

# Exact boundary controllability of a nonlinear KdV equation with critical lengths

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***Exact boundary controllability of a nonlinear KdV  
equation with critical lengths.***

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## **Exact boundary controllability of a nonlinear KdV equation with critical lengths.**

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Thème 4 — Simulation et optimisation  
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**Abstract:** We study the boundary controllability of a nonlinear Korteweg-de Vries equation with the Dirichlet boundary condition on an interval with a critical length for which it has been shown by Rosier that the linearized control system around the origin is not controllable. We prove that the nonlinear term gives the local controllability around the origin

**Key-words:** Exact boundary controllability, Korteweg-de Vries, nonlinear equation, missed directions

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## **Contrôlabilité frontière exacte de l'équation de KdV non linéaire pour des longueurs critiques.**

**Résumé :** Cet article étudie la contrôlabilité exacte de l'équation de Korteweg-de Vries non linéaire ; les conditions au bord sont des conditions de Dirichlet prises sur un intervalle appartenant à un ensemble de longueurs critiques pour lesquelles Rosier a montré que le système de contrôle linéarisé autour de zéro n'était pas contrôlable. On montre que le terme non linéaire donne le résultat de contrôlabilité localement autour de zéro.

**Mots-clés :** Contrôlabilité exacte, Korteweg-de Vries, équation non linéaire, directions manquantes

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# 1 Introduction

Let us consider the following Korteweg-de Vries control system

$$(\text{KdV}) \begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t, 0) = y(t, L) = 0. \end{cases}$$

For this control system,  $L > 0$  is given, the state is  $y(\cdot) : [0, L] \rightarrow \mathbb{R}$ , and for the control one can take, for example,  $u(\cdot) = y_x(\cdot, L) \in \mathbb{R}$ . The Korteweg-de Vries equation serves to model various physical phenomena (see e.g. [19]), for example the propagation of small amplitude long water waves in a uniform channel. Let us recall that Bona and Winther have pointed out in [3] that the term  $y_x$  in (KdV) has to be added to model the water waves when  $x$  denotes the spatial coordinate in a *fixed* frame. We are interested in the local controllability of (KdV) around 0. Rosier has proved in [12] that the control system (KdV) is locally controllable around 0 provided that the length of the spatial domain is not critical.

**Theorem 1** ([12, Theorem 1.3].) *Let  $T > 0$ , and let us assume that*

$$L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{j^2 + l^2 + jl}{3}}; j, l \in \mathbb{N}^* \right\}. \quad (1.1)$$

*Then there exists  $r_0 > 0$  such that, for every  $(y_0, y_T) \in L^2(0, L)^2$  with  $\|y_0\|_{L^2(0, L)} < r_0$  and  $\|y_T\|_{L^2(0, L)} < r_0$ , there exists*

$$y \in C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$$

*satisfying (KdV) such that  $y(0, \cdot) = y_0$  and  $y(T, \cdot) = y_T$ .*

The aim of this paper is to study the local exact controllability around 0 of the nonlinear KdV equation when  $L = 2k\pi \in \mathcal{N}$  (take  $j = l = k$  in (1.1)). Our main theorem is the following one.

**Theorem 2** *Let  $k$  be a positive integer and let  $T > 0$ . There exists  $r_1 > 0$  such that, for every  $(y_0, y_T) \in L^2(0, 2k\pi)^2$  with  $\|y_0\|_{L^2(0, 2k\pi)} < r_1$  and  $\|y_T\|_{L^2(0, 2k\pi)} < r_1$ , there exists*

$$y \in C([0, T], L^2(0, 2k\pi)) \cap L^2(0, T, H^1(0, 2k\pi))$$

*satisfying (KdV) with  $L = 2k\pi$  such that  $y(0, \cdot) = y_0$  and  $y(T, \cdot) = y_T$ .*

When  $L = 2k\pi$  the linearized control system of (KdV) around 0 is

$$(\text{KdVL}) \begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, 2k\pi) = 0. \end{cases}$$

It has been shown by Rosier in [12] that this linear control system is not controllable. To prove that the nonlinear term  $yy_x$  gives the local controllability, a first approach could be to use the exact controllability of the nonlinear equation around nontrivial stationary solutions proved in [8] and to apply the method introduced in [6] (that is, use the return method [4, 5] together with quasi-static deformations; see also [7] for this last point). But, with this method, it seems that one can only obtain the local exact controllability for *large* time. To prove Theorem 2 we use a different strategy that we briefly describe now. We first point out that in this theorem we may assume that  $y_0 = 0$ : this follows easily from the invariance of the control system (KdV) by the change of variables  $\tau = T - t$ ,  $\xi = 2k\pi - x$ . Then we use the following result, due to Rosier, for the linearized control system (KdVL).

**Theorem 3** ([12, Remark 3.6].) *Let  $T > 0$ ,  $H = \left\{ y \in L^2(0, 2k\pi); \int_0^{2k\pi} (1 - \cos(x))y dx = 0 \right\}$ . For every  $(y_0, y_T) \in H \times H$ , there exists  $y \in C([0, T], L^2(0, 2k\pi)) \cap L^2(0, T, H^1(0, 2k\pi))$  satisfying (KdVL) such that  $y(0, \cdot) = y_0$  and  $y(T, \cdot) = y_T$ .*

Then, as we shall prove in section 2, the nonlinear term  $yy_x$  allows us to “go” in the two directions  $\pm(1 - \cos(x))$  which are missed by the linearized control system (KdVL). Finally in section 3 we derive Theorem 2 from section 2 by means of a fixed point theorem.

