# A convex parametrization of suboptimal hinfini controllers

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**Abstract:** A new parametrization is proposed for suboptimal $H_\infty$ controllers of order no larger than the plant order. Here such controllers are generated from pairs of symmetric matrices $(X, Y)$ constrained by two Riccati matrix inequalities and some positivity requirements. Interestingly, the Riccati expressions and the positivity conditions are exactly those arising in the usual state-space solution of suboptimal $H_\infty$ problems.

When working with the inverses $R$ and $S$ of $X$ and $Y$, respectively, the constraints can be rewritten as linear matrix inequalities which define a convex parameter set. This sets up a convenient framework to handle design objectives which can be reflected in terms of $(R, S)$. Examples of such objectives include reduced-order $H_\infty$ design and the avoidance of pole/zero cancellation between the plant and the controller.

# Une Paramétrisation Convexe des Contrôleurs $H_\infty$ Sous-Optimaux

**Résumé:** Cet article introduit une nouvelle paramétrisation des contrôleurs $H_\infty$ sous-optimaux d'ordre au plus égal à celui du système à commander. Ici les paramètres sont des pairs de matrices $(X, Y)$ contraintes par des inégalités de Riccati et des contraintes de positivité. Il y a une analogie totale entre cette charactérisation et celle des performances $H_\infty$ atteignables [3].

En travaillant avec les inverses $R$ et $S$ de $X$ et $Y$, les contraintes deviennent convexes. Cette propriété est favorable à l'application d'algorithmes d'optimisation convexe avec convergence garantie. Deux applications sont abordées: la synthèse de contrôleurs $H_\infty$ d'ordre réduit et l'élimination des simplifications pôle/zéro entre le système et le contrôleur.

# 1 Introduction

DGKF's state-space results [3] offer simple and numerically appealing means of designing $H_\infty$ controllers. Indeed, computations essentially reduce to solving the well-known matrix Riccati equation. Moreover, explicit state-space formulas are given for some particular $H_\infty$ controller called the "central controller." Finally, all suitable controllers are parametrized via a linear fractional transformation depending on a free dynamical parameter $Q(s)$ [3]. These attractive features all contribute to the growing popularity of state-space-oriented $H_\infty$ synthesis.

Yet, the diversity in $H_\infty$ controllers is hardly exploited in DGKF's state-space approach. As a result, applications make exclusive use of the central controller in spite of certain undesirable properties. For instance, this particular controller tends to cancel stable plant poles [10]. In mixed-sensitivity problems with flexible modes, this often results in unacceptable designs from an engineering point of view. Also, the order of the central controller matches that of the augmented plant and may therefore be quite high. Finally, the central controller is the optimal choice with respect to some entropy criterion [6], but may well prove a bad choice with respect to more desirable design objectives. Such objectives include the $H_2$ norm of certain transfer functions and the time-domain behavior (overshoot, rise- and settling time).

The $Q$-parametrization of all suboptimal $H_\infty$ controllers could help select controllers which are better-suited to the overall design requirements. Unfortunately, the linear-fractional nature of this parametrization does not blend nicely with the state-space formulation. As a result, there is no obvious way of choosing the free parameter $Q(s)$ in order to, e.g., reduce the controller order or prevent plant/controller cancellations.

This paper introduces a state-space counterpart of the $Q$-parametrization which is appealing both on computational and analytical grounds. Here the parameter set consists of pairs of symmetric matrices constrained by Riccati inequalities and positivity requirements. Interestingly, these constraints completely parallel DGKF's characterization of feasible $\gamma$'s, except that Riccati equations are replaced by Riccati inequalities. To be specific, recall that the "standard" suboptimal $H_\infty$ problem of parameter $\gamma$ considered in [3] is solvable if and only if the two Riccati equations

$$
\begin{aligned}
A^T X + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 &= 0\,; &(1.1)\\
AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T &= 0 &(1.2)
\end{aligned}
$$

have stabilizing solutions $X_\infty$ and $Y_\infty$ which further satisfy:

$$X_\infty \geq 0; \qquad Y_\infty \geq 0; \qquad \rho(X_\infty Y_\infty) < \gamma^2. \tag{1.3}$$

In comparison, this paper shows that $\gamma$-suboptimal controllers of order no larger than the plant order can be "parametrized" by the set of pairs $(X, Y)$ of symmetric matrices constrained by:

$$
\left\{
\begin{array}{l}
A^T X + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 < 0\\
AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T < 0\\
X > 0, \quad Y > 0, \quad \rho(XY) \leq \gamma^2
\end{array}
\right. \tag{1.4}
$$

There is an obvious analogy between the two results. In fact, DGFK's solvability conditions are easily deduced from our parametrization (see Subsection 4.3 below). Note that the requirement $\rho(XY) < \gamma^2$ is relaxed to $\rho(XY) \leq \gamma^2$, equality corresponding to reduced-order $H_\infty$ controllers.

The paper is organized as follows. The next section gives a precise statement of the suboptimal $H_\infty$ problem and recalls the instrumental Bounded Real Lemma. Section 3 is an intermediate step toward the advertised parametrization. There, insight is gained into the connection between suboptimal $H_\infty$ controllers and pairs $(X, Y)$ satisfying (1.4). This will prove useful for the reconstruction of $H_\infty$ controllers from such $(X, Y)$'s. Section 4 presents the main result which is a parametrization of suboptimal $H_\infty$ controllers by the set of pairs $(X, Y)$ satisfying (1.4). The constraints (1.4) are shown to define a convex set when reformulated in terms of $R := X^{-1}$ and $S := Y^{-1}$. Since convexity is highly desirable in optimization problems, all theorems are stated in terms of $R, S$ instead of $X, Y$. A simple algorithm is also proposed for the construction of suboptimal controllers from admissible pairs $(R, S)$. Only elementary matrix computations are involved in this construction. In Section 5 finally, two applications illustrate the potential of this parametrization as a design tool. Firstly, the problem of preventing plant/controller cancellations in $H_\infty$ design is turned into a convex optimization problem in the $(R, S)$-space. Secondly, applications to reduced-order $H_\infty$ design are discussed.

## 2 The Suboptimal $H_\infty$ Control Problem

As usual in $H_\infty$ control problems, consider a proper plant $G(s)$ which maps exogenous inputs $w$ and control inputs $u$ to controlled outputs $z$ and measured outputs $y$. That is, $\begin{pmatrix} z \\ y \end{pmatrix} = G(s) \begin{pmatrix} w \\ u \end{pmatrix}$ where

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1} (B_1, \ B_2). \tag{2.1}$$

This realization is taken minimal and $n$ denotes its order ($A \in \mathbb{R}^{n \times n}$). The vectors $z$, $y$, $w$, and $u$ are of size $p_1$, $p_2$, $m_1$, and $m_2$, respectively, with the assumption that $m_1 \geq p_2$ and $p_1 \geq m_2$.

The suboptimal $H_\infty$ control problem of parameter $\gamma$ consists of finding a dynamic real-rational output feedback law $u = K(s)y$ such that:

- the closed-loop system is internally stable,
- the $H_\infty$ norm of the closed-loop transfer function from $w$ to $z$ is strictly less than $\gamma$.

Controllers solving this problem (if any) will be called **$\gamma$-suboptimal**. Observing that the closed-loop transfer function from $w$ to $z$ is given by the linear fractional transformation (LFT):

$$\mathcal{F}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}, \tag{2.2}$$

$K$ is $\gamma$-suboptimal if and only if the closed-loop system is internally stable and $\|\mathcal{F}(G, K)\|_\infty < \gamma$. The optimal $H_\infty$ attenuation $\gamma_{opt}$ is defined as the smallest (asymptotically) achievable $\gamma$, that is, the infimum of all $\gamma > 0$ for which there exist $\gamma$-suboptimal controllers.

The following assumptions on the state-space realization (2.1) of the plant $G$ are made throughout the paper:

**(A1)** $(A, B_2, C_2)$ is stabilizable and detectable.

**(A2)** $D_{12}^T(D_{12}, C_1) = (I, 0)$ and $D_{21}(D_{21}^T, B_1^T) = (I, 0)$.

**(A3)** $D_{22} = 0$ and $D_{11} = 0$.

Along with **(A1)**, full column rank for $D_{12}$ and full row rank for $D_{21}$ are the only hard requirements for the validity of our results. With these assumptions standing, **(A2)**-**(A3)** are only introduced for the resulting notational simplification. Note that the customary requirement that $G_{12}$ and $G_{21}$ have no $j\omega$-axis transmission zero is not needed in our approach.

