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Abstract: The most general H_∞ control problem is solved by elementary state-space manipulations. Here the characterization of feasible closed-loop gains γ is in terms of Riccati inequalities rather than equations. This allows treatment within a single framework of both regular and singular continuous- or discrete-time H_∞ problems.

An interesting by-product of this approach is a convex state-space parametrization of all H_∞-suboptimal controllers, including reduced-order ones. Here the free parameters are pairs of positive definite matrices solving the Riccati inequalities and satisfying some coupling constraint. Such pairs form a convex set and given any of them, the controller reconstruction amounts to solving a linear matrix inequality (LMI). Applications of these results to the improvement of classical H_∞ design techniques are discussed.

Une Solution Complète du Problème H_∞ par les Inégalités de Riccati

Résumé: Une solution complète du problème de contrôle H_∞ est obtenue par des manipulations élémentaires basées sur les représentations d'état. Ici la faisabilité d'un gain en boucle fermée γ est caractérisée en terme d'inéquations de Riccati au lieu des équations de Riccati habituelles. Ceci permet un traitement unifié des cas continu et discret ainsi que des problèmes H_∞ singuliers.

On obtient de plus une paramétrisation convexe de tous les compensateurs H_∞ sous-optimaux, y compris ceux d'ordre réduit. Les paramètres libres sont des pairs de matrices définies positives solutions des inéquations de Riccati et satisfaisant une contrainte de couplage. L'ensemble de ces pairs est convexe et étant donné un élément quelconque de cet ensemble, les paramètres du compensateur se calculent par la résolution d'une inégalité matricielle linéaire. Quelques applications de ces résultats à l'amélioration des techniques de synthèse H_∞ sont discutées en fin du rapport.

1 Introduction

DGKF's state-space formulas [3] are widely accepted as an efficient and numerically sound way of computing H_∞ controllers. Indeed, solving two algebraic Riccati equations (**ARE**) is all it takes to test for existence of adequate controllers. In addition, explicit state-space formulas are given for some particular solution called the "central controller." Finally, all suitable controllers are parametrized via a linear fractional transformation built around the central controller and involving a free dynamical parameter $Q(s)$ [3].

Unfortunately, DGKF's solution also suffers from shortcomings which limit the scope and performance of current state-space design techniques. First, it only applies to plants which satisfy certain restrictive assumptions called "regularity" assumptions. Even though extensions to singular problems have been proposed by [17, 15, 18], their numerical implementation is far less immediate. Secondly, DGKF's approach overemphasizes the "central" solution. Indeed, the Q-parametrization is impractical as a design tool since there is no obvious connection between the free parameter $Q(s)$ and the controller or closed-loop properties. As a result, the diversity in H_∞ controllers is hardly exploited and applications rely almost exclusively on the central controller despite certain undesirable properties. For instance, its tendency to cancel the stable poles of the plant [16] which leads to unacceptable designs for flexible structures. Finally, the order of the central controller matches that of the augmented plant which may be relatively high. Reduced-order H_∞ design would therefore be desirable and the Q-parametrization seems inadequate for this purpose.

An alternative to the Q-parametrization is the concept of convex state-space parametrization of H_∞ controllers introduced in [7]. This parametrization consists of replacing the usual H_∞ Riccati equations by Riccati inequalities and of using the solution set of these inequalities to recover all suboptimal H_∞ controllers, including reduced-order ones. Specifically, controllers are generated from the set of pairs (X, Y) of symmetric matrices satisfying (with the assumptions of [3]):

$$\left\{\begin{array}{l} A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 \ < \ 0 \\ AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T \ < \ 0 \\ X > 0, \qquad Y > 0, \qquad \rho(XY) \leq \gamma^2. \end{array}\right.$$

As it turns, certain desirable objectives such as controller order reduction or damping of the closed-loop modes have a clear interpretation in terms of X, Y (see Section 9). Unlike the Q-parametrization, this formulation therefore allows practical design of better H_∞ controllers.

The present paper generalizes the results of [7] to singular continuous-time problems and to regular or singular discrete-time problems. We are only concerned with infinite-horizon problems for linear time-invariant systems. Our approach is conceptually straightforward and solely relies on the following two facts: (1) via the Bounded Real Lemma, H_∞-like constraints can be converted into algebraic Riccati inequalities (**ARI**) and in fact into linear matrix inequalities (**LMI**); (2) the controller parameters enter the LMI linearly; hence they can be eliminated to obtain solvability conditions which depend only on the plant data and two extra parameters R and S. Insighted by [7], Iwasaki & Skelton have independently obtained results similar to ours for the continuous-time case [10]. Finally, there are analogies between our derivation technique and the manipulations in [13] (see Lemma 3.1 below and its applications).

The paper is organized as follows. Section 2 gives a precise statement of the suboptimal H_∞ control problem and recalls its state-space formulation. Section 3 contains an instrumental lemma which allows the elimination of the controller parameters. Solvability conditions for the most general suboptimal H_∞ problem are derived in Sections 4 and 5 in the continuous- and discrete-time cases, respectively. In Section 6, these conditions are turned into LMI's which define a convex set of free parameters. The issue of constructing adequate controllers from these free parameters is addressed in Section 7. Finally, Section 8 compares this approach to the classical ARE-based results of [9] and applications to the refinement of current H_∞ design techniques are discussed in Section 9.

The following notation will be used throughout the paper: $\sigma_{max}(M)$ for the largest singular value of a matrix M, and $\operatorname{Ker} M$ and $\operatorname{Im} M$ for the null space and range of the linear operator associated with M.

2 Suboptimal H_∞ Problem

Consider a proper continuous- or discrete-time plant P which maps exogenous inputs w and control inputs u to controlled outputs q and measured outputs y. That is,

$$\begin{pmatrix} Q(\sigma) \\ Y(\sigma) \end{pmatrix} = P(\sigma) \begin{pmatrix} W(\sigma) \\ U(\sigma) \end{pmatrix}$$

where σ stands for the Laplace variable s in the continuous-time case and for the Z-transform variable z in the discrete-time case. Given some dynamic output feedback law $u = K(\sigma)y$ and with the partitioning

$$P(\sigma) = \begin{pmatrix} P_{11}(\sigma) & P_{12}(\sigma) \\ P_{21}(\sigma) & P_{22}(\sigma) \end{pmatrix} \qquad (\sigma = s, z), \tag{2.1}$$

the closed-loop transfer function from disturbance w to controlled output q is:

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}. \tag{2.2}$$

The suboptimal H_∞ control problem of parameter γ consists of finding a controller $K(\sigma)$ such that:

- the closed-loop system is internally stable,
- the H_∞ norm of $\mathcal{F}(P, K)$ (the maximum gain from w to q) is strictly less than γ.

Solutions of this problem (if any) will be called **γ-suboptimal** controllers.

As usual in state-space approaches to H_∞ control, introduce some minimal realization of the plant P:

$$P(\sigma) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (\sigma I - A)^{-1} (B_1,\ B_2) \qquad (\sigma = s, z) \tag{2.3}$$

where the partitioning is conformable to (2.1). The problem dimensions are summarized by:

$$A \in \mathbb{R}^{n\times n}; \qquad D_{11} \in \mathbb{R}^{p_1\times m_1}; \qquad D_{22} \in \mathbb{R}^{p_2\times m_2}.$$

Throughout the paper, the only assumptions on the plant parameters are:

(A1) (A, B_2, C_2) is stabilizable and detectable,

(A2) $D_{22} = 0$.

The first assumption is necessary and sufficient to allow stabilization of the plant by dynamic output feedback. As of **(A2)**, it incurs no loss of generality while considerably simplifying calculations [9]. None of the customary "regularity" assumptions on the rank of D_{12} and D_{21} and on the invariant zeros of $P_{12}(s)$ and $P_{21}(s)$ [3] is needed in this approach.

Assuming **(A2)** and given any proper real-rational controller $K(\sigma)$ of realization

$$K(\sigma) = D_K + C_K(\sigma I - A_K)^{-1}B_K; \qquad A_K \in \mathbb{R}^{k\times k} \qquad (\sigma = s, z), \tag{2.4}$$

a (not necessarily minimal) realization of the closed-loop transfer function from w to z is obtained as:

$$\mathcal{F}(G, K)(\sigma) = D_{cl} + C_{cl}(\sigma I - A_{cl})^{-1}B_{cl} \tag{2.5}$$

where

$$\begin{aligned} A_{cl} &= \begin{pmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{pmatrix}; \qquad B_{cl} = \begin{pmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{pmatrix}; \\ C_{cl} &= (C_1 + D_{12}D_KC_2\,,\ D_{12}C_K); \qquad D_{cl} = D_{11} + D_{12}D_KD_{21}. \end{aligned} \tag{2.6}$$

Gathering all controller parameters into the single variable

$$\Theta := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix} \tag{2.7}$$

and introducing the shorthands:

$$\begin{aligned} A_0 &= \begin{pmatrix} A & 0 \\ 0 & 0_k \end{pmatrix}; \qquad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}; \qquad C_0 = (C_1\,,\ 0); \\ \mathcal{B} &= \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix}; \quad \mathcal{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \end{pmatrix}; \quad \mathcal{D}_{12} = (0\,,\ D_{12}); \quad \mathcal{D}_{21} = \begin{pmatrix} 0 \\ D_{21} \end{pmatrix}, \end{aligned} \tag{2.8}$$

the dependence on K of $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ takes the affine form:

$$A_{cl} = A_0 + \mathcal{B}\,\Theta\,\mathcal{C}; \quad B_{cl} = B_0 + \mathcal{B}\,\Theta\,\mathcal{D}_{21}; \quad C_{cl} = C_0 + \mathcal{D}_{12}\,\Theta\,\mathcal{C}; \quad D_{cl} = D_{11} + \mathcal{D}_{12}\Theta\mathcal{D}_{21}. \tag{2.9}$$

Note that all parameters in (2.8) involve only plant data and that $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ depend linearly on the controller data Θ. This fact is instrumental to the solution derived in Section 4.

