Regularization in state space

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REGULARISATION DANS L'ESPACE D'ETAT

REGULARIZATION IN STATE SPACE

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Résumé

Nous introduisons et analysons la régularisation dans l'espace d'état pour les problèmes inverses non linéaires. Nous donnons des applications aux problèmes d'estimation de paramètre, ainsi que des résultats numériques.

Abstract

This paper is devoted to the introduction and analysis of regularization in state space for nonlinear illposed inverse problems. Applications to parameter estimation problems are given and numerical experiments are described.

Mots Clefs

Régularisation, problème inverse, estimation de paramètre, moindres carrés non-linéaires.

Keywords

Regularization, inverse problems, parameter estimation, non linear least squares.

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Regularization in State Space

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Abstract

This paper is devoted to the introduction and analysis of regularization in state space for nonlinear illposed inverse problems. Applications to parameter estimation problems are given and numerical experiments are described.

1 Introduction

The objective of this paper is the study of a regularized least squares formulation for nonlinear illposed inverse problems. Specifically, let φ be a mapping from a subset C of a space E (parameter space) into a space F (state space), and let A be a linear operator from F into a space G (measurement space). The mapping φ may typically be the parameter-to-solution mapping of a partial differential equation, and A may represent point evaluation or it can be an injection operator from a function space with finer into a function space with coarser norm, realizing the fact that in applications it may be more realistic to assume (accurate) measurements in the coarser rather than the finer norm.

A nonlinear least squares formulations employing regularization in parameter space is given by :

$$(1.1) \qquad \inf_{x \in C} \mid A\varphi(x) - z \mid_G^2 + \varepsilon^2 \mid x - x_{est} \mid_E^2,$$

where $\varepsilon \in \mathbb{R}$ is the regularization parameter, z is the measurement and x_{est} represents an a-priori guess to the "generalized inverse" x^* of $A\varphi$ at z, i.e. $\hat{x}$ is defined as the solution to :

$$(1.2) \qquad \min_{x \in C} \mid A\varphi(x) - z \mid_G^2,$$

provided, of course, it exists. Investigation concentrating on (1.1) (with $F = G$, $A = id$) were carried out in ([2, 7, 13, 16, 17]), for example.

The regularization technique to be studied in this paper will not be in parameter, but rather in the state space. It is motivated by the following consideration. In applications the nonlinear mapping $\varphi : C \subset E \to F$ may be wellposed, with the only illposedness arising due to the linear mapping A which may be compact or may even have finite dimensional range. Thus we consider the situation :

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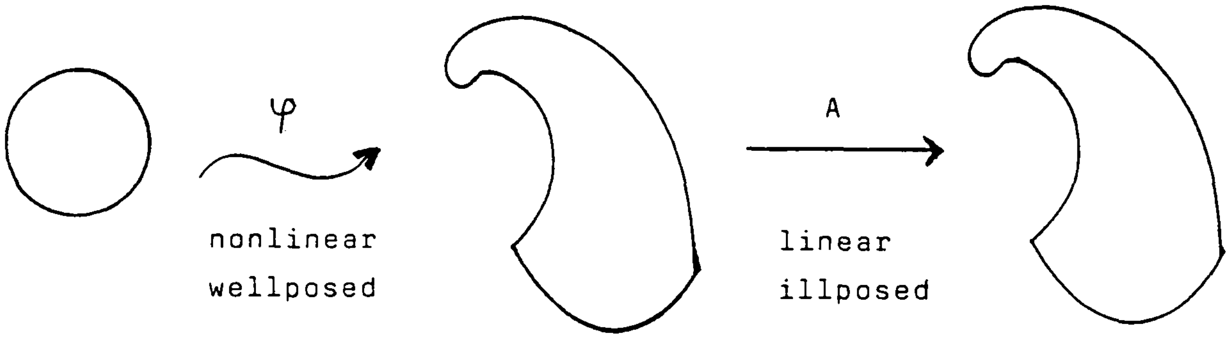

It is then natural to regularize in the domain of A rather than the domain of φ. In this way we arrive at a formulation of the inverse problem which involves regularization in state space :

$$\inf_{x\in G} \mid A\varphi(x) - z \mid_G^2 + \varepsilon^2 \mid \varphi(x) - \tilde{z} \mid_F^2, \tag{1.3}$$

where $z \in G$ and $\tilde{z} \in F$. Of course, z could be chosen to be $A\tilde{z}$, but we rather think of $z \in G$ as the available observation and of $\tilde{z}$ constructed from z. For example, if z represents pointwise data in a finite dimensional space G and if F is a function space, then $\tilde{z}$ can be an interpolation in F of the pointwise data. If both G and F are function spaces with F strictly embedded in G and $z \in G$ but $z \notin F$, then $\tilde{z}$ would arise from Z by a smoothing process.

We shall concentrate on a study of (1.3). In section 2 it will be justified to call (1.3) a *regularization technique for determining* $\hat{x}$ *from the data* z, by analyzing the properties of the solutions x^ε of (1.3) as $\varepsilon \to 0$ and as $(\tilde{z}, z)$ varies in $F \times G$. These results are based on a wellposedness assumption of the nonlinear mapping $\varphi : E \to F$. Examples satisfying this assumption are given in Section 3. Many results in the abstract as well as in the numerical treatment of optimization problems rely on the "second order sufficient optimality condition" which, roughly speaking, is the positivity of the Hessian of the cost functional at the minimizer. Section 4 is therefore devoted to a study of the second order sufficient optimality condition for the regularized least squares problem with regularization in state space.

Numerical experiments were carried out demonstrating that regularization in state space can be an effective tool for solving nonlinear illposed inverse problems. These results are given in Section 5.

2 Basic Properties

We consider the problem :

$$(P^\varepsilon) \qquad \min \mid A\varphi(x) - z \mid_G^2 + \varepsilon^2 \mid \varphi(x) - \tilde{z} \mid_F^2 \quad \text{over } x \in C,$$

where $\varepsilon \in \mathbb{R}$, $z \in G$, $\tilde{z} \in F$. E and F are reflexive Banach spaces. G is a normed linear space. C is a bounded subset of E and $A : F \to G$ is a continuous linear operator.

In applications A may be an embedding, a restriction or a point evaluation operator. The following hypotheses will be referred to :

$(H1)$ $\varphi : C \subset E \to F$ is weakly sequentially closed, i.e. $x_n \rightharpoonup x$ in E with $x_n \in C$, and $\varphi(x_n) \rightharpoonup \hat{\varphi}$ in F, imply $x \in C$ and $\varphi(x) = \hat{\varphi}$.

$(H2)$ φ is continuously invertible at $\hat{x} \in C$, i.e. if $\varphi(x_n) \to \varphi(\hat{x})$ in F, then $x_n \to \hat{x}$ in E.

Further we introduce the attainable set $\mathcal{V} = \{A\varphi(x) : x \in C\}$.

Proposition 1 *(Existence). For all $\varepsilon \neq 0$ there exists a solution x^ε of (P^ε), provided that $(H1)$ holds. If in addition $\varphi(C)$ is bounded in F, then there exists a solution $\hat{x}$ of the unregularized problem (P^0).*

Proof

Let $\varepsilon \neq 0$ and let $\{x\}_{n=1}^{\infty}$ be a minimizing sequence for (P^ε). Then $\{(x_n, \varphi(x_n))\}_{n=1}^{\infty}$ is bounded in $E \times F$. Therefore a subsequence, again denoted by $\{(x_n, \varphi(x_n))\}_{n=1}^{\infty}$ converges weakly in $E \times F$ to a limit $(x^\varepsilon, \varphi^\varepsilon) \in E \times F$. Due to $(H1)$ we have $x^\varepsilon \in C$ and $\varphi(x^\varepsilon) = \varphi^\varepsilon$. Weak lower semicontinuity of the norms implies that x^ε is the desired solution of (P^ε).

The set of solutions to (P^ε) is denoted by X^ε.

Proposition 2 *(Monotonicity). Let $\varepsilon_2 > \varepsilon_1 \geq 0$ and let x^{ε_i} denote a solution of (P^{ε_i}). Then :*

$$\text{(i)} \quad \sup | \varphi(x^{\varepsilon_2}) - \bar{z} |_F \leq \inf | \varphi(x^{\varepsilon_1}) - \bar{z} |_F,$$

$$\text{(ii)} \quad \sup | A\varphi(x^{\varepsilon_1}) - z |_G \leq \inf | A\varphi(x^{\varepsilon_2}) - z |_G .$$

If the unregularized problem has a solution $\hat{x}$ then moreover :

$$\text{(iii)} \quad \left(| A\varphi(x^{\varepsilon_2}) - z |_G^2 + \varepsilon_2^2 | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2\right) \leq dist_G(z, \mathcal{V})^2 + \varepsilon_2^2 | \varphi(\hat{x}) - \bar{z} |_F^2,$$

for all $x^{\varepsilon_2} \in X^{\varepsilon_2}$.

The sup *and* inf *are taken over $x^{\varepsilon_i} \in X^{\varepsilon_i}, i = 1, 2$, and $dist_G(z, \mathcal{V}) = \inf\{| z - v |_G : v \in \mathcal{V}\}$.*

Proof

Let $x^{\varepsilon_1} \in X^{\varepsilon_1}$ be arbitrary. Adding $(\varepsilon_2^2 - \varepsilon_1^2) | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2$ to both sides of :

$$| A\varphi(x^{\varepsilon_1}) - z |_G^2 + \varepsilon_1^2 | \varphi(x^{\varepsilon_1}) - \bar{z} |_F^2 \leq | A\varphi(x^{\varepsilon_2}) - z |_G^2 + \varepsilon_1^2 | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2$$

yields :

$$\begin{aligned} &| A\varphi(x^{\varepsilon_1}) - z |_G^2 + \varepsilon_2^2 | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2 + \varepsilon_1^2 (| \varphi(x^{\varepsilon_1}) - \bar{z} |_F^2 - | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2) \\ &\leq | A\varphi(x^{\varepsilon_2}) - z |_G^2 + \varepsilon_2^2 | \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2 \leq | A\varphi(x^{\varepsilon_1}) - z |_G^2 + \varepsilon_2^2 | \varphi(x^{\varepsilon_1}) - \bar{z} |_F^2 . \end{aligned} \tag{2.1}$$

Estimating the first term by the last gives :

$$\varepsilon_2^2 \left(| \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2 - | \varphi(x^{\varepsilon_1}) - \bar{z} |_F^2\right) \leq \varepsilon_1^2 \left(| \varphi(x^{\varepsilon_2}) - \bar{z} |_F^2 - | \varphi(x^{\varepsilon_1}) - \bar{z} |_F^2\right)$$

Since $\varepsilon_1^2 < \varepsilon_2^2$ we find :

$$| \varphi(x^{\varepsilon_2}) - \tilde{z} |_F \leq | \varphi(x^{\varepsilon_1}) - \tilde{z} |_F,$$

which implies (i). The first inequality in (2.1) implies :

$$| A\varphi(x^{\varepsilon_1}) - z |_G^2 + \varepsilon_1^2 | \varphi(x^{\varepsilon_1}) - \tilde{z} |_F^2 \leq | A\varphi(x^{\varepsilon_2}) - z |_G^2 + \varepsilon_1^2 | \varphi(x^{\varepsilon_2}) - \tilde{z} |_F^2$$

and due to (i) this gives :

$$| A\varphi(x^{\varepsilon_1}) - z |_G \leq | A\varphi(x^{\varepsilon_2}) - z |_G,$$

and (ii) follows. Assertion (iii) follows directly from the fact that x^{ε_2} is a solution of (P^{ε_2}).