The parametrization introduced in this paper is state-space oriented and specialized to controllers whose order is no larger than the plant order. Given the state-space realization

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K; \qquad A_K \in \mathbb{R}^{k\times k};\ k \le n \tag{2.3}$$

of some controller $K$, a (non necessarily minimal) realization of the closed-loop transfer function from $w$ to $z$ is:

$$\mathcal{F}(G,K)(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl} \tag{2.4}$$

where

$$\begin{aligned} A_{cl} &= \begin{pmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{pmatrix}; \qquad B_{cl} = \begin{pmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{pmatrix}; \\ C_{cl} &= \begin{pmatrix} C_1 + D_{12}D_KC_2 & D_{12}C_K \end{pmatrix}; \qquad D_{cl} = D_{12}D_KD_{21}. \end{aligned} \tag{2.5}$$

In terms of these state-space parameters, the controller $K$ is $\gamma$-suboptimal if and only if

$$A_{cl} \text{ is stable and } \|D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_\infty < \gamma. \tag{2.6}$$

Interestingly, these two requirements can be lumped into a single condition involving a Riccati inequality of order $n + k$. This instrumental reformulation is known as the Bounded Real Lemma. When expressed in terms of Riccati inequalities, this lemma does not require minimality of the realization and takes the fully general form given next.

**Lemma 2.1 (Bounded Real Lemma)** *Consider some (non necessarily minimal) realization $T(s) = D + C(sI - A)^{-1}B$ of some (non necessarily square) transfer function $T(s)$. Then the following statements are equivalent:*

*(i) $A$ is stable and $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$,*

*(ii) $\sigma_{max}(D) < \gamma$ and there exists a symmetric positive definite solution $X$ to the ARI:*

$$A^TX + XA + \gamma^{-2}C^TC + (\gamma^{-2}C^TD + XB)(I - \gamma^{-2}D^TD)^{-1}(\gamma^{-2}C^TD + XB)^T < 0. \tag{2.7}$$

# 3 Suboptimal Controllers and Riccati Inequalities

In this section, the conditions (2.6) for $\gamma$-suboptimality are converted into Riccati inequality constraints via the Bounded Real Lemma. The Riccati inequalities involve the plant state-space parameters, but also some of the controller parameters. Hence this reformulation is not readily useful for controller synthesis. Nonetheless, it constitutes an insightful intermediate step toward the convex parametrization announced in Section 1. In particular, it brings out the connection between suboptimal controllers and the constraints (1.4).

The next theorem gives necessary conditions for $\gamma$-suboptimality in terms of matrix inequalities involving two matrix parameters $R$ and $S$. Sufficiency of these conditions is addressed in the subsequent Theorem 3.2. These two results heavily rely on the Bounded Real Lemma and strengthen Theorem 6.5, p. 222 of [9]. Note that $X, Y$ in (1.4) correspond to $R^{-1}$ and $S^{-1}$, respectively, for reasons which will become clear in Section 4.

**Theorem 3.1 (Necessity Part)** *Assume (A1)-(A3) and suppose the controller $K(s)$ is $\gamma$-suboptimal of order no larger than the plant order ($k \leq n$). Let*

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K; \qquad A_K \in \mathbb{R}^{k\times k} \tag{3.1}$$

*denote a minimal realization of $K$. Then*

**(C1)** $\sigma_{max}(D_K) < \gamma$

*and there exist symmetric matrices $R, S$ in $\mathbb{R}^{n\times n}$ and $M, N$ in $\mathbb{R}^{n\times k}$ such that*

**(C2)** *$R, S, M$, and $N$ satisfy*

$$\Delta_R := AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T + M_R(I - \gamma^{-2}D_KD_K^T)^{-1}M_R^T < 0; \tag{3.2}$$

$$\Delta_S := A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2 \ + \ N_S(I - \gamma^{-2}D_K^TD_K)^{-1}N_S^T < 0; \tag{3.3}$$

$$MN^T = \gamma^{-2}I - RS \tag{3.4}$$

*with the shorthands:*

$$M_R := B_2 + RC_2^TD_K^T + MC_K^T; \qquad N_S := C_2^T + SB_2D_K + NB_K. \tag{3.5}$$

**(C3)** $R > 0, \ S > 0,$ *and* $\lambda_{\min}(RS) \geq \gamma^{-2}$.

**Proof:** See Appendix B.

∎

The matrices $R, S, M, N$ are constructed by applying the Bounded Real Lemma to the realization (2.4) of the closed-loop system $\mathcal{F}(G, K)$. Specifically, given any solution $X_{cl} \in \mathbb{R}^{(n+k)\times(n+k)}$ of the Riccati inequality (2.7) written for $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, suitable $R, S, M, N$ are readily obtained by partitioning:

$$X_{cl} =: \begin{pmatrix} S & N \\ N^T & \star \end{pmatrix}; \qquad \gamma^{-2}X_{cl}^{-1} =: \begin{pmatrix} R & M \\ M^T & \star \end{pmatrix}; \qquad R, S \in \mathbb{R}^{n\times n}, \ \ M, N \in \mathbb{R}^{n\times k}. \tag{3.6}$$

Hence **(C2)** expresses the fact that $\|\mathcal{F}(G, K)\|_\infty < \gamma$ since it has to do with existence of some real symmetric solution $X_{cl}$ to (2.7). Meanwhile, **(C3)** expresses internal stability of the closed-loop system since it is equivalent to $X_{cl} > 0$. Note that the correspondence $K(s) \to (R, S)$ is not univocal; because of the inequalities, to each suboptimal $K(s)$ corresponds a family of pairs $(R, S)$ (see Subsection 4.2 for detail).

Concerning the controller order $k$, note that if $k < n$ then $\lambda_{\min}(RS) = \gamma^{-2}$ or more precisely, $rank(\gamma^{-2}I - RS) = k$ (recall that $M, N \in \mathbb{R}^{n\times k}$). Equivalently in terms of $(X, Y) := (R^{-1}, S^{-1})$, we have $\rho(XY) = \gamma^2$ and $rank(\gamma^{-2}XY - I) = k$. As confirmed in the sequel, rank deficiency of $\gamma^{-2}I - RS$ is characteristic of reduced-order $H_\infty$ controllers and the rank of this "coupling matrix" determines the controller order (see Subsection 4.4).

We now turn to the converse of Theorem 3.1. Observe that the controller state matrix $A_K$ does not appear in **(C1)-(C3)**. Hence the main concern is whether the knowledge of $R, S, M, N$ and $B_K, C_K, D_K$ satisfying **(C1)-(C3)** is sufficient to reconstruct the $A_K$ matrix of some $\gamma$-suboptimal controller. The next theorem confirms the feasibility of this operation.

**Theorem 3.2 (Sufficiency Part)** *With the notation and assumptions of Theorem 3.2, consider any matrices $R, S, M, N, B_K, C_K, D_K$ such that*

- $R = R^T \in \mathbb{R}^{n\times n}$, $S = S^T \in \mathbb{R}^{n\times n}$, *and* $M, N \in \mathbb{R}^{n\times k}$ *have full column rank* ($k \leq n$).
- $R, S, M, N, B_K, C_K, D_K$ *jointly satisfy* **(C1)-(C3)**.