3 Useful Results

The following Lemma plays a central role in our approach.

Lemma 3.1 *Given a symmetric matrix $\Psi \in \mathbb{R}^{m\times m}$ and two matrices P, Q of column dimension m, consider the problem of finding some matrix Θ of compatible dimensions such that*

$$\Psi + P^T\Theta^TQ + Q^T\Theta P < 0. \tag{3.1}$$

With W_P, W_Q denoting any matrices whose columns form bases of the null spaces of P and Q, respectively, this problem has a solution if and only if

$$\begin{cases} W_P^T\,\Psi\,W_P < 0 \\ W_Q^T\,\Psi\,W_Q < 0. \end{cases} \tag{3.2}$$

Proof: Necessity of (3.2) is clear; for instance, from $P W_P = 0$ it follows that (3.1) reduces to $W_P^T \Psi W_P < 0$ when pre- and post-multiplied by W_P^T and W_P, respectively. For details on the sufficiency part, see Appendix A. An analogous result can also be found in [13].

■

The proofs of our main theorems will also make extensive use of the following standard result on Schur complements and negative definite 2×2 block matrices.

Lemma 3.2 *The block matrix* $\begin{pmatrix} P & M \\ M^T & Q \end{pmatrix}$ *is negative definite if and only if*

$$\begin{cases} Q < 0 \\ P - MQ^{-1}M^T < 0 \end{cases} \tag{3.3}$$

In the sequel, $P - MQ^{-1}M^T$ *will be referred to as the Schur complement of* Q.

■

4 Solvability of Continuous-Time Problems

We first recall the Bounded Real Lemma for continuous-time systems. This lemma helps turning the H_∞ suboptimal constraints into an LMI.

Lemma 4.1 *Consider a continuous-time transfer function* $T(s)$ *of (not necessarily minimal) realization* $T(s) = D + C(sI - A)^{-1}B$. *The following statements are equivalent:*

(i) $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$ *and* A *is stable in the continuous-time sense* $(\mathrm{Re}\,(\lambda_i(A)) < 0)$*;*

(ii) there exists a symmetric **positive definite** *solution* X *to the LMI:*

$$\begin{pmatrix} A^T X + XA & XB & \gamma^{-1}C^T \\ B^T X & -I & \gamma^{-1}D^T \\ \gamma^{-1}C & \gamma^{-1}D & -I \end{pmatrix} < 0. \tag{4.1}$$

Proof: See, e.g., citeSch90Ric, p. 82.

■

Note that the LMI (4.1) is equivalent to

$$\begin{cases} \sigma_{max}(D) < \gamma \\ A^T X + XA + \gamma^{-2}C^T C + (XB + \gamma^{-2}C^T D)(I - \gamma^{-2}D^T D)^{-1}(B^T X + \gamma^{-2}D^T C) < 0 \end{cases}$$

where we recognize the more familiar algebraic Riccati inequality associated with the Bounded Real Lemma.

Combining the Bounded Real Lemma 4.1 and Lemma 3.1, we obtain the following necessary and sufficient conditions for the existence of γ-suboptimal controllers of order k.

Theorem 4.2 *Consider a proper plant* $P(s)$ *of minimal realization (2.3) and assume (A1)-(A2). With the notation (2.8), define*

$$\mathcal{P} := (\mathcal{B}^T,\ 0_{(k+m_2)\times m_1},\ \mathcal{D}_{12}^T); \qquad \mathcal{Q} := (\mathcal{C},\ \mathcal{D}_{21},\ 0_{(k+p_2)\times p_1}) \tag{4.2}$$

and let $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ be two matrices whose columns span $\mathrm{Ker}\,\mathcal{P}$ *and* $\mathrm{Ker}\,\mathcal{Q}$*, respectively.*

Then the set of γ-suboptimal controllers of order k is non empty if and only if there exists some $(n+k)\times(n+k)$ **positive definite** *matrix $X_{c\ell}$ such that:*

$$W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0; \qquad W_{\mathcal{Q}}^T \Psi_{X_{c\ell}} W_{\mathcal{Q}} < 0 \tag{4.3}$$

where

$$\begin{aligned}
\Psi_{X_{c\ell}} &:= \begin{pmatrix} A_0^T X_{c\ell} + X_{c\ell} A_0 & \vdots & X_{c\ell} B_0 & \gamma^{-1} C_0^T \\ \cdots & \vdots & \cdots & \cdots \\ B_0^T X_{c\ell} & \vdots & -I & \gamma^{-1} D_{11}^T \\ \gamma^{-1} C_0 & \vdots & \gamma^{-1} D_{11} & -I \end{pmatrix} \\
\Phi_{X_{c\ell}} &:= \begin{pmatrix} A_0 (\gamma^{-2} X_{c\ell}^{-1}) + (\gamma^{-2} X_{c\ell}^{-1}) A_0^T & \vdots & \gamma^{-1} B_0 & (\gamma^{-2} X_{c\ell}^{-1}) C_0^T \\ \cdots & \vdots & \cdots & \cdots \\ \gamma^{-1} B_0^T & \vdots & -I & \gamma^{-1} D_{11}^T \\ C_0 (\gamma^{-2} X_{c\ell}^{-1}) & \vdots & \gamma^{-1} D_{11} & -I \end{pmatrix}.
\end{aligned} \tag{4.4}$$

Proof: From the Bounded Real Lemma, $K(s) = D_K + C_K(sI - A_K)^{-1} B_K$ is a kth-order γ-suboptimal controller if and only if the LMI

$$\begin{pmatrix} A_{c\ell}^T X_{c\ell} + X_{c\ell} A_{c\ell} & X_{c\ell} B_{c\ell} & \gamma^{-1} C_{c\ell}^T \\ B_{c\ell}^T X_{c\ell} & -I & \gamma^{-1} D_{c\ell}^T \\ \gamma^{-1} C_{c\ell} & \gamma^{-1} D_{c\ell} & -I \end{pmatrix} < 0 \tag{4.5}$$

holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$. This LMI depends linearly on the controller parameters as seen when $A_{c\ell}, B_{c\ell}, C_{c\ell}, D_{c\ell}$ are replaced by their expressions (2.9). The inequality (4.5) then reads:

$$\Psi_{X_{c\ell}} + \mathcal{Q}^T \Theta^T \mathcal{P}_{X_{c\ell}} + \mathcal{P}_{X_{c\ell}}^T \Theta \mathcal{Q} < 0 \tag{4.6}$$

where $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$, $\Psi_{X_{c\ell}}$ and $\mathcal{Q}$ are defined above, and

$$\mathcal{P}_{X_{c\ell}} := (\mathcal{B}^T X_{c\ell},\, 0,\, \mathcal{D}_{12}^T). \tag{4.7}$$

Invoking Lemma 3.1, we can now eliminate Θ and derive existence conditions involving only $X_{c\ell}$ and the plant parameters. Specifically, let $W_{\mathcal{P}_{X_{c\ell}}}$ and $W_{\mathcal{Q}}$ denote matrices whose columns span $\mathrm{Ker}\,\mathcal{P}_{X_{c\ell}}$ and $\mathrm{Ker}\,\mathcal{Q}$, respectively. Then (4.6) holds for some Θ if and only if

$$W_{\mathcal{P}_{X_{c\ell}}}^T \Psi_{X_{c\ell}} W_{\mathcal{P}_{X_{c\ell}}} < 0; \qquad W_{\mathcal{Q}}^T \Psi_{X_{c\ell}} W_{\mathcal{Q}} < 0. \tag{4.8}$$

Hence the set of kth-order γ-suboptimal controllers is nonempty if and only if (4.8) holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$.