Proposition 3 *(Stability). Assume that $(H1)$ holds. Let $\varepsilon \neq 0$ be fixed, let $\{(\tilde{z}_n, z_n)\}$ be a sequence in $F \times G$ with $\lim(\tilde{z}_n, z_n) = (\tilde{z}, z)$ in $F \times G$ and let x_n be a solution of (P^ε) with $(\tilde{z}, z)$ replaced by $(\tilde{z}_n, z_n)$. Then there exists a weakly convergent subsequence, every weak limit x^* of such a subsequence $\{x_{n_k}\}$ is a solution of (P^ε) and $\varphi(x_{n_k}) \to \varphi(x^*)$ in F. If moreover $(H2)$ holds at x^*, then the sequence x_{n_k} converges strongly in E to x^*.*

Proof

Let $\{x_n\}$ be a sequence of solutions to the problems (P^ε) with $(\tilde{z}, z)$ replaced by $(\tilde{z}_n, z_n)$. Since C and $\{(\tilde{z}_n, z_n)\}_{n=1}^\infty$ are bounded it follows that $\{(x_n, \varphi(x_n))\}_{n=1}^\infty$ is bounded in $E \times F$ with $x_n \in C$. Thus there exists a subsequence denoted by the same symbol with a weak limit (x^*, φ^*) in $E \times F$. Due to $(H1)$ we have $x^* \in C$ and $\varphi(x^*) = \varphi^*$. Note that :

$$(2.2) \quad \begin{aligned} &| A\varphi(x^*) - z |_G^2 + \varepsilon^2 | \varphi(x^*) - \tilde{z} |_F^2 \\ &\leq \underline{\lim}_{n\to\infty} (| A\varphi(x_n) - z_n |_G^2 + \varepsilon^2 | \varphi(x_n) - \tilde{z}_n |_F^2) \\ &\leq \lim_{n\to\infty} (| A\varphi(x) - z_n |_G^2 + \varepsilon^2 | \varphi(x) - \tilde{z}_n |_F^2) \\ &= | A\varphi(x) - z |_G^2 + \varepsilon^2 | \varphi(x) - \tilde{z} |_F^2 \end{aligned}$$

for all $x \in C$. Therefore x^* is a solution of (P^ε). From (2.2) it also follows that :

$$(2.3) \quad \lim_{n\to\infty} \left(| A\varphi(x_n) - z_n |_G^2 + \varepsilon^2 | \varphi(x_n) - \tilde{z}_n |_F^2\right) = | A\varphi(x^*) - z |_G^2 + \varepsilon^2 | \varphi(x^*) - \tilde{z} |_F^2 .$$

We show next that $\lim \varphi(x_n) = \varphi(x^*)$ in F. Since $\varphi(x_n) \rightharpoonup \varphi(x^*)$ in F it suffices to show that $\overline{\lim} | \varphi(x_n) - \tilde{z} |_F \leq | \varphi(x^*) - \tilde{z} |_F$. So suppose to the contrary that $\overline{\lim} | \varphi(x_n) - \tilde{z} |_F = \varphi_1 > | \varphi(x^*) - \tilde{z} |_F$. Then there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $\lim | \varphi(x_{n_k}) - \tilde{z} |_F = \varphi_1$. Using (2.3) we find :

$$\begin{aligned} &\lim | A\varphi(x_{n_k}) - z |_G^2 = \lim (| A\varphi(x_{n_k}) - z_{n_k} |_G^2 + \varepsilon^2 | \varphi(x_{n_k}) - \tilde{z}_n |_F^2) - \varepsilon^2 \lim | \varphi(x_{n_k}) - \tilde{z} |_F^2 \\ &= | A\varphi(x^*) - z |_G^2 + \varepsilon^2 | \varphi(x^*) - \tilde{z} |_F^2 - \varepsilon^2 \varphi_1^2 < | A\varphi(x^*) - z |_G^2 . \end{aligned}$$

This contradicts the fact that $A\varphi(x_n)$ converges weakly in G to $A\varphi(x^*)$. Therefore $\lim \varphi(x_n) = \varphi(x^*)$ in F and the first part of the proposition is verified. The second part follows from $(H2)$ at $\hat{x}$.

Proposition 4 *(Convergence). Assume that $(H1)$ holds and that $\varphi(C)$ is bounded in F. Let $\varepsilon_n \to 0$ and let x^{ε_n} be a sequence of solution to (P^{ε_n}). Then there exists a weakly convergent subsequence, every weak limit $\hat{x}$ of such a sequence $\{x^{\varepsilon_{n_k}}\}$ is a solution of the unregularized problem (P^0) and $\varphi(x^{\varepsilon_{n_k}}) \to \varphi(\hat{x})$ strongly in F. If moreover $(H2)$ holds then $x^{\varepsilon_{n_k}}$ converges strongly in E to $\hat{x}$.*

Proof

By assumption $\{(x_{n_k}, \varphi(x_n))\}_{n=1}^{\infty}$ is bounded in $E \times F$ and hence, without change in notation, there exists a weakly convergent subsequence in $E \times F$ with weak limit $(\hat{x}, \hat{\varphi})$. Due to $(H1)$ we have $\hat{x} \in C$ and $F(\hat{x}) = \hat{\varphi}$. Note that for each n :

$$| A\varphi(x^{\varepsilon_n}) - z |_G^2 + \varepsilon_n^2 | \varphi(x^{\varepsilon_n}) - \bar{z} |_F^2 \leq | A\varphi(x) - z |_G^2 + \varepsilon_n^2 | \varphi(x) - \bar{z} |_F^2$$

for all $x \in C$. Taking lim inf we find :

$$| A\varphi(\hat{x}) - z |_G \leq | A\varphi(x) - z |_G,$$

and hence $\hat{x}$ is a solution to (P^0). From Proposition (1) (i) we further have :

$$\limsup | \varphi(x^{\varepsilon_n}) - \bar{z} |_F \leq | \varphi(\hat{x}) - \bar{z} |_F$$

and therefore $\lim \varphi(\hat{x}^{\varepsilon_n}) = \varphi(\hat{x})$ in F. The second part of the proposition follows from $(H2)$.

Remark 1 From Proposition (2)(i) and Proposition (4) it follows that :

$$| \varphi(\hat{x}) - \bar{z} |_F = \min\{| \varphi(x) - \bar{z} |_F : | A\varphi(x) - z |_G = dist_G(z, \mathcal{V})\}.$$

which shows that $\hat{x}$ is an state space $\bar{z}$-minimum norm solution of the unregularized problem (P^o).

Proposition 5 *Assume that $(H1)$ holds and that $\varphi(C)$ is bounded in F. Then :*

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \sup_{x^\varepsilon \in X^\varepsilon} [| A\varphi(x) - z |_G^2 - dist_G(z, \mathcal{V})] = 0.$$

Observe that the expression in brackets in nonnegative.

Proof

The second inequality in (2.1) implies :

(2.4) $$0 \leq | A\varphi(x^\varepsilon) - z |_G^2 - dist_G(z, \mathcal{V})^2 \leq \varepsilon^2(| \varphi(x^*) - \bar{z} |_F^2 - | \varphi(x^\varepsilon) - \bar{z} |_F^2),$$

where x^* is chosen such that :

$$| \varphi(x^*) - \bar{z} |_F = \min\{| \varphi(x) - \bar{z} |_F : | \varphi(x) - z |_G = dist_G(z, \mathcal{V})\}.$$

If Proposition (5) was false, then there would exist $\delta > 0$ and a sequence of solutions x^{ε_n} of (P^{ε_n}) with $\varepsilon_n \to 0$, such that :

(2.5) $$\varepsilon_n^{-2} \left(| A\varphi(x^{\varepsilon_n}) - z |_G^2 - dist_G(z, \mathcal{V})^2\right) \geq \delta > 0 \text{ for all } n.$$

Due to Proposition (4), and Remark (1), there exists a subsequence $\{x^{\varepsilon_{n_k}}\}$ with limit $\hat{x}$ such that :

$$\lim_{k \to \infty} | \varphi(x^{\varepsilon_{n_k}}) - \bar{z} |_F = | \varphi(\hat{x}) - \bar{z} |_F = \{| \varphi(x) - \bar{z} |_F : | A\varphi(x) - z |_G = dist_G(z, \mathcal{V})\}.$$

It follows that (2.5) contradicts (2.4) and the claim follows.

In the following result we consider the case where the error in the data as well as the regularization parameter converge to zero. The optimization problems are :

(P_n) $\quad \min \mid A\varphi(x) - z_n \mid_G^2 + \varepsilon_n^2 \mid \varphi(x) - \bar{z} \mid_F^2$ over $x \in C$,
where $\{\varepsilon_n\}$ and $\{z_n\}$ are sequences in $\mathbb{R}$ and G satisfying

$(H3)$ $\quad \mid z_n - z \mid_G \leq \delta_n$ with $z \in \mathcal{V}$
and
$(H4)$ $\quad \varepsilon_n \to 0, \left\{\frac{\delta_n}{\varepsilon_n}\right\}$ bounded .