*Then there exists* $A_K \in \mathbb{R}^{k\times k}$ *for which the controller* $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$ *is* $\gamma$-*suboptimal and of order at most* $k$. *Such a matrix* $A_K$ *is constructed as follows.*

*(i) Compute matrices* $\Delta_{12} \in \mathbb{R}^{n\times k}$ *and* $\Delta_{22} \in \mathbb{R}^{k\times k}$ *such that* $\begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix} > 0$ *and*

$$\Delta_R + \gamma^2(R, M)\begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix}\begin{pmatrix} R \\ M^T \end{pmatrix} = 0. \tag{3.7}$$

*(ii) Compute* $A_K \in \mathbb{R}^{k\times k}$ *as the unique solution of*

$$\begin{aligned} -NA_KM^T &= \gamma^{-2}A^T + S(\gamma^{-2}B_1B_1^T + B_2C_KM^T) + (\gamma^{-2}C_1^TC_1 + NB_KC_2)R + \\ &\quad S(A + B_2D_KC_2)R + \gamma^{-2}N_S(I - \gamma^{-2}D_K^TD_K)^{-1}D_K^TM_R^T - \Delta_SR + \Delta_{12}M^T. \end{aligned} \tag{3.8}$$

**Proof:** See Appendix B.

■

Hence some suitable controller state matrix $A_K$ can be computed by solving two systems of linear equations. Note that $A_K$ is directly sought of dimensions $k \times k$ in the reduced-order case. Consequently, no switching to descriptor form is needed when $\gamma^{-2}I - RS$ or equivalently $I - \gamma^{-2}XY$ is rank-deficient. Interestingly, this scheme also applies to optimal central controller design when $I - \gamma^{-2}X_\infty Y_\infty$ is singular at $\gamma_{opt}$ [4]. Finally, observe that (3.8) can also be written (with the notation $Z := \gamma^{-2}I - RS$):

$$\begin{aligned} -NA_KM^T &= Z^T(A + RC_1^TC_1)^T + NB_KC_2R + S\Delta_R - \Delta_SR + \Delta_{12}M^T \\ &\quad \{\gamma^{-2}NB_KD_K^T + Z^TC_2^TD_K^T - SMC_K^T\}(I - \gamma^{-2}D_KD_K^T)^{-1}M_R^T, \end{aligned} \tag{3.9}$$

or equivalently as:

$$\begin{aligned} -NA_KM^T &= (A + B_1B_1^TS)^TZ^T + SB_2C_KM^T + \Delta_{12}M^T + \\ &\quad N_S(I - \gamma^{-2}D_K^TD_K)^{-1}\{\gamma^{-2}MC_K^TD_K + ZB_2D_K - RNB_K\}^T. \end{aligned} \tag{3.10}$$

Simple algebra involving (3.2)-(3.3) accounts for these reformulations (see Appendix B for detail).

At first glance, the trading of the unknown $A_K$ against the four unknowns $R, S, M, N$ seems rather unfavorable. However, much structure has been gained in the process. In the full-order case for instance, $M$ and $N$ are square invertible and can be absorbed in $C_K, B_K$ without loss of generality. The two Riccati inequalities then decouple and given any $R, S$ satisfying **(C3)** and

$$\begin{aligned} AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T &< 0; \\ A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2 &< 0, \end{aligned}$$

some $B_K, C_K, D_K$ can be constructed such that (3.2)-(3.3) hold. This simple observation is the gateway to the state-space parametrization introduced in the next section.

## 4 Convex Parametrization of Suboptimal Controllers

The results of Section 3 are awkward for design purposes because the unknown controllers parameters $B_K, C_K, D_K$ still appear in (3.2)-(3.3). In this section, $M, N, B_K, C_K, D_K$ are removed altogether from the formulation. This leads to our main result: a parametrization of $\gamma$-suboptimal controllers (of order no larger than the plant order) by all pairs $(R, S)$ in the set

$$\mathcal{A}_\gamma := \{(R,S):\ R = R^T \text{ and } S = S^T \text{ and } R, S \text{ satisfy the constraints } (\mathcal{C}_\gamma)\ \} \tag{4.1}$$

where

$$(\mathcal{C}_\gamma) \qquad \begin{cases} AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T \ < \ 0 \\ A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2 \ < \ 0 \\ R > 0, \quad S > 0, \quad \lambda_{\min}(RS) \geq \gamma^{-2} \end{cases} \tag{4.2}$$

Note that (4.2) is equivalent to (1.4) in the transformation $(R, S) \to (X, Y) := (R^{-1}, S^{-1})$.

Clearly, all pairs $(R, S)$ associated with $\gamma$-suboptimal controllers via Theorem 3.1 lie in $\mathcal{A}_\gamma$. Conversely, the next theorem shows that given any $(R, S)$ in $\mathcal{A}_\gamma$, we can reconstruct some $\gamma$-suboptimal controller of order $k \leq n$ by some elementary matrix computations. This confirms the relevance of Riccati inequalities in $H_\infty$ design (see also, e.g., [11, 12]).

**Theorem 4.1** *Assume* **(A1)-(A3)** *and introduce the Riccati residuals:*

$$\mathcal{R}_R \ := \ AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T; \tag{4.3}$$

$$\mathcal{R}_S \ := \ A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2. \tag{4.4}$$

*Given any pair $(R, S)$ in $\mathcal{A}_\gamma$, a $\gamma$-suboptimal controller $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ of order no larger than $k := rank(\gamma^{-2}I - RS)$ can be reconstructed as follows.*

*(a) Compute full column rank matrices $M, N \in \mathbf{R}^{n\times k}$ such that $MN^T = \gamma^{-2}I - RS$.*

*(b) Select $D_K$ as follows. If $k = n$, pick any matrix satisfying $\sigma_{max}(D_K) < \gamma$. Otherwise, chose $D_K$ so that $\sigma_{max}(D_K) < \gamma$ and*

$$\begin{pmatrix} I & \gamma^{-1}D_K \\ \gamma^{-1}D_K^T & I \end{pmatrix} > \begin{pmatrix} -B_2^T \\ \gamma C_2R \end{pmatrix} V_2 \left\{V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2\right\}^{-1} V_2^T\left(-B_2, \gamma RC_2^T\right) \tag{4.5}$$

*where $V_2$ denotes an orthonormal basis of* $\operatorname{Ker}(\gamma^{-2}I - SR)$*. A possible choice is*

$$D_K = -\gamma^2B_2^TV_2\left\{V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2\right\}^{-1} V_2^TRC_2^T. \tag{4.6}$$

*(c) For $B_K, C_K$, chose any matrices compatible with*

$$\Delta_R := \mathcal{R}_R + (B_2 + RC_2^TD_K^T + MC_K^T)(I - \gamma^{-2}D_KD_K^T)^{-1}(B_2 + RC_2^TD_K^T + MC_K^T)^T \ < \ 0; \tag{4.7}$$

$$\Delta_S := \mathcal{R}_S + (C_2^T + SB_2D_K + NB_K)(I - \gamma^{-2}D_K^TD_K)^{-1}(C_2^T + SB_2D_K + NB_K)^T \ < \ 0. \tag{4.8}$$

*(d) Use Theorem 3.2 to reconstruct $A_K$.*

**Proof:** See Appendix C.

■

Summing up, there is an exhaustive correspondence between the set of $\gamma$-suboptimal controllers of order $k \leq n$ and the set $\mathcal{A}_\gamma$ defined by (4.1)-(4.2). Properties and interpretation of this result are discussed in the remainder of the section.

## 4.1 Properties of $\mathcal{A}_\gamma$

The parameter set $\mathcal{A}_\gamma$ has nice properties which make this formulation computationally appealing. First of all, $\mathcal{A}_\gamma$ is a convex subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. This is easily seen when observing that the constraints $(\mathcal{C}_\gamma)$ are equivalent to the three linear matrix inequalities:

$$\begin{pmatrix} AR + RA^T + \gamma^{-2}B_1B_1^T - B_2B_2^T & RC_1^T \\ C_1R & -I \end{pmatrix} < 0\,; \tag{4.9}$$

$$\begin{pmatrix} A^TS + SA + \gamma^{-2}C_1^TC_1 - C_2^TC_2 & SB_1 \\ B_1^TS & -I \end{pmatrix} < 0\,; \tag{4.10}$$

$$\begin{pmatrix} R & \gamma^{-1}I \\ \gamma^{-1}I & S \end{pmatrix} \geq 0\,, \tag{4.11}$$

each of which define a convex subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$. Incidentally, convexity does not hold for the set of pairs $(X,Y)$ defined by (1.4). This is our main motivation for using the parameters $R,S$ instead of $X,Y$.

With $\gamma_{opt}$ denoting the smallest feasible $\gamma$, the family of sets $\{\mathcal{A}_\gamma\}_{\gamma>0}$ has the following properties:

- $\mathcal{A}_\gamma \neq \emptyset$ if and only if $\gamma > \gamma_{opt}$ ;
- $\mathcal{A}_{\gamma_1} \subset \mathcal{A}_{\gamma_2}$ whenever $\gamma_1 < \gamma_2$.

In other words, the set $\mathcal{A}_\gamma$ grows larger as $\gamma$ increases.

Another interesting property concerns extremal points of $\mathcal{A}_\gamma$ for $\gamma > \gamma_{opt}$. The next lemma shows that the stabilizing solutions $X_\infty$ and $Y_\infty$ of the usual $H_\infty$ Riccati equations (1.1)-(1.2) are minimizers among all $X,Y$ satisfying (1.4).