To complete the proof, we just need to rewrite the first inequality in (4.8) as $W_{\mathcal{P}}^T \Phi_{X_{c\ell}} W_{\mathcal{P}} < 0$ where $W_{\mathcal{P}}$ is any basis of $\mathrm{Ker}\,\mathcal{P}$. To this end, observe that $\mathcal{P}_{X_{c\ell}} = \mathcal{P} \begin{pmatrix} X_{c\ell} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$

with $\mathcal{P}$ as in (4.2). Given any basis $W_{\mathcal{P}}$ of $\operatorname{Ker}\mathcal{P}$, it follows that

$$W_{\mathcal{P}_{X_{cl}}} := \begin{pmatrix} \gamma^{-1}X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} W_{\mathcal{P}}$$

forms a basis of $\operatorname{Ker}\mathcal{P}_{X_{cl}}$. Hence $W_{\mathcal{P}_{X_{cl}}}^T \Psi_{X_{cl}} W_{\mathcal{P}_{X_{cl}}} < 0$ is equivalent to

$$W_{\mathcal{P}}^T \left\{ \begin{pmatrix} \gamma^{-1}X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \Psi_{X_{cl}} \begin{pmatrix} \gamma^{-1}X_{cl}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \right\} W_{\mathcal{P}} \;=\; W_{\mathcal{P}}^T \Phi_{X_{cl}} W_{\mathcal{P}} < 0.$$

■

The characterization of Theorem 4.2 is awkward because both X_{cl} and its inverse are involved and the role of each specific plant parameter is somewhat blurred. Fortunately, the conditions (4.3) can be reduced to a pair of Riccati inequalities of lower dimensions which exactly parallel the usual H_∞ Riccati equations. To derive this simpler characterization, it suffices to partition X_{cl} and $\gamma^{-2}X_{cl}^{-1}$ conformably to A_{cl}, calculate explicit bases $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ of $\operatorname{Ker}\mathcal{P}$ and $\operatorname{Ker}\mathcal{Q}$, and carry out the block matrix multiplications. Formulas and calculations are simpler when introducing the following shorthands:

$$\begin{aligned} \hat{A} &:= A - B_2 D_{12}^{+} C_1; \quad \hat{B}_1 := B_1 - B_2 D_{12}^{+} D_{11}; \quad \hat{B}_2 := B_2 D_{12}^{+}; \\ \hat{C}_1 &:= (I - D_{12} D_{12}^{+}) C_1; \qquad \hat{D}_{11} := (I - D_{12} D_{12}^{+}) D_{11}, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \tilde{A} &:= A - B_1 D_{21}^{+} C_2; \quad \tilde{C}_1 := C_1 - D_{11} D_{21}^{+} C_2; \quad \tilde{C}_2 := D_{21}^{+} C_2; \\ \tilde{B}_1 &:= B_1 (I - D_{21}^{+} D_{21}); \qquad \tilde{D}_{11} := D_{11}(I - D_{21}^{+} D_{21}). \end{aligned} \tag{4.10}$$

Note that this approach is valid for both regular and singular H_∞ problems.

Theorem 4.3 (γ-suboptimal controllers for continuous-time plants)
Consider a proper continuous-time plant $P(s)$ of order n and minimal realization (2.3) and assume **(A1)-(A2)**. *Let W_{12} and W_{21} denote bases of the null spaces of $(I - D_{12}^{+} D_{12}) B_2^T$ and $(I - D_{21} D_{21}^{+}) C_2$, respectively. With the notation (4.9)-(4.10), the suboptimal H_∞ problem of parameter γ is solvable if and only if*

(i) $\gamma > \max\,(\sigma_{max}(\hat{D}_{11}), \sigma_{max}(\tilde{D}_{11}))$,

(ii) there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ such that

$$W_{12}^T \left\{ \hat{A}R + R\hat{A}^T - \hat{B}_2\hat{B}_2^T + \begin{pmatrix} \hat{C}_1 R \\ \gamma^{-1}\hat{B}_1^T \end{pmatrix}^T \begin{pmatrix} I & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1 R \\ \gamma^{-1}\hat{B}_1^T \end{pmatrix} \right\} W_{12} < 0 \tag{4.11}$$

$$W_{21}^T \left\{ \tilde{A}^T S + S\tilde{A} - \tilde{C}_2^T\tilde{C}_2 + \begin{pmatrix} \tilde{B}_1^T S \\ \gamma^{-1}\tilde{C}_1 \end{pmatrix}^T \begin{pmatrix} I & -\tilde{D}_{11}^T/\gamma \\ -\tilde{D}_{11}/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{B}_1^T S \\ \gamma^{-1}\tilde{C}_1 \end{pmatrix} \right\} W_{21} < 0 \tag{4.12}$$

$$R > 0; \qquad S > 0; \qquad \lambda_{\min}(RS) \geq \gamma^{-2}. \tag{4.13}$$

Moreover, the set of γ-suboptimal controllers of order k is non empty if and only if (ii) holds for some R, S which further satisfy the rank constraint:

$$\mathrm{Rank}\,(\gamma^{-2}I - RS) \leq k. \tag{4.14}$$

Prior to a formal proof of this result, we give some insight into its meaning and implications. First, if D_{12} and D_{21}^T have full column rank, the projections $I - D_{12}^+D_{12}$ and $I - D_{21}D_{21}^+$ are identically zero and W_{12} and W_{21} can be taken as the identity matrix. The constraints (4.11)-(4.12) then reduce to a pair of algebraic Riccati inequalities. Solutions R, S to these ARI's are further constrained by the positivity and coupling conditions (4.13). With the notation $X := R^{-1}$ and $Y := S^{-1}$, and the simplifying assumptions of [3]:

$$D_{11} = 0; \qquad D_{12}^T(D_{12}, C_1) = (I, 0); \qquad D_{21}(D_{21}^T, B_1^T) = (I, 0),$$

(4.11)-(4.13) reduce to

$$A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 \;<\; 0 \tag{4.15}$$

$$AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T \;<\; 0 \tag{4.16}$$

$$X > 0; \qquad Y > 0; \qquad \rho(XY) \leq \gamma^2 \tag{4.17}$$

In the left-hand sides of (4.15)-(4.16) we recognize the usual H_∞ Riccati expressions. The constraints (4.17) are also similar to those arising in [3]. Further details on the connection with the classical results in terms of Riccati *equations* can be found in Section 8.

When the H_∞ problem is singular, the ARI constraints are relaxed in some directions via projections on the ranges of $(I - D_{12}^+D_{12})B_2^T$ and $(I - D_{21}D_{21}^+)C_2$, respectively. Since DGFK's results are not applicable to singular problems, the characterization of Theorem 4.3 coupled with convex optimization techniques is a computationally appealing substitute (see Section 6). Note that [17] also solves singular H_∞ problems by means of Riccati inequalities. Yet, the characterization of [17] contains additional rank and stability constraints which destroy convexity and hence are detrimental to numerical tractability. For more details on singular vs. regular problems, see [8].

Finally, another novelty in Theorem 4.3 is the rank condition (4.14) for existence of reduced-order H_∞ controllers. If $k \geq n$ (full order or higher), this condition is trivially satisfied and (4.11)-(4.13) are necessary and sufficient for the existence of γ-suboptimal controllers of order k. This confirms the well-known fact that whenever a suboptimal H_∞ problem is solvable, we can find adequate controllers of order equal to the plant order n. Yet, γ-suboptimal controllers are not necessarily of order n. In fact, there will exist reduced-order controllers ($k < n$) whenever (4.11)-(4.13) hold for some pair (R, S) which further satisfies $\mathrm{Rank}\,(\gamma^{-2}I - RS) = I$. Note that $\lambda_{\min}(RS) = \gamma^{-2}$ for such pairs, or equivalently in terms of $X := R^{-1}$ and $Y := S^{-1}$, $\rho(XY) = \gamma^2$. Hence equality in $\lambda_{\min}(RS) \geq \gamma^{-2}$ corresponds to pairs (R, S) generating reduced-order H_∞ controllers. This is consistent with the order reduction experienced in optimal central controllers when $\rho(X_\infty Y_\infty) = \gamma^2$ at the optimum.

Proof of Theorem 4.3: From Theorem 4.2, the set of γ-suboptimal controllers of order k is non empty if and only if (4.3) holds for some $X_{c\ell} > 0$ in $\mathbb{R}^{(n+k)\times(n+k)}$. To express (4.3) in terms of the plant parameters, introduce the block partitions:

$$X_{c\ell} := \begin{pmatrix} S & N \\ N^T & * \end{pmatrix}; \qquad \gamma^{-2}X_{c\ell}^{-1} := \begin{pmatrix} R & M \\ M^T & * \end{pmatrix} \tag{4.18}$$

where $R, S \in \mathbb{R}^{n\times n}$ and $M, N \in \mathbb{R}^{n\times k}$. Consider the first constraint $W_{\mathcal{P}}^T\, \Phi_{X_{cl}}\, W_{\mathcal{P}} < 0$ of (4.3), for instance. With the notation (4.18), $\Phi_{X_{cl}}$ defined by (4.4) reads:

$$\Phi_{X_{cl}} = \left(\begin{array}{cc:cc} AR + RA^T & AM & \gamma^{-1}B_1 & RC_1^T \\ M^TA^T & 0 & 0 & M^TC_1^T \\ \cdots & \cdots & \cdots & \cdots \\ \gamma^{-1}B_1^T & 0 & -I & \gamma^{-1}D_{11}^T \\ C_1R & C_1M & \gamma^{-1}D_{11} & -I \end{array} \right). \tag{4.19}$$

Meanwhile, from

$$\mathcal{P} = \begin{pmatrix} 0 & I_k & 0 & 0 \\ B_2^T & 0 & \underbrace{0}_{m_1} & D_{12}^T \end{pmatrix}$$

it follows that bases of the $\operatorname{Ker}\mathcal{P}$ are of the form:

$$W_{\mathcal{P}} = \begin{pmatrix} W_1 & 0 \\ 0 & 0 \\ 0 & I_{m_1} \\ W_4 & 0 \end{pmatrix} \tag{4.20}$$

where $\begin{pmatrix} W_1 \\ W_4 \end{pmatrix}$ denotes any basis of the null space of $(B_2^T\ ,\ D_{12}^T)$. Observing that the second row in the block expression (4.20) is identically zero, the condition $W_{\mathcal{P}}^T\, \Phi_{X_{cl}}\, W_{\mathcal{P}} < 0$ reduces to:

$$\begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_4 & 0 \end{pmatrix}^T \begin{pmatrix} AR + RA^T & \gamma^{-1}B_1 & RC_1^T \\ \gamma^{-1}B_1^T & -I & \gamma^{-1}D_{11}^T \\ C_1R & \gamma^{-1}D_{11} & -I \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_4 & 0 \end{pmatrix} < 0.$$

By a Schur complement argument with respect to the (2,2) block, this is also equivalent to:

$$\begin{pmatrix} W_1 \\ W_4 \end{pmatrix}^T \left\{ \begin{pmatrix} AR + RA^T & RC_1^T \\ C_1R & -I \end{pmatrix} + \gamma^{-2} \begin{pmatrix} B_1 \\ D_{11} \end{pmatrix} (B_1^T\ ,\ D_{11}^T) \right\} \begin{pmatrix} W_1 \\ W_4 \end{pmatrix} < 0,$$

that is,

$$\mathcal{N}_R^T \begin{pmatrix} AR + RA^T + \gamma^{-2}B_1B_1^T & RC_1^T + \gamma^{-2}B_1D_{11}^T \\ C_1R + \gamma^{-2}D_{11}B_1^T & -I + \gamma^{-2}D_{11}D_{11}^T \end{pmatrix} \mathcal{N}_R < 0 \tag{4.21}$$

where the columns of $\mathcal{N}_R$ span the null space of $(B_2^T\ ,\ D_{12}^T)$.