**Remark 4** *For the other critical lengths, we believe that the same result holds. Note that the situation is more complicated in these other cases: there are now four uncontrollable directions for the linearized control system around 0 (see [12, Proof of Lemma 3.5 and Remark 3.6]).*

**Remark 5** *The method we use here (try to move in the directions which are missed by the linearized control system) is classical to study the local controllability of control system in finite dimension. Here we fix the time and perform a power series expansion, with the same scaling on the state and on the control. In finite dimension, much more subtle tools have been introduced: for example different scalings on the components of the control and of the state as well as scaling on time. See e.g. [1, 2, 9, 10, 17, 18, 20] and the references therein.*

**Remark 6** *One can find other results on the controllability of KdV control systems in [11, 13, 14, 15, 16] and the references therein.*

## 2 Motion in the $\pm(1 - \cos(x))$ directions.

Let  $L > 0$ . We first recall some properties proved by Rosier in [12] for the following linear KdV Cauchy problem

$$y_t + y_x + y_{xxx} = f, \quad (2.1)$$

$$y(t, 0) = y(t, L) = 0, \quad (2.2)$$

$$y_x(t, L) = h(t), \quad (2.3)$$

$$y(T_0, x) = y_0(x). \quad (2.4)$$

We adopt the notations of [12]. Let  $A$  denote the operator  $Aw = -w''' - w'$  defined on  $\mathcal{D}(A) := \{w \in H^3(0, L); w(0) = w(L) = w_x(L) = 0\}$  and let  $(S(t)_{t \geq 0})$  denote the semi-group of contractions associated with  $A$  (see [12, Proposition 3.1]). For  $T_0 < T_1$ , let

$$B_{T_0, T_1} := C([T_0, T_1], L^2(0, L)) \cap L^2(T_0, T_1, H^1(0, L))$$

endowed with the norm

$$\|y\|_{B_{T_0, T_1}} = \text{Max} \{ \|y(t, \cdot)\|_{L^2(0, L)}; t \in [T_0, T_1] \} + \left( \int_{T_0}^{T_1} \|y(t, \cdot)\|_{H^1(0, L)}^2 dt \right)^{\frac{1}{2}}.$$

Rosier has proved the following proposition.

**Proposition 7** ([12, (Proofs of) Propositions 3.2 and 3.7].) *Let  $T_0 < T_1$ . There exist unique continuous linear maps  $\Psi_{T_0, T_1}$  and  $\delta_{T_0, T_1}$*

$$\begin{aligned} \Psi_{T_0, T_1} : L^2(0, L) \times L^2(T_0, T_1) \times L^1(T_0, T_1, L^2(0, L)) &\longrightarrow B_{T_0, T_1} \\ (y_0, h, f) &\longmapsto \Psi_{T_0, T_1}(y_0, h, f), \end{aligned}$$



$$\begin{aligned} \delta_{T_0, T_1} : L^2(0, L) \times L^2(T_0, T_1) \times L^1(T_0, T_1, L^2(0, L)) &\longrightarrow L^2(T_0, T_1) \\ (y_0, h, f) &\longmapsto \delta_{T_0, T_1}(y_0, h, f), \end{aligned}$$

such that, for  $y_0 \in \mathcal{D}(A)$ ,  $h \in C^2([T_0, T_1])$  with  $h(T_0) = 0$  and  $f \in C^1([T_0, T_1], L^2(0, L))$ ,

$$\begin{aligned} \Psi_{T_0, T_1}(y_0, h, f) \text{ is the unique classical solution of (2.1)-(2.2)-(2.3)-(2.4),} \\ \delta_{T_0, T_1}(y_0, h, f)(t) = (\Psi_{T_0, T_1}(y_0, h, f))_x(t, 0). \end{aligned}$$

The function  $\Psi_{T_0, T_1}(y_0, h, f)$  is called the mild solution of (2.1) to (2.4). For simplicity, we write  $B$  for  $B_{T_0, T_1}$  and  $\Psi$  for  $\Psi_{T_0, T_1}$  when  $(T_0, T_1) = (0, T)$ . Note that the existence of the continuous linear map  $\delta_{T_0, T_1}$  shows that, with  $y := \Psi_{T_0, T_1}(y_0, h, f)$ , “ $y_x(t, 0)$ ” makes sense in  $L^2(T_0, T_1)$ . For simplicity we shall write  $y_x(t, 0)$  instead of  $\delta_{T_0, T_1}(y_0, h, f)(t)$ . Let  $f \in L^2(T_0, T_1, L^2(0, L))$ . We say that  $y : [T_0, T_1] \times [0, L] \rightarrow \mathbb{R}$  is a mild solution of

$$\begin{aligned} y_t + y_x + y_{xxx} &= f, \\ y(t, 0) = y(t, L) &= 0, \end{aligned}$$

if there exists  $h \in L^2(T_0, T_1)$  such that  $y$  is the mild solution of (2.1) to (2.4) with  $y_0(x) := y(T_0, x)$ . Note that it follows from the proof of Theorem 3 given in [12] that this theorem holds for mild solutions of (KdVL).

Until the end of this section we assume that

$$L \in \{2k\pi; k \in \mathbb{N}^*\}. \quad (2.5)$$

The aim of this section is to prove the following result.

**Proposition 8** *Let  $T > 0$ . There exists  $(u_+, v_+, w_+)$  in  $L^2(0, T)^3$  and  $(u_-, v_-, w_-)$  in  $L^2(0, T)^3$  such that, if  $\alpha_\pm, \beta_\pm, \gamma_\pm$  are the mild solutions of*

$$\alpha_{\pm t} + \alpha_{\pm x} + \alpha_{\pm xxx} = 0, \quad (2.6)$$

$$\alpha_\pm(t, 0) = \alpha_\pm(t, L) = 0, \quad (2.7)$$

$$\alpha_{\pm x}(t, L) = u_\pm(t), \quad (2.8)$$

$$\alpha_\pm(0, x) = 0, \quad (2.9)$$

$$(2.10)$$

of

$$\beta_{\pm t} + \beta_{\pm x} + \beta_{\pm xxx} = -\alpha_\pm \alpha_{\pm x}, \quad (2.11)$$

$$\beta_\pm(t, 0) = \beta_\pm(t, L) = 0, \quad (2.12)$$

$$\beta_{\pm x}(t, L) = v_\pm(t), \quad (2.13)$$

$$\beta_\pm(0, x) = 0, \quad (2.14)$$

and of

$$\gamma_{\pm t} + \gamma_{\pm x} + \gamma_{\pm xxx} = -(\alpha_\pm \beta_\pm)_x, \quad (2.15)$$

$$\gamma_\pm(t, 0) = \gamma_\pm(t, L) = 0, \quad (2.16)$$

$$\gamma_{\pm x}(t, L) = w_\pm(t), \quad (2.17)$$

$$\gamma_\pm(0, x) = 0, \quad (2.18)$$

then

$$\alpha_\pm(T, x) = 0, \beta_\pm(T, x) = 0 \text{ and } \gamma_\pm(T, x) = \pm(1 - \cos(x)). \quad (2.19)$$