**Lemma 4.2** *Assume $\gamma > \gamma_{opt}$ and let $X_\infty$ and $Y_\infty$ denote the stabilizing solutions of (1.1)-(1.2), respectively. Then for any pair of $n \times n$ symmetric matrices $(X,Y)$ satisfying*

$$A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 \quad < \quad 0\,; \tag{4.12}$$

$$AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T \quad < \quad 0\,, \tag{4.13}$$

*we have*

$$X_\infty < X; \qquad\qquad Y_\infty < Y\,. \tag{4.14}$$

**Proof:** The proof is easily adapted from standard monotonicity results on Riccati solutions (see, e.g., [13]).

■

Defining $R_\infty := X_\infty^{-1}$ and $S_\infty := Y_\infty^{-1}$, it follows that $(R_\infty, S_\infty)$ is maximal among all $(R,S)$ in $\mathcal{A}_\gamma$. That is, $R < R_\infty$ and $S < S_\infty$ for all $(R,S) \in \mathcal{A}_\gamma$. Note that $(R_\infty, S_\infty)$ always lies on the boundary of $\mathcal{A}_\gamma$ associated with the two Riccati inequalities. Also, $R_\infty$ (respectively, $S_\infty$) is unbounded whenever $X_\infty$ (respectively, $Y_\infty$) is singular. The set $\mathcal{A}_\gamma$ is then unbounded toward infinity. To gain geometrical insight into the nature of $\mathcal{A}_\gamma$, we now discuss a simple example for a plant of order one.

**Example 4.3** Consider the plant $G(s)$ of order one and state-space parameters (cf. (2.5)):

$$A = 1; \;\; B_1 = C_1 = 0; \;\; B_2 = 1; \;\; C_2 = 2; \;\; B_2 = 1; \;\; D_{12} = D_{21} = 1; \;\; D_{11} = D_{22} = 0.$$

The lowest achievable $\gamma$ for this plant is $\gamma_{opt} = 1$. For $\gamma > 1$, the set $\mathcal{A}_\gamma$ is defined by the constraints:

$$2r - 1 < 0; \qquad 2s - 4 < 0; \qquad r > 0; \qquad s > 0; \qquad rs \geq \gamma^{-2}.$$

In Figure 4.4, this domain is plotted in the $(r, s)$-plane for $\gamma = 2$. Note that the Riccati inequalities define two strips while the coupling constraint $rs \geq \gamma^{-2}$ selects the region above the hyperbola $rs = \gamma^{-2}$. Since $X_\infty = 2$ and $Y_\infty = 1/2$ independently of $\gamma$, $(R_\infty, S_\infty) := (X_\infty^{-1}, Y_\infty^{-1}) = (1/2, 2)$ is an extremal point of $\mathcal{A}_\gamma$ for all $\gamma > 1$. This corner point corresponds to the central controller design while pairs $(r, s)$ sitting on the hyperbola part of the boundary correspond to reduced-order suboptimal controllers.

As $\gamma$ decreases from $+\infty$, the region $\mathcal{A}_\gamma$ shrinks further and further. A particularity of this problem is that the Riccati constraints are independent of $\gamma$. As a result, only the boundary associated with $rs \geq \gamma^{-2}$ is moving. As $\gamma \rightarrow \gamma_{opt}$, this boundary approaches the corner point $(R_\infty, S_\infty)$ and $\mathcal{A}_\gamma$ shrinks to an empty set as illustrated on Figure 4.5. At $\gamma = \gamma_{opt}$, $\mathcal{A}_\gamma$ is empty since no controller can internally stabilize the plant while enforcing the strict constraint $\|\mathcal{F}(G, K)\|_\infty < \gamma_{opt}$. Nevertheless, the corner point $(R_\infty, S_\infty)$ generally yields a solution of the "optimal" $H_\infty$ problem where $\|\mathcal{F}(G, K)\|_\infty = \gamma_{opt}$. Due to the triviality of this example, the central controller design is the only feasible design at the optimum. In more realistic problems however, there often remain degrees of freedom in the choice of $(R, S)$ at $\gamma_{opt}$.

## 4.2 Nature of the Parametrization

The word "parametrization" should be taken in a loose sense here since the correspondence $K(s) \leftrightarrow (R, S)$ is not univocal. Indeed, given any $(R, S)$ in $\mathcal{A}_\gamma$ we can reconstruct a family of $\gamma$-suboptimal controllers where state-space parameters vary in open sets. Conversely, to each suboptimal controller corresponds a convex subset of $\mathcal{A}_\gamma$. To see this, recall from Section 3 that $S$ is constructed from solutions $X_{cl}$ of the Bounded Real Lemma inequality (2.7) written for the closed-loop system. For fixed $(A_K, B_K, C_K, D_K)$, the solution set of this inequality is convex since (2.7) can be rewritten as

$$\begin{pmatrix} A^T X + XA + \gamma^{-2} C^T C & \gamma^{-2} C^T D + XB \\ \gamma^{-2} D^T C + B^T X & \gamma^{-2} D^T D - I \end{pmatrix} < 0$$

which is affine in $X$. The resulting set of $S$ matrices is therefore convex. Similar conclusions hold for the $R$ component upon observing that $\gamma^{-2} X_{cl}^{-1}$ solves the Riccati inequality dual of (2.7) in the transformation $(A, B, C, D) \rightarrow (A^T, C^T, B^T, D^T)$.

This topological rather than pointwise correspondence between suboptimal controllers and pairs $(R, S)$ may seem a handicap at first. Fortunately, certain important properties of the controller or the closed-loop system can be monitored through $(R, S)$ independently of the particular controller reconstructed (see Section 5). Nevertheless, more research is needed to characterize the family of controllers associated with a given $(R, S) \in \mathcal{A}_\gamma$ and to exploit these additional degrees of freedom.

