A quick construction of a retraction of all retractions for stable bifinites

Roberto M. Amadio

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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Roberto M. AMADIO

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A Quick Construction of a Retraction of all Retractions for Stable Bifinites [1]

Une Construction Rapide de la Rétraction de toutes les Rétractions pour les Bifinis Stables

Roberto M. Amadio
CNRS-CRIN, INRIA-Lorraine, Nancy

Abstract
In the framework of stable domain theory we show that the space of retractions over a bifinite is a retract of the functional space.

Résumé
Dans le cadre de la théorie stable des domaines nous montrons que l'espace des rétractions sur un bifini est une rétraction de l'espace fonctionnel.

[1] This note has been accepted for publication on Information and Computation.

Introduction

Scott (1980) shows that the collection of finitary retractions over a bounded complete algebraic cpo D is the image of a finitary retraction over the space of continuous functions D→D. Rothe (1991) extends this result to a class of FS-domains (see Jung (1991)). In the stable case Berardi (1991) was apparently the first to observe that when working over dI-domains (Berry (1979)) the image of a stable retraction is still a dI-domain. It was then possible to adapt Scott's technique to show that the space of retractions over a dI-domain is a retract of the functional space In this note we give the corresponding of Berardi's results for the 'stable bifinites' studied in Amadio (1991). The proof is new and shorter, this seems due to the fact that stable bifinites can be described in a synthetic way as 'directed colimits of stable projections'.

We refer to Amadio and Longo (1987), Berardi (1991) for the origins of the problem. We will simply mention here that when the domain at hand is a model of the type free lambda calculus it is easy to build models for type theories with a type of all types, by interpreting types-as-retractions. We refer to Amadio (1991) for some background on stable domain theory, omitted proofs, a characterization of stable bifinites, and a comparison with other categories of 'stable domains'. Interestingly Heckmann (1992) shows that stable bifinites are closed under the convex powerdomain construction.

Preliminaries

A cpo is a directed complete partial order with a least element. When we speak of continuity we always refer to Scott topology. '$\sqcup$' is the join, '$\wedge$' is the meet, and '$\uparrow$' predicates the existence of an upper bound.

We concentrate on a category of cpos with continuous meets of pairs of compatible elements and continuous maps preserving such meets. We briefly refer to such cpos as *meet cpos*. Formally D is a meet cpo if:

(1) D is a cpo, (2) $\forall d, e \in D.(d\uparrow e \Rightarrow \exists\, d\wedge e)$,
(3) $\forall X \subseteq D$, directed $(d\uparrow \sqcup X \Rightarrow d\wedge \sqcup X = \sqcup_{x\in X}(d\wedge x))$.

- Let D, E be meet cpos. A map f: D→E is *stable* if
(1) it is Scott continuous, and (2) $\forall d, e\in D.(d\uparrow e \Rightarrow f(d\wedge e) = f(d)\wedge f(e))$.

- Stable maps are *ordered* as: $f \leq g \Leftrightarrow \forall d, d'\in D.(d \leq d' \Rightarrow f(d) = f(d') \wedge g(d))$.

Meet cpos and stable maps form a *ccc* which roughly plays in the stable case the same role as the category of cpos and continuous maps in the continuous case. In particular the *exponent* object is given by the collection of stable maps with the stable order. One will remark that in the exponent meets and directed joins are computed point-wise.

A *projection* is a retraction (i.e. idempotent) below the identity. Stable projections have wonderful properties, in particular it is easy to show that stable projections over a meet cpo D are exactly the functions p: D→D such that $p\leq id_D$ (hint: $p, q \leq id \Rightarrow p\circ q = p\wedge q$). This fact simplifies definition 1 and proposition 2.

1. Definition *(Stable Bifinite)*

Let D be a meet cpo. D is a stable bifinite, and we will write $D \in \mathrm{Bif}_\wedge$, if there is a directed set $\{p_i\}_{i \in I}$ of stable functions over D such that: (i) $p_i \leq id_D$. (ii) $im(p_i)$ is finite, and (iii) $\bigsqcup_{i \in I} p_i = id_D$.

Amadio (1991) shows that the category $\mathbf{Bif}_\wedge$ of stable bifinites and stable maps is cartesian closed. In the case we restrict $\mathbf{Bif}_\wedge$ to the countably based objects one obtains the *largest known (!) ccc of ω-algebraic meet cpos and stable maps* (so countably based dI-domains are stable bifinites). Our results here immediately extend to countably based stable bifinites, and to stable bifinites that may fail to have a least element.