**Remark 9** It would have been quite natural to look for the existence of  $(u_+, v_+)$  in  $L^2(0, T)^2$  and of  $(u_-, v_-)$  in  $L^2(0, T)^2$  such that, if  $\alpha_\pm, \beta_\pm$  are the mild solutions of (2.6) to (2.14), then

$$\alpha_\pm(T, x) = 0 \text{ and } \beta_\pm(T, x) = \pm(1 - \cos(x)).$$

The existence of such  $(u_\pm, v_\pm)$  would have also implied Theorem 2. Unfortunately, as it is proved in Corollary 19 below, such  $(u_\pm, v_\pm)$  do not exist. Roughly speaking, an expansion to the 2<sup>nd</sup> order is not sufficient. We must go to the 3<sup>rd</sup> order to get local controllability.

In order to prove Proposition 8, let us first remark that  $(u_+, v_+, w_+)$  satisfies the required properties if and only if  $(u_-, v_-, w_-) := (-u_+, v_+, -w_+)$  satisfies the required properties. Moreover, in order to prove the existence of  $(u_+, v_+, w_+)$  it suffices to prove the existence of  $(u_+, v_+, \tilde{w}_+)$  in  $L^2(0, T)^3$  such that, if  $\alpha_+, \beta_+$  and  $\tilde{\gamma}_+$  are the mild solution of (2.6) to (2.18) with  $\gamma_+ := \tilde{\gamma}_+$  and  $w_+ := \tilde{w}_+$ , then

$$\alpha_+(T, \cdot) = 0, \beta_+(T, \cdot) = 0 \text{ and } \int_0^L \tilde{\gamma}_+(T, x)(1 - \cos(x))dx = \|1 - \cos(x)\|_{L^2(0, L)}^2.$$

Indeed, by Theorem 3 (for mild solutions), there exists  $w_+^*$  in  $L^2(0, T)$  such that the mild solution  $\gamma_+^*$  of

$$\begin{aligned} \gamma_{+t}^* + \gamma_{+x}^* + \gamma_{+xxx}^* &= 0, \\ \gamma_+^*(t, 0) = \gamma_+^*(t, L) &= 0, \\ \gamma_{+x}^*(t, L) &= w_+^*(t), \\ \gamma_+^*(0, x) &= 0, \end{aligned}$$

satisfies

$$\gamma_+^*(T, \cdot) = -\mathcal{P}_H(\tilde{\gamma}_+(T, \cdot)),$$

where  $\mathcal{P}_H$  denotes the orthogonal projection on  $H$  for the  $L^2$ -scalar product. Then  $u_+, v_+$  and  $w_+ := \tilde{w}_+ + w_+^*$  satisfy the properties required by Proposition 8 (with  $\gamma_+ := \tilde{\gamma}_+ + \gamma_+^*$ ). Similarly, in order to prove the existence of  $(u_-, v_-, w_-)$  it suffices to prove the existence of  $(u_-, v_-, \tilde{w}_-)$  in  $L^2(0, T)^3$  such that, if  $\alpha_-, \beta_-$  and  $\tilde{\gamma}_-$  are the mild solution of (2.6) to (2.18) with  $\gamma_- := \tilde{\gamma}_-$  and  $w_- := \tilde{w}_-$ , then

$$\alpha_-(T, \cdot) = 0, \beta_-(T, \cdot) = 0 \text{ and } \int_0^L \gamma_-(T, x)(1 - \cos(x))dx = -\|1 - \cos(x)\|_{L^2(0, L)}^2.$$

From (2.15), (2.16) and (2.18), one gets, using integration by parts (which can be easily justified by density arguments),

$$\begin{aligned} \int_0^L \gamma_\pm(T, x)(1 - \cos(x))dx &= \int_0^T \int_0^L \gamma_{\pm t}(t, x)(1 - \cos(x))dxdt \\ &= \int_0^T \int_0^L (-\gamma_{\pm x} - \gamma_{\pm xxx} - (\alpha_\pm \beta_\pm)_x)(1 - \cos(x))dxdt \\ &= \int_0^T \int_0^L \alpha_\pm \beta_\pm \sin(x)dxdt. \end{aligned}$$

Hence Proposition 8 is a consequence of the following proposition.

**Proposition 10** *Let  $T > 0$ . There exists  $(u, v)$  in  $L^2(0, T)^2$  such that, if  $\alpha, \beta$  are the mild solutions of*

$$\alpha_t + \alpha_x + \alpha_{xxx} = 0, \quad (2.20)$$

$$\alpha(t, 0) = \alpha(t, L) = 0, \quad (2.21)$$

$$\alpha_x(t, L) = u(t), \quad (2.22)$$

$$\alpha(0, x) = 0, \quad (2.23)$$

$$(2.24)$$

and of

$$\beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \quad (2.25)$$

$$\beta(t, 0) = \beta(t, L) = 0, \quad (2.26)$$

$$\beta_x(t, L) = v(t), \quad (2.27)$$

$$\beta(0, x) = 0, \quad (2.28)$$

then

$$\alpha(T, \cdot) = 0, \beta(T, \cdot) = 0, \quad (2.29)$$

$$\int_0^T \int_0^L \alpha\beta \sin(x) dx dt \neq 0. \quad (2.30)$$

Let  $T > 0$ . Let  $\alpha_1 \in B$  be a mild solution of

$$\alpha_{1t} + \alpha_{1x} + \alpha_{1xxx} = 0, \quad (2.31)$$

$$\alpha_1(t, 0) = \alpha_1(t, L) = 0, \quad (2.32)$$

such that

$$\int_0^L \alpha_1(0, x)(1 - \cos(x)) dx = 0. \quad (2.33)$$

Let us multiply (2.31) by  $(1 - \cos(x))$  and integrate the resulting equality on  $[0, T] \times [0, L]$ . Then, using integrations by parts together with (2.32) and (2.33), one gets

$$\int_0^L \alpha_1(T, x)(1 - \cos(x)) dx = 0. \quad (2.34)$$

By Theorem 3, (2.33) and (2.34),  $\alpha_1$  can be extended to  $[-T, 2T] \times [0, L]$  in such a way that this extension is still a mild solution of (2.31)-(2.32) and satisfies