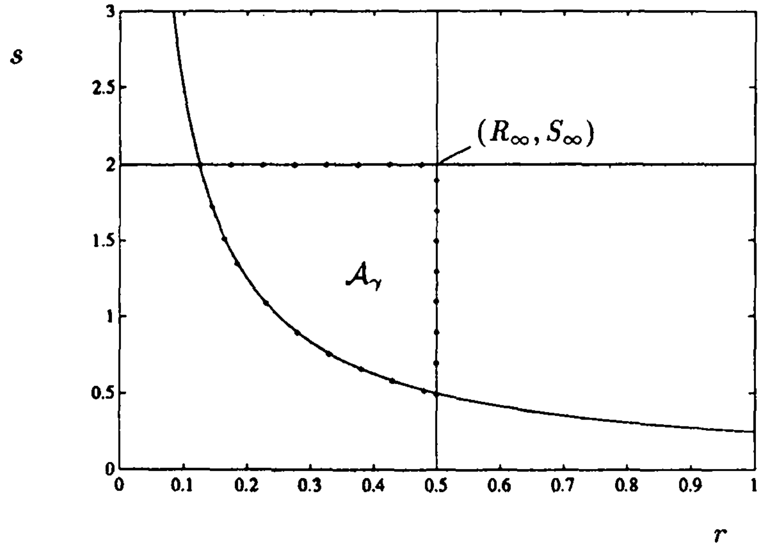


Figure 4.4

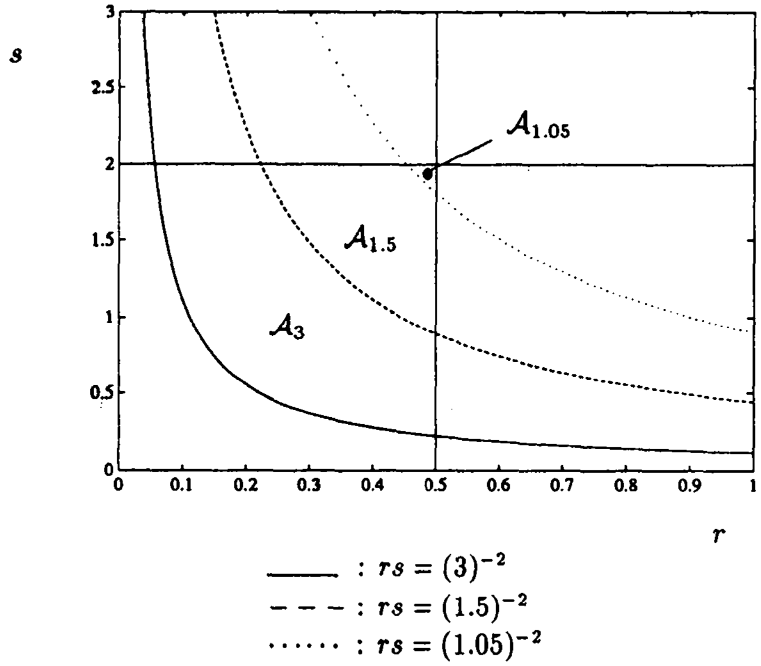


Figure 4.5

### 4.3 Connection with DGKF's Results

DGKF's characterization of suboptimal $\gamma$'s (see Section 1) is easily deduced from the previous results. Indeed, it can be shown that the Riccati equations (1.1)-(1.2) do have stabilizing solution $X_\infty$ and $Y_\infty$ whenever $\mathcal{A}_\gamma$ is non empty (see [5] for details). Invoking monotonicity results comparable to those of Lemma 4.2, we further have $X_\infty^{-1} \geq R$ and $Y_\infty^{-1} \geq S$ for all $(R, S)$ in $\mathcal{A}_\gamma$. Hence $X_\infty$ and $Y_\infty$ satisfy (1.3). As for the converse, it readily follows from continuity properties of stabilizing Riccati solutions under small perturbations (see e.g., [2]). Specifically, if (1.1)-(1.2) have stabilizing solutions then for $\epsilon$ small enough, the perturbed equations

$$\begin{aligned} A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 + \epsilon I &= 0\,; \\ AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T + \epsilon I &= 0 \end{aligned}$$

retain stabilizing solutions which continuously depend on $\epsilon$. Consequently, $\mathcal{A}_\gamma$ is nonempty and there exist suboptimal $H_\infty$ controllers for this $\gamma$. Finally, note that the extremal pair $(X_\infty, Y_\infty)$ corresponds to the central controller. This is best seen on the characterization of Theorem 3.1 once (3.2)-(3.3) are rewritten in terms of $X, Y$ as:

$$\begin{aligned} A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 + M_X(I - \gamma^{-2}D_KD_K^T)^{-1}M_X^T < 0; \\ AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T \;+\; N_Y(I - \gamma^{-2}D_K^TD_K)^{-1}N_Y^T < 0 \end{aligned}$$

where

$$M_X := XB_2 + C_2^TD_K^T + \bar{M}C_K^T; \qquad N_Y := YC_2^T + B_2D_K + \bar{N}B_K; \qquad \bar{M}\bar{N}^T = \gamma^{-2}XY - I.$$

When $(X, Y)$ approaches $(X_\infty, Y_\infty)$, the quadratic tail terms must approach zero. At the limit, we have $M_X = 0$ and $N_Y = 0$. That is,

$$X_\infty B_2 + C_2^TD_K^T + \bar{M}C_K^T = 0; \qquad Y_\infty C_2^T + B_2D_K + \bar{N}B_K; \qquad \bar{M}\bar{N}^T = \gamma^{-2}X_\infty Y_\infty - I. \quad (4.15)$$

Taking $D_K = 0$ for strict properness and, e.g., $(\bar{M}, \bar{N}) = (I, \gamma^{-2}X_\infty Y_\infty - I)$, the matrices $B_K, C_K$ are easily obtained from (4.15) and $A_K$ can be computed from (3.9) with $\Delta_{12} = 0$. This yields

$$\begin{aligned} A_K = A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty - (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^TC_2; \\ B_K = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^T; \qquad C_K = -B_2^TX_\infty \end{aligned}$$

which are exactly the central controller formulas [3]. More detail can be found in [5]. Finally, the notion of central controller can be extended by allowing $D_K$ to be nonzero. This extension proves useful for the design of nearly optimal central controllers (see [4] for details).

## 5 Applications

### 5.1 Preventing Pole/Zero Cancellations between the Plant and the Controller

The central controller has the undesirable property of cancelling all stable poles of the plant which are $(A, B_1)$-uncontrollable or $(C_1, A)$-unobservable. This behavior is frequently encountered in mixed-sensitivity design and leads to unacceptable designs in the presence of flexible modes. Cancellations in the mixed-sensitivity context have been thoroughly studied in [10]. Various remedies have been proposed which generally consist of modifying the criterion to penalize cancellations [8, 10]. Yet no general and direct remedy is available in the usual framework. By contrast, the parametrization introduced above offers direct and numerically tractable means of preventing cancellations of lightly damped modes. Indeed, if all controllers obtained from a given $(R,S) \in \mathcal{A}_\gamma$ involve cancellations, then $(R,S)$ must satisfy one the following:

$$\lambda_{\max}(R) \gg 1; \qquad \lambda_{\max}(S) \gg 1; \qquad |\lambda_{\max}(\mathcal{R}_R)| \ll 1; \qquad |\lambda_{\max}(\mathcal{R}_S)| \ll 1. \tag{5.16}$$

Here $\mathcal{R}_R$ and $\mathcal{R}_S$ denote the Riccati residuals defined in (4.4). "Bad" pairs $(R,S)$ therefore lie near the boundary of $\mathcal{A}_\gamma$ corresponding to the Riccati inequalities.

A qualitative justification of this claim goes as follows. Consider some $(R,S) \in \mathcal{A}_\gamma$, suppose the norms of $R,S$ are comparable to the scale of the problem, and consider some controller $K(s)$ associated with $(R,S)$ via Theorem 4.1. Assuming for simplicity that $D_{cl} = 0$, $X_{cl} > 0$ determined by (3.6) satisfies the closed-loop Bounded Real Lemma equation

$$A_{cl}^T X_{cl} + X_{cl} A_{cl} + X_{cl} B_{cl} B_{cl}^T X_{cl} + \gamma^{-2} C_{cl}^T C_{cl} + \Delta = 0 \tag{5.17}$$

where $\Delta := \begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix} > 0$ satisfies (3.7). Recall that cancelled modes are nonminimal modes of $(A_{cl}, B_{cl}, C_{cl})$. If $K(s)$ cancels some lightly-damped mode of the plant, $A_{cl}$ is nearly unstable and the solution $H$ of

$$A_{cl}^T H + H A_{cl} + I = 0$$

has large norm [7]. In fact, this is also true of the upper-left $n \times n$ subblock of $H$ because of the particular structure of the nonminimal subspace here. Observing that $X_{cl} \geq \lambda_{\min}(\Delta) H$, it follows that either $\|S\| \gg 1$ or $\lambda_{\min}(\Delta) \ll 1$. Since $S$ was assumed of relatively small norm, $\Delta$ must therefore be nearly singular.

Summing up, controllers causing cancellations correspond to nearly singular $\Delta$'s in (5.17) (provided that $R,S$ are of small norm). Now, recall from Theorem 4.1 that $\Delta$ is not uniquely determined by $(R,S)$. Thus, for $(R,S)$ to be unacceptable in the sense that all resulting controllers involve cancellations, all $\Delta$'s compatible with (3.7) must be nearly singular. When $R$ and hence $M$ have small norm, this requires that either $\Delta_R$ or $\Delta_S$ be nearly singular (cf. (3.7) and Lemma A.1 below). In turn, this must extend to $\mathcal{R}_R$ or $\mathcal{R}_S$ because of the remaining degrees of freedom in the choice of $\Delta_R, \Delta_S$ given $\mathcal{R}_R, \mathcal{R}_S$. Hence (5.16) holds when all designs obtained via Theorem 4.1 involve cancellations of flexible modes.

Steering clear of such pairs $(R,S)$ can be done in a number of ways. For instance, we can seek "good" pairs $(R,S)$ by solving

$$\min_{(R,S)\in\mathcal{A}_\gamma} \; Trace(R+S)$$

while placing a steep barrier on the Riccati inequality constraints. This will drive $\lambda_{\max}(\mathcal{R}_R)$ and $\lambda_{\max}(\mathcal{R}_S)$ away from zero and the criterion will ensure that the norms of $R$ and $S$ remain small. Another possible approach consists of finding the analytic center [1] of the intersection

$$\mathcal{A}_\gamma \cap \{ (R,S) \; : \; Trace(R+S) \leq constant \}.$$

Here again steep barriers should be used for the Riccati inequality constraints. Note that in both cases the resulting problem is convex and can be handled by standard convex optimization algorithms.