We say that an algebraic cpo has *property I* if the principal ideals generated by compact elements are finite. When looking for an algebraic cpo, such that the image of its retractions are again algebraic, property I plays a crucial role, as suggested by the following example (it can be shown that stable bifinites have property I; hint: the image of a stable projection is downward closed).

Example

Consider the following collection of intervals over the real line ordered by inclusion: $\{[0, r[\mid r \in \mathbb{R} \cup \{+\infty\}\} \cup \{[0, q] \mid q \in \mathbb{Q}\}$. This is an algebraic complete linear order. However the image of the following stable projection p is *not* algebraic: $p([0,r[) = [0, r[$, $p([0, q]) = [0,q[$.

2. Proposition *($D \in Bif_\wedge$, r retraction $\Rightarrow$ $r(D) \in Bif_\wedge$)*

Let D be a stable bifinite and r: D→D be a stable retraction then the image, r(D), is a stable bifinite.

Proof

Let $\{p_i\}_{i \in I}$ be a directed collection of projections associated to D. Define $q_i = r \circ p_i \circ r$, for $i \in I$. Observe $q_i \leq r \circ id \circ r = r$, and $im(q_i)$ is finite. Since $\bigsqcup_{i \in I} q_i = r$, we conclude r(D) is a stable bifinite. □

We give a simple proof of the fact that the collection of stable retractions over a stable bifinite D, say Ret(D), is a retract of its stable functional space D→D. The *keyvault of the construction* is to observe that given f: D→D, with im(f) finite, there is a natural way to associate to f a retraction, namely iterate f a finite number of times (this point is also used in Rothe (1991)).

3. Theorem *($D \in Bif_\wedge$ $\Rightarrow$ $Ret(D) \triangleleft D \to D$)*

Given a stable bifinite D the collection of stable retractions, Ret(D), is a retract of the functional space D→D.

4. Lemma

Let D be a stable bifinite with the relative directed set of projections $\{p_i\}_{i\in I}$. Then for any $f: D\to D$, $\#\{(f\circ p_i\circ f)^k \mid k\geq 1\}\cap Ret(D) = 1$.

Proof

First let us recall a simple fact about combinatorics.

Let X be a set, $g: X\to X$ a function, and $im(g)$ finite then $\#\{g^k \mid k\geq 1\}\cap Ret(X) = 1$.

Because: $\forall k\geq 1.\ im(g^{k+1}) \subseteq im(g^k)$. Since $im(g)$ is finite the following is well defined: $h \triangleq \min\{k\geq 1 \mid im(g^{k+1}) = im(g^k)\}$. So $g_{|im(g^h)}$ is a permutation. If $\#im(g^h) = n$ then $(g^h)^{n!}$ is the identity on $im(g^h)$, and therefore a retraction over X. As for the uniqueness observe that if $g^i\circ g^i = g^i$ and $g^j\circ g^j = g^j$ for $i, j\geq 1$ then $g^i = g^{ij} = g^j$.

Next observe that $im(p_i)$ finite implies $im(f\circ p_i\circ f)$ is finite, and we can apply the previous fact. □

Conventions: In the hypotheses of the previous lemma we write:

$$f_i \equiv f\circ p_i\circ f, \qquad k_i \equiv \#im(p_i)\ !$$

Note that k_i is an upper bound on the least k such that $f_i^k\in Ret(D)$, and it is independent from f.

Next the crucial remark is that: $r\in Ret(D) \Rightarrow r_i\in Ret(D)$, because by the definition of stable order: $r_i\leq r,\ r_i d\leq rd \Rightarrow r_i(r_i d) = r_i(rd) \wedge r(r_i d) = r_i d \wedge r_i d = r_i d$. It is then appealing trying to define:

$$\rho: (D\to D) \to Ret(D), \qquad \rho(f) = \bigsqcup_{i\in I} (f_i)^{k_i},$$

as: (i) the join of a directed set of retractions is a retractions, and (ii) $r\in Ret(D) \Rightarrow \rho(r) = \bigsqcup_{i\in I} (r_i)^{k_i} = \bigsqcup_{i\in I} r_i = r$. The following lemma concludes our argument by verifying that ρ is a stable retraction over $D\to D$.

5. Lemma

(1) Function composition is a stable operation.

(2) If D is a meet cpo then Ret(D) is a meet cpo (with the order given by $D\to D$).

(3) ρ is a stable map.

(4) $im(\rho) = Ret(D)$, and $\rho\circ\rho = \rho$.