$$\alpha_1(-T, x) = \alpha_1(2T, x) = 0.$$

Let  $\beta_1 : [0, T] \times [0, L] \rightarrow \mathbb{R}$  be a mild solution of

$$\beta_{1t}^* + \beta_{1x}^* + \beta_{1xxx}^* = -\alpha_1\alpha_{1x}, \quad (2.35)$$

$$\beta_1^*(t, 0) = \beta_1^*(t, L) = 0. \quad (2.36)$$

By Theorem 3, there exists  $\theta_1 \in \mathbb{R}$  and a mild solution  $\beta_1 \in B_{-T, 2T}$  of

$$\beta_{1t} + \beta_{1x} + \beta_{1xxx} = -\alpha_1 \alpha_{1x}, \quad (2.37)$$

$$\beta_1(t, 0) = \beta_1(t, L) = 0, \quad (2.38)$$

such that

$$\beta_1(t, x) = \beta_1^*(t, x) + \theta_1(1 - \cos(x)), \quad \forall t \in [0, T], \quad (2.39)$$

$$\beta_1(-T, x) = 0, \mathcal{P}_H(\beta_1(2T, \cdot)) = 0. \quad (2.40)$$

By Corollary 19 below, it follows that

$$\beta_1(2T, x) = 0.$$

Let now  $\alpha_2 : [0, T] \times [0, L] \rightarrow \mathbb{R}$  be a mild solution of

$$\alpha_{2t} + \alpha_{2x} + \alpha_{2xxx} = 0, \quad (2.41)$$

$$\alpha_2(t, 0) = \alpha_2(t, L) = 0, \quad (2.42)$$

such that

$$\alpha_2(0, x) = \alpha_2(T, x) = 0. \quad (2.43)$$

By Theorem 3, there exists a mild solution  $\beta_2 : [0, T] \times [0, L] \rightarrow \mathbb{R}$  of

$$\beta_{2t} + \beta_{2x} + \beta_{2xxx} = -\alpha_2 \alpha_{2x},$$

$$\beta_2(t, 0) = \beta_2(t, L) = 0,$$

such that

$$\beta_2(0, x) = 0, \mathcal{P}_H(\beta_2(T, \cdot)) = 0.$$

By Corollary 19 again,

$$\beta_2(T, x) = 0.$$

Similarly, by Theorem 3 and Corollary 19, there exists a mild solution  $\beta_3 : [0, T] \times [0, L] \rightarrow \mathbb{R}$  of

$$\beta_{3t} + \beta_{3x} + \beta_{3xxx} = -(\alpha_1 \alpha_2)_x, \quad (2.44)$$

$$\beta_3(t, 0) = \beta_3(t, L) = 0, \quad (2.45)$$

such that

$$\beta_3(0, x) = 0, \beta_3(T, x) = 0. \quad (2.46)$$

We extend  $\alpha_2$ ,  $\beta_2$  and  $\beta_3$  to  $[-T, 2T] \times [0, L]$  by requiring

$$\alpha_2(t, x) = \beta_2(t, x) = \beta_3(t, x) = 0, \quad \forall t \in [-T, 0] \cup [T, 2T].$$

Let us consider, for  $(\rho_1, \rho_2) \in \mathbb{R}^2$ ,  $\alpha := \rho_1 \alpha_1 + \rho_2 \alpha_2$  and  $\beta := \rho_1^2 \beta_1 + \rho_2^2 \beta_2 + \rho_1 \rho_2 \beta_3$ . Let  $u(t) := \alpha_x(t, L)$ ,  $v(t) := \beta_x(t, L)$  for  $t \in [-T, 2T]$ . Then (2.20) to (2.22) and (2.25) to (2.27) hold in the mild sense and  $\alpha(-T, x) =$

$\beta(-T, x) = \alpha(2T, x) = \beta(2T, x) = 0$ . We shall look at the contrapositive and assume that Proposition 10 is false. Then, for every  $(\rho_1, \rho_2) \in \mathbb{R}^2$ ,

$$\int_{-T}^{2T} \int_0^L \alpha \beta \sin(x) dx dt = \int_0^T \int_0^L (\rho_1 \alpha_1 + \rho_2 \alpha_2) (\rho_1^2 \beta_1 + \rho_2^2 \beta_2 + \rho_1 \rho_2 \beta_3) \sin(x) dx dt = 0. \quad (2.47)$$

By looking at the coefficient of  $\rho_1^2 \rho_2$  in (2.47), we get

$$\int_0^T \int_0^L (\alpha_1 \beta_3 + \alpha_2 \beta_1) \sin(x) dx dt = 0. \quad (2.48)$$

For the time being the functions  $\alpha_1$ ,  $\beta_1$ ,  $\beta_2$  and  $\beta_3$  were assumed to be real valued. But, looking at the real and imaginary parts and applying the same trick we have used to get (2.48), we also see that (2.48) holds if these four functions take their values in  $\mathbb{C}$ . (Introduce arbitrary  $(\rho_1, \rho_2)$ . If  $\alpha_1 = a_1 + ib_1$ , consider  $\tilde{\alpha}_1 := \rho_1 a_1 + \rho_2 b_1$  etc.) Of course one says that  $y = y^1 + iy^2 : [T_0, T_1] \times [0, L] \rightarrow \mathbb{C}$  is a mild solution of

$$\begin{aligned} y_t + y_x + y_{xxx} &= f, \\ y(t, 0) = y(t, L) &= 0, \end{aligned}$$

with  $f = f^1 + if^2 \in L^1(T_0, T_1, L^2(0, L, \mathbb{C}))$ , if  $y^1$  and  $y^2$  are mild solutions of

$$\begin{aligned} y_t^1 + y_x^1 + y_{xxx}^1 &= f^1, \\ y^1(t, 0) = y^1(t, L) &= 0, \end{aligned}$$

$$\begin{aligned} y_t^2 + y_x^2 + y_{xxx}^2 &= f^2, \\ y^2(t, 0) = y^2(t, L) &= 0. \end{aligned}$$