### 5.2 Reduced-Order Design

Reduced-order $H_\infty$ synthesis is a promising application of the parametrization introduced above. Indeed, $\gamma$-suboptimal controllers of order $k < n$ have a simple characterization in this framework: they correspond to pairs $(R, S)$ of $\mathcal{A}_\gamma$ for which $rank(\gamma^{-2}I - RS) = k$. Such pairs lie on the boundary of $\mathcal{A}_\gamma$ attached to the constraint $\lambda_{\min}(RS) \geq \gamma^{-2}$ and saturate this constraint in $n - k$ directions. Hence the reduced-order design problem has a clear formulation in terms of the parameters $(R, S)$: it consists of decreasing the rank of $\gamma^{-2}I - RS$ as much as possible without leaving $\mathcal{A}_\gamma$.

For feasible $\gamma$'s and with $\lambda_1(RS) \leq \cdots \leq \lambda_{n-k}(RS)$ denoting the $n - k$ smallest eigenvalues of $RS$, the synthesis of controllers of order $k < n$ amounts to minimizing for $(R, S) \in \mathcal{A}_\gamma$ the criterion:

$$\Psi(R, S) = \sum_{i=1}^{n-k} \lambda_i(RS).$$

Indeed, there exist suboptimal controllers of order $k$ if and only if the global minimum of $\Psi(R, S)$ is $(n-k)\gamma^{-2}$. This objective function $\Psi$ is not convex but in fact concave. Hence global convergence is not guaranteed. Nevertheless, the structural properties of the problem should help monitor gradient descent methods so as to obtain significant order reductions upon convergence.

## 6 Conclusions

A state-space-oriented parametrization of suboptimal $H_\infty$ controllers of order no larger than the plant order has been introduced. Here parameters are positive solutions of Riccati inequalities subject to a coupling constraint. The formalism parallels DGKF's characterization of feasible $\gamma$'s except that Riccati equations become inequalities. Some significant design problems are easily formulated in this new framework and the convexity of the parameter set favors the use of convex optimization techniques.

This algebraic parametrization opens new perspectives for improving $H_\infty$ design by making the most out of the suboptimal controller diversity. Applications to plant/controller cancellations and controller order reduction have been discussed. Further research is however needed to explore and exploit the full potential of this design tool.

## Appendix A

This appendix lists a few technical results which are useful in proving the theorems of Sections 3 and 4.

**Lemma A.1** *Let $P$ and $Q$ be two positive definite $n \times n$ matrices, $R \in \mathbb{R}^{n\times n}$ be nonsingular, and $M \in \mathbb{R}^{n\times k}$ have full column rank ($k \leq n$). If $k < n$, also assume that $V_2^T(Q - RPR^T)V_2 = 0$ where $V_2$ denotes any basis of the null space of $M^T$. Then there exist $X \in \mathbb{R}^{k\times k}$ and $Y \in \mathbb{R}^{n\times k}$ such that*

- $(R, M)\begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix}\begin{pmatrix} R^T \\ M^T \end{pmatrix} = Q.$
- $\begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix}$ *is positive definite.*

**Proof:** Up to redefining $R$ and $M$ as $Q^{-1/2}R$ and $Q^{-1/2}M$, respectively, assume without loss of generality that $Q = I$. Also assume $V_2$ is an orthonormal basis of $\text{Ker}\, M^T$ and complete it into an orthogonal matrix $V = (V_1, V_2)$. Finally, introduce the notation:

$$\begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} M = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

To ensure the positive definiteness of $\begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix}$, seek $X$ of the form $Y^T P^{-1} Y + Z$ with $Z > 0$. When pre- and postmultiplied by $V^T$ and $V$, respectively, the first constraint becomes

$$\begin{pmatrix} (R_1P + M_1Y)P^{-1}(R_1P + M_1Y)^T + M_1^T Z M_1 & (R_1P + M_1Y)R_2^T \\ R_2(R_1P + M_1Y)^T & R_2PR_2^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{A.1}$$

Solvability for $Y, Z$ requires that $R_2PR_2^T = I$ which is precisely ensured by the assumption $V_2^T(Q - RPR^T)V_2 = 0$. Observing that $M_1$ is square invertible since $M$ is full rank, it follows that $Y := -M_1^{-1}R_1P$ and $Z := (M_1M_1^T)^{-1}$ solve (A.1). Equivalently, the original requirements are met when chosing $Y = -M_1^{-1}R_1P$ and $X = M_1^{-1}(I + R_1PR_1^T)M_1^{-T}$.

■

**Lemma A.2** *Let $E, Q$ be two positive definite matrices and $M \in \mathbb{R}^{n\times k}$ have full column rank. Given an arbitrary matrix $L$ of compatible dimensions, the problem of finding a matrix $X$ such that*

$$(L + MX)E(L + MX)^T < Q. \tag{A.2}$$

*is solvable if and only if $V_2^T LEL^T V_2 < V_2^T Q V_2$ whenever $M^T V_2 = 0$.*

**Proof:** It suffices to consider the case where the columns of $V_2$ span the null space of $M^T$. As in Lemma A.1, assume without loss of generality that $Q = I$, complete $V_2$ into an orthogonal matrix $V = (V_1, V_2)$, and introduce the partitions:

$$\begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}; \qquad \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} M = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$$

where $M_1$ is square invertible. Pre- and postmultiplying by $V^T$ and $V$, respectively, (A.2) becomes:

$$\begin{pmatrix} (L_1 + M_1X)E(L_1 + M_1X)^T & (L_1 + M_1X)EL_2^T \\ L_2E(L_1 + M_1X)^T & L_2EL_2^T \end{pmatrix} < \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{A.3}$$

Clearly, solvability requires $L_2EL_2^T < I$ or equivalently, $V_2^T LEL^T V_2 < V_2^T Q V_2$. Conversely, this condition is also sufficient since (A.3) is then satisfied for $X = -M_1^{-1}L_1$.

■

**Lemma A.3** *Consider two symmetric matrices $R, S \in \mathbb{R}^{n\times n}$ such that $\text{rank}(\gamma^{-2}I - RS) = k < n$. Let $V_2$ denote an orthonormal basis of* $\text{Ker}\,(\gamma^{-2}I - SR)$ *and introduce the Riccati residuals:*

$$\begin{aligned} \mathcal{R}_R &:= AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T \,; \\ \mathcal{R}_S &:= A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2 \,. \end{aligned}$$

*Then*

$$V_2^T \left\{\mathcal{R}_R + B_2B_2^T - \gamma^2 R(\mathcal{R}_S + C_2^TC_2)R\right\} V_2 = 0. \tag{A.4}$$

**Proof:** From the definition of $\mathcal{R}_R$ and $\mathcal{R}_S$,

$$\mathcal{R}_R S - R\mathcal{R}_S = (A + RC_1C_1^T)(RS - \gamma^{-2}I) - (RS - \gamma^{-2}I)(A + B_1B_1^TS) + RC_2^TC_2 - B_2B_2^TS.$$

Pre- and post-multiplying by $V_2^T$ and $RV_2$, respectively, and recalling that $SRV_2 = \gamma^{-2}V_2$, it follows that

$$V_2^T(\gamma^{-2}\mathcal{R}_R - R\mathcal{R}_S R)V_2 = V_2^T(RC_2^TC_2R - \gamma^{-2}B_2B_2^T)V_2$$

which is exactly (A.4).

■

## Appendix B

Proof of Theorems 3.1 and 3.2:

**Necessity:** Suppose the controller $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ is $\gamma$-suboptimal and apply the Bounded Real Lemma 2.1 to the realization (2.5) of the closed-loop transfer function $\mathcal{F}(G, K)$. It follows that $\sigma_{max}(D_{cl}) < \gamma$ which is equivalent to **(C1)** in virtue of **(A2)**. In addition, the Riccati equation

$$A_{cl}^TX_{cl} + X_{cl}A_{cl} + \gamma^{-2}C_{cl}^TC_{cl} + (\gamma^{-2}C_{cl}^TD_{cl} + X_{cl}B_{cl})(I - \gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}C_{cl}^TD_{cl} + X_{cl}B_{cl})^T + \Delta = 0 \tag{B.1}$$

must have a solution $X_{cl} > 0$ for some $\Delta > 0$. This algebraic Riccati inequality (**ARI**) will readily yield (3.2)-(3.3) upon rewriting it in terms of the plant and controller parameters.