Proposition 6 *(Convergence-Stability). Assume $(H1)$, $(H3)$ and $(H4)$ to hold and let x_n be a solution for (P_n), $n = 1, 2, \ldots$. Then there exists a weakly convergent subsequence of $\{x_n\}$, every weak limit $\hat{x}$ of such a sequence $\{x_{n_k}\}$ satisfies :*

$$A\varphi(\hat{x}) = z, \qquad \varphi(x_{n_k}) \rightharpoonup \varphi(\hat{x}) \text{ in } F$$

and :

$$\frac{1}{\varepsilon_{n_k}} \mid A\varphi(x_{n_k}) - z \mid_G = O(1).$$

If $\frac{\delta_n}{\varepsilon_n} = o(1)$, then :

$$\varphi(x_{n_k}) \to \varphi(\hat{x}) \text{ in } F \text{ and } \mid \varphi(\hat{x}) - \bar{z} \mid_F = \min \{\mid \varphi(x) - \bar{z} \mid_F : A\varphi(x) = z\}.$$

If in addition $(H2)$ holds, then $x_{n_k} \to \hat{x}$ in E.

Proof
Due to $(H3)$ there exists $x^* \in C$ with $A\varphi(x^*) = z$. For each n we find :

$$(2.6) \qquad \begin{aligned} \mid A\varphi(x_n) - z_n \mid_G^2 + \varepsilon_n^2 \mid \varphi(x_n) - \bar{z} \mid_F^2 &\leq \mid z - z_n \mid_G^2 + \varepsilon_n^2 \mid \varphi(x^*) - \bar{z} \mid_F^2 \\ &\leq \delta_n^2 + \varepsilon_n^2 \mid \varphi(x^*) - \bar{z} \mid_F^2 . \end{aligned}$$

Since $z_n \to z$ in G and $\varepsilon_n \to 0$ it follows that $A\varphi(x_n) \to z$ in G. Boundedness of $\left\{\frac{\delta_n}{\varepsilon_n}\right\}$ implies that $\{\varphi(x_n)\}$ is bounded in F. Since C is bounded as well there exists a weakly convergent subsequence of $\{x_n\}$, again denoted by x_n, with $\{x_n, \varphi(x_n)\} \rightharpoonup (\hat{x}, \hat{\varphi})$ in $E \times F$. As before $(H1)$ implies that $\hat{x} \in C$ and $\varphi(\hat{x}) = \hat{\varphi}$. Since $A\varphi(x_n) \to z$ in G we also find that $A\varphi(\hat{x}) = z$. From (2.6) it follows that $\frac{1}{\varepsilon_n} \mid A\varphi(x_n) - z \mid_G = O(1)$. The first part of the proposition is thus verified. Next we assume that $\frac{\delta_n}{\varepsilon_n} = o(1)$. Analogous to (2.6) one finds :

$$\mid \varphi(x_n) - \bar{z} \mid_F^2 \leq \frac{\delta_n^2}{\varepsilon_n^2} + \mid \varphi(\hat{x}) - \bar{z} \mid_F^2.$$

and consequently :

$$\overline{\lim} \mid \varphi(x_n) - \bar{z} \mid_F \leq \mid \varphi(\hat{x}) - \bar{z} \mid_F \leq \underline{\lim} \mid \varphi(x_n) - \bar{z} \mid_F .$$

This implies that $\varphi(x_n) \to \varphi(\hat{x})$ in F. For x^* an arbitrary element satisfying $A\varphi(x^*) = z$ one obtains :

$$| \varphi(\hat{x}) - \tilde{z} |_F^2 = \lim | \varphi(x_n) - \tilde{z} |_F^2$$

$$\leq \lim \left[\frac{1}{\varepsilon_n^2} \left(| z - z_n |_G^2 - | A\varphi(x_n) - z_n |_G^2 \right) + | \varphi(\hat{x}) - \tilde{z} |_F^2 \right] \leq | \varphi(\hat{x}) - \tilde{z} |_F^2,$$

and therefore :

$$| \varphi(\hat{x}) - \tilde{z} |_F = \min\{ | \varphi(x^*) - \tilde{z} |_F : A\varphi(x^*) = z \}.$$

This ends the proof.

The conclusions of Proposition (6) remain valid if $\tilde{z}$ in (P_n) is replaced by $\tilde{z}_n$ with $\tilde{z}_n \to \tilde{z}$ in F.

Remark 2 If the problems (P_n) are not solved exactly, but rather x_n satisfies :

$$| A\varphi(x_n) - z_n |_G^2 + \varepsilon_n^2 | \varphi(x_n) - \tilde{z} |_F^2 \leq | A\varphi(x) - z_n |_G^2 + \varepsilon_n^2 | \varphi(x_n) - \tilde{z} |_F^2 + \eta_n^2, \text{ for all } x \in C,$$

then the conclusions of Proposition (6) remain correct, provided that $\frac{\eta_n}{\varepsilon_n} = o(1)$.

Remark 3 Proposition (6) can be used to argue that the constraints involved in defining C need not be active. In fact, assume that *int* C, the interior of C, is not empty and that there exists a unique element $\hat{x} \in \text{int } C$ with $z = A\varphi(\hat{x})$. Then, with $(H1)$-$(H4)$ holding and $\frac{\delta_n}{\varepsilon_n} = o(1)$, there exists an index N_0 such that $x_n \in$ *int* C for all $n \geq N_0$.

3 Examples

In this section we give some examples illustrating the applicability of the assumptions in section 2.

Example 3.1

In ([3]) we considered the problem of estimating the diffusion coefficient a in the one dimensional equation :

$$-(au_x)_x = f.$$

We considered the problem in reparametrized form with specific boundary conditions :

$$(3.1) \qquad \begin{aligned} -(\tfrac{1}{b} u_x)_x &= f \\ u(0) = u(1) &= 0. \end{aligned}$$

where $f \in H^{-1}$ and $b \in C = \{b \in L^2(0,1) \ : \ 0 < b_m \leq b(x) \leq b_M, \text{ a.e. } a.[0,1]\}$

In the context of the theory developed in section 2 we take $E = L^2, F = H_0^1, G = L^2$, with all function spaces considered over the interval $(0,1)$ and $A = id$, the identity operator. The solution $u = u(b)$ to (3.1) is given by :

$$(3.2) \qquad u(x) = -\int_0^x b(y)\{H(y) - \bar{H}_b\}\, dy.$$

where :

$$H(y) = \int_0^x f(s)\, ds,$$

and :

$$\bar{H}_b = \frac{\int_0^1 b(y)H(y)\, dy}{\int_0^1 b(y)\, dy},$$

and $\varphi : C \to H_0^1$ is given by $\varphi(b) = u(b)$. Clearly $(H1)$ holds in this case and $\varphi(C)$ is bounded, so that the unregularized problem has a solution. We turn to $(H2)$. From (3.2) and the fact that $u(b)$ satisfies homogenous Dirichlet boundary conditions, we conclude that u_x will vanish at least at one point if f is smooth, or it will have discontinuities if f is e.g. a linear combination of δ-functions. The set of admissible coefficients will be modified such that the coefficients are held constant in the neighborhood of such points. Let I_j be finitely many pairwise disjoint open intervals in $(0,1)$ and define :

$$\tilde{C} = \{b \in C : b = b_j = \text{ unknown constant on } I_j\}.$$

Clearly $(H1)$ holds with C replaced by $\tilde{C}$. In [([3]), Lemma 2, Theorem 6.1] we specified conditions on f and I_j which guarantee that $\varphi : \tilde{C} \to H_0^1$ has a Lipschitz continuous inverse when restricted to its range. This gives $(H2)$ at every $x \in \tilde{C}$.

One could equally well replace the L^2-observation by a pointwise observation by choosing $G = \mathbb{R}^n$ and $Au = \{u(x_i)\}_{i=1}^n$, with $0 \le x_1 < \ldots < x_n \le 1$.

Example 3.2

Here we consider the multidimensional analog of the problem in Example 3.1 :

$$(3.3) \qquad \begin{cases} -div(a\ grad\ u) = f \text{ in } \Omega \\ u \mid \partial\Omega = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$, $f \in L^2$, and $a \in C$, where :

$$C = \{a \in W^{1,4} : a(x) \ge \alpha, \quad \mid a \mid_{W^{1,4}} \le \gamma\} \subset L^2,$$

all function spaces being considered over the domain Ω. In the context of section 2 we choose $E = G = L^2$, $F = H^2$, $A = id$, and $\varphi(a) = u(a)$ with $u(a)$ the solution to (3.3). It is simple to argue that C is bounded and weakly closed and that $(H1)$ holds. Clearly $\varphi(C)$ is bounded in H^2.

For $(H2)$ to hold additional hypotheses are needed. Rather general conditions are given in [([11]), Theorem (4.1)]. Here we consider only a specific case :

$$(3.4) \qquad \begin{array}{l} \text{there exist } \lambda < 0 \text{ and } k > 0 \text{ such that} \\ \lambda \mid \nabla u(\hat{a})(x) \mid^2 - \frac{1}{2}\Delta u(\hat{a})(x) \le -k \quad \text{a.e. on } \Omega, \\ \text{and } \frac{\partial u(\hat{a})}{\partial n} \mid \partial\Omega \le 0. \end{array}$$

With (3.4) holding, there exists a constant K such that :

$$\mid \hat{a} - a \mid_{L^2} \le K \mid u(\hat{a}) - u(a) \mid_{H^2}, \text{ for all } a \in C.$$

so that $(H2)$ holds.

