x \to \infty$; $g(x) = \Theta(f(x))$ means $g(x)$ grows asymptotically as $f(x)$ as $x \to \infty$. See [8] for a full description of these notations.

We now prove

Theorem 1 *If a lion pursues a man inside a circular region of radius r, and both have maximum speed s, then the time needed for the lion to get within a distance c of the man is*

$$\Omega\left(\frac{r}{s}\sqrt{\frac{\log\frac{r}{c}}{\log\log\frac{r}{c}}}\right). \tag{1}$$

Proof. We give an algorithm for the man to follow; first we determine the number of steps until the man reaches the edge of the circle, while at the same time we determine the angle (with respect to the lion) at which the man will move. Then we prove that no matter what the lion does, capture cannot occur in time less than the bound in (1). The man's algorithm is to choose, at discrete time steps, a direction and move along a straight line of length l, whose value we determine later. The direction chosen is at an angle of $90° + \alpha$ to the line joining the lion and man at the beginning of the step; α is a small value that we will also determine later. As shown in Figure 2, there are

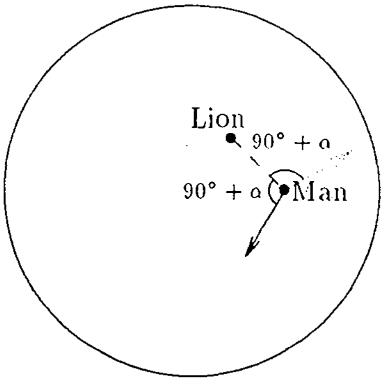


Figure 2: Configuration of the lion and man at the beginning of step i. The man moves as shown by the solid arrow, at an angle of $90° + \alpha$ with respect to the lion, such that he moves at an angle at most $90° + \alpha$ with respect to the line segment connecting the center of the circle and the man.

two possible directions for the man to move with this algorithm; the man moves so that the chosen direction makes an angle of at most $90° + \alpha$ with the line segment between the center of the circle and the position of the man at the beginning of the ith step (that is, the man moves along the solid arrow in Figure 2, not the dotted arrow).

Let m be the number of steps taken if the man employs this algorithm and consider M_i, the distance from the man to the center at the start of the ith step. Initially, $M_1 = r/2 < r$. ¿From Figure 3 we see that the distance from the man to the center at the beginning of the $(i+1)$st step satisfies

$$\begin{aligned} M_{i+1} &\leq \sqrt{M_i^2 + l^2 - 2M_i l \cos(90° + \alpha)} \\ &= \sqrt{M_i^2 + l^2 + 2M_i l \sin \alpha}. \end{aligned}$$

By choosing α such that $\sin \alpha = \beta l / r$ for some constant β we have

$$M_{i+1} \leq \sqrt{M_i^2 + (1 + 2\beta) l^2},$$

and expanding the square root with a Taylor expansion gives

$$\begin{aligned} M_{i+1} &\leq M_i + (1 + 2\beta) l^2 / (2M_i) \\ &\leq M_i + (1 + 2\beta) l^2 / (2r). \end{aligned}$$

Therefore

$$M_{i+1} \leq r/2 + (i+1)(1 + 2\beta) l^2 / (2r). \tag{2}$$

m is chosen so that the man will reach the edge of the circle just as the lion captures him so $M_m \approx r > 3r/4$ and applying (2) to the $i = m$th step gives

$$r/4 \leq (m+1)(1 + 2\beta) l^2 / (2r).$$

Since β is a constant,

$$m = \Omega(r^2 / l^2). \tag{3}$$

To give a lower bound on the time for the lion to capture a man who employs this algorithm, we give a counter-strategy for the lion to follow. First, we show that the counter-strategy is within

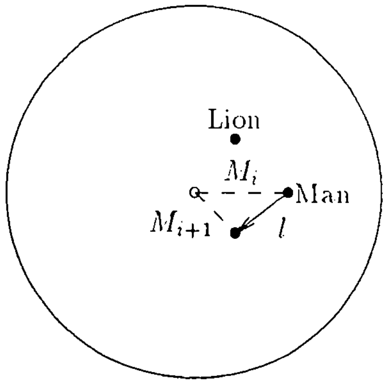


Figure 3: Man's movement following lower bound strategy. At the beginning of step i the man is at a distance M_i from the center of the circle. During step i the man moves a distance l at an angle less than or equal to $90° + \alpha$ with respect to the line segment connecting the center of the circle and the man. At the beginning of step $i + 1$ the man is at a distance M_{i+1} from the center of the circle.