Let  $\lambda \in \mathbb{C}$ . Let  $y_\lambda \in C^\infty([0, L], \mathbb{C})$  be such that

$$\lambda y_\lambda + y_{\lambda x} + y_{\lambda xxx} = 0, \quad (2.49)$$

$$y_\lambda(0) = y_\lambda(L) = 0. \quad (2.50)$$

We take, for  $t \in [0, T]$  and  $x \in [0, L]$ ,

$$\alpha_1(t, x) = e^{\lambda t} y_\lambda(x). \quad (2.51)$$

From (2.49), (2.50) and (2.51), we get (2.31) and (2.32). Multiplying (2.49) by  $(1 - \cos(x))$ , integrating the resulting equality on  $[0, L]$  and using integrations by parts together with (2.50), we get

$$\lambda \int_0^L (1 - \cos(x)) y_\lambda dx = 0. \quad (2.52)$$

By (2.51) and (2.52), if

$$\lambda \neq 0, \quad (2.53)$$

which will be assumed until the end of this section, (2.33) holds. Let  $\tilde{A}$  denote the operator  $Aw = -w''' - w'$  defined on  $\mathcal{D}(\tilde{A}) := \{w \in H^3(0, L); w(0) = w(L) = 0, w_x(0) = w_x(L)\}$ . Then  $i\tilde{A}$  is a self-adjoint operator on  $L^2(0, L)$  with compact resolvent. Hence, the spectrum  $\sigma(\tilde{A})$  of  $\tilde{A}$  is a discrete subset of  $i\mathbb{R}$ . Let us assume that

$$\lambda \notin \sigma(\tilde{A}). \quad (2.54)$$

Then there exists one (and only one)  $\phi_\lambda \in C^\infty([0, L], \mathbb{C})$  such that

$$\lambda\phi_\lambda + \phi_{\lambda x} + \phi_{\lambda xxx} = y_\lambda \sin(x), \quad (2.55)$$

$$\phi_\lambda(0) = \phi_\lambda(L) = 0, \phi_{\lambda x}(0) = \phi_{\lambda x}(L). \quad (2.56)$$

We multiply (2.55) by  $y_\lambda(L-x)$ , integrate on  $[0, L]$ , use integrations by parts together with (2.5), (2.49), (2.50) and (2.56). We get

$$\phi_{\lambda x}(L)(y_{\lambda x}(L) - y_{\lambda x}(0)) = 0. \quad (2.57)$$

From now on we assume that

$$y_\lambda \neq 0. \quad (2.58)$$

By (2.49), (2.50), (2.54) and (2.58),

$$y_{\lambda x}(L) \neq y_{\lambda x}(0), \quad (2.59)$$

which, with (2.57), gives

$$\phi_{\lambda x}(L) = 0. \quad (2.60)$$

From (2.44), (2.45), (2.51), (2.55), (2.56) and (2.60), we get

$$\frac{d}{dt} \int_0^L e^{\lambda t} \phi_\lambda \beta_3 dx = \int_0^L \alpha_1 \beta_3 \sin(x) dx + \int_0^L \alpha_1 \alpha_2 \phi_{\lambda x} e^{\lambda t} dx. \quad (2.61)$$

We also assume that

$$2\lambda \notin \sigma(\tilde{A}). \quad (2.62)$$

Then there also exists one and only one  $z_\lambda \in C^\infty([0, L], \mathbb{C})$  such that

$$2\lambda z_\lambda + z_{\lambda x} + z_{\lambda xxx} = -y_\lambda y_{\lambda x}, \quad (2.63)$$

$$z_\lambda(0) = z_\lambda(L) = 0, z_{\lambda x}(0) = z_{\lambda x}(L). \quad (2.64)$$

Let us define  $\beta_1^*$  by

$$\beta_1^*(t, x) = e^{2\lambda t} z_\lambda(x). \quad (2.65)$$

From (2.51) and (2.51) to (2.65), (2.35) and (2.36) hold. Let also (use (2.62) again)  $\psi_\lambda \in C^\infty([0, L], \mathbb{C})$  be the unique solution of

$$2\lambda\psi_\lambda + \psi_{\lambda x} + \psi_{\lambda xxx} = z_\lambda \sin(x), \quad (2.66)$$

$$\psi_\lambda(0) = \psi_\lambda(L) = 0, \psi_{\lambda x}(0) = \psi_{\lambda x}(L). \quad (2.67)$$

Let

$$\mu(x) = -\frac{1}{2}x \sin(x) + \frac{1}{12}(1 - \cos(2x)).$$

Then

$$\mu_x + \mu_{xxx} = (1 - \cos(x)) \sin(x), \quad (2.68)$$

$$\mu(0) = \mu(L) = \mu_x(0) = 0, \quad (2.69)$$

$$\mu_x(L) = -\frac{L}{2}. \quad (2.70)$$

From (2.39), (2.41), (2.42), (2.55), (2.56) and (2.65) to (2.70), we get

$$\begin{aligned} \frac{d}{dt} \int_0^L (e^{2\lambda t} \psi_\lambda + \theta_1 \mu) \alpha_2 dx &= e^{2\lambda t} \psi_{\lambda x}(L) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) \\ &\quad - \frac{L}{2} \theta_1 \alpha_{2x}(t, L) + \int_0^L \alpha_2 \beta_1 \sin(x) dx. \end{aligned} \quad (2.71)$$

By (2.43) and (2.46),

$$\int_0^T \left( \frac{d}{dt} \int_0^L (e^{2\lambda t} \psi_\lambda + \theta_1 \mu) \alpha_2 + e^{\lambda t} \phi_\lambda \beta_3 dx \right) dt = 0. \quad (2.72)$$

From (2.61), (2.71) and (2.72), we get

$$\begin{aligned} \int_0^T \int_0^L (\alpha_1 \beta_3 + \alpha_2 \beta_1) \sin(x) dx dt &= - \int_0^T \int_0^L e^{\lambda t} \alpha_1 \alpha_2 \phi_{\lambda x} dx dt \\ &\quad - \int_0^T \left( e^{2\lambda t} \psi_{\lambda x}(L) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) - \frac{L}{2} \theta_1 \alpha_{2x}(t, L) \right) dt. \end{aligned} \quad (2.73)$$