Example 3.3
Here we consider the estimation of the coefficient c in :

$$(3.5) \qquad \begin{aligned} -\Delta u + cu &= f \text{ in } \Omega \\ u \mid \partial\Omega &= 0 \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Let :

$$a_c(u, v) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

be the bilinear form defined by :

$$a_c(u, v) = < \nabla u, \nabla v >_{L^2(\Omega)} + < cu, v >_{L^2(\Omega)},$$

let c_0 be a reference coefficient and K a constant such that :

$$a_{c_0}(u, u) \geq 2K \mid u \mid_{H^1}^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Due to continuous embedding of $H_0^1(\Omega)$ into $L^4(\Omega)$ there exists $\gamma > 0$ such that :

$$a_c(u, u) \geq K \mid u \mid_{H_0^1}^2 \quad \text{for all } u \in H_0^1(\Omega)$$

and c with $\mid c - c_0 \mid_{L^2} \leq \gamma$. It follows that (3.4) has a solution $u(c) \in H_0^1(\Omega)$ for each such c and moreover $u \in H^2 \cap H_0^1$ [LU]. We define :

$$C = \left\{ c \in L^2(\Omega) : \mid c - c_0 \mid_{L^2} \leq \gamma, \quad c = \text{ unknown const on } \Omega \backslash \Omega' \right\}$$

where Ω' is a subdomain of Ω with $\overline{\Omega'} \subset \Omega$. A third subdomain Ω'' strictly containing Ω' and strictly contained in Ω will be used :

$$\Omega \supset \overline{\Omega''} \text{ and } \Omega'' \supset \overline{\Omega'}.$$

This hypothesis implies that : restricted to C the norms $\mid c \mid_{L^2(\Omega)}$ and $\mid c \mid_{L^2(\Omega'')}$ are equivalent.
In the context of section 2 we define :

$$E = L^2(\Omega), \quad F = H^2(\Omega), \quad G = L^2(\Omega), \quad A = id. \quad \text{and } \varphi(c) = u(c).$$

Assumption $(H1)$ is clearly satisfied and $\varphi(C)$ is bounded in $H^2(\Omega)$, see ([14],p.189). We turn to $(H2)$ and assume that $u(\hat{c})$ satisfies :

$$(3.6) \qquad u(\hat{c})(x) \geq k > 0 \quad \text{for all } x \in \Omega''.$$

Then we have for any $c \in C$:

$$(\hat{c} - c)u(\hat{c}) = \Delta(u(\hat{c}) - u(c)) + c(u(c) - u(\hat{c})),$$

and by (3.6) there exists a constant $\tilde{k} > 0$ independent of $c \in C$ such that :

$$| \hat{c} - c |_{L^2(\Omega'')} \leq \tilde{k} \mid u(c) - u(\hat{c}) |_{H^2(\Omega)} .$$

This gives $(H2)$. Note that in the present example the set C has nonempty interior, compare Remark (3).

Example 3.4
In the final example we consider the estimation of (c, τ) in :

$$\text{(3.7)} \qquad \begin{cases} -\Delta u + cu = f & \text{in } \Omega \\ \frac{\partial u}{\partial n} + \tau(u - g) = 0 & \text{on } \partial\Omega, \end{cases}$$

with a bounded domain in $\mathbb{R}^2$ with smooth boundary, $f \in L^2(\Omega)$ and $g \in H^1(\partial\Omega)$. This example is motivated by the practical situation where the heat transfer coefficient between the body Ω and the outside through the boundary (τ) and the lateral side (c) has to be estimated. For simplicity we put $\Gamma = \partial\Omega$. The variational form of (3.7) is given by :

$$\text{(3.8)} \qquad \begin{aligned} &< \nabla u, \nabla v >_{L^2(\Omega)} + < cu, v >_{L^2(\Omega)} + < \tau\gamma_0 u, \gamma_0 v >_{L^2(\Gamma)} \\ &=< f, v >_{L^2(\Omega)} + < \tau g, \gamma_0 v >_{L^2(\Gamma)} \ \text{ for } v \in H^1(\Omega), \end{aligned}$$

where γ_0 denotes the zero order trace operator. For $\tau_0 \geq 0$ there exists $\gamma > 0$ such that the bilinear form :

$$b_{(c,\tau)}(u, v) =< \nabla u, \nabla v >_{L^2(\Omega)} + < cu, v >_{L^2(\Omega)} + < \tau\gamma_0 u, \gamma_0 v >_{L^2(\Gamma)}$$

is continuous and coercive for all $(c, \tau) \in C$, where :

$$C = \left\{(c, \tau) \in L^\infty(\Omega) \times H^1(\Gamma) : 0 < c_m \leq c(x) \leq c_M \text{ a.e. on } \Omega, \text{ and } \mid \tau - \tau_0 |_{H^1(\Gamma)} \leq \gamma \right\}.$$

Hence for each $(c, \tau) \in C$ there exists a unique solution $u(c, \tau)$ of (3.8). Moreover $u \in H^2(\Omega)$ and it satisfies the boundary condition in (3.7) in the sense of $H^{1/2}(\Gamma)$. In terms of the notation of section 2 we take :

$$E = L^2(\Omega) \times L^2(\Gamma), \quad F = H^2(\Omega), \quad G = L^2(\Omega), \quad A = id, \text{ and } \varphi(c, \tau) = u(c, \tau).$$

Clearly $(H1)$ is satisfied and since $\Omega \subset \mathbb{R}^2$ it can be shown that $\varphi(C)$ is bounded in $H^2(\Omega)$ ([8]). If $u(\hat{c}, \hat{\tau})$ satisfies :

$$\text{(3.9)} \qquad u(\hat{c}, \hat{\tau})(x) \geq k > 0 \quad \text{for all } x \in \bar{\Omega},$$

then :

$$| (c, \tau) - (\hat{c}, \hat{\tau}) |_{L^2(\Omega) \times L^2(\Gamma)} \leq K \mid u(c, \tau) - u(\hat{c}, \hat{\tau}) |_{H^2(\Omega)},$$

with K independent of $(c, \tau) \in C$. Thus (3.8) implies $(H2)$.

4 Second order analysis

4.1 A general result for problems with bilinear structure

We give conditions which guarantee positive definiteness of the Hessian of :

$$J_\varepsilon(x) = \frac{1}{2} \mid A\varphi(x) - z \mid_G^2 + \frac{\varepsilon^2}{2} \mid \varphi(x) - \bar{z} \mid_F^2 .$$

Positive definiteness of the Hessian of J_ε is the essential assumption required to guarantee local uniqueness and stability with respect to $(\bar{z}, z) \in F \times G$ of the solutions to (P^ε), see ([1, 4, 5]) and the references given there. It is also essential for arguing convergence and rate of convergence results for numerical methods to solve (P^ε), compare ([10, 12, 13, 15]).

Throughout this section it is assumed that $\varphi : C \subset E \to F$ is twice continuously Fréchet differentiable. The following additional hypotheses will be used.

There exist constants $k_1 > 0$ and k_2 such that for all $x \in C$ and $h \in E$:

$(H5)$ $\quad \mid \varphi'(x)h \mid_F \geq k_1 \mid h \mid_E,$

$(H6)$ (i) $\mid A\varphi''(x)(h,h) \mid_G \leq k_2 \mid A\varphi'(x)h \mid_G \mid h \mid_E$, and
(ii) $\mid \varphi''(x)(h,h) \mid_F \leq k_2 \mid \varphi'(x)h \mid_F \mid h \mid_E$.

Before stating the main result of this subsection, the applicability of $(H5)$ and $(H6)$ is shown by means of the estimation of c in Examples 3.3 and 3.4. The linear structure in which the unknown coefficient c and the state variable u appear in (3.5) and (3.7) allows to verify $(H6)$.

__Remark__ 4 We return to Example 3.3, the problem of identifying c in (3.5) from data for u. We ask the reader to recall the notation of that example, and we repeat only that $C \subset E = L^2$, $F = H^2$ and $G = L^2$. Here all function spaces are considered over Ω. For $c \in C$ we define $A(c) : H^2 \cap H_0^1 \to L^2$ by :

$$A(c)u = -\Delta u + cu.$$

One can argue that the mapping $\varphi : C \subset L^2 \to H^2$ is twice continuously Frechet differentiable with :

$$\varphi'(c)h = -A^{-1}(c)(hu(c))$$

and :

$$\varphi''(c)(h,h) = -2A^{-1}(c)(h\varphi'(c)h),$$

for $c \in C$ and $h \in L^2$. The assumptions on C imply the existence of constants $K_1 > 0$ and K_2 such that :

(4.1) $$K_1 \mid f \mid_{L^2} \leq \mid A^{-1}(c)f \mid_{H^2} \leq K_1^{-1} \mid f \mid_{L^2}$$

and :

(4.2) $$\mid A^{-1}(c)f \mid_{L^2} \leq K_2 \mid f \mid_{L^1} .$$

Let us again assume (3.6) to hold. This implies the existence of $\bar{\gamma} > 0$ such that :

(4.3) $$u(c)(x) \geq \frac{\kappa}{2} \text{ for all } x \in \Omega'' \text{ and } c \in C_{\tilde{\gamma}},$$

where :

$$C_{\tilde{\gamma}} = \left\{ c \in L^2 \; : | \, c - \hat{c} \, |_{L^2}, \quad c = \text{ unknown constant on } \Omega \backslash \Omega' \right\}$$

and $C_{\tilde{\gamma}} \subset C$. This is a consequence of the fact that $c_n \to c$ in L^2 with $c_n \in C$ implies that $u(c_n) \to u(c)$ in H^2 and $u(c_n) \to u(c)$ in L^∞.

We now verify $(H5)$ and $(H6)$ with C chosen to be $C_{\tilde{\gamma}}$. Let $c \in C_{\tilde{\gamma}}$ and $h \in L^2$. By (4.1) and (4.3) we find :

$$| \varphi'(c)h |_{H^2} = | A^{-1}(c)(hu(c)) |_{H^2} \geq K_1 \, | \, hu(c) \, |_{L^2} \geq \frac{\kappa}{2} K_1 \, | \, h \, |_{L^2},$$

which gives $(H5)$. Next we use (4.1) and (4.2) :

$$| \varphi''(c)(h,h) |_{L^2} \leq 2 \, | \, A^{-1}(c)(h\varphi'(c)h) \, |_{L^2} \leq 2K_2 \, | \, h \, |_{L^2} | \, \varphi'(c)h \, |_{L^2},$$

so that $(H6)(i)$ holds. Similarly :

$$| \varphi''(c)(h,h) |_{H^2} \leq 2 \, | \, A^{-1}(c)(h\varphi'(c)h) \, |_{H^2} \leq 2K_1^{-1} \, | \, h \, |_{L^2} | \, \varphi'(c)h \, |_{L^\infty}$$

$$\leq 2K_1^{-1} K_3 \, | \, h \, |_{L^2} | \, \varphi'(c)h \, |_{H^2},$$

where K_3 is the embedding constant of H^2 into L^∞, and $(H6)(ii)$ holds as well. ■

Remark 5 We turn to the verification of $(H5)$ and $(H6)$ for the problem of identifying c in :