To see this, partition $X_{cl}$ and $\gamma^{-2}X_{cl}^{-1}$ conformably to (2.5) as

$$X_{cl} = \begin{pmatrix} S & N \\ N^T & \star \end{pmatrix}; \qquad \gamma^{-2}X_{cl}^{-1} = \begin{pmatrix} R & M \\ M^T & \star \end{pmatrix}; \qquad R, S \in \mathbb{R}^{n\times n};\ M, N \in \mathbb{R}^{n\times k}. \tag{B.2}$$

With this representation, $X_{cl} > 0$ is equivalent to **(C3)** and $MN^T = \gamma^{-2}I - RS$. To establish (3.2)-(3.3) and complete the proof, we could proceed as in [5] and block-partition (B.1) as well as its dual solved by $\gamma^{-2}X_{cl}^{-1}$. The (1,1) blocks of these partitioned ARI's would then yield (3.2)-(3.3). With the sufficiency part in mind, it is however preferable to follow a slightly different route. Specifically, observe that $R, S, M, N$ fully determine $X_{cl}$ as the unique solution of $\Pi_2 = X_{cl}\,\Pi_1$ where

$$\Pi_1 := \begin{pmatrix} I & \gamma R \\ 0 & \gamma M^T \end{pmatrix}; \qquad \Pi_2 := \begin{pmatrix} S & \gamma^{-1}I \\ N^T & 0 \end{pmatrix}. \tag{B.3}$$

That is, $X_{cl} = \Pi_2\Pi_1^\dagger$. When pre- and post-multiplied by $\Pi_1^T$ and $\Pi_1$, respectively, (B.1) thus becomes:

$$\begin{aligned}&\Pi_1^TA_{cl}^T\Pi_2 + \Pi_2^TA_{cl}\Pi_1 + \gamma^{-2}\Pi_1^TC_{cl}^TC_{cl}\Pi_1 + \\ &\quad(\gamma^{-2}\Pi_1^TC_{cl}^TD_{cl} + \Pi_2^TB_{cl})(I - \gamma^{-2}D_{cl}^TD_{cl})^{-1}(\gamma^{-2}\Pi_1^TC_{cl}^TD_{cl} + \Pi_2^TB_{cl})^T + \Pi_1^T\Delta\Pi_1 = 0.\end{aligned} \tag{B.4}$$

Note that (B.1) and (B.4) are equivalent since $M^T$ and hence $\Pi_1$ are full row rank.

Next, replace $A_{cl}, B_{cl}, C_{cl}, D_{cl}, \Pi_1, \Pi_2$ with their expressions in (2.5) and (B.3), expand the matrix products, and simplify the resulting block expressions. Introducing the notation (3.5) and $\Upsilon_K := I - \gamma^{-2}D_K^TD_K$, simple algebra based on **(A2)** shows that

$$\gamma^{-2}\Pi_1^TC_{cl}^TD_{cl} + \Pi_2^TB_{cl} = \begin{pmatrix} SB_1 + (N_S - C_2^T\Upsilon_K)D_{21} \\ \gamma^{-1}\{B_1 + M_RD_KD_{21}\} \end{pmatrix}.$$

Together with **(A2)**, this allows to simplify the quadratic term in (B.4) to:

$$(\gamma^{-2}\Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})(I - \gamma^{-2} D_{cl}^T D_{cl})^{-1}(\gamma^{-2}\Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})^T =$$

$$\begin{pmatrix} SB_1B_1^TS + N_S\Upsilon_K^{-1}N_S^T - N_SC_2 - C_2^TN_S^T + C_2^T\Upsilon_KC_2 & \gamma^{-1}\left\{SB_1B_1^T + (N_S\Upsilon_K^{-1} - C_2^T)D_K^TM_R^T\right\} \\ \gamma^{-1}\left\{B_1B_1^TS + M_RD_K(\Upsilon_K^{-1}N_S^T - C_2)\right\} & \gamma^{-2}\left\{B_1B_1^T + M_RD_K\Upsilon_K^{-1}D_K^TM_R^T\right\} \end{pmatrix}. \tag{B.5}$$

Completing the calculations with $\Delta$ partitioned as $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix}$, it follows that (B.4) and hence (B.1) are equivalent to the following system of three matrix equations:

$$AR + RA^T + RC_1^TC_1R + \gamma^{-2}B_1B_1^T - B_2B_2^T + M_R(I - \gamma^{-2}D_KD_K^T)^{-1}M_R^T + \gamma^2(R, M)\Delta\begin{pmatrix} R \\ M^T \end{pmatrix} = 0; \tag{B.6}$$

$$A^TS + SA + SB_1B_1^TS + \gamma^{-2}C_1^TC_1 - C_2^TC_2 + N_S(I - \gamma^{-2}D_K^TD_K)^{-1}N_S^T + \Delta_{11} = 0; \tag{B.7}$$

$$\begin{aligned} NA_KM^T \; + S(A + B_2D_KC_2)R + S(\gamma^{-2}B_1B_1^T + B_2C_KM^T) + (\gamma^{-2}C_1^TC_1 + NB_KC_2)R + \\ \gamma^{-2}A^T + \gamma^{-2}N_S(I - \gamma^{-2}D_K^TD_K)^{-1}D_K^TM_R^T + \Delta_{11}R + \Delta_{12}M^T = 0. \end{aligned} \tag{B.8}$$

Recalling that $\Delta > 0$, the ARI's (3.2)-(3.3) readily follow from (B.6)-(B.7). Hence (B.1) provides matrices $R, S, M, N$ which satisfy **(C2)-(C3)**.

**Sufficiency:** Conversely, suppose we are given matrices $R, S, M, N, B_K, C_K, D_K$ which jointly satisfy **(C1)-(C3)**. Without loss of generality, $M$ and $N$ can be assumed full column rank. Our goal is to reconstruct some $\gamma$-suboptimal controller from this data. In light of the necessity part, a natural approach consists of

(i) constructing $\Delta > 0$ for which (B.6)-(B.7) hold;

(ii) finding a solution $A_K \in \mathbb{R}^{k\times k}$ to (B.8).

When feasible, this will ensure that $X_{cl} := \begin{pmatrix} S & \gamma^{-1}I \\ N^T & 0 \end{pmatrix}\begin{pmatrix} I & \gamma R \\ 0 & \gamma M^T \end{pmatrix}^\dagger$ solves the Bounded Real Lemma equation (B.1). Since this $X_{cl}$ is positive definite in virtue of **(C3)**, Lemma 2.1 will in turn guarantee that $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$ is a $k$th-order $\gamma$-suboptimal controller. Hence it suffices to establish the feasibility of (i)-(ii).

<u>Feasibility of (i):</u> From (B.6) and (B.7), adequate $\Delta$'s should satisfy (with the partition $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix}$):

$$\Delta > 0; \qquad \Delta_R + \gamma^2(R, M)\Delta\begin{pmatrix} R \\ M^T \end{pmatrix} = 0; \qquad \Delta_S + \Delta_{11} = 0. \tag{B.9}$$

Take $\Delta_{11} = -\Delta_S$. Feasibility of the first two constraints in (B.9) is then addressed by Lemma A.1 with $P := -\Delta_S$ and $Q := -\Delta_R$. Since $-\Delta_R > 0$, $-\Delta_S > 0$, and $M$ is full column rank, this lemma is applicable provided that

$$V_2^T(\Delta_R - \gamma^2R\Delta_SR)V_2 = 0 \tag{B.10}$$

where $V_2$ denotes any orthonormal basis of $\operatorname{Ker} M^T = \operatorname{Ker}(\gamma^{-2}I - SR)$. To check this condition, observe from Lemma A.3 that

$$V_2^T\left\{\mathcal{R}_R + B_2B_2^T - \gamma^2R(\mathcal{R}_S + C_2^TC_2)R\right\}V_2 = 0 \tag{B.11}$$

holds for the Riccati residuals

$$\mathcal{R}_R = \Delta_R - M_R(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T\,; \qquad \mathcal{R}_S = \Delta_S - N_S(I - \gamma^{-2}D_K^T D_K)^{-1}N_S^T.$$

Using the identities $V_2^T M_R = V_2^T(B_2 + RC_2^T D_K^T)$ and $V_2^T R N_S = V_2^T(RC_2 + \gamma^{-2}B_2 D_K)$, (B.10) is easily deduced by expanding (B.11) and simplifying the resulting expression. Hence Lemma A.1 is applicable and guarantees the existence of some matrix $\Delta$ satisfying (B.9).