To derive the ARI (4.11), it now suffices to calculate $\mathcal{N}_R$ explicitly. To this end, introduce a basis W_{12} of the null space of $(I - D_{12}^+D_{12})B_2^T$ and an orthonormal basis U_{12} of $(\operatorname{Im} D_{12})^\perp$; that is, a matrix U_{12} such that $[D_{12}, U_{12}]$ is invertible and $\begin{pmatrix} D_{12}^T \\ U_{12}^T \end{pmatrix} U_{12} = \begin{pmatrix} 0 \\ I \end{pmatrix}$. Elementary linear algebra then shows that $\mathcal{N}_R$ can be taken as:

$$\mathcal{N}_R = \begin{pmatrix} W_{12} & 0 \\ -\hat{B}_2^T W_{12} & U_{12} \end{pmatrix}. \tag{4.22}$$

Carrying out the matrix products in (4.21), observing that $\hat{B}_2 U_{12} = 0$, and using the notation (4.9) to simplify the resulting expression, we obtain:

$$\begin{pmatrix} W_{12}^T \left(\hat{A}R + R\hat{A}^T + \gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T\right) W_{12} & W_{12}^T(RC_1^T + \gamma^{-2}\hat{B}_1 D_{11}^T)\, U_{12} \\ U_{12}^T\, (C_1 R + \gamma^{-2} D_{11}\hat{B}_1^T) & U_{12}^T\, (-I + \gamma^{-2} D_{11} D_{11}^T)\, U_{12} \end{pmatrix} < 0.$$

Now, U_{12} can be eliminated upon remarking that $U_{12}U_{12}^T = I - D_{12}D_{12}^+$. With the notation (4.9), it follows that

$$\begin{pmatrix} W_{12}^T \left(\hat{A}R + R\hat{A}^T + \gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T\right) W_{12} & W_{12}^T(R\hat{C}_1^T + \gamma^{-2}\hat{B}_1 \hat{D}_{11}^T) \\ (\hat{C}_1 R + \gamma^{-2} \hat{D}_{11}\hat{B}_1^T)W_{12} & -I + \gamma^{-2} \hat{D}_{11} \hat{D}_{11}^T \end{pmatrix} < 0$$

which is equivalent to (4.11) by another Schur complement argument.

Similarly, the condition $W_Q^T\, \Psi_{X_{cl}}\, W_Q < 0$ is equivalent to S satisfying (4.12). Hence X_{cl} satisfies (4.3) if and only if R, S satisfy (4.11)-(4.12). Moreover, $X_{cl} \in \mathbb{R}^{(n+k)\times(n+k)}$ and $X_{cl} > 0$ is equivalent to R, S satisfying (4.13) and (4.14) (see, e.g., [13, 7]).

Summing up, if $X_{cl} > 0$ of dimension $n+k$ solves (4.3) then (4.11)-(4.14) hold for the symmetric matrices R, S given by (4.18). Conversely, if the system (4.11)-(4.14) admits a solution (R, S), then $X_{cl} > 0$ of dimension $n+k$ can be reconstructed from R, S to satisfy (4.18) [13]. From (4.11)-(4.12), this X_{cl} must further solve (4.3): the proof is complete.

■

5 Solvability of Discrete-Time Problems

The machinery developed in the previous section for continuous-time problems is easily transposed to the discrete-time context and leads to qualitatively similar results as shown next. We begin by recalling the Bounded Real Lemma for discrete-time systems.

Lemma 5.1 *Consider a discrete-time transfer function $T(z)$ of (not necessarily minimal) realization $T(z) = D + C(zI - A)^{-1}B$. The following statements are equivalent:*

(i) $\|D + C(sI - A)^{-1}B\|_\infty < 1$ *and A is stable in the discrete-time sense ($|\lambda_i(A)| < 1$);*

(ii) $\inf_{T \text{ invertible}} \sigma_{max} \begin{pmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{pmatrix} < 1,$

(iii) there exists $X = X^T > 0$ such that

$$\begin{pmatrix} A^T X A - X & A^T X B & C^T \\ B^T X A & B^T X B - I & D^T \\ C & D & -I \end{pmatrix} < 0 \tag{5.1}$$

(iv) there exists $X = X^T > 0$ such that

$$\begin{pmatrix} -X^{-1} & A & B & 0 \\ A^T & -X & 0 & C^T \\ B^T & 0 & -I & D^T \\ 0 & C & D & -I \end{pmatrix} < 0 \tag{5.2}$$

Proof: See, e.g., [4]. ∎

Applying this lemma to the realization (2.6), the controller

$$K(z) = D_K + C_K(zI - A_K)^{-1}B_K \; ; \qquad A_K \in \mathbb{R}^{k\times k}$$

is γ-suboptimal if and only if the LMI

$$\begin{pmatrix} -X_{cl}^{-1} & A_{cl} & B_{cl} & 0 \\ A_{cl}^T & -X_{cl} & 0 & \gamma^{-1}C_{cl}^T \\ B_{cl} & 0 & -I & \gamma^{-1}D_{cl}^T \\ 0 & \gamma^{-1}C_{cl} & \gamma^{-1}D_{cl} & -I \end{pmatrix} < 0. \tag{5.3}$$

With $\Theta := \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$ and the decompositions (2.9), this reads:

$$\Psi_{X_{cl}} + \mathcal{Q}^T\Theta^T\mathcal{P} + \mathcal{P}^T\Theta\,\mathcal{Q} < 0 \tag{5.4}$$

where

$$\Psi_{X_{cl}} = \begin{pmatrix} -\gamma^{-2}X_{cl}^{-1} & \vdots & \gamma^{-1}A_0 & \vdots & \gamma^{-1}B_0 & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ \gamma^{-1}A_0^T & \vdots & -X_{cl} & \vdots & 0 & \gamma^{-1}C_0^T \\ \cdots & \vdots & \cdots & \vdots & \cdots & \cdots \\ \gamma^{-1}B_0^T & \vdots & 0 & \vdots & -I & \gamma^{-1}D_{11}^T \\ 0 & \vdots & \gamma^{-1}C_0 & \vdots & \gamma^{-1}D_{11} & -I \end{pmatrix} \tag{5.5}$$

$$\mathcal{P} = \gamma^{-1/2}(\mathcal{B}^T\,,\,0\,,\,0\,,\,\mathcal{D}_{12}^T) = \gamma^{-1/2}\begin{pmatrix} 0 & I & \vdots & 0 & 0 & \vdots & 0 & 0 \\ B_2^T & 0 & \vdots & 0 & 0 & \vdots & 0 & D_{12}^T \end{pmatrix} \tag{5.6}$$

$$\mathcal{Q} = \gamma^{-1/2}(0\,,\,\mathcal{C}\,,\,\mathcal{D}_{21}\,,\,0) = \gamma^{-1/2}\begin{pmatrix} 0 & 0 & \vdots & 0 & I & \vdots & 0 & 0 \\ 0 & 0 & \vdots & C_2 & 0 & \vdots & D_{21} & 0 \end{pmatrix} \tag{5.7}$$

From Lemma 3.1, (5.4) is feasible in Θ if and only if

$$W_{\mathcal{P}}^T\,\Psi_{X_{cl}}\,W_{\mathcal{P}} < 0\,; \qquad W_{\mathcal{Q}}^T\,\Psi_{X_{cl}}\,W_{\mathcal{Q}} < 0 \tag{5.8}$$

where $W_{\mathcal{P}}$ and $W_{\mathcal{Q}}$ denote bases of $\operatorname{Ker}\mathcal{P}$ and $\operatorname{Ker}\mathcal{Q}$.

As in the continuous-time case, focus on the first constraint and observe that, conformably to (5.6), $W_{\mathcal{P}}$ is of the form:

$$W_{\mathcal{P}} = \begin{pmatrix} W_1 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 0 & 0 & 0 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & \vdots & I & 0 & 0 \\ 0 & \vdots & 0 & I & 0 \\ \cdots & \vdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & 0 & I \\ W_4 & \vdots & 0 & 0 & 0 \end{pmatrix} \tag{5.9}$$

where $\begin{pmatrix} W_1 \\ W_4 \end{pmatrix}$ is any basis of the null space of $(B_2^T\,,\,D_{12}^T)$.