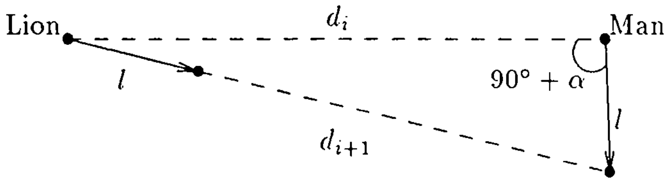


Figure 4: Closeup of the lion employing the counter-strategy in the proof of Theorem 1. The distance between the lion and man is d_i at the beginning of the ith and d_{i+1} at the beginning of the $(i + 1)$st step; l is the distance the man moves during the ith step.

an additive factor of time $O(c/s)$ of being the quickest of all possible counter-strategies to capture; then we determine how long it takes for capture. The counter-strategy is that during the man's ith step the lion moves directly toward where the man will be at the beginning of the $(i + 1)$st step.

We must show that this counter-strategy minimizes the distance between the lion and the man (except on the last step). Let d_i be the distance separating the lion and man at the beginning of the ith step. We prove by induction that d_i is the closest (over all strategies) that the lion can be to the man at the beginning of the ith step. For the basis of the induction, note that by definition d_1 is the closest the lion can be to the man at the beginning of the second step. Assuming that the hypothesis is true at step i, then d_{i+1} is the closest the lion can be to the man by the triangle inequality (see Figure 4). If step i is the last step, then there might exist a move for the lion to capture the man more quickly than moving to where the man would be at the end of the ith step (see Figure 5); however, it would only be an improvement of at most time $l/s = O(c/s)$.

To determine how long the counter-strategy takes until capture, we note (see Figure 4) that

$$\begin{aligned}(d_{i+1} + l)^2 &= d_i^2 + l^2 - 2d_i l \cos(90° + \alpha) \\ &= d_i^2 + l^2 + 2d_i l \sin \alpha,\end{aligned}$$

which can be rewritten as

$$d_{i+1} = \sqrt{d_i^2 + l^2 + 2\beta d_i l^2/r} - l, \tag{4}$$

by definition of α.

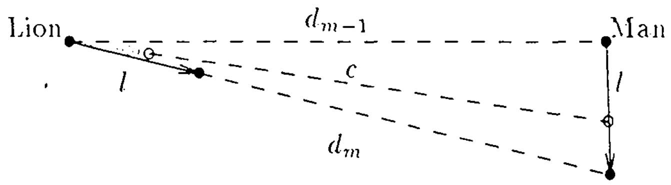


Figure 5: The last step when the lion employs the counter-strategy in proof of Theorem 1. If the lion moves along the dotted line, it will capture the man before it will capture the man if it moves along the solid line.

We partition the steps into three groups. First, those for which $d_i > l$, ending with step n for which $d_n > l$ and $d_{n+1} \le l$. Second, those for which $l \ge d_i > l^2/r$, ending with step p for which $d_p > l^2/r$ and $d_{p+1} \le l^2/r$. Finally, those for which $l^2/r \ge d_i$ up to $d_m \le c$. We show that most steps are in the third group and we then solve l and m in terms of r and c.

For steps in the first group, $d_i \ge l$ and applying Taylor's theorem to (4) gives

$$d_i - l \le d_{i+1} \le d_i - l + l^2/(2d_i) + \beta l^2/r.$$

Since $d_i > l$, we get

$$d_i - d_{i+1} = \Theta(l)$$

and therefore, by summing,

$$d_1 - d_n = \Theta(nl).$$

Since $d_1 = r/2 \gg d_n \approx l$ we get

$$n = \Theta(r/l) = o(m).$$

For steps in the second group, $l \ge d_i \ge l^2/r$. Letting $f_i = d_i/l$, equation (4) becomes

$$f_{i+1} = \sqrt{f_i^2 + 2\beta f_i l/r + 1} - 1.$$

We are only interested in steps beyond the nth and therefore we rewrite the above equation as

$$f_{i+n+1} = \sqrt{f_{i+n}^2 + 2\beta l f_{i+n}/r + 1} - 1.$$

Because $f_{i+n} < 1$, we get from Taylor's theorem

$$f_{i+n+1} \le f_{i+n}^2/2 + \beta l f_{i+n}/r. \tag{5}$$

The first term dominates in this part, so

$$f_{i+n}^2 > f_{i+n+1} > f_{i+n}^2/4. \tag{6}$$

To get a bound on f_i, we note that the recurrence relation

$$g_i = h g_{i-1}^2$$

has the solution

$$g_i = \frac{1}{h}(h g_k)^{2^{i-k}},$$

for $i \ge k \ge 1$. Therefore (6) gives us

$$f_{i+n} > 4\left(\frac{f_n}{4}\right)^{2^i}. \tag{7}$$

Noting that $f_n > 1$, we can rewrite (7) in terms of d_i as

$$d_i = \Theta\left(l2^{-2^i}\right)$$

and therefore

$$\begin{aligned} p &= \Theta\left(\log\log\frac{r}{l}\right) \\ &= o(m). \end{aligned}$$