Let (see (2.62))  $\delta_\lambda \in C^\infty([0, L], \mathbb{C})$  be the unique solution of

$$2\lambda \delta_\lambda + \delta_{\lambda x} + \delta_{\lambda xxx} = y_\lambda \phi_{\lambda x}, \quad (2.74)$$

$$\delta_\lambda(0) = \delta_\lambda(L) = 0, \quad \delta_{\lambda x}(0) = \delta_{\lambda x}(L). \quad (2.75)$$

From (2.41), (2.42), (2.51), (2.74) and (2.75), we get

$$\frac{d}{dt} \int_0^L e^{2\lambda t} \delta_\lambda \alpha_2 dx = e^{2\lambda t} \delta_{\lambda x}(L) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) + \int_0^L e^{\lambda t} \phi_{\lambda x} \alpha_1 \alpha_2 dx, \quad (2.76)$$

which, with (2.43), gives

$$\int_0^T \int_0^L e^{\lambda t} \phi_{\lambda x} \alpha_1 \alpha_2 dx dt + \int_0^T e^{2\lambda t} \delta_{\lambda x}(L) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) dt = 0. \quad (2.77)$$

From (2.48), (2.73) and (2.77), we get

$$\int_0^T \left( e^{2\lambda t} (\delta_{\lambda x}(L) - \psi_{\lambda x}(L)) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) + \frac{L}{2} \theta_1 \alpha_{2x}(t, L) \right) dt = 0. \quad (2.78)$$

Let us restrict ourselves to the case where

$$\alpha_2 = 0 \text{ on } [3T/4, T] \times [0, L]. \quad (2.79)$$

This allows us to perform a time translation of  $\epsilon \in [0, T/4]$ : if we define

$$\alpha_{2\epsilon}(t, x) := \alpha_2(t - \epsilon, x), \quad \forall t \in [\epsilon, T], \quad (2.80)$$

$$\alpha_{2\epsilon}(t, x) := 0, \quad \forall t \in [0, \epsilon], \quad (2.81)$$

then  $\alpha_{2\epsilon}$  also satisfies (2.41), (2.42) (in the mild sense) and (2.43). Hence, by (2.78) associated to  $\alpha_{2\epsilon}$ ,

$$\int_0^T \left( e^{2\lambda t} (\delta_{\lambda x}(L) - \psi_{\lambda x}(L)) (\alpha_{2x}(t - \epsilon, L) - \alpha_{2x}(t - \epsilon, 0)) + \frac{L}{2} \theta_1 \alpha_{2x}(t - \epsilon, L) \right) dt = 0, \quad \forall \epsilon \in [0, T/4],$$

which is equivalent to

$$\int_0^T \left( e^{2\lambda(t+\epsilon)} (\delta_{\lambda x}(L) - \psi_{\lambda x}(L)) (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0)) + \frac{L}{2} \theta_1 \alpha_{2x}(t, L) \right) dt = 0, \quad \forall \epsilon \in [0, T/4].$$

This last property, with (2.53), implies that

$$(\delta_{\lambda x}(L) - \psi_{\lambda x}(L)) \int_0^T (e^{2\lambda t} (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0))) dt = 0. \quad (2.82)$$

Let  $a \in \mathbb{R} \setminus [-1/\sqrt{3}, 1/\sqrt{3}]$ . We take  $\lambda := 2ia(4a^2 - 1)$ . Let

$$y_\lambda(x) := \kappa e^{(-\sqrt{3a^2-1}-ia)x} + (1 - \kappa) e^{(\sqrt{3a^2-1}-ia)x} - e^{2iax}, \quad (2.83)$$

with

$$\kappa := \frac{e^{2iaL} - e^{(\sqrt{3a^2-1}-ia)L}}{e^{(-\sqrt{3a^2-1}-ia)L} - e^{(\sqrt{3a^2-1}-ia)L}}. \quad (2.84)$$

One easily checks that such a  $y_\lambda$  satisfies (2.55), (2.56) and (2.58). Let, with  $\lambda := 2ia(4a^2 - 1)$ ,

$$\Sigma := \{a \in \mathbb{R} \setminus [-1/\sqrt{3}, 1/\sqrt{3}]; \lambda \notin \sigma(\tilde{A}) \text{ and } 2\lambda \notin \sigma(\tilde{A})\}.$$

Then, the function  $S : \Sigma \rightarrow \mathbb{C}$ ,  $S(a) = \delta_{\lambda x}(L) - \psi_{\lambda x}(L)$  is continuous (and even analytic). In Appendix C we prove the following lemma

**Lemma 11** *The function  $S$  is not identically equal to 0.*

This lemma and (2.82) imply that

$$\alpha_{2x}(t, L) - \alpha_{2x}(t, 0) = 0. \quad (2.85)$$

Indeed, let  $\mathcal{L} : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\lambda \mapsto \int_0^T (e^{2\lambda t} (\alpha_{2x}(t, L) - \alpha_{2x}(t, 0))) dt$ . The function  $\mathcal{L}$  is holomorphic. Hence the zeros of  $\mathcal{L}$  are isolated if  $\mathcal{L} \neq 0$ . However, by Lemma 11, (2.82) and the continuity of  $S$ , there exists a nonempty open subset of  $i\mathbb{R}$  on which  $\mathcal{L}$  vanishes (let us recall that  $\sigma(\tilde{A})$  is a discrete subset of  $i\mathbb{R}$ ). Hence  $\mathcal{L} = 0$ , which implies (2.85).

We multiply (2.41) by  $\bar{\alpha}_2$ , take the real part, integrate on  $[0, L]$ . Then, using integrations by parts and (2.42) together with (2.85), we get

$$\frac{d}{dt} \int_0^L |\alpha_2|^2 dx = 0,$$

which, with (2.43), implies that

$$\alpha_2 = 0. \quad (2.86)$$

But, by Theorem 3 (for mild solutions), there are mild solutions of (2.41) and (2.42) satisfying (2.43) and (2.79) such that (2.86) does not hold. This ends the proof of Proposition 10 and therefore of Proposition 8.