With this $W_{\mathcal{P}}$ and the partitioning (4.18) of $X_{c\ell}$ and $X_{c\ell}^{-1}$, $W_{\mathcal{P}}^T \Psi_{X_{c\ell}} W_P < 0$ reduces to:

$$\begin{pmatrix} W_1^T & \vdots & 0 & \vdots & 0 & W_4^T \\ 0 & \vdots & I_{n+k} & \vdots & 0 & 0 \\ 0 & \vdots & 0 & \vdots & I_{m_1} & 0 \end{pmatrix} \begin{pmatrix} -R & \vdots & \gamma^{-1}A & 0 & \vdots & \gamma^{-1}B_1 & 0 \\ \cdots & \vdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ \gamma^{-1}A^T & \vdots & & & \vdots & 0 & \gamma^{-1}C_1^T \\ 0 & \vdots & & -X_{c\ell} & \vdots & 0 & 0 \\ \cdots & \vdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ \gamma^{-1}B_1^T & \vdots & 0 & 0 & \vdots & -I & \gamma^{-1}D_{11}^T \\ 0 & \vdots & \gamma^{-1}C_1 & 0 & \vdots & \gamma^{-1}D_{11} & -I \end{pmatrix} \begin{pmatrix} W_1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & I_{n+k} & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & I_{m_1} \\ W_4 & 0 & 0 \end{pmatrix} < 0.$$

Carrying out block multiplications and forming the Schur complement of the block $-X_{c\ell}$, this is equivalent to $X_{c\ell} > 0$ and

$$\begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_4 & 0 \end{pmatrix}^T \left\{ \begin{pmatrix} -R & \gamma^{-1}B_1 & 0 \\ \gamma^{-1}B_1^T & -I & \gamma^{-1}D_{11}^T \\ 0 & \gamma^{-1}D_{11} & -I \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \\ C_1 & 0 \end{pmatrix} (\gamma^{-2}X_{c\ell}^{-1}) \begin{pmatrix} A^T & 0 & C_1^T \\ 0 & 0 & 0 \end{pmatrix} \right\} \begin{pmatrix} W_1 & 0 \\ 0 & I_{m_1} \\ W_4 & 0 \end{pmatrix} < 0$$

Further simplification of the bracketed expression and computation of the Schur complement of the (2,2) block then yield:

$$\begin{pmatrix} W_1 \\ W_4 \end{pmatrix}^T \begin{pmatrix} ARA^T - R + \gamma^{-2}B_1B_1^T & ARC_1^T + \gamma^{-2}B_1D_{11}^T \\ C_1RA^T + \gamma^{-2}D_{11}B_1^T & -I + \gamma^{-2}D_{11}D_{11}^T + C_1RC_1^T \end{pmatrix} \begin{pmatrix} W_1 \\ W_4 \end{pmatrix} < 0. \tag{5.10}$$

Now, $\begin{pmatrix} W_1 \\ W_4 \end{pmatrix}$ can be replaced by the explicit expression (4.22) with identical definitions of W_{12} and U_{12}. It then suffices to eliminate U_{12} as in the proof of Theorem 4.3 to finally obtain:

$$\begin{pmatrix} W_{12}^T \left(\hat{A}R\hat{A}^T - R + \gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T\right) W_{12} & W_{12}^T \left(\hat{A}R\hat{C}_1^T + \gamma^{-2}\hat{B}_1\hat{D}_{11}^T\right) \\ \left(\hat{C}_1R\hat{A}^T + \gamma^{-2}\hat{D}_{11}\hat{B}_1^T\right) W_{12} & -I + \gamma^{-2}\hat{D}_{11}\hat{D}_{11}^T + \hat{C}_1R\hat{C}_1^T \end{pmatrix} < 0.$$

Summing up, we can state the following counterpart of Theorem 4.3 for discrete-time systems.

Theorem 5.2 (γ-suboptimal controllers for discrete-time plants)
Consider a proper discrete-time plant $P(z)$ of order n and minimal realization (2.3) and assume **(A1)-(A2)**. *Let W_{12} and W_{21} denote bases of the null spaces of $(I - D_{12}^+D_{12})B_2^T$ and $(I - D_{21}D_{21}^+)C_2$, respectively. With the notation (4.9)-(4.10), the suboptimal H_∞ problem of parameter γ is solvable if and only if*

(i) $\gamma > \max\left(\sigma_{max}(\hat{D}_{11}), \sigma_{max}(\tilde{D}_{11})\right)$,

(ii) there exist pairs of symmetric matrices (R, S) in $\mathbb{R}^{n\times n}$ such that

$$\hat{C}_1R\hat{C}_1^T + \gamma^{-2}\hat{D}_{11}\hat{D}_{11}^T < I; \qquad \tilde{B}_1^T S\tilde{B}_1 + \gamma^{-2}\tilde{D}_{11}^T\tilde{D}_{11} < I \tag{5.11}$$

$$W_{12}^T \left\{ \hat{A}R\hat{A}^T - R - \hat{B}_2\hat{B}_2^T + \begin{pmatrix} \hat{C}_1R\hat{A}^T \\ \gamma^{-1}\hat{B}_1^T \end{pmatrix}^T \begin{pmatrix} I - \hat{C}_1R\hat{C}_1^T & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1R\hat{A}^T \\ \gamma^{-1}\hat{B}_1^T \end{pmatrix} \right\} W_{12} < 0 \tag{5.12}$$

$$W_{21}^T \left\{ \tilde{A}^T S \tilde{A} - S - \tilde{C}_2^T \tilde{C}_2 + \begin{pmatrix} \tilde{B}_1^T S \tilde{A} \\ \gamma^{-1} \tilde{C}_1 \end{pmatrix}^T \begin{pmatrix} I - \tilde{B}_1^T S \tilde{B}_1 & -\tilde{D}_{11}^T/\gamma \\ -\tilde{D}_{11}/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{B}_1^T S \tilde{A} \\ \gamma^{-1} \tilde{C}_1 \end{pmatrix} \right\} W_{21} < 0 \quad (5.13)$$

$$R > 0; \qquad S > 0; \qquad \lambda_{\min}(RS) \geq \gamma^{-2}. \qquad (5.14)$$

Moreover, the set of γ-suboptimal controllers of order k is non empty if and only if (ii) holds for some R, S which further satisfy the rank constraint:

$$\text{Rank}\,(\gamma^{-2} I - RS) \leq k. \qquad (5.15)$$

■

Clearly the continuous- and discrete-time characterizations are similar in nature. Discrete-time Riccati expressions replace continuous-time ones, and the only qualitative difference lies in the additional constraints (5.11) on R and S. Even this difference becomes immaterial when (5.12)-(5.13) are written as LMI's in R and S (see Section 6).

6 LMI Formulation and Convexity Properties

The solvability conditions obtained in Theorems 4.3 and 5.2 involve Riccati inequalities instead of equations. At first this could seem a drawback since ARI's cannot be solved by the standard numerical techniques for Riccati equations [11]. Fortunately, these ARI's as well as the positivity and coupling constraints turn out to depend convexly on the unknown variables R, S. In fact, they can be rewritten as matrix inequalities linear in R and S. Hence this characterization is not only numerically tractable, but also falls within the scope of efficient convex optimization algorithms such as [1, 12].

The convex LMI reformulation is a by-product of the proofs of Theorems 4.3 and 5.2. For continuous-time systems for instance, we have shown that $W_{\mathcal{P}}^T \Phi_{X_{cl}} W_{\mathcal{P}} < 0$ was equivalent to the LMI (4.21). In addition, the positivity and coupling constraints (4.17) are equivalent to $\begin{pmatrix} R & \gamma^{-1} I \\ \gamma^{-1} I & S \end{pmatrix} \geq 0$. Hence the LMI reformulation of Theorem 4.3 is as follows.

The continuous-time suboptimal H_∞ problem of parameter γ is solvable if and only if

$$\mathcal{N}_R^T \begin{pmatrix} AR + RA^T + \gamma^{-2} B_1 B_1^T & RC_1^T + \gamma^{-2} B_1 D_{11}^T \\ C_1 R + \gamma^{-2} D_{11} B_1^T & -I + \gamma^{-2} D_{11} D_{11}^T \end{pmatrix} \mathcal{N}_R < 0 \qquad (6.1)$$

$$\mathcal{N}_S^T \begin{pmatrix} A^T S + SA + \gamma^{-2} C_1^T C_1 & SB_1 + \gamma^{-2} C_1^T D_{11} \\ B_1^T S + \gamma^{-2} D_{11}^T B_1 & -I + \gamma^{-2} D_{11}^T D_{11} \end{pmatrix} \mathcal{N}_S < 0 \qquad (6.2)$$

$$\begin{pmatrix} R & \gamma^{-1} I \\ \gamma^{-1} I & S \end{pmatrix} \geq 0 \qquad (6.3)$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively. In addition, there exists γ-suboptimal controllers of order $k < n$ (reduced order) if and only if (6.1)-(6.3) hold for some R, S which further satisfy:

$$\text{Rank}\,(\gamma^{-2} I - RS) \leq k. \qquad (6.4)$$

Note that $\mathcal{N}_R$ and $\mathcal{N}_S$ should be chosen orthonormal for numerical stability. Such bases are easily computed via SVD's of $\begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}$ and (C_2, D_{21}). The following counterpart is obtained for discrete-time systems based on (5.10).