For steps of the third type, $l^2/r \geq d_i$ and the second term dominates in (5) so

$$\beta l f_{i+p}/(2r) \leq f_{i+p+1} \leq \beta l f_{i+p}/r,$$

and therefore

$$f_{i+p+1} = \Theta((l/r)^{i-p}).$$

At the last step,

$$\begin{aligned} f_m &= c/l \\ &= \Theta((l/r)^m). \end{aligned}$$

This last equation and (3) give

$$l = \Theta\left(r\frac{\sqrt{\log\log(r/c)}}{\sqrt{\log(r/c)}}\right)$$

and hence

$$m = \Omega\left(\frac{\log\frac{r}{c}}{\log\log\frac{r}{c}}\right).$$

The total time taken is the distance traveled during the m steps, divided by the speed:

$$\begin{aligned} T &= \frac{1}{s}\sum_{i=1}^{m} l \\ &= \frac{ml}{s} \\ &= \Omega\left(\frac{r}{s}\sqrt{\frac{\log\frac{r}{c}}{\log\log\frac{r}{c}}}\right), \end{aligned}$$

as desired. □

We now prove

Theorem 2 *If a lion pursues a man inside a circular region of radius r and both have maximum speed s, then the time needed for the lion to get within a distance c of the man is*

$$O\left(\frac{r}{s}\log\frac{r}{c}\right).$$

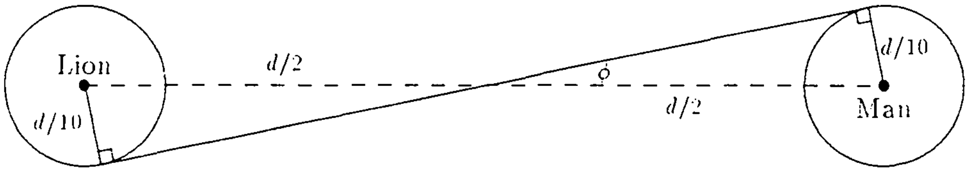


Figure 6: The maximum change in angle between the lion and man, initially a distance d apart, if both move a distance $d/10$, is $\phi = \arcsin(1/5)$.

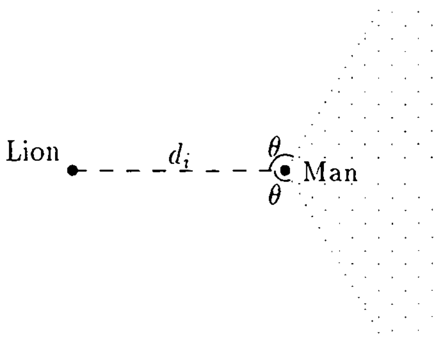


Figure 7: Lion chasing the man using the first part of the algorithm given in Theorem 2. If the man moves at an angle in the shaded region, the lion moves parallel to the man; otherwise the lion moves toward the man.

Proof. We present a two-part algorithm for the lion, and show that no matter what the man does, the lion can capture the man within the stated time. The lion chooses an arbitrary angle θ, $90° + \phi < \theta < 180°$, where $\phi = \arcsin(1/5)$ (see Figure 6). In the first part of the algorithm, if the man moves at an angle greater than $180° - \theta$ and less than $180° + \theta$ with respect to the line connecting the lion and man, the lion heads continuously toward the man, otherwise the lion moves parallel to the man (see Figure 7). However, it may happen that the lion cannot move parallel to the man, as shown in Figure 8, in which case the lion employs the second part of the strategy. In the second part, the lion repeatedly moves straight toward the man until one of the following three conditions occurs:

- The lion and man have the same y coordinate, where the x and y axes are defined so that the lion and man had the same y coordinate when the second part started.
- The line from the lion to man makes an angle of $180° - \theta + \phi$ with respect to the x axis
- The lion has gotten within distance ad of the man, where d is the distance between the lion and man at the start of the second part and a is defined below.

This algorithm requires that the lion react to the man's actions and we assume that the reaction time of the lion is either zero (so the lion knows where the man will move) or so close to zero as not to affect the calculations.