(4.4) $$\begin{aligned} -\Delta u + cu &= f \text{ in } \Omega \\ \tfrac{\partial u}{\partial n} + \tau_0(u - g) &= 0 \text{ on } \Gamma, \end{aligned}$$

with Ω as in Example 3.4 and $\tau_0 \in H^1(\Gamma)$, $f \in L^2(\Omega)$, $g \in H^1(\Gamma)$ fixed. Let $E = G = L^2$, $F = H^2$, and :

$$C = \{ c \in L^\infty(\Omega) \; : 0 < c_m \leq c(x) \leq c_M \quad \text{a.e. on } \Omega \},$$

and assume that (3.9) holds for $\hat{c}$, i.e.

$$u(\hat{c})(x) \geq \kappa > 0 \quad \text{on } \Omega.$$

Then there exists $\tilde{\gamma} > 0$ such that :

(4.5) $$u(c)(x) \geq \frac{\kappa}{2} \text{ on } \Omega \text{ for all } c \in C_{\tilde{\gamma}},$$

where $C_{\tilde{\gamma}} = \{ c \in C \; : | \, c - \hat{c} \, |_{L^2} \leq \tilde{\gamma} \}$. The mapping $c \to \varphi(c) = u(c)$, $c \in C$, with $u(c)$ the solution of (4.4) is twice continuously differentiable with $\varphi'(c)h =: \eta$ and $\varphi''(c)(h,h) =: \xi$ characterized by :

(4.6) $$A(c)\eta = -hu(c) \text{ and } A(c)\xi = -2h\eta,$$

where $A(c)$ is defined by

$$\text{dom } A(c) = \{\varphi \in H^2 \ : \frac{\partial \varphi}{\partial n} + \tau_0 \varphi = 0 \text{ on } \Gamma\}$$

and :

$$A(c)\varphi = -\Delta\varphi + c\varphi.$$

From (4.5) and (4.6) it follows that $(H5)$ holds for $c \in C_{\bar{\gamma}}$. Since H^2 is continuously embeded in L^∞ it is simple to verify $(H6)(ii)$ and it remains to consider $(H6)(i)$. We find :

$$| \xi |_{L^2} = \sup < \xi, A(c)\varphi >_{L^2} = \sup < A(c)\xi, \varphi >_{L^2} = 2 < h\eta, \varphi >_{L^2},$$

where the sup is taken over $\varphi \in$ dom $A(c)$ with $| A(c)\varphi |_{L^2} = 1$. Consequently :

$$| \xi |_{L^2} \leq 2 \, | h\eta |_{L^1} \sup | \varphi |_{L^\infty} \leq K \, | h |_{L^2} | \eta |_{L^2},$$

where K is independent of $c \in C$. This is $(H6)(i)$, and thus we have shown that $(H5)$, $(H6)$ holds with $C = C_{\bar{\gamma}}$. Notice that when trying to include the estimation of τ in the framework of this section, one encounters the difficulty that $(c, \tau) \to u(c, \tau)$ is not welldefined on $L^2(\Omega) \times L^2(\Gamma)$. This can be circumvented by choosing $L^2(\Omega) \times H^1(\Gamma)$ for E. Then φ is welldefined and differentiable for $(c, \tau) \in (L^2(\Omega) \times H^1(\Gamma)) \cap C$, with C as in Example 3.4 and $(H6)(ii)$ holds. But for $(H5)$ only an estimate of the type :

$$| \varphi'(c,\tau)(h,v) |_{H^2} \geq k_1 \, | (h,v) |_{L^2(\Omega) \times L^2(\Gamma)}$$

with $(h, v) \in L^2(\Omega) \times H^1(\Gamma)$ and appropriately chosen (c, τ) is feasible.

A similar problem arises if one was to consider Example 3.1 without reparametrization. This issue is addressed in detail in Section 4.2. ■

__Theorem__ **4.1** *Let $(H1)$, $(H5)$ and $(H6)$ hold, and assume that $\varphi(C)$ is bounded. Moreover, let $\hat{x}$ be an state space $\bar{z}$- minimum norm solution of the unregularized problem satisfying :*

$$| \varphi(\hat{x}) - \bar{z} |_F = \min \{| \varphi(x) - \bar{z} |_F \, : \, | A\varphi(x) - z |_G = dist_G(z, \mathcal{V})\},$$

and define :

$$\underline{\varepsilon}^2 = \frac{4}{k_1} k_2^2 dist_G(z, \mathcal{V})^2.$$

If :

$$(4.7) \qquad | \varphi(\hat{x}) - \bar{z} |_F \leq \frac{\sqrt{k_1}}{2k_2},$$

then :

$$(4.8) \qquad J_\varepsilon''(x^\varepsilon)(h,h) \geq \frac{1}{2} \, | A\varphi'(x^\varepsilon)h |_G^2 + \frac{\varepsilon^2}{4} k_1 \, | h |_E^2 \quad \textit{for all } h \in E,$$

and all solutions x^ε of (P^ε) with $\varepsilon^2 \geq \underline{\varepsilon}^2$. In particular, if z is attainable, then $\underline{\varepsilon} = 0$.

<u>Proof</u>

Using $(H6)$ and $(H5)$ we find for every solution x^ε of (P^ε) :

$$\begin{aligned} J_\varepsilon''(x^\varepsilon)(h,h) &= \mid A\varphi'(x^\varepsilon)h \mid_G^2 + < A\varphi(x^\varepsilon) - z, A\varphi''(x^\varepsilon)(h,h) >_G + \varepsilon^2 \mid \varphi'(x^\varepsilon)h \mid_F^2 \\ &\quad + \varepsilon^2 < \varphi(x^\varepsilon) - \bar{z}, \varphi''(x^\varepsilon)(h,h) >_F \\ &\geq \tfrac{1}{2} \mid A\varphi'(x^\varepsilon)h \mid_G^2 - \tfrac{1}{2}k_2^2 \mid A\varphi(x^\varepsilon) - z \mid_G^2 \mid h \mid_E^2 + \tfrac{\varepsilon^2}{2} \mid \varphi'(x^\varepsilon)h \mid_F^2 - \tfrac{\varepsilon^2}{2}k_2^2 \mid \varphi(x^\varepsilon) - \bar{z} \mid_F \mid h \mid_E^2 \\ &= \tfrac{1}{2} \mid A\varphi'(x^\varepsilon)h \mid_G^2 + \tfrac{\varepsilon^2}{2} \mid h \mid_E^2 \left[k_1 - k_2^2 \left(\frac{\mid A\varphi(x^\varepsilon) - z \mid_G^2}{\varepsilon^2} + \mid \varphi(x^\varepsilon) - \bar{z} \mid_F^2\right)\right]. \end{aligned}$$

We next use (2.4) and (4.7) :

$$\begin{aligned} J_\varepsilon''(x^\varepsilon)(h,h) &\geq \tfrac{1}{2} \mid A\varphi'(x^\varepsilon)h \mid_G^2 + \tfrac{\varepsilon^2}{2} \mid h \mid_E^2 \left[k_1 - k_2^2 \left(\mid \varphi(\hat{x}) - \bar{z} \mid_F^2 + \frac{dist_G(z,\mathcal{V})^2}{\varepsilon^2}\right)\right] \\ &\geq \tfrac{1}{2} \mid A\varphi'(x^\varepsilon)h \mid_G^2 + \tfrac{\varepsilon^2}{2} \mid h \mid_E^2 \left[\tfrac{3}{4}k_1 - k_2^2 \frac{dist_G(z,\mathcal{V})^2}{\underline{\varepsilon}^2}\right] \\ &= \tfrac{1}{2} \mid A\varphi'(x^\varepsilon)h \mid_G^2 + \tfrac{\varepsilon^2}{4} \mid h \mid_E^2 \end{aligned}$$

which holds for all solutions x^ε of (P^ε) with $\varepsilon^2 \geq \underline{\varepsilon}^2$. This ends the proof.

<u>Remark</u> **6** The two terms on the right hand side of (4.8) express the degree of wellposedness. They separate the weak wellposedness which is present in the problem itself without regularization from the uniform wellposedness due to regularization.

Let us also compare (4.8) of Theorem (4.1) to analogous estimates when regularization in parameter space is used ([4, 5]). To obtain positivity of the Hessian in the case of regularization in parameter space the regularization parameter has to be chosen in an interval, the lower bound of which is zero or positive depending on whether z is attainable or not. For regularization in state space there is no upper bound on the regularization parameter. If one regularizes in parameter space with a term of the form $\varepsilon^2 \mid x - x_{est} \mid^2$ where x_{est} represents an a-priori guess to the unknown parameter, then this information does not enhance the positivity of the lower bound to the Hessian. For regularization in state space the role of $\bar{x}$ is -in some sense- taken by $\bar{z}$. For nonlinear problems the term containg $\bar{z}$ does not vanish and $\bar{z}$ must contain sufficient information -compare (4.7)- to guarantee positivity of the Hessian.

4.2 Second order analysis for Example 3.1

In section 4.1 we gave conditions guaranting positivity of the Hessian. We recall that the topology for F has to be fine enough for $(H5)$ to hold and it has to be sufficiently coarse so that φ allows a second derivative and $(H6)(ii)$ holds. This may lead to difficulties as was seen in Remark (5). If we consider the mapping $a \to \varphi(a) = u(a)$ from $L^2(0,1)$ to $H^1(0,1)$, with a positive and $u = u(a)$ the solution of :

$$-(au_x)_x = f, \quad u(0) = u(1) = 0.$$

then $(H5)$ holds under appropriate conditions on f, but φ'' is not even well defined ([3]). In this subsection we shall show that the reparametrization $a \to \frac{1}{b}$ as in (3.1) provides a technique to circumvent this problem and to obtain positivity of the Hessian of the regularized cost functional.