The discrete-time suboptimal H_∞ problem of parameter γ is solvable if and only if

$$\mathcal{N}_R^T \begin{pmatrix} ARA^T - R + \gamma^{-2}B_1B_1^T & ARC_1^T + \gamma^{-2}B_1D_{11}^T \\ C_1RA^T + \gamma^{-2}D_{11}B_1^T & -I + \gamma^{-2}D_{11}D_{11}^T + C_1RC_1^T \end{pmatrix} \mathcal{N}_R < 0 \tag{6.5}$$

$$\mathcal{N}_S^T \begin{pmatrix} A^TSA - S + \gamma^{-2}C_1^TC_1 & A^TSB_1 + \gamma^{-2}D_{11}^TB_1 \\ B_1^TSA\gamma^{-2}D_{11}^TB_1 & -I + \gamma^{-2}D_{11}^TD_{11} + B_1^TSB_1 \end{pmatrix} \mathcal{N}_S < 0 \tag{6.6}$$

$$\begin{pmatrix} R & \gamma^{-1}I \\ \gamma^{-1}I & S \end{pmatrix} \geq 0 \tag{6.7}$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ denote bases of the null spaces of (B_2^T, D_{12}^T) and (C_2, D_{21}), respectively. In addition, there exists γ-suboptimal controllers of order $k < n$ (reduced order) if and only if (6.5)-(6.7) hold for some R, S which further satisfy:

$$\operatorname{Rank}(\gamma^{-2}I - RS) \leq k.$$

The 2×2 block matrices in (6.1)-(6.2) and (6.5)-(6.6) depend only on the open-loop plant parameters A, B_1, C_1, D_{11}. Meanwhile, the control interconnection parameters B_2, C_2, D_{12}, D_{21} specify the projections $\mathcal{N}_R$ and $\mathcal{N}_S$.

The constraints for solvability of the suboptimal H_∞ problem are linear in R, S and therefore define a convex set of pairs (R, S). Hence efficient convex optimization algorithms such as [1, 12] can be used to test whether this set is nonempty and to generate particular members. Applications to the improvement of classical H_∞ designs are discussed in Section 9. By contrast, the reduced-order problem is nonconvex due to the additional rank constraint (6.4). Devising appropriate optimization techniques for this problem constitutes a challenge for future research.

7 Controller Reconstruction and Related Computational Issues

The theorems of Sections 4 and 5 are existence theorems which do not address the computation of the controller itself. This issue is now discussed in detail. Suppose we are given some solution (R, S) of, for instance, the set of constraints (6.1)-(6.4). To recover adequate controllers from this data, the first step consists of reconstructing a positive definite matrix $X_{cl} \in \mathbb{R}^{(n+k)\times(n+k)}$ compatible with (4.18). For simplicity, consider the case $k \leq n$ (controllers of order no larger than the plant order) and compute two full-column-rank matrices $M, N \in \mathbb{R}^{n\times k}$ such that

$$MN^T = \gamma^{-2}I - RS. \tag{7.1}$$

An adequate X_{cl} is then obtained as the unique solution of the linear equation:

$$\begin{pmatrix} S & \gamma^{-1}I \\ N^T & 0 \end{pmatrix} = X_{cl} \begin{pmatrix} I & \gamma R \\ 0 & \gamma M^T \end{pmatrix}. \tag{7.2}$$

By definition of R, S and by Theorem 4.3, this X_{cl} satisfies (4.3). Looking at the proof of Theorem 4.2, it follows that the Bounded Real Lemma inequality (4.5) or equivalently the LMI (4.6) hold for some controller data $\Theta = \begin{pmatrix} A_K & B_K \\ C_K & D_K \end{pmatrix}$. In fact, any solution Θ of (4.6) yields a γ-suboptimal controller of order k. Conversely, to any such controller we can associate a pair (R, S) satisfying (6.1)-(6.4) via the Bounded Real Lemma. Hence there is an exhaustive correspondence between the set of γ-suboptimal controller of order k and the convex set of pairs (R, S) satisfying (6.1)-(6.4). The case $k > n$ is analogous except for additional degrees of freedom in the reconstruction of X_{cl}.

From a numerical point of view, there are at least two ways of computing solutions Θ of (4.6) for a given X_{cl}. The first approach consists of linear algebra manipulations and leads to an explicit description of the solution set. Specifically, look at the proof of Lemma 3.1 and particularly at condition (A.6). Defining

$$\Lambda_{22} := -\Psi_{22} + \Psi_{12}^T \Psi_{11}^{-1} \Psi_{12} > 0; \qquad \Lambda_{33} := -\Psi_{33} + \Psi_{13}^T \Psi_{11}^{-1} \Psi_{13} > 0, \tag{7.3}$$

and using the usual Schur complement argument, Θ_{11} is thereby constrained by

$$-\Lambda_{22} + (\Theta_{11} + \Lambda_{32})^T \Lambda_{33}^{-1} (\Theta_{11} + \Lambda_{32}) < 0.$$

With the notation $\tilde{\Theta}_{11} := \Lambda_{33}^{-1/2} \Theta_{11} \Lambda_{22}^{-1/2}$ and $\tilde{\Lambda} := \Lambda_{33}^{-1/2} \Lambda_{32} \Lambda_{22}^{-1/2}$, this also reads

$$(\tilde{\Theta}_{11} + \tilde{\Lambda})^T (\tilde{\Theta}_{11} + \tilde{\Lambda}) < I$$

and thus $\tilde{\Theta}_{11}$ must be of the form $-\tilde{\Lambda} + U$ where U is any matrix of compatible dimensions such that $\sigma_{max}(U) < 1$.

Summing up, the solution set of (4.6) is obtained by selecting the Θ_{ij}'s in (A.2) as follows:

- $\Theta_{11} = -\Lambda_{32} + \Lambda_{33}^{1/2} U \Lambda_{22}^{1/2}$ with U arbitrary subject to $\sigma_{max}(U) < 1$;
- Θ_{12} and Θ_{21}: arbitrary;
- $\Theta_{22} = \Sigma + \Upsilon$ where $\Upsilon + \Upsilon^T = 0$ and

$$\Sigma < \frac{1}{2} \left\{ -\Psi_{44} + \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix}^T \Pi^{-1} \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix} \right\}. \tag{7.4}$$

Note that this approach involves a preliminary congruence transformation which might be ill-conditioned.

The second approach simply consists of solving the LMI (4.6) by standard convex optimization algorithms. Though more costly, it is potentially more stable numerically, and has also the advantage of allowing for extra constraints on the controller parameters. For instance, particular structures can be imposed to (A_K, B_K, C_K) in order to decouple certain input/output pairs.

8 Comparison with Classical Results

The classical H_∞ state-space formulas of [9, 3] are only applicable to plants which satisfy the following restrictive assumptions:

(A3) D_{12} has full column rank and D_{21} has full row rank,

(A4) $P_{12}(s)$ and $P_{21}(s)$ have no invariant zero on the imaginary axis.

Under these assumptions, [9, 3] provide a characterization of feasible γ's in terms of the stabilizing solutions of two H_∞ Riccati equations which parallel our Riccati inequalities. To better understand why these stabilizing ARE solutions play a special role, we find the following monotonicity result useful.

Lemma 8.1 *Consider a plant $P(s)$ satisfying (A1)-(A4) and suppose the ARI*

$$\hat{A}^T X + X\hat{A} - X\hat{B}_2\hat{B}_2^T X + \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix}^T \begin{pmatrix} I & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix} < 0 \tag{8.1}$$

has a solution $X_0 = X_0^T \in \mathbb{R}^{n\times n}$. Then:

(i) The Hamiltonian matrix

$$H_\gamma = \begin{pmatrix} \hat{A} & -\hat{B}_2\hat{B}_2^T \\ 0 & -\hat{A}^T \end{pmatrix} + \begin{pmatrix} 0 & \gamma^{-1}\hat{B}_1 \\ -\hat{C}_1^T & 0 \end{pmatrix} \begin{pmatrix} I & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1 & 0 \\ 0 & \gamma^{-1}\hat{B}_1^T \end{pmatrix} \tag{8.2}$$

has no eigenvalue on the imaginary axis.

(ii) If moreover $X_0 > 0$, the ARE

$$\hat{A}^T X + X\hat{A} - X\hat{B}_2\hat{B}_2^T X + \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix}^T \begin{pmatrix} I & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix} = 0 \tag{8.3}$$

has a stabilizing solution X_∞ satisfying

$$0 \le X_\infty < X_0. \tag{8.4}$$

Proof: See Appendix B.

■

This lemma shows that if the set of *positive definite* solutions of the ARI (8.1) is nonempty, then the corresponding ARE has a nonnegative stabilizing solution which is *minimal* in this set. This result together with Theorem 4.3 explains the special role played by the stabilizing solutions X_∞ and Y_∞ of the ARE's (8.3) and

$$\tilde{A}Y + Y\tilde{A}^T - Y\tilde{C}_2^T\tilde{C}_2 Y + \begin{pmatrix} \tilde{B}_1^T \\ \gamma^{-1}\tilde{C}_1 Y \end{pmatrix}^{-1} \begin{pmatrix} I & -\tilde{D}_{11}^T/\gamma \\ -\tilde{D}_{11}/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{B}_1^T \\ \gamma^{-1}\tilde{C}_1 Y \end{pmatrix} = 0. \tag{8.5}$$

Specifically, solvability of the γ-suboptimal H_∞ problem implies the existence of symmetric matrices R_0 and S_0 satisfying (4.11)-(4.13). Observing that $W_{12} = I$ and $W_{21} = I$ under the assumption **(A3)**, it follows that $X_0 := R_0^{-1}$ and $Y_0 := S_0^{-1}$ solve the ARI counterparts of (8.3) and (8.5) and further satisfy:

$$X_0 > 0; \qquad Y_0 > 0; \qquad \rho(X_0Y_0) < \gamma^2. \tag{8.6}$$

Invoking Lemma 8.1, the ARE's (8.3) and (8.5) must then have stabilizing solutions X_∞ and Y_∞ such that:

$$0 \le X_\infty < X_0; \qquad 0 \le Y_\infty < Y_0. \tag{8.7}$$

Remarking that (8.6)-(8.7) imply $\rho(X_\infty Y_\infty) < \gamma^2$, we exactly obtain the necessary conditions of [3].