Proof

(1) Let $f, g \leq h$, and $f', g' \leq h'$, so that $dom(f) = dom(g) = cod(f') = cod(g')$. We have to show: $(f\wedge g)(f'\wedge g')(d) = ((f\circ f')\wedge(g\circ g'))(d)$, any d. We compute:

$$(f\wedge g)(f'\wedge g')(d) = (f\wedge g)(f'd\wedge g'd) = f(f'd) \wedge f(g'd) \wedge g(f'd) \wedge g(g'd).$$

Observe: $f \leq h,\ g'd \leq h'd \Rightarrow f(g'd) = f(h'd) \wedge h(g'd) \geq f(f'd) \wedge g(g'd)$.

$g \leq h,\ f'd \leq h'd \Rightarrow g(f'd) = g(h'd) \wedge h(f'd) \geq g(g'd) \wedge f(f'd)$.

Hence: $(f\wedge g)(f'\wedge g')(d) = f(f'd) \wedge g(g'd) = ((f\circ f')\wedge(g\circ g'))(d)$, any d.

It is worth pointing out that for $f = f'$, $g = g'$ we get: $(f\wedge g)^k = f^k\wedge g^k$, for $k\geq 1$.

(2) Note that $Ret(D) = Fixpoints(\lambda f.\ f \circ f)$. In complete analogy with the continuous case it is easy to show that the collection of fix-points of a stable map over a meet cpo is a meet cpo.

(3) First observe that for any given $f: D \to D$, $\{ (f_i)^{k_i} \}_{i \in I}$ is a *directed set*. Because:

$$p_i \preceq p_j \Rightarrow (f_i)^{k_i} = (f_i)^{k_i k_j} \preceq (f_j)^{k_i k_j} = (f_j)^{k_j}.$$

Therefore by (2) it follows that $\rho(f)$ is well-defined and it is a retraction. A similar argument shows that ρ is *monotone*, as: $f \preceq g \Rightarrow (f_i)^{k_i} \preceq (g_i)^{k_i}$, for any i.

Next we wish to show that ρ is *meet-preserving*. Suppose f, g $\preceq$ h. Observe:

$(f \wedge g)_i = (f \wedge g) \circ p_i \circ (f \wedge g) = (f \circ p_i \circ f) \wedge (f \circ p_i \circ g) \wedge (g \circ p_i \circ f) \wedge (g \circ p_i \circ g)$. Also, any d:

$$f \preceq h,\ p_i(gd) \leq p_i(hd) \Rightarrow f(p_i(gd)) = f(p_i(hd)) \wedge h(p_i(gd)) \geq f(p_i(fd)) \wedge g(p_i(gd)).$$

$$g \preceq h,\ p_i(fd) \leq p_i(hd) \Rightarrow g(p_i(fd)) = g(p_i(hd)) \wedge h(p_i(fd)) \geq g(p_i(gd)) \wedge f(p_i(fd)).$$

Hence: $(f \wedge g)_i = f_i \wedge g_i$. If we combine this with the stability of iteration (1), and property (3) of meet cpos we get:

$$\rho(f \wedge g) = \sqcup_{i \in I} ((f \wedge g)_i)^{k_i} = \sqcup_{i \in I} (f_i \wedge g_i)^{k_i} = \sqcup_{i \in I} ((f_i)^{k_i} \wedge (g_i)^{k_i}) =$$
$$\sqcup_{i \in I} (f_i)^{k_i} \wedge \sqcup_{i \in I} (g_i)^{k_i} = \rho(f) \wedge \rho(g).$$

It remains to show that ρ *preserves directed sets*. Let $\{ f_v \}_{v \in V}$ be a directed set in $D \to D$ (do not confuse f_v and f_i !). First observe:

$(\sqcup_{v \in V} f_v)_i = (\sqcup_{v \in V} f_v) \circ p_i \circ (\sqcup_{v \in V} f_v) = \sqcup_{v \in V} (f_v \circ p_i \circ f_v) = \sqcup_{v \in V} (f_v)_i$. Also:

$((\sqcup_{v \in V} f_v)_i)^{k_i} = (\sqcup_{v \in V} (f_v)_i)^{k_i} = \sqcup_{v \in V} ((f_v)_i)^{k_i}$. Hence:

$$\rho(\sqcup_{v \in V} f_v) = \sqcup_{i \in I} ((\sqcup_{v \in V} f_v)_i)^{k_i} = \sqcup_{i \in I} \sqcup_{v \in V} ((f_v)_i)^{k_i} =$$
$$\sqcup_{v \in V} \sqcup_{i \in I} ((f_v)_i)^{k_i} = \sqcup_{v \in V} \rho(f_v).$$

(4) We have already argued that $r \in Ret(D) \Rightarrow \rho(r) = r$. This immediately implies the second claim, as: $\rho(f) \in Ret(D) \Rightarrow \rho(\rho(f)) = \rho(f)$. $\square$

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