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Développements en polynômes orthogonaux via la transformée de Fourier

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Abstract

Using techniques of operational calculus we present methods for computing the generalized Fourier coefficients for certain families of orthogonal polynomials, specifically Meixner classes of polynomials. In particular, Krawtchouk transforms are considered.

Résumé

Nous présentons des méthodes faisant appel au calcul opérationnel, permettant de calculer les coefficients de Fourier généralisés pour certaines familles de polynômes orthogonaux, spécifiquement des classes de polynômes de Meixner. Les transformées de Krawtchouk font l'objet d'une étude détaillée.

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Orthogonal polynomial expansions via Fourier transform

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Abstract. Using techniques of operational calculus we present methods for computing the generalized Fourier coefficients for certain families of orthogonal polynomials, specifically Meixner classes of polynomials. In particular, Krawtchouk transforms are considered.

Introduction.

The Meixner polynomials are special families of orthogonal polynomials closely related to Lie algebras. (See [4] for details.) A particular class, the KRAWTCHOUK polynomials, arise as functions on the finite abelian group $\mathbf{Z}_2^n$ and thus the calculation of the KRAWTCHOUK transform is of particular interest. Diaconis and Rockmore [3] mention the question of rapid calculation of the KRAWTCHOUK transform. In this paper, we use the close relationship between Meixner polynomials and representations of the Heisenberg algebra to give expressions for the generalized Fourier coefficients of a function expanded in a series of orthogonal polynomials of Meixner type. From these formulas, we obtain, in conjunction with the fast Fourier transform (FFT), efficient methods for the calculation of the coefficients. Applications of these polynomials in numerical analysis have shown their efficacy and usefulness. See [8] for example. We should mention that the algebraic/analytic/operator structure of these polynomials has been discussed, e.g., in [1] and [10] whose work is very close in spirit to ours in many points.

The paper is organized as follows. Section 1 presents the basic facts concerning Meixner polynomials and their connection with Heisenberg algebras. Section 2 gives a detailed discussion of the role of the lowering operator V. In Section 3 we discuss the KRAWTCHOUK case. Section 4 deals with the general Meixner case, and finally in section 5 we return to the KRAWTCHOUK expansions. The essential point of our approach is the reduction to multiplication by a Vandermonde matrix and the FFT.

I. Meixner polynomials and Heisenberg algebras

Here we show how to define representations of the Heisenberg algebra on a vector space. Then the realizations of the basic operators for the Meixner polynomial classes are given.

1.1 HEISENBERG ALGEBRAS

A set of three operators A, B, C is a basis for the Heisenberg algebra if they satisfy $[A, B] = C$ and $[A, C] = [B, C] = 0$, i.e., C commutes with A and B.

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Given a vector space with basis ψ_n a representation of the Heisenberg algebra may be given in terms of the *raising and lowering operators* R and V defined by their action on the basis vectors:

$$\begin{aligned} R\psi_n &= \psi_{n+1} \\ V\psi_n &= n\psi_{n-1} \end{aligned} \tag{1.1.1}$$

It is readily seen that, with the Lie bracket given by the commutator $VR - RV$,

$$[V, R] = I \tag{1.1.2}$$

I denoting the identity operator.

1.1.1 Proposition. *The operators V and R defined in (1.1.1) generate a Heisenberg algebra.*

The Meixner polynomials are orthogonal polynomials such that V is expressed by an analytic function of $D = d/dx$, where x is the variable in which the polynomials are given [1] [6] [7] [9]. After suitable normalizations, one finds six families [2] of orthogonal polynomials as follows with the corresponding functions V (the functions $L(s)$ listed here will be explained in the next section).

1.1.2 Proposition. *For the Meixner classes of polynomials the V operators take the form:*

Meixner	$V(s) = \dfrac{\tanh qs}{q - \alpha \tanh qs}$	$L(s) = -\dfrac{\alpha}{\beta}s - \log \dfrac{qV(s)}{\sinh qs}$
Meixner − Pollaczek	$V(s) = \tan s$	$L(s) = \log \sec s$
KRAWTCHOUK	$V(s) = \tanh s$	$L(s) = \log \cosh s$
Charlier	$V(s) = e^s - 1$	$L(s) = e^s - 1 - s$
Laguerre	$V(s) = s/(1-s)$	$L(s) = -\log(1-s) - s$
Hermite	$V(s) = s$	$L(s) = s^2/2$

where, for the general case, α, β are given parameters and $q^2 = \alpha^2 - \beta$.