Conversely, suppose the ARE's (8.3) and (8.5) have stabilizing solutions satisfying

$$X_\infty \geq 0; \qquad Y_\infty \geq 0; \qquad \rho(X_\infty Y_\infty) < \gamma^2.$$

By standard results on the continuity of ARE stabilizing solutions under perturbation [2], the ARE

$$\hat{A}^T X + X\hat{A} - X\hat{B}_2\hat{B}_2^T X + \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix}^T \begin{pmatrix} I & -\hat{D}_{11}/\gamma \\ -\hat{D}_{11}^T/\gamma & I \end{pmatrix}^{-1} \begin{pmatrix} \hat{C}_1 \\ \gamma^{-1}\hat{B}_1^T X \end{pmatrix} + \epsilon I = 0 \tag{8.8}$$

retains a stabilizing solution X_ϵ for $\epsilon > 0$ small enough. Remarking that solutions of (8.8) cannot be singular, we must have $X_\epsilon > 0$ by continuity. Hence X_ϵ is a solution of the ARI (8.1) such that $X_\infty < X_\epsilon$. Solutions of the Y-ARI are similarly constructed and by continuity, $\rho(X_\infty Y_\infty) < \gamma^2$ implies $\rho(X_\epsilon Y_\epsilon) < \gamma^2$ for ϵ small enough. A solution of the system (4.11)-(4.13) is then obtained as $(X_\epsilon^{-1}, Y_\epsilon^{-1})$ and from Theorem 4.3 we can conclude that the γ-suboptimal H_∞ problem is solvable.

For regular H_∞ problems, the solvability tests of [3] are computationally more efficient since solving ARE's is of lesser complexity than solving LMI's. For singular problems however, the convex LMI characterization offers a numerically sound alternative where DGKF's solution breaks down and discontinuities render solvability assessment delicate [8]. Because of the convexity, our solvability test is also competitive with that of [17]. Note that the solution of [17] can be seen as an extension of [3] to the singular case since it also emphasizes extremal points and stabilizing solutions, this time of nonstrict ARI's.

Finally, DGKF's approach revolves around the "central" solution and the Q-parametrization seems inadequate for improving on the central controller design. By contrast, our synthesis framework can take into account other desirable properties for the controller or the closed-loop system (see Section 9), and thus generate better-suited H_∞ controllers. It therefore constitutes a valuable alternative to the ARE-based approach of [3].

9 Applications

In the sequel $\mathcal{A}_\gamma$ denotes the set of pairs (R, S) satisfying (6.1)-(6.3).

9.1 Preventing Pole/Zero Cancellations between the Plant and the Controller

The central controller has the undesirable property of cancelling all stable poles of the plant which are $(\tilde{A}, \tilde{B}_1)$-uncontrollable or $(\hat{C}_1, \hat{A})$-unobservable. That is, all stable invariant zeros of $P_{12}(s)$ and $P_{21}(s)$. Such exact cancellations are frequently encountered in mixed-sensitivity problems and lead to unacceptable designs in the presence of flexible modes [16]. Various remedies have been proposed which generally consist of modifying the criterion to penalize cancellations [19, 16]. Yet, no general remedy is available outside of the loop shaping context.

By contrast, the parametrization introduced above offers direct and numerically tractable means of preventing cancellations of poorly damped modes. Indeed, if all controllers obtained from a given $(R, S) \in \mathcal{A}_\gamma$ involve cancellations, then (R, S) must satisfy one the following:

$$\lambda_{\max}(R) \gg 1; \;\; \lambda_{\max}(S) \gg 1; \;\; |\lambda_{\max}(\mathcal{R}_R)| \ll 1; \;\; |\lambda_{\max}(\mathcal{R}_S)| \ll 1. \tag{9.1}$$

Here $\mathcal{R}_R$ and $\mathcal{R}_S$ denote the Riccati residuals, that is, the left-hand sides of (4.11)-(4.12). "Bad" pairs (R, S) therefore lie near the part of the boundary of $\mathcal{A}_\gamma$ associated with the Riccati inequalities.

A qualitative justification of this claim can be found in [7]. Here the essential fact is that (R, S) pairs come from solutions of the Bounded Real Lemma inequality for the closed-loop system, whence their connection with closed-loop properties.

Steering clear of such pairs (R, S) can be done in a number of ways. For instance, we can seek "good" pairs (R, S) by solving

$$\min_{(R,S)\in\mathcal{A}_\gamma} Trace(R+S)$$

while placing a steep barrier on the Riccati inequality constraints. This will drive $\lambda_{\max}(\mathcal{R}_R)$ and $\lambda_{\max}(\mathcal{R}_S)$ away from zero and the criterion will ensure that the norms of R and S remain small. Another possible approach consists of finding the analytic center [1] of the intersection

$$\mathcal{A}_\gamma \cap \{\, (R,S) \; : \; Trace(R+S) \leq constant \,\}.$$

Here again steep barriers should be used for the Riccati inequality constraints. Note that in both cases the resulting problem is convex and can be handled by standard convex optimization algorithms.

9.2 Reduced-Order Design

Reduced-order H_∞ synthesis is a promising application of the parametrization introduced above. Indeed, γ-suboptimal controllers of order $k < n$ have a simple characterization in this framework: they correspond to pairs (R, S) of $\mathcal{A}_\gamma$ for which $rank(\gamma^{-2}I - RS) = k$. Such pairs lie on the part of the boundary of $\mathcal{A}_\gamma$ attached to the constraint $\lambda_{\min}(RS) \geq \gamma^{-2}$ and they saturate this constraint in $n-k$ directions. Hence the reduced-order design problem has a clear formulation in terms of the parameters (R, S): it consists of decreasing the rank of $\gamma^{-2}I - RS$ as much as possible without leaving $\mathcal{A}_\gamma$.

For feasible γ's and with $\lambda_1(RS) \leq \cdots \leq \lambda_{n-k}(RS)$ denoting the $n-k$ smallest eigenvalues of RS, the synthesis of controllers of order $k < n$ amounts to minimizing for $(R, S) \in \mathcal{A}_\gamma$ the criterion:

$$\Psi(R,S) = \sum_{i=1}^{n-k} \lambda_i(RS).$$

There will exist suboptimal controllers of order k if and only if the global minimum of $\Psi(R, S)$ is $(n-k)\gamma^{-2}$. This objective function Ψ is not convex but in fact concave. Hence global convergence is not guaranteed. Nevertheless, the structural properties of the problem should help monitor gradient descent methods so as to obtain significant order reductions upon convergence.

9.3 Reliable Computation of Optimal Central Controllers

When the H_∞ optimal gain γ_{opt} is characterized by

$$\rho(X_\infty Y_\infty) = \gamma_{opt}^2,$$

the computation of the central controller K_c is ill-conditioned near and at the optimum. This is due to the cancellation(s) at infinity which occur in the pole/zero structure of K_c. Such cancellations induce a feedthrough term in K_c as well as some order reduction at γ_{opt}.

These numerical difficulties can be eliminated altogether by allowing for a feedthrough term in suboptimal K_c's as well. This is easily done by extending the notion of central controller on the basis of the parametrization derived above. Finite instead of infinite pole/zero cancellations can then be obtained at γ_{opt} by appropriate choice of the feedthrough matrix. Only numerically stable computations are involved in the process. More details on this approach can be found in [5].

10 Conclusions

We have presented a complete solution of the most general continuous- and discrete-time (suboptimal) H_∞ problems. Our feasibility conditions parallel the usual ones except that Riccati inequalities replace Riccati equations. This inequality formulation also provides a complete parametrization of all H_∞-suboptimal controllers. Here the free parameters are the pairs (R, S) of positive definite matrices solving the Riccati inequalities and satisfying some coupling constraint. Both the computation of adequate parameters and the controller reconstruction lead to convex optimization problems. Because of the connection between (R, S), the controller order, and the closed-loop properties, this approach holds promises for the improvement of current H_∞ design techniques.