### 3 Local exact controllability.

In this section we still assume that (2.5) holds and we end the proof of Theorem 2. As pointed out in section 1, the invariance of the control system (KdV) by the change of variables  $\tau = T - t$ ,  $\xi = L - x$  allows us to prove only that, for every  $T > 0$ , there exists  $r'_1 > 0$  such that, for every  $y_T \in L^2(0, L)$  with  $\|y_T\|_{L^2(0, L)} \leq r'_1$ , there exists  $u \in L^2(0, T)$  such that the mild solution  $y$  of

$$y_t + y_x + y_{xxx} + yy_x = 0, \quad (3.1)$$

$$y(t, 0) = y(t, L) = 0, \quad (3.2)$$

$$y_x(t, L) = u(t), \quad (3.3)$$

$$y(0, x) = 0, \quad (3.4)$$

satisfies  $y(T, \cdot) = y_T$ . Of course, by “ $y$  is a mild solution of (3.1) to (3.4)”, we mean that  $y$  is in  $B$  and is the mild solution of

$$y_t + y_x + y_{xxx} = f,$$

$$y(t, 0) = y(t, L) = 0,$$

$$y_x(t, L) = u(t),$$

$$y(0, x) = 0,$$

with  $f := -yy_x$  (note that, if  $y$  is in  $B$ , then  $yy_x \in L^1(0, T, L^2(0, L))$ ). We use similar natural conventions until the end of this paper. It follows from Proposition 14 and Proposition 15 below that, for a given  $u \in L^2(0, T)$ , there exists at most one mild solution of (3.1) to (3.4) and that such a solution exists if  $\|u\|_{L^2(0, T)}$  is small enough (the smallness depending on  $T$  and  $L$ ).

By (the proof of) Theorem 3, there exists a continuous linear map  $\Gamma$

$$\Gamma : h \in H \subset L^2(0, L) \longmapsto \Gamma(h) \in L^2(0, T) \quad (3.5)$$

such that the mild solution of

$$y_t + y_x + y_{xxx} = 0,$$

$$y(t, 0) = y(t, L) = 0,$$

$$y_x(t, L) = \Gamma(h)(t),$$

$$y(0, x) = 0,$$

satisfies  $y(T, x) = h(x)$ . (One can take for  $\Gamma$  the control obtained by means of HUM; see [12, Remark 3.10].)

Let  $y_T \in L^2(0, L)$  be such that  $\|y_T\|_{L^2(0, L)} \leq r$ ,  $r > 0$  to be chosen later, small enough so that the maps introduced below are well defined in a neighborhood of 0. Let  $\mathcal{T}_{y_T}$  denote the map,

$$\begin{aligned} \mathcal{T}_{y_T} : L^2(0, L) &\longrightarrow L^2(0, L) \\ z &\longmapsto z + y_T - F(G(z)), \end{aligned}$$

with

$$\begin{aligned} F : L^2(0, T) &\longrightarrow L^2(0, L) \\ u &\longmapsto y(T, \cdot), \end{aligned}$$

where  $y$  is the mild solution of (3.1) to (3.4) and  $G : L^2(0, L) \longrightarrow L^2(0, T)$  is defined as follows. We decompose  $z = \mathcal{P}_H(z) + \rho(z)(1 - \cos(x))$ . Then

1. If  $\rho(z) \geq 0$ ,  $G(z) = \Gamma(\mathcal{P}_H(z)) + \rho^{1/3}(z)u_+ + \rho^{2/3}(z)v_+ + \rho(z)w_+$ ,
2. If  $\rho(z) < 0$ ,  $G(z) = \Gamma(\mathcal{P}_H(z)) + |\rho(z)|^{1/3}u_- + |\rho(z)|^{2/3}v_- + |\rho(z)|w_-$ .

(The functions  $u_{\pm}$ ,  $v_{\pm}$  and  $w_{\pm}$  are fixed as in Proposition 8.)

Clearly, each fixed point  $z^*$  of  $\mathcal{T}_{y_T}$  satisfies  $F(G(z^*)) = y_T$ , and the control  $u = G(z^*)$  is a solution to our problem.

Until the end of this paper, we adopt the following notations:

- For  $z \in L^2(0, T, H^1(0, L))$ ,  $\|z\|_{L^2(H^1)} = \|z\|_{L^2(0, T, H^1(0, L))}$ ,
- For  $z \in L^1(0, T, L^2(0, L))$ ,  $\|z\|_{L^1(L^2)} = \|z\|_{L^1(0, T, L^2(0, L))}$ ,
- $\mathcal{B}_R = \{z \in L^2(0, L); \|z\|_{L^2(0, L)} \leq R\}$ .

First of all we prove a lemma about the map  $\mathcal{T}_0$ .

**Lemma 12** *There exist  $C_1 = C_1(T, L) > 0$  and  $\epsilon_1 = \epsilon_1(T, L) > 0$  such that, for every  $z \in \mathcal{B}_{\epsilon_1}$ ,*

$$\|\mathcal{T}_0 z\|_{L^2(0, L)} \leq C_1 \|z\|_{L^2(0, L)}^{4/3}. \quad (3.6)$$

Let  $z \in L^2(0, L)$ . Let  $(\tilde{u}, \tilde{v}, \tilde{w}) = (u_+, v_+, w_+)$  if  $\rho(z) \geq 0$  and  $(\tilde{u}, \tilde{v}, \tilde{w}) = (u_-, v_-, w_-)$  if  $\rho(z) < 0$ . Let  $y$  be the mild solution of

$$y_t + y_x + y_{xxx} + yy_x = 0, \quad (3.7)$$

$$y(t, 0) = y(t, L) = 0, \quad (3.8)$$

$$y_x(t, L) = \Gamma(\mathcal{P}_H(z))(t) + |\rho(z)|^{1/3}\tilde{u}(t) + |\rho(z)|^{2/3}\tilde{v}(t) + |\rho(z)|\tilde{w}(t), \quad (3.9)$$

$$y(0, x) = 0. \quad (3.10)$$

By Proposition 14 and by Proposition 15, there exist  $\epsilon_2 = \epsilon_2(T, L) > 0$  and  $C_2 = C_2(T, L) > 0$ , such that, for every  $z \in L^2(0, L)$  with  $\|z\|_{L^2(0, L)} \leq \epsilon_2$ , there exists a unique mild solution of (3.7) to (3.10) and this mild solution satisfies

$$\|y\|_B \leq C_2 \|z\|_{L^2(0, L)}^{1/3}. \quad (3.11)$$

Let  $\tilde{y}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$  be the mild solutions of