(Note the normalizations $V(0) = L'(0) = 0$, $V'(0) = 1$.)

1.2 GENERATING FUNCTIONS

The role of V as lowering operator may be seen from the generating functions. These have a particular structure that is determined by the function $L(s)$, the logarithm of the Fourier transform of the measure of orthogonality. Each class of polynomials corresponds to a convolution family of measures, p_t. They satisfy the relation

$$e^{tL(is)} = \int e^{isx}\, p_t(dx) \tag{1.2.1}$$

where L is analytic in a neighborhood of the origin (in the complex plane). Let U be the functional inverse of V (in a neighborhood of 0) and set $M(s) = L(U(s))$. The generating functions have the form:

$$e^{xU(s) - tM(s)} = \sum_0^\infty \frac{s^n}{n!} J_n(x, t) \tag{1.2.2}$$

the function V is given as the derivative of L. (This is discussed in detail below). The measures p_t are a convolution family, p_t corresponding to the t th power of p_1: $p_t = p_1^{*t}$.

II. The operator V

2.1 V AND FOURIER TRANSFORM

The operator V is the lowering operator, the action of which on the polynomials is given by

$$V J_n = n\, J_{n-1} \tag{2.1.1}$$

that is, V acts as a generalized derivative operator. We use J_n to denote $J_n(x, t)$. (See [5] for more related to the discussion below.)

From orthogonality, one finds that V satisfies a Riccati equation [7], which in standard form may be written

$$V' = 1 + 2\alpha V + \beta V^2 \tag{2.1.2}$$

for real constants α, β, the prime denoting differentiation. Consider the measure of orthogonality $p(dx)$ (i.e., $p_1(dx)$). With s replacing is in (1.2.1) we express the moment generating function in the form

$$e^{L(s)} = \int e^{sx}\, p(dx) \tag{2.1.3}$$

Differentiating with respect to s on both sides we have

$$V(s)\, e^{L(s)} = \int e^{sx}\, x\, p(dx) \tag{2.1.4}$$

with $V(s) = L'(s)$. By the Riccati equation it follows that repeated differentiation leads to a relation of the form

$$V(s)^n e^{L(s)} = \int e^{sx}\, \phi_n(x)\, p(dx) \tag{2.1.5}$$

Thus, from the Fourier point of view, V^n corresponds to the operator of multiplication by $\phi_n(x)$. We will see that $\phi_n(x)$ is proportional to $J_n(x, 1)$.

2.1.1 Proposition. *The polynomials defined via (2.1.5) satisfy*

$$\phi_n(x) = \frac{n!}{\gamma_n}\, J_n(x, 1) \tag{2.1.6}$$

where $\gamma_n = \langle J_n^2 \rangle = n!\beta^n (1/\beta)_n$.

Proof: Replacing s by is in (2.1.5), Fourier inversion yields the distributional relation, writing $p(dx)$ as $p(x)\, dx$,

$$\begin{aligned} \phi_n(x) p(x) &= \int e^{-isx}\, V(is)^n\, e^{L(is)}\, ds/2\pi \\ &= V^*(D)^n p(x) \end{aligned} \tag{2.1.7}$$

where $V^*(z) = V(-z)$, the $*$ denoting adjoint. In fact,

$$\int J_m(x,1)\, V^*(D)^n p(x)\, dx = \langle V^n J_m \rangle = n!\, \delta_{nm} \tag{2.1.8}$$

Comparing with the orthogonality relations of the J's shows the Rodrigues-type formula

$$V^*(D)^n p(x) = \frac{n!}{\gamma_n} J_n(x,1)\, p(x) \tag{2.1.9}$$

with $\gamma_n = \langle J_n^2 \rangle$. Now compare with (2.1.7). (See [7], pp. 26–27.) ∎

2.2 V AND THE GENERATING FUNCTION

In the generating function

$$e^{xU(s)-tM(s)} = \sum_0^\infty \frac{s^n}{n!} J_n(x,t) \tag{2.2.1}$$

replacing $s \to V(s)$ gives

$$e^{xs-tL(s)} = \sum_0^\infty \frac{V(s)^n}{n!} J_n(x,t) \tag{2.2.2}$$

In this form the action of V is clear. Multiplication by $V(D)$ on the left multiplies by $V(s)$ on the right, resulting in $J_n \to nJ_{n-1}$.