Appendix A

Proof of Lemma 3.1: Let U_{PQ} be a basis of $\operatorname{Ker} P \cap \operatorname{Ker} Q$ and introduce matrices U_P, U_Q such that $W_P := [U_{PQ}, U_P]$ and $W_Q := [U_{PQ}, U_Q]$ are bases of $\operatorname{Ker} P$ and $\operatorname{Ker} Q$, respectively. Observing that $[U_{PQ}, U_P, U_Q]$ is then a basis of $\operatorname{Ker} P \oplus \operatorname{Ker} Q$, complete it into a basis $T = [U_{PQ}, U_P, U_Q, V]$ of $\mathbb{R}^m$. The matrix T is nonsingular and therefore (3.1) is equivalent to:

$$T^T \Psi\, T + (PT)^T \Theta^T\, (QT) + (QT)^T \Theta\, (PT) < 0. \tag{A.1}$$

Block-partition PT, QT and $T^T \Psi T$ conformably to the partition $[U_{PQ}, U_P, U_Q, V]$ of T. By construction, we have

$$PT = (0, 0, P_1, P_2); \qquad QT = (0, Q_1, 0, Q_2)$$

and $[P_1, P_2]$ and $[Q_1, Q_2]$ have full column rank. With the notation

$$\begin{pmatrix} P_1^T \\ P_2^T \end{pmatrix} \Theta\, (Q_1, Q_2) = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \tag{A.2}$$

and

$$T^T \Psi\, T = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & \Psi_{34} \\ \Psi_{14}^T & \Psi_{24}^T & \Psi_{34}^T & \Psi_{44} \end{pmatrix},$$

(A.1) reads:

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} + \Theta_{11}^T & \Psi_{24} + \Theta_{21}^T \\ \Psi_{13}^T & \Psi_{23}^T + \Theta_{11} & \Psi_{33} & \Psi_{34} + \Theta_{12} \\ \Psi_{14}^T & \Psi_{24}^T + \Theta_{21} & \Psi_{34}^T + \Theta_{12}^T & \Psi_{44} + \Theta_{22} + \Theta_{22}^T \end{pmatrix} < 0. \tag{A.3}$$

Here the Θ_{ij}'s are arbitrary since Θ is arbitrary and $[P_1, P_2]$ and $[Q_1, Q_2]$ have full column rank. Hence our problem reduces to finding conditions on the Ψ_{ij}'s which ensure feasibility of (A.3) for some Θ_{ij}'s.

By a Schur complement argument, (A.3) is equivalent to

$$\Pi = \begin{pmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} + \Theta_{11}^T \\ \Psi_{13}^T & \Psi_{23}^T + \Theta_{11} & \Psi_{33} \end{pmatrix} < 0 \tag{A.4}$$

$$\Psi_{44} + \Theta_{22} + \Theta_{22}^T - \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix}^T \Pi^{-1} \begin{pmatrix} \Psi_{14} \\ \Psi_{24} + \Theta_{21}^T \\ \Psi_{34} + \Theta_{12} \end{pmatrix} < 0. \tag{A.5}$$

Given Θ_{11}, Θ_{12}, and Θ_{21}, we can always find Θ_{22} such that (A.5) is satisfied. Hence (3.1) is feasible if and only if (A.4) is feasible for some Θ_{11}.

Now, (A.4) is equivalent to

$$\begin{pmatrix} I & 0 & 0 \\ -\Psi_{12}^T\Psi_{11}^{-1} & I & 0 \\ -\Psi_{13}^T\Psi_{11}^{-1} & 0 & I \end{pmatrix} \Pi \begin{pmatrix} I & -\Psi_{11}^{-1}\Psi_{12} & -\Psi_{11}^{-1}\Psi_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} < 0.$$

That is,

$$\begin{pmatrix} \Psi_{11} & 0 & 0 \\ 0 & \Psi_{22} - \Psi_{12}^T\Psi_{11}^{-1}\Psi_{12} & \Theta_{11}^T + \Lambda_{32}^T \\ 0 & \Theta_{11} + \Lambda_{32} & \Psi_{33} - \Psi_{13}^T\Psi_{11}^{-1}\Psi_{13} \end{pmatrix} < 0 \tag{A.6}$$

where

$$\Lambda_{32} := \Psi_{23}^T - \Psi_{13}^T\Psi_{11}^{-1}\Psi_{12}. \tag{A.7}$$

Since Θ_{11} is arbitrary, this is feasible if and only if

$$\begin{cases} \Psi_{11} & < 0 \\ \Psi_{22} - \Psi_{12}^T\Psi_{11}^{-1}\Psi_{12} & < 0 \\ \Psi_{33} - \Psi_{13}^T\Psi_{11}^{-1}\Psi_{13} & < 0 \end{cases}$$

or equivalently, if and only if

$$\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22} \end{pmatrix} < 0\,; \qquad \begin{pmatrix} \Psi_{11} & \Psi_{13} \\ \Psi_{13}^T & \Psi_{33} \end{pmatrix} < 0.$$

This last condition is exactly (3.2) upon recalling the definition of W_P, W_Q, and the Ψ_{ij}'s. ∎

Appendix B

Proof of Lemma 8.1:

(i): For simplicity, assume $D_{11} = 0$ and drop the hats on $A, B_1, B_2, \ldots$ The ARI (8.1) then reads

$$A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 < 0 \tag{B.1}$$

and

$$H_\gamma = \begin{pmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{pmatrix}.$$

The proof is by contradiction. Suppose $H_\gamma \begin{pmatrix} u \\ v \end{pmatrix} = j\omega \begin{pmatrix} u \\ v \end{pmatrix}$ with $(u, v) \neq (0,0)$. That is,

$$
\begin{aligned}
Au + Fv &= j\omega\, u && \text{(B.2)}\\
-C_1^T C_1 u - A^T v &= j\omega\, v && \text{(B.3)}
\end{aligned}
$$

where $F := \gamma^{-2} B_1 B_1^T - B_2 B_2^T$. Observing that solutions of (B.1) cannot be singular and defining $R_0 := X_0^{-1}$, (B.1) is equivalent to

$$\mathcal{R} := AR_0 + R_0 A^T + R_0 C_1^T C_1 R_0 + F < 0. \qquad \text{(B.4)}$$

From (B.2), we get $v^H F v = j\omega\, v^H u - v^H A u$ and from (B.3): $A^T v = -C_1^T C_1 u - j\omega\, v$. Consequently,

$$
\begin{aligned}
v^H \mathcal{R} v &= (v^H A) R_0 v + v^H R_0 (A^T v) + v^H R_0\, C_1^T C_1\, R_0 v + v^H F v \\
&= \{-u^H C_1^T C_1 + j\omega\, v^H\}\, R_0 v + v^H R_0 \{-C_1^T C_1 u - j\omega\, v\} + \\
&\qquad\qquad v^H R_0\, C_1^T C_1\, R_0 v + \{j\omega\, v^H u - (v^H A) u\} \\
&= -u^H C_1^T C_1 R_0 v - v^H R_0 C_1^T C_1 u + v^H R_0 C_1^T C_1 R_0 v + j\omega\, v^H u + \{u^H C_1^T C_1 - j\omega\, v^H\}\, u \\
&= (R_0 v - u)^H\, C_1^T C_1\, (R_0 v - u)\ \geq 0
\end{aligned}
$$

which contradicts $\mathcal{R} < 0$.

(ii): Since H_γ has no $j\omega$-axis eigenvalue, its stable invariant subspace $\begin{pmatrix} P \\ Q \end{pmatrix}$ is of dimension n. Assume that (C_1, A) has no stable unobservable mode. By standard results on H_∞ Riccati equations (see, e.g., [6]), Q is then invertible and $R_\infty := PQ^{-1}$ is an antistabilizing solution of:

$$AR + RA^T + RC_1^T C_1 R + F = 0. \qquad \text{(B.5)}$$

Subtracting (B.5) from (B.4), we obtain

$$A(R_0 - R_\infty) + (R_0 - R_\infty)A^T + R_0 C_1^T C_1 R_0 - R_\infty C_1^T C_1 R_\infty < 0$$

or equivalently,

$$(A + R_\infty C_1^T C_1)(R_0 - R_\infty) + (R_0 - R_\infty)(A + R_\infty C_1^T C_1)^T + (R_0 - R_\infty) C_1^T C_1 (R_0 - R_\infty) < 0.$$

From $A + R_\infty C_1^T C_1$ antistable and Lyapunov's theorem, we conclude that

$$R_0 < R_\infty \qquad \text{(B.6)}$$

which together with our assumption $X_0 > 0$ ensures that $R_\infty > 0$. Consequently, P is invertible and $X_\infty := QP^{-1} = R_\infty^{-1}$ is a stabilizing solution of the ARE (8.3). In addition, (B.6) and $R_0 > 0$ also give $0 < R_\infty^{-1} < R_0^{-1}$, that is,

$$0 < X_\infty < X_0. \qquad \text{(B.7)}$$

The proof is complete upon removing the assumption on (C_1, A). Continuity under small perturbations of the data can be used to this end. Specifically, we can always perturb (C_1, A) to $(C_1^{(\epsilon)}, A^{(\epsilon)})$ in such a way that the "no stable unobservable mode" assumption holds for $\epsilon > 0$ small enough and that X_0 remains a solution of the perturbed ARI. Now, the stable invariant subspace $\begin{pmatrix} P_\epsilon \\ Q_\epsilon \end{pmatrix}$ of $H_\gamma^{(\epsilon)}$ depends continuously on ϵ and from the discussion above we have

$$0 < X_\infty^{(\epsilon)} = Q_\epsilon P_\epsilon^{-1} < X_0$$

for $\epsilon > 0$. Consequently, as $\epsilon \to 0$, $X_\infty^{(\epsilon)}$ has a finite limit X_∞ which is clearly a stabilizing solution of (8.3) and satisfies

$$0 \leq X_\infty \leq X_0.$$

Finally, the second inequality is easily strengthened upon replacing X_0 by $(1-\epsilon)X_0$ in the previous argument.

■

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