Feasibility of (ii): Given any $\Delta$ solving (B.9), (B.6)-(B.7) hold. To exhibit some controller state matrix $A_K \in \mathbb{R}^{k\times k}$ for which the Bounded Real Lemma equation (B.1) holds, it now suffices to show that the third equation (B.8) is solvable for $A_K$. Note that (B.8) is exactly (3.8) since $\Delta_{11} = -\Delta_S$.

When $k = n$ (full-order controller), solvability of (B.8) is immediate since $M$ and $N$ are invertible. By contrast, the reduced-order case ($k < n$) deserves some attention due to the row rank deficiency of $M$ and $N$. Let $\mathcal{T}$ denote the right-hand side of (3.8):

$$\begin{aligned}\mathcal{T} := \gamma^{-2}A^T + S(\gamma^{-2}B_1 B_1^T + B_2 C_K M^T) + (\gamma^{-2}C_1^T C_1 + N B_K C_2)R +&\\ S(A + B_2 D_K C_2)R + \gamma^{-2}N_S(I - \gamma^{-2}D_K^T D_K)^{-1}D_K^T M_R^T - \Delta_S R + \Delta_{12}M^T&\end{aligned}$$

and consider some basis $V_2$ of $\operatorname{Ker}(\gamma^{-2}I - SR) = \operatorname{Ker} M^T$. Observing that $RV_2$ is then a basis of $\operatorname{Ker}(\gamma^{-2}I - RS) = \operatorname{Ker} N^T$, it is immediate that (B.8) is solvable if and only if

$$\mathcal{T}V_2 = 0; \qquad V_2^T R\mathcal{T} = 0.$$

Checking these requirements is best done on the reformulations (3.9)-(3.10) of (B.8). To obtain (3.10), simply replace the term $SAR + \gamma^{-2}SB_1B_1^T$ of (B.8) by

$$-S\left\{RA^T + RC_1^T C_1 R - B_2 B_2^T + M_R(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T + \gamma^2(R, M)\Delta\begin{pmatrix} R \\ M^T\end{pmatrix}\right\}$$

in virtue of (B.6). A similar manipulation based on (B.7) yields (3.9).

Using the right-hand side of (3.10) as equivalent expression of $\mathcal{T}$, the condition $\mathcal{T}V_2 = 0$ is readily verified when observing that $M^T V_2 = 0$, $N^T R V_2 = 0$, and $ZV_2 = 0$. Similarly, (3.9) allows to write

$$V_2^T R\,\mathcal{T} = -V_2^T RSMC_K^T(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T + V_2^T R(S\Delta_R - \Delta_S R + \Delta_{12}M^T)$$

which also evaluates to zero since $V_2^T RSM = \gamma^{-2}V_2^T M = 0$ and

$$\begin{aligned} S\Delta_R - \Delta_S R + \Delta_{12}M^T &= -\gamma^2 S(R, M)\Delta\begin{pmatrix} R \\ M^T\end{pmatrix} - \Delta_S R + \Delta_{12}M^T \\ &= -\gamma^2 S\left\{-R\Delta_S R + R\Delta_{12}M^T + M\Delta_{12}^T R + M\Delta_{22}M^T\right\} - \Delta_S R + \Delta_{12}M^T \\ &= \gamma^2(\gamma^{-2}I - SR)(-\Delta_S R + \Delta_{12}M^T) + \gamma^2 SM(\Delta_{12}^T R + \Delta_{22}M^T).\end{aligned}$$

Hence (B.8) is always solvable and uniquely determines $A_K$ since $M$ and $N$ are full column rank. ∎

# Appendix C

Proof of Theorem 4.1: Assume $(R, S) \in \mathcal{A}_\gamma$. First consider the full-order case ($k = n$) and let $D_K$ be any matrix such that $\sigma_{max}(D_K) < \gamma$. Since $\mathcal{R}_R, \mathcal{R}_S$ are negative definite and $M, N$ are square invertible, $B_K$ and $C_K$ can always be chosen so that both $\Delta_R$ and $\Delta_S$ are negative definite. Invoking Theorem 3.2, $\gamma$-suboptimal controllers $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ can then be derived by reconstructing their state matrix $A_K \in \mathbf{R}^{n\times n}$.

The reduced-order case ($k < n$) is more delicate because $M, N$ are not invertible but only full column rank. As a result, $M_R$ and $N_S$ cannot be arbitrarily set through the choice of $B_K$ and $C_K$. In turn, this constrains the choice of $D_K$ through (4.7)-(4.8). To be specific, introduce some orthonormal basis $V_2$ of $\mathrm{Ker}\,(\gamma^{-2}I - SR)$ as in Lemma A.3. Then the columns of $V_2$ and $RV_2$ span $\mathrm{Ker}\,M^T$ and $\mathrm{Ker}\,N^T$ and Lemma A.2 shows that (4.7)-(4.8) is feasible for some $B_K, C_K$ if and only if $D_K$ satisfies

$$V_2^T\left\{\mathcal{R}_R + (B_2 + RC_2^TD_K^T)(I - \gamma^{-2}D_KD_K^T)^{-1}(B_2 + RC_2^TD_K^T)^T\right\}V_2 < 0\,; \tag{C.1}$$

$$V_2^TR\left\{\mathcal{R}_S + (C_2^T + SB_2D_K)(I - \gamma^{-2}D_K^TD_K)^{-1}(C_2^T + SB_2D_K)^T\right\}RV_2 < 0. \tag{C.2}$$

Using the identity (A.4) of Lemma A.3, it is easily verified that these two inequalities are equivalent. Hence (C.1) summarizes the constraint on $D_K$. Observing that

$$\begin{pmatrix} I & \gamma^{-1}D_K \\ \gamma^{-1}D_K^T & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - \gamma^{-2}D_KD_K^T)^{-1} & -\gamma^{-1}(I - \gamma^{-2}D_KD_K^T)^{-1}D_K \\ -\gamma^{-1}D_K^T(I - \gamma^{-2}D_KD_K^T)^{-1} & (I - \gamma^{-2}D_K^TD_K)^{-1} \end{pmatrix},$$

rewrite (C.1) as

$$V_2^T(\mathcal{R}_R - \gamma^2RC_2^TC_2R)V_2 + V_2^T\left(-B_2, \gamma RC_2^T\right)\begin{pmatrix} I & \gamma^{-1}D_K \\ \gamma^{-1}D_K^T & I \end{pmatrix}^{-1}\begin{pmatrix} -B_2^T \\ \gamma C_2R \end{pmatrix}V_2 < 0.$$

By a well-known result on $2\times 2$ symmetric block matrices, this last inequality is equivalent to (4.5).

Summing up, (4.7)-(4.8) are feasible if and only if there exists some $D_K$ satisfying (4.5) together with $\sigma_{max}(D_K) < \gamma$. Now, we claim that the $D_K$ matrix proposed in (4.6) does satisfy these two requirements. To see this, observe that $\mathcal{R}_R < 0$ implies that

$$\gamma^2C_2RV_2\left\{V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2\right\}^{-1}V_2^TRC_2^T < I. \tag{C.3}$$

Similarly,

$$B_2^TV_2\left\{V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2\right\}^{-1}V_2^TB_2 < I \tag{C.4}$$

since $V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2 = V_2^T(B_2B_2^T - \gamma^2R\,\mathcal{R}_SR)V_2$ from Lemma A.3 and $R\,\mathcal{R}_SR < 0$. With the shorthand $\Phi = V_2^T(\gamma^2RC_2^TC_2R - \mathcal{R}_R)V_2$, it follows that

$$\sigma_{max}(\gamma^2B_2^TV_2\Phi^{-1}V_2^TRC_2^T) \le \gamma\,\sigma_{max}(B_2^TV_2\Phi^{-1/2})\,\sigma_{max}(\gamma\Phi^{-1/2}V_2^TRC_2^T) < \gamma.$$

Consequently, the expression (4.6) satisfies $\sigma_{max}(D_K) < \gamma$. Moreover, (4.5) holds for this $D_K$ in virtue of (C.3)-(C.4).

Hence matrices $D_K, B_K, C_K$ can always be computed which satisfy (4.6)-(4.7). The construction of $K(s)$ is then completed as in the full-order case.

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