III. KRAWTCHOUK polynomials

First we see how the KRAWTCHOUK polynomials fit into the above scheme. Then calculation of the KRAWTCHOUK expansions is considered.

3.1 KRAWTCHOUK STRUCTURES

For the binomial distribution, the measure p_t is given by the discrete weights $p_N(x) = 2^{-N}\binom{N}{\pi}$ with $\pi = (N+x)/2$. We have the moment generating function:

$$\int e^{sx}\, p_N(dx) = 2^{-N}(e^s + e^{-s})^N = \cosh^N s \tag{3.1.1}$$

where the time-parameter t is replaced by the discrete index N. Thus

$$\begin{aligned} L(s) &= \log\cosh s \\ L'(s) = V(s) &= \tanh s \end{aligned} \tag{3.1.2}$$

If we think of the KRAWTCHOUK polynomials as (modified) elementary symmetric functions in the quantities π $+1's$ and ν $-1's$ thought of as the steps of a random walker on the line starting from 0, then we have the generating function

$$(1+v)^{(N+x)/2}(1-v)^{(N-x)/2} = \sum_0^\infty \frac{v^n}{n!} K_n(x, N) \tag{3.1.3}$$

where $N = \pi + \nu$ is the total number of steps and $x = \pi - \nu$ is the position of the walker after N steps. This may be rewritten in the form

$$\left(\frac{1+v}{1-v}\right)^{x/2} = (1-v^2)^{-N/2} \sum_0^\infty \frac{v^n}{n!} K_n(x,N) \tag{3.1.4}$$

Now substitute $v = \tanh s$ to get

3.1.1 Proposition.

$$e^{sx} = \cosh^N s \sum_0^\infty \frac{\tanh^n s}{n!} K_n(x,N) \tag{3.1.5}$$

3.2 CALCULATION OF KRAWTCHOUK EXPANSIONS

There are two ways of formulating the basic approach:

1) Replace $s \to is$ in eq. (3.1.5). This gives

$$e^{isx} = i^n \sum_{n=0}^N \frac{\cos^{N-n} s \sin^n s}{n!} K_n(x,N) \tag{3.2.1}$$

where x runs from $-N,\ldots,N$ in steps of two. We have

$$s = \frac{2\pi(N-2k)}{1+N}, \qquad 0 \le k \le N \tag{3.2.2}$$

If f has KRAWTCHOUK-expansion $\sum f_n K_n/n!$, then, denoting the *finite* Fourier transform of f by $\tilde{f}$,

$$\tilde{f}(s) = (N+1)^{-1} \sum_x e^{-isx} f(x) \tag{3.2.3}$$

where the sum on x runs from $-N,\ldots,N$ in steps of two, (3.2.1) yields

3.2.1 Proposition.

$$f_n = i^n \sum_s \tilde{f}(s) \cos^{N-n} s \sin^n s \tag{3.2.4}$$

2) Denote here $\frac{d}{ds}$ by D. Thus, $f(x) = e^{xD} f(s)|_0$, which may be denoted just $e^{xD} f(0)$, as we will do below. From (3.1.5) we have

$$e^{xD} f(s) = \sum_0^N \frac{\cosh^{N-n} D \sinh^n D}{n!} f(s) K_n(x,N) \tag{3.2.5}$$

Setting $s = 0$ gives the desired expansion

3.2.2 Proposition.

$$f(x) = 2^{-N} \sum_{0}^{N} \frac{K_n(x,N)}{n!} (e^D + e^{-D})^{N-n}(e^D - e^{-D})^n f(0) \tag{3.2.6}$$

As alternative forms we have:

3.2.3 Proposition. *For the* KRAWTCHOUK *polynomials we have the following expressions for the expansion coefficients:*

1) As in (3.2.4):

$$f_n = i^n \sum \cos^N s \, \tan^n s \, \tilde{f}(s) \tag{3.2.7}$$

2) The formula (3.2.6), written in the form

$$f_n = 2^{-N}(e^{2D} - e^{-2D})^{\min(n,N-n)}(e^D + \operatorname{sgn}(N-2n)e^{-D})^{|N-2n|} f(0) \tag{3.2.8}$$

Observe that the exponent in the first factor in (3.2.8) may be computed as $\frac{1}{2}(N - |N - 2n|)$.