We consider here the problem :

$$(4.9) \qquad \begin{cases} -\left(\frac{1}{b}u_x\right)_x = \sum_{j=1}^J f_j \delta(x - x_j), & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

with $x_j < x_{j+1}$, $x \in (0,1)$, $f_j \in \mathbb{R}$ and $\delta(x)$ the delta function with impuls at x. Let $\eta = col\,(\eta_1, \ldots, \eta_J)$ be such that $I_j = (x_j - \eta_j, x_j + \eta_j)$ satisfy $I_j \cap I_{j+1} = \emptyset$ for $j = 1, \ldots, J-1$, and define :

$$D_\eta = \{b \in H^1(0,1) \; : 0 < b_m \leq b(x), \text{ a.e. on } (0,1), |\, b\,|_{H^1} \leq b_M, b = b_j \text{ on } I_j\},$$

where $b_j, j = 1, \ldots, J$ are unknown constants. Unless indicated otherwise the function spaces are taken over the interval $(0,1)$ in this subsection. We ask the reader to recall the definitions of H and $\overline{H}_b$ of Example 4.1. The following assumption will be used :

$(H7)$ there exist constants $0 < H_m < H_M$ such that :

$$H_m \leq |\, H(x) - \overline{H}_b \,| \leq H_M \text{ for all } x \in [0,1] \text{ and } b \in D_\eta.$$

In the context of the general theory :

$$E = G = L^2(0,1), \quad F = H_0^1, \quad A = id.$$

and :

$$\varphi(b) = u(b) = -\int_0^x b(y)[H(y) - \overline{H}_b]\, dy.$$

The mapping φ is twice continuously Fréchet differentiable and for $b \in D_\eta$, $h \in L^2$

$$A(\frac{1}{b})u'(b)h = -(\frac{h}{b^2}u_x)_x, \tag{4.10}$$

$$A(\frac{1}{b})u''(b)(h,h) = 2[\frac{h^2}{b^3}u_x - \frac{h}{b^2}(u'(b)h)_x]_x, \tag{4.11}$$

where $A(\frac{1}{b}) \; : H_0^1 \to H^{-1}$ is given by $A(\frac{1}{b})v = -(\frac{1}{b}v_x)_x$, and $u = A^{-1}(\frac{1}{b})f$.
The regularized cost functional is given by :

$$J_\varepsilon(b) = \frac{1}{2} \,|\, u(b) - z \,|_{L^2}^2 + \frac{\varepsilon^2}{2} \,|\, u(b) - \tilde{z} \,|_{H_0^1}^2,$$

for $b \in D_\eta$.

<u>Theorem</u> 4.2 *Assume that $(H7)$ holds, let $z \in L^2$, $\tilde{z} \in H_0^1$, $\varepsilon \in \mathbb{R}$ and denote by b^ε any solution of :*

$$\min_{b \in D_\eta} |\, u(b) - z \,|_{L^2}^2 + \varepsilon^2 \,|\, u(b) - \tilde{z} \,|_{H_0^1}^2.$$

with $u(b)$ a solution of (4.9). Assume that the unregularized problem has a solution and let $\hat{b}$ be one such solution satisfying :

$$|\, u(\hat{b}) - \tilde{z} \,|_{H_0^1} = \min\left\{|\, u(b) - \tilde{z} \,|_{H_0^1} : |\, u(b) - z \,|_{L^2} = dist_{L^2}(z, \mathcal{V})\right\}.$$

where $\mathcal{V} = \{u(b) \; : b \in D_\eta\} \subset L^2$. Then there exist constants $\kappa_1, \kappa_2, \kappa_3 > 0$ and $\kappa_4 > 0$ independent of $h \in L_c^2$ such that :

$$|\, u(b^\varepsilon) - \tilde{z} \,|_{H_0^1} \leq \kappa_1 \text{ and } \varepsilon \geq \kappa_2 dist_{L^2}(z, \mathcal{V})$$

imply :

$$J_\varepsilon''(b^\varepsilon)(h,h) \geq \frac{\kappa_3}{2} \mid A^{-1}(\frac{1}{b^\varepsilon})(\frac{h}{b^\varepsilon})_x \mid_{L^2}^2 + \varepsilon^2 \kappa_4 \mid h \mid_{L^2}^2,$$

for all :

$$h \in L_c^2 = \{h \in L^2 : h = h_j \in \mathbb{R} \text{ on } I_j, \quad j = 1, \ldots, J\}.$$

Proof

In the first estimate we utilise two lemmas which are stated and proved below. We find :

$$\begin{aligned} J_\varepsilon''(b^\varepsilon)(h,h) &= \mid u'(b^\varepsilon)h \mid_{L^2}^2 + < u(b^\varepsilon) - z, u''(b^\varepsilon)(h,h) >_{L^2} \\ &+ \varepsilon^2 \mid u'(b^\varepsilon)h \mid_{H_0^1}^2 + \varepsilon^2 < u''(b^\varepsilon)(h,h), u(b^\varepsilon) - \tilde{z} >_{H_0^1} \\ &\geq \kappa_3 \mid A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x \mid_{L^2}^2 + 2\int_0^1 \tfrac{h}{(b^\varepsilon)^2} u_x \, dx < u(b^\varepsilon) - z, A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x >_{L^2} \\ &+ \varepsilon^2 \mid u'(b^\varepsilon)h \mid_{H_0^1}^2 + 2\varepsilon^2 \int_0^1 \tfrac{h}{(b^\varepsilon)^2} u_x \, dx < A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x, u(b^\varepsilon) - \tilde{z} >_{H_0^1}. \end{aligned}$$

We choose constants $\kappa_5, \kappa_6, \kappa_7$ such that :

$$(4.12) \qquad \begin{cases} \mid \int_0^1 \frac{h}{b^2} u_x(b) \, dx \mid \leq \kappa_5 \mid h \mid_{L^2}, \quad \mid A^{-1}(\frac{1}{b}(\frac{h}{b})_x \mid_{H_0^1} \leq \kappa_6 \mid h \mid_{L^2}, \\ \mid u_x(b)h \mid_{H_0^1} \geq \sqrt{\kappa_7} \mid h \mid_{L^2}, \text{ for all } b \in D_\eta \text{ and } h \in L_c^2, \end{cases}$$

see ([3]). Using (4.12) we find :

$$\begin{aligned} J_\varepsilon''(b^\varepsilon)(h,h) &\geq \tfrac{\kappa_3}{2} \mid A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x \mid_{L^2}^2 - \tfrac{2}{\kappa_3}\kappa_5^2 \mid h \mid_{L^2}^2 \mid u(b^\varepsilon) - z \mid_{L^2}^2 \\ &+ \varepsilon^2 \kappa_7 \mid h \mid_{L^2}^2 - 2\varepsilon^2 \kappa_5 \kappa_6 \mid h \mid_{L^2}^2 \mid u(b_\varepsilon) - \tilde{z} \mid_{H_0^1}^2. \end{aligned}$$

Let $\kappa_8 = \max\left(\frac{2}{\kappa_3}\kappa_5^2, 2\kappa_5\kappa_6\right)$, then :

$$\begin{aligned} J_\varepsilon''(b^\varepsilon)(h,h) &\geq \tfrac{\kappa_3}{2} \mid A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x \mid_{L^2}^2 + \varepsilon^2 \mid h \mid_{L^2}^2 + \left[\kappa_7 - \kappa_8 \left(\tfrac{|u(b^\varepsilon) - z|_{L^2}^2}{\varepsilon^2} + \mid u(b^\varepsilon) - \tilde{z} \mid_{H_0^1}^2\right)\right] \\ &\geq \tfrac{\kappa_3}{2} \mid A^{-1}(\tfrac{1}{b^\varepsilon})(\tfrac{h}{b^\varepsilon})_x \mid_{L^2}^2 + \varepsilon^2 \mid h \mid_{L^2}^2 \left[\kappa_7 - \kappa_8 \left(\tfrac{dist_{L^2}(z,\mathcal{V})^2}{\varepsilon^2} + \mid u(\hat{b}) - \tilde{z} \mid_{L^2}^2\right)\right]. \end{aligned}$$

Let $\hat{b}$ satisfy :

$$\mid u(\hat{b}) - \tilde{z} \mid_{H_0^1}^2 \leq \kappa_1 := \frac{\kappa_7}{4\kappa_8}.$$

Then :

$$J_\varepsilon''(b^\varepsilon)(h,h) \geq \frac{\kappa_3}{2} \mid A^{-1}(\frac{1}{b^\varepsilon})(\frac{h}{b^\varepsilon})_x \mid_{L^2}^2 + \varepsilon^2 \mid h \mid_{L^2}^2 \left[\frac{3}{4}\kappa_7 - \frac{\kappa_8}{\varepsilon^2} dist_{L^2}(z,\mathcal{V})^2\right].$$

For $\kappa_2 := \kappa_1^{-1}$, and $\varepsilon^2 \geq \kappa_2 dist_{L^2}^2(z,\mathcal{V})$ one finds :

$$J_\varepsilon''(b^\varepsilon)(h,h) \geq \frac{\kappa_3}{2} \mid A^{-1}(\frac{1}{b^\varepsilon})(\frac{h}{b^\varepsilon})_x \mid_{L^2}^2 + \varepsilon^2 \frac{\kappa_7}{2} \mid h \mid_{L^2}^2,$$

for all $h \in L_c^2$. Thus the claim holds with $\kappa_4 = \frac{\kappa_7}{2}$.

Remark 7 We point out that a less precise estimate than the one used for the second term in $J_\varepsilon''(b^\varepsilon)$, given by :

$$< u(b^\varepsilon) - z, u''(b^\varepsilon)(h,h) >_{L^2} \leq const \mid u(b^\varepsilon) - z \mid_{L^2} \mid h \mid_{L^2}^2,$$

would not allow to draw the same conclusion, since in general $\mid u(b^\varepsilon) - z \mid_{L^2}$ converges too slowly (e.g. like $O(\varepsilon)$, if $z \in \mathcal{V}$, see Proposition 5).

Lemma 1 *For $b \in D_\eta$ and $h \in L^2$, let $\xi := u''(b)(h,h)$. Then ξ is characterized as the unique solution of :*

$$A(\frac{1}{b})\xi = 2\int_0^1 \frac{h}{b^2}u_x\ dx\ (\frac{h}{b})_x.$$

Proof

Using (4.11) we find :

$$A(\frac{1}{b})\xi = 2\left[\frac{h}{b}\left(F + \frac{1}{b}\left(A^{-1}(\frac{1}{b})F_x\right)_x\right)\right]_x,$$

where $F = \frac{h}{b^2}u_x$, and further :

$$A(\frac{1}{b})\xi = 2\left[\frac{h}{b}(I-P)F\right]_x,$$

where $P = -\frac{1}{b}DA^{-1}(\frac{1}{b})D$, is considered as a bounded linear operator on L^2 and D denotes differentiation. It is simple to show that P is a projection, with $\ker P$ = *set* of constant functions and $(I-P)F = \int_0^1 F\ dx$. It follows that $A(\frac{1}{b})\xi = 2\left[\frac{h}{b}\int_0^1 \frac{h}{b^2}u_x\ dx\right]_x$, as desired.