We bound the time needed until capture under this algorithm by dividing the time into steps and then proving that there are $O(\log \frac{r}{c})$ steps each of time $O(\frac{r}{s})$. Let d_i be the distance between

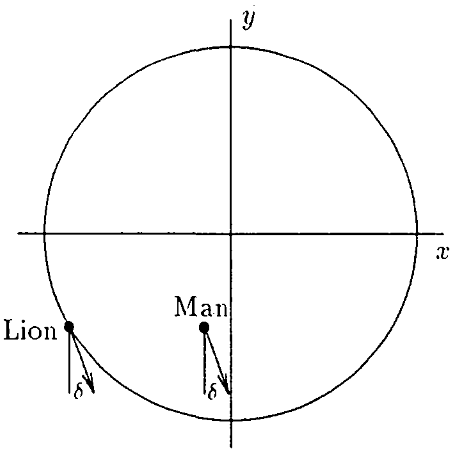


Figure 8: Beginning of the second part of the lion's strategy for Theorem 2. The x and y axes are defined so that the lion and man have the same y coordinate. The lion has reached the edge of the circle and the man moves along the arrow with $90° + \delta > \theta$. The lion will repeatedly look at the man's position and move there until either the lion and man have the same y coordinate, the line from the lion to man makes an angle of $180° - \theta + \phi$ with respect to the x axis, or the lion has gotten within distance ad_i of the man and starts a new step.

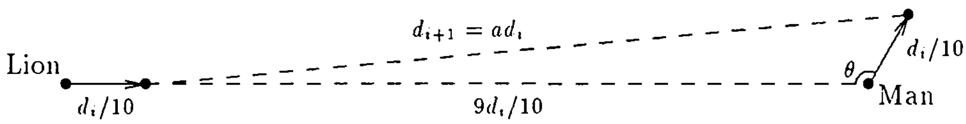


Figure 9: Closeup of the lion and man in which the man moves at an angle of $180° - \theta$ from the lion for distance $d_i/10$. If the lion moves directly to the right as shown, the distance between the lion and man is d_i before the move and is $ad_i = d_{i+1}$ afterwards where $a < 1$ is fixed and depends only on the initial choice of θ.

the lion and the man at the beginning of step i. Define step $i + 1$ to start as soon as the distance between the lion and man is $ad_i = d_{i+1}$, where $a < 1$ is defined as follows. Suppose that the man moves at angle $180° - \theta$ for distance $d_i/10$ during step i, and that the lion moves toward the man's position at the beginning of step i (see Figure 9). After these movements, the distance between the lion and man is taken to be ad_i, thus defining the value of a.

Solving the simple recurrence from the definition of d_i, we find that $d_i = a^{i-1}d_1$ and n, the number of steps until capture, is bounded by

$$a^n d_1 \leq c,$$

and therefore

$$\begin{aligned} n &\leq \log_{1/a} \frac{d_1}{c} \\ &= O\left(\log \frac{r}{c}\right). \end{aligned} \tag{8}$$

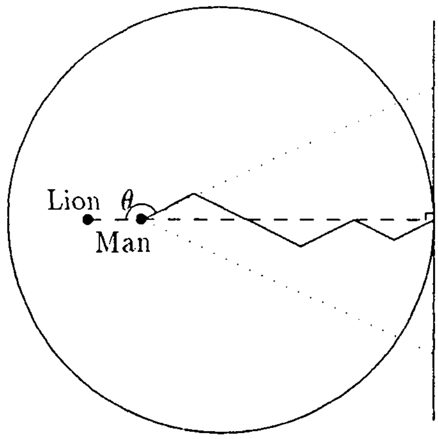


Figure 10: Lion chasing man in circle. If the man moves between angles of $180° - \theta$ and $180° + \theta$ he must stay between the dotted lines. The longest path he can follow is a zig-zag line at angles of $180° - \theta$ and $180° + \theta$, as shown by the solid line.

To determine the time of each step, we consider the time spent in each part. In considering the first part, we assume the second part does not occur. Consider separately the time the lion spends moving toward the man and the time the lion spends moving parallel to the man. ¿From Figure 9, the man can move at angles of at least $180° - \theta$ and at most $180° + \theta$ from the lion, that is the unshaded region of Figure 7, for a distance less than $d_i/10$ until the next step. If the lion moves directly to the right for distance $d_i/10$, as shown, the distance between the lion and man is at most ad_i. Thus the time in which the lion moves toward the man is less than $d_i/(10s) = O(r/s)$.