IV. Calculation of Fourier coefficients

In this section we present two methods for calculating the expansion of a function for the general Meixner case. The first method is adapted for numerical computation; the second for symbolic calculation.

Remark. Throughout the discussion Fourier transform is conventionally defined as

$$\hat{f}(s) = \int e^{-isx} f(x)\, dx/2\pi$$

4.1 FOURIER TRANSFORM METHOD

Let f have the expansion

$$f = \sum f_n J_n / n! \tag{4.1.1}$$

Then, since $V\,J_n = nJ_{n-1}$ and the J_n are orthogonal with $J_0 = 1$

$$f_n = \langle V^n f \rangle / n! \tag{4.1.2}$$

4.1.1 Proposition. *We have the expression*

$$\langle V^n f \rangle = \int e^{tL(is)}\, V(is)^n \hat{f}(s)\, ds \tag{4.1.3}$$

Proof: Recall that the (inverse) Fourier transform of the measure p_t given by (1.2.1).

$$\int e^{isx}\, p_t(dx) = e^{tL(is)} \tag{4.1.4}$$

Thus,

$$\hat{p}_t(s) = \overline{e^{tL(is)}}/2\pi \tag{4.1.5}$$

Using Parseval's formula, for any function g we have, in the sense of distributions,

$$\langle g \rangle = \int g(x)p_t(dx) = 2\pi \int \hat{g}(s)\overline{\hat{p}_t(s)}\, ds \tag{4.1.6}$$

From the Fourier inversion formula we have the action of the operator $V(d/dx)$

$$\begin{aligned} f(x) &= \int e^{isx}\, \hat{f}(s)\, ds \\ V^n f(x) &= \int V(is)^n e^{isx}\, \hat{f}(s)\, ds \end{aligned} \tag{4.1.7}$$

Combining this with formula (4.1.6) for calculating expected values yields the result. (4.1.3). ∎

4.2 OPERATOR CALCULUS METHOD

From the generating function, (2.2.2) we have:

$$e^{sx-tL(s)} = \sum_0^\infty \frac{V(s)^n}{n!}\, J_n(x,t) \tag{4.2.1}$$

We extend (3.1.5) for the KRAWTCHOUK case to the general Meixner case.

4.2.1 Proposition. *For the Meixner polynomials we have the expansion*

$$e^{sx} = e^{tL(s)} \sum_0^\infty \frac{V(s)^n}{n!}\, J_n(x,t) \tag{4.2.2}$$

Using the operational formula $e^{aD} f(x) = f(x+a)$, put $x \to a$, $s \to D$, then $x \to 0$, $a \to x$, which gives the expansion

$$f(x) = \sum_0^\infty \frac{J_n(x,t)}{n!}\, e^{tL(D)}\, V(D)^n f(0) \tag{4.2.3}$$

Thus (cf. Rota, Kahaner, & Odlyzko, in [10]),

4.2.2 Proposition. *For the Meixner polynomials the expansion coefficients are given by*

$$f_n = e^{tL(D)}\, V(D)^n f(0) \tag{4.2.4}$$

V. Comments on complexity

The Vandermonde matrix multiplication here is quite easy to implement. The efficiency of the computations may be improved due to the simplicity of the structures involved in this approach. Practical computation of the Fourier transform may be done by standard FFT methods. An alternative approach is to represent the function by a polynomial and use a symbolic manipulation package to implement the operator calculus. Difference methods may be applied directly as well. An algorithm based on formulas (3.2.4), (3.2.7) has been implemented by D. Sheeran using the software FOURIER developed by D. Kammler of SIUC. An algorithm based on (3.2.5), (3.2.6) has also been programmed by D. Sheeran.

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