Lemma 2 *Let (H7) hold. Then there exists a constant $\kappa_3 > 0$ such that :*

$$|\ u'(b)h\ |_{L^2} \geq \kappa_3\ |\ A^{-1}(\frac{1}{b})(\frac{h}{b})_x\ |_{L^2}$$

for all $b \in D_\eta$ and $h \in L^2$.

Proof

For simplicity of exposition we assume that $J = 1$, and we write η in place of η_1. The general case requires only minor technical modifications. We choose $\alpha \in (0,\eta)$, and we put $\tilde{\Omega} = (0, x_1 - \alpha) \cup (x_1 + \alpha, 1)$. Throughout K denotes a generic constant that is independent of $b \in D_\eta$ and $h \in L^2$. We find :

$$\begin{aligned} |\ u'(b)h\ |_{L^2} &= |\ A^{-1}(\tfrac{1}{b})\left(\tfrac{h}{b^2}u_x\right)\ |_{L^2} = |\ A^{-1}(\tfrac{1}{b})\left[\tfrac{h}{b}(H - \bar{H}_b)\right]_x\ |_{L^2} \\ &\geq K\ |\ \left[\tfrac{h}{b}(H-\bar{H}_b)\right]_x\ |_{(H^2\cap H_0^1)^*} = K \sup < \tfrac{h}{b}(H - \bar{H}_b), \varphi_x >_{L^2(0,1)}, \end{aligned}$$

where the sup is taken over all : $\varphi \in H_0^1 \cap H^2$ with $|\ \varphi\ |_{H_0^1\cap H^2} = 1$.

For the inequality we made use of the fact that $A(\frac{1}{b})$ is an isomorphism between $H_0^1 \cap H^2$ and L^2 uniformly in $b \in D_\eta$; here we also used the assumption that $b \in H^1$. We further obtain :

$$(4.13)\quad |\ u'(b)h\ |_{L^2} \geq K \sup_{\substack{\varphi \in H_0^2(\tilde{\Omega}) \\ |\varphi|_{H^2(\tilde{\Omega})}}} < \frac{h}{b}(H - \bar{H}_b), \varphi_x >_{L^2(\tilde{\Omega})} = K\ |\ [\frac{h}{b}(H - \bar{H}_b)]_x\ |_{H_0^2(\tilde{\Omega})^*},$$

where :

$$H_0^2(\tilde{\Omega}) = \{\varphi \in H^2(\tilde{\Omega}) : \varphi(0) = \varphi(x_1 - \alpha) = \varphi(x_1 + \alpha) = \varphi(1) = \varphi_x(x_1 - \alpha) = \varphi_x(x_1 + \alpha) = 0\}.$$

Next it will be shown that :

$$| A^{-1}(\frac{1}{b})(\frac{h}{b})_x |_{L^2} \leq \frac{1}{K} | [\frac{h}{b}(H - \bar{H}_b)]_x |_{H_0^2(\hat{\Omega})^*} . \tag{4.14}$$

The assertion of the lemma then follows directly from (4.13) and (4.14). To verify (4.14) we first observe that there are constants H_1 and H_2 such that :

$$H - \bar{H}_b = \begin{cases} H_1 \text{ on } (0, x_1) \\ H_2 \text{ on } (x_1, 1). \end{cases}$$

Further $\frac{h}{b}$ equals a constant $\bar{k} = \bar{k}(h, b)$ on $(x_1 - \eta, x_1 + \eta)$. Let $\bar{\varphi} \in H_0^1 \cap H^2$ with $| \bar{\varphi} |_{H_0^1 \cap H^2} = 1$ be chosen such that :

$$| (\frac{h}{b})_x |_{(H_0^1 \cap H^2)^*} = < \frac{h}{b}, \bar{\varphi}_x >_{L^2} .$$

We then find :

$$\begin{aligned} &| A^{-1}(\tfrac{1}{b})(\tfrac{h}{b})_x |_{L^2} \leq K | (\tfrac{h}{b})_x |_{(H_0^1 \cap H^2)^*} = K < \tfrac{h}{b}, \bar{\varphi}_x >_{L^2(0,1)} \\ &\leq K \left[H_1^{-1} < H_1 \tfrac{h}{b}, \bar{\varphi}_x >_{L^2(0,x_1-\eta)} + \bar{k} < 1, \bar{\varphi}_x >_{L^2(x_1-\eta,x_1+\eta)} + \tfrac{1}{H_2} < H_2 \tfrac{h}{b}, \bar{\varphi}_x >_{L^2(x_1+\eta,1)} \right] . \end{aligned} \tag{4.15}$$

We choose functions :

$$\bar{\varphi}^1 \in H_0^1(0, x_1 - \alpha) \cap H^2(0, x_1 - \alpha) \text{ and } \bar{\varphi}^2 \in H_0^1(x_1 + \alpha, 1) \cap H^2(x_1 + \alpha, 1)$$

such that :

$$\bar{\varphi}^1 | [0, x_1 - \eta] = \bar{\varphi} | [0, x_1 - \eta], \quad \bar{\varphi}_x^1(x_1 - \alpha) = 0,$$

$$\bar{\varphi}^2 | [x_1 + \eta, 1] = \bar{\varphi} | [x_1 + \eta, 1], \quad \bar{\varphi}_x^2(x_1 + \alpha) = 0,$$

and :

$$| \bar{\varphi}^1 |_{H^2(0,x_1-\alpha)} \leq K, \quad | \bar{\varphi}^2 |_{H^2(x_1+\alpha,1)} \leq K.$$

Using these functions in (4.15) we obtain :

$$\begin{aligned} | A^{-1}(\tfrac{1}{b})(\tfrac{h}{b})_x |_{L^2(0,1)} &\leq H_1^{-1} < H_1 \tfrac{h}{b}, \bar{\varphi}_x^1 >_{L^2(0,x_1-\alpha)} - \bar{k} < 1, \bar{\varphi}_x^1 >_{L^2(x_1-\eta,x_1-\alpha)} \\ + \bar{k} < 1, \bar{\varphi}_x >_{L^2(x_1-\eta,x_1+\eta)} &+ H_2^{-1} < H_2 \tfrac{h}{b}, \bar{\varphi}_x^2 >_{L^2(x_1+\alpha,1)} - \bar{k} < 1, \bar{\varphi}_x^2 >_{L^2(x_1+\alpha,x_1+\eta)} \\ &\leq K \left(| (H_1 \tfrac{h}{b})_x |_{H_0^2(0,x_1-\alpha)^*} + | (H_2 \tfrac{h}{b})_x |_{H_0^2(x_1+\alpha,1)^*} \right) \\ &\leq K | (H - \bar{H}_b)(\tfrac{h}{b})_x |_{H_0^2(\hat{\Omega})^*} . \end{aligned}$$

where $H_0^2(0,x_1-\alpha)^*$ and $H_0^2(x_1+\alpha,1)^*$ are defined analogously to $H_0^2(\hat{\Omega})^*$. This gives (4.14), for an appropriate choice of K, and the proof is finished.

5 Numerical experiments

We describe some numerical experiments for estimating the diffusion coefficient a in :

$$\begin{cases} -(au_x)_x = f \text{ in } (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

by minimizing the cost functional :

$$J(a) = | u - z^\delta |_{L^2}^2 + \varepsilon^2 | u_x - \tilde{z}_x |_{L^2}^2 + \beta^2 | a_x |_{L^2}^2 + K | a_x - a_x^0 |_{L^2(I)}^2, \tag{5.2}$$

where z^δ denotes noisy data in $L^2 = L^2(0,1)$; ε, β and K are constants, $a^0 \in H^1$, I is a subset of $(0,1)$ and u is a solution of (5.1). Numerically we did not use explicit constraints.

For $a \to u(a)$ from $C = \{a \in L^2(0,1) : 0 < a_m \le a(x) \le a_M\} \subset L^2 \to H^1$ to have a continuous inverse, see condition $(H2)$, it is necessary to constrain the class of admissible coefficients a in neighborhoods of the singular points of the observations, i.e. in neighborhoods of zeroes of $u(\hat{a})_x$ if $u(\hat{a}) \in C^1$ and of discontinuities of $u(\hat{a})_x$ otherwise, see Example 3.1 and ([3]). This can be accomplished with $K > 0$, for example. If $a^0 = 0$ the coefficients are forced to be almost constant on I. The term $\beta^2 | a_x |_{L^2}^2$ can also be thought of as eliminating underdetermination due to singularities of $u(\hat{a})_x$. It is a regularization term in parameter space which also has the effect of penalizing oscillations of the coefficient.

The optimization problem was solved by the augmented Lagrangian technique described in ([9]). The discretization of the state variable u was carried out by linear splines on the grid $\left\{\frac{i}{2N}\right\}_{i=0}^{2N}$, and of the coefficient a by linear splines on the grid $\left\{\frac{i}{N}\right\}_{i=0}^{N}$. Noisy data z^δ were produced by adding uniformly distributed random numbers from the interval $(-\delta,\delta)$ to the unperturbed observation $z^0 = u(\hat{a})$ at the gridpoints $\left\{\frac{i}{2N}\right\}_{i=1}^{2N-1}$. The specific choices for the unperturbed observation z^0 and the "unknown" coefficient $\hat{a}$ were made as follows :

Example 5.1

$$\begin{aligned} z^0 &= u(\hat{a}) = \exp(x)\sin(\pi x), \\ \hat{a} &= \tfrac{1}{2}\arctan(40(x - \tfrac{3}{5})) + \tfrac{3}{2}. \end{aligned}$$

Example 5.2

$$z^0 = u(\hat{a}) = \begin{cases} \frac{x}{.65} & \text{for} \quad x \le .65 \\ \frac{1-x}{.35} & \text{for} \quad x > .65. \end{cases}$$

$$\hat{a} = \frac{1}{2}\arctan(40(x - \frac{3}{5})) + \frac{3}{2}.$$

With $\hat{a}$ and $u(\hat{a})$ given, f is calculated from (5.1). We point out that with these choices for $\hat{a}$ and z^0 the resulting inverse problem is (numerically) not a simple one, since :

(i) the maximum of the slope of $\hat{a}$ occurs near a singular point of z^0 in both examples.

(ii) in our calculations we chose $N = 32$, which is a reasonably fine resolution allowing for many undesirable oscillations.

(iii) absolute, not relative noise was used.