To bound the time the lion spends moving parallel to the man, note that the man cannot move forever at an angle between $180° - \theta$ and $180° + \theta$; eventually he will reach the edge of the circle since $90° + \phi < \theta < 180°$. As shown in Figure 10, if the man moves only at angles between $180° - \theta$ and $180° + \theta$ then the longest distance he can move before reaching the edge of the circle is in a zig-zag path at angles of $180° - \theta$ and $180° + \theta$. Furthermore, the length of such a path is the same as the length of the dotted lines in Figure 10. By the definition of a step, the man cannot, in one step, change the angle between the lion and man by $90°$ (see Figure 11). So the distance the man moves in one step, with the lion moving parallel to him, is bounded by

$$\frac{(2r - d_i) + d_i/10}{-\cos\theta} = O(r),$$

where $2r - d_i$ is the furthest a line can extend from the lion through the man to the circle's edge, and $d_i/10$ is the maximum distance the man can move with the lion moving toward him. Thus, the time the lion spends moving parallel to the man is $O(r/s)$.

For the second part, as shown in Figure 8, we assume that the lion is moving downwards and to the right (the circle can be rotated and/or reflected). Since the second part ends if the man is ever at an angle greater than $180° - \theta + \phi$ with respect to the x axis and $\theta > 90° + \phi$, the lion always moves at an angle less than $90° - \epsilon$ with respect to the x axis for some positive constant $\epsilon = 90° - \theta + \phi$. Therefore the time spent in any one instance of the second part is clearly $O(r/s)$.

We divide the analysis into sections based on how the second part ends. Clearly if the second part ends because the lion gets within distance ad there is no problem.

If the second part ends because the line between the lion and man is at an angle of $180° - \theta + \phi$ with respect to the x axis then, we restart the step. The second part of the lion's algorithm cannot

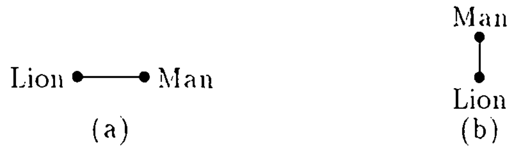
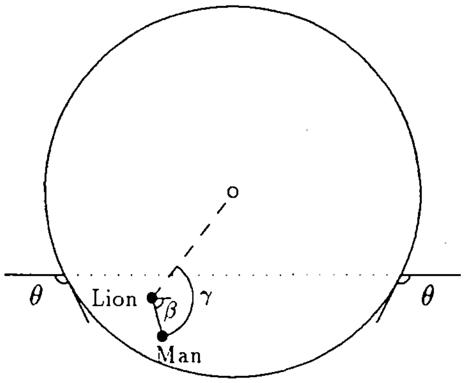


Figure 11: A rotation of 90° of the man's position to the lion's position from (a) to (b) is not possible in one step by the definition of a step (see Figure 9) .



$$\gamma \geq \theta > 90° + \phi$$

$$\beta = 180° - \theta + \phi$$

Figure 12: When second step cannot occur. If the lion ever starts a step from a position below the dotted line with the man at $180° - \theta + \phi$, then the second part of the lion's strategy in Theorem 2 will not occur in that step. The angles θ on the outside of the circle are with respect to the tangent of the circle.

occur during this new step because, as shown in Figure 6, the man cannot change the lion's direction by an angle of ϕ and, as shown in Figure 12, the center and the man are on opposite sides of the line through the lion at an angle of $90° + \phi$ with respect to the line segment between the lion and man.

Otherwise, the second part ends because the lion has the same y coordinate as the man. This condition is similar to the first part with the lion moving parallel with respect to the man, except for two changes. First, the distance between the lion and the man is not d, but lies between ad and d. Second, the angle moved to the right is between $180° - \theta + \phi$ and $180° + \theta - \phi$ instead of between $180° - \theta$ and $180° + \theta$. However, these are minor changes and the analysis of the lion moving parallel with respect to the man in the analysis of the first part can include any distance moved in the second part.

Since the times in the first and second parts are each $O(r/s)$, the total time per step is $O(r/s)$ and combining this with (8), we see that the time the lion needs to capture the man is

$$O\left(\frac{r}{s} \log \frac{r}{c}\right).$$

□

There are many interesting variations of this continuous pursuit problem that might be considered. What algorithm should the lion employ if it moves faster than the man [6]? What happens if the arena is unbounded on one side [7]? How long does it take n lions to capture a man? If we introduce a hole (a region in which neither the man nor lion may move), one lion cannot always capture the man if c is less than the radius of the hole; two lions, however, are sufficient and can even capture the man with $c = 0$. By adapting an algorithm of Aigner and Fromme [1], it can be shown that three lions can capture the man for any finite number of holes—are three lions necessary or do two suffice?

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