In the numerical results below, the values for ε^2, β^2 and K are zero, unless specified otherwise, and "L^2-error" denotes the L^2-distance between the numerical result for the coefficient and $\hat{a}$. In all cases, where $K > 0$, we took $I = (.56, .64)$ for Example 5.1 and $I = (.61, .69)$ for Example 5.2. We begin with numerical results for Example 5.1 with noiseless observations.

Table 1

Ex 5.1 $\delta = 0$	$\epsilon^2 = 0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = 0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$, $K = 10$
L^2-error	.23	.15	.09	.04

For the last entry in Table 1, a_x^0 was chosen as a constant function with value equal to the slope of the tangent to $\hat{a}$ at $\frac{3}{5}$. The graphs of $\hat{a}$ and the numerical result corresponding to the last entry in Table 1 are given in Plot 1. Next we consider noisy observations:

Table 2

Ex 5.1 $\delta = .02$	$\epsilon^2 = 0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = 0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^\delta$, $K = 10$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$, $K = 10$
L^2-error	.44	.16	.74	.13

Here $a_x^0 = 0$. For the same specifications as in the last column but with $\tilde{z} = 0$ the L^2-error is .15. The graph corresponding to the last entry in Table 2 is included in Plot 2. Note the difference in the behavior of the numerical solutions of Plot 1 and Plot 2 on the interval I. This is primarily due to the different choice of a^0. The result of the next to last column shows that the choice of $\tilde{z}$ as the noisy data does (of course) not give a good result. In fact, $\tilde{z} = 0$ is preferable, as can be seen from column 3. If one desires to use the information of the noisy data z^δ for the choice of $\tilde{z}$, then one must regularize or precondition z^δ. In the best possible situation one would obtain the noisefree data z^0. The result for $\tilde{z} = z^0$ is given in the last column of Table 2.

Before turning to numerical results for Example 5.2 we point out that the singular sets of the unperturbed observations in Examples 5.1 and 5.2 are very different. The unperturbed observation of Example 5.2 has one isolated singular point at which the derivative is not defined. The coefficient a is identifiable from z^0 in the class of H^1 functions, but not in the class of L^2 functions. In Example 5.1 the unperturbed observation is such that its derivative is zero at one point. Moreover z_x^0 is small in its neighborhood, which causes additional numerical instabilities. In the class of H^1 functions, a is uniquely determined by z^0. We first give the results for noiseless data.

Table 3

Ex 5.2 $\delta = 0$	$\epsilon^2 = 0$	$\epsilon^2 = 10^{-4}$	$\epsilon^2 = 10^{-4}$ $\beta^2 = 10^{-6}$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$, $K = 10$
L^2-error	.166	.169	.03	.121	.036

For the last entry in Table 3 we chose a_x^0 as a constant function with value equal to the slope of the tangent to $\hat{a}$ at $\frac{3}{5}$. The graphs for the results of entry one and four a given in Plot 3 and the graph for the last entry is given in Plot 4. A comparison of these graphs suggests that there is nonuniqueness (possibly in H^{-1} for the infinite dimensional problem) which is eliminated by $K > 0$; compare also the numerical result for $\beta > 0$. Additional numerical experiments showed that the range of successfull numerical results with $\beta > 0$ is enlarged by choosing $\varepsilon > 0$. Results with noisy data are given next.

Table 4

Ex 5.2 $\delta = .02$	$\epsilon^2 = 0$	$\epsilon^2 = 10^{-4}$	$\epsilon^2 = 10^{-4}$ $\beta^2 = 10^{-6}$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$	$\epsilon^2 = 10^{-4}$ $\tilde{z} = 0$, $K = 10$	$\epsilon^2 = 10^{-3}$ $\tilde{z} = z^0$, $K = 10$
L^2-error	.67	.33	.09	.23	.27	.06

For the last entry a^0 is again chosen to be the tangent to $\hat{a}$ at $\frac{3}{5}$.

In Table 3 and 4 some results are given for $\epsilon^2 = 10^{-3}$ and others for $\epsilon^2 = 10^{-4}$. In all cases the algorithm converges for both choices of ϵ^2, but the results are better for that value of ϵ^2 which is shown in the tables.

While the primary importance of the numerical results here is to demonstrate that regularization in state space is effective, we also carried out numerical tests with two other cost functionals, which we briefly report upon. In the first we changed the ϵ^2-regularization term and took

$$J_1(a) = |u - z^\delta|_{L^2}^2 + \epsilon^2 |u_{xx} - z_{xx}^2|_{L^2}^2.$$

For Example 5.1 with $\delta = .02, \tilde{z} = 0$ and $\epsilon^2 = 10^{-3}$ the L^2-error is .106 (compare Table 2).

For the second cost functional we combine regularization in state- with regularizations in parameter space. This leads us to consider

$$J_2(a) = |u - z^\delta|_{L^2}^2 + \epsilon^2 |u_x - \tilde{z}_x|_{L^2}^1 + \beta^2 \left| \frac{a_x}{|z_x(\cdot)| + .01} \right|_{L^2}^2,$$

where the ϵ^2-term regularizes noise and the β^2-term regularies the effects of small values of $|z_x(\cdot)|$. For Example 5.1 with $\epsilon^2 = 10^{-3}, \beta^2 = 10^{-7}$, and $\tilde{z} = 0$ ($\tilde{z} = z^0$) the L^2-error is .074(.077), compare Table 2.

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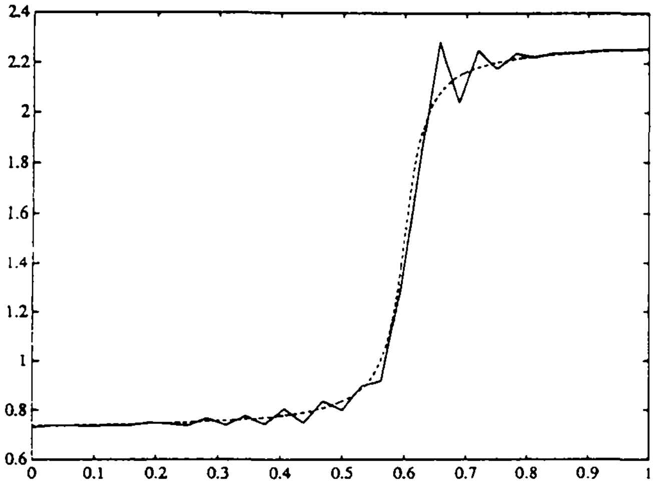
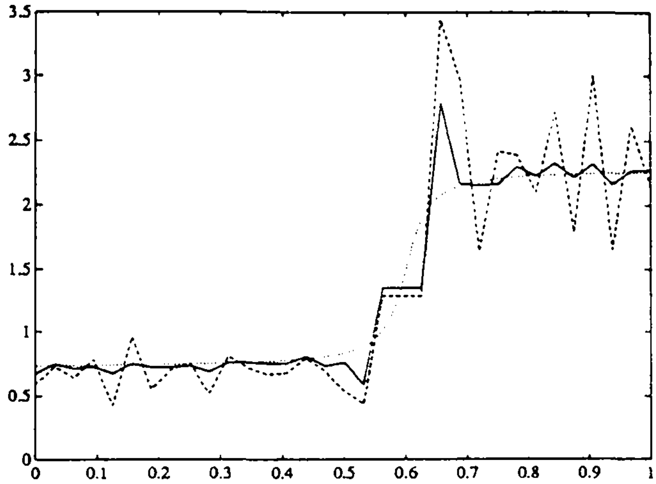
Plot 1



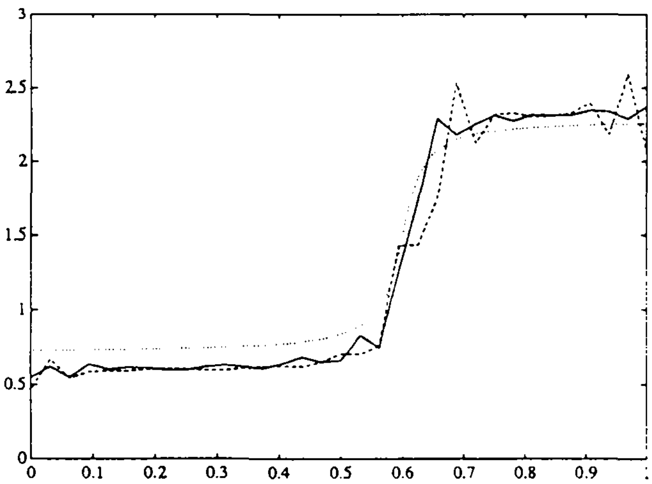
$\varepsilon^2 = 10^{-3}$, $K = 10$
a^0 = tangent
$\hat{a}$ --------, $a_{\varepsilon^2 = 10^{-3}}$ ———

Plot 2



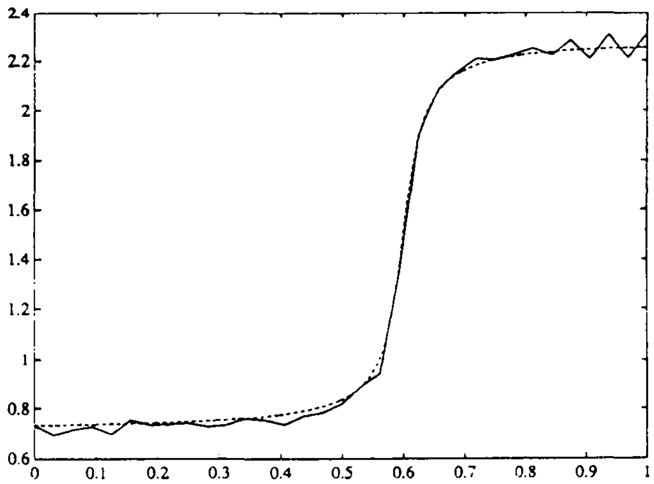
$\varepsilon^2 = 0$, $\varepsilon^2 = 10^{-3}$, $K = 10$
a^0 = constant
$\hat{a}$ ··········, $a_{\varepsilon^2 = 0}$ --------, $a_{\varepsilon^2 = 10^{-3}}$ ———

Plot 3



$\varepsilon^2 = 0$, $\varepsilon^2 = 10^{-3}$
$\hat{a}$ ··········, $a_{\varepsilon^2 = 0}$ --------, $a_{\varepsilon^2 = 10^{-3}}$ ———

Plot 4



$\varepsilon^2 = 10^{-3}$, $K = 10$
a^0 = tangent
$\hat{a}$ --------, $a_{\varepsilon^2 = 10^{-3}}$ ———

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