$$\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \quad (3.12)$$

$$\tilde{y}(t, 0) = \tilde{y}(t, L) = 0, \quad (3.13)$$

$$\tilde{y}_x(t, L) = \Gamma(\mathcal{P}_H(z))(t), \quad (3.14)$$

$$\tilde{y}(0, x) = 0, \quad (3.15)$$

$$\tilde{\alpha}_t + \tilde{\alpha}_x + \tilde{\alpha}_{xxx} = 0, \quad (3.16)$$

$$\tilde{\alpha}(t, 0) = \tilde{\alpha}(t, L) = 0, \quad (3.17)$$

$$\tilde{\alpha}_x(t, L) = \tilde{u}(t), \quad (3.18)$$

$$\tilde{\alpha}(0, x) = 0, \quad (3.19)$$

$$\tilde{\beta}_t + \tilde{\beta}_x + \tilde{\beta}_{xxx} = -\tilde{\alpha}\tilde{\alpha}_x, \quad (3.20)$$

$$\tilde{\beta}(t, 0) = \tilde{\beta}(t, L) = 0, \quad (3.21)$$

$$\tilde{\beta}_x(t, L) = \tilde{v}(t), \quad (3.22)$$

$$\tilde{\beta}(0, x) = 0. \quad (3.23)$$

$$\tilde{\gamma}_t + \tilde{\gamma}_x + \tilde{\gamma}_{xxx} = -(\tilde{\alpha}\tilde{\beta})_x, \quad (3.24)$$

$$\tilde{\gamma}(t, 0) = \tilde{\gamma}(t, L) = 0, \quad (3.25)$$

$$\tilde{\gamma}_x(t, L) = \tilde{w}(t), \quad (3.26)$$

$$\tilde{\gamma}(0, x) = 0. \quad (3.27)$$

Let

$$\phi := y - \tilde{y} - |\rho(z)|^{1/3}\tilde{\alpha} - |\rho(z)|^{2/3}\tilde{\beta} - |\rho(z)|\tilde{\gamma}, \quad (3.28)$$

$$a := \tilde{y} + |\rho(z)|^{1/3}\tilde{\alpha} + |\rho(z)|^{2/3}\tilde{\beta} + |\rho(z)|\tilde{\gamma}, \quad (3.29)$$

$$\begin{aligned} b := & \tilde{y}\tilde{y}_x + (\tilde{y}(|\rho(z)|^{1/3}\tilde{\alpha} + |\rho(z)|^{2/3}\tilde{\beta} + |\rho(z)|\tilde{\gamma}))_x \\ & + |\rho(z)|^{4/3}(\tilde{\alpha}\tilde{\gamma})_x + |\rho(z)|^{4/3}\tilde{\beta}\tilde{\beta}_x + |\rho(z)|^{5/3}(\tilde{\beta}\tilde{\gamma})_x + |\rho(z)|^2\tilde{\gamma}\tilde{\gamma}_x. \end{aligned} \quad (3.30)$$

By Proposition 7, (3.11) to (3.30) and standard estimates, there exists  $C_3 = C_3(T, L) > 0$  such that, for every  $z \in L^2(0, L)$  with  $\|z\|_{L^2(0, L)} \leq \epsilon_2$ ,

$$\|a\|_B \leq C_3 \|z\|_{L^2(0, L)}^{1/3}, \quad (3.31)$$

$$\|b\|_{L^1(L^2)} \leq C_3 \|z\|_{L^2(0, L)}^{4/3}, \quad (3.32)$$

$$\|\phi\|_B \leq C_3 \|z\|_{L^2(0, L)}^{1/3}. \quad (3.33)$$

Similarly standard estimates give the existence of  $C_4 = C_4(T, L) > 0$  such that

$$\|(\phi a)_x\|_{L^1(L^2)} \leq C_4 \|\phi\|_B \|a\|_B. \quad (3.34)$$

Note that, by (2.19), (3.7) to (3.10) and (3.12) to (3.28),

$$\phi(T, \cdot) = F(G(z)) - z = -\mathcal{T}_0 z. \quad (3.35)$$

Moreover, by (3.7) to (3.10) and (3.12) to (3.30),  $\phi$  is a mild solution of

$$\phi_t + \phi_x + \phi_{xxx} + \phi\phi_x = -(\phi a)_x - b, \quad (3.36)$$

$$\phi(t, 0) = \phi(t, L) = \phi_x(t, L) = 0, \quad (3.37)$$

$$\phi(0, x) = 0. \quad (3.38)$$

From (3.32), (3.34), (3.36), (3.37), (3.38) and Proposition 15, there exists  $C_5 = C_5(T, L) > 0$  such that

$$\|\phi\|_B^2 \leq (\|z\|_{L^2(0, L)}^{8/3} + \|\phi\|_B^2 \|a\|_B^2) e^{C_5(1 + \|\phi\|_B^2)}. \quad (3.39)$$

From (3.31), (3.33) and (3.39), one gets the existence of  $\epsilon_3 = \epsilon_3(T, L) > 0$  and of  $C_6 = C_6(T, L) > 0$  such that, for every  $z \in L^2(0, L)$  with  $\|z\|_{L^2(0, L)} \leq \epsilon_3$ ,

$$\|\phi\|_B \leq C_6 \|z\|_{L^2(0, L)}^{4/3}, \quad (3.40)$$

which, with (3.35), ends the proof of Lemma 12.

We now study  $\mathcal{P}_H \circ \mathcal{T}_{y_T}$  on the space  $H$ . Let, for  $\omega \in \mathbb{R}$  and  $y_T \in L^2(0, L)$ ,

$$\begin{aligned} \Pi : H &\longrightarrow H \\ g &\longmapsto g + \mathcal{P}_H(y_T) - \mathcal{P}_H(F(G(g + \omega e))), \end{aligned}$$

where  $e(x) := 1 - \cos(x)$  and  $\omega \in \mathbb{R}$ . (In fact we should write, for example,  $\Pi_{y_T, \omega}$ , but, for simplicity we omit the indices  $y_T$  and  $\omega$ .) To prove the existence of a fixed point for  $\Pi$ , we apply the Banach fixed-point theorem to the restriction of  $\Pi$  to the closed ball  $\mathcal{B}_R \cap H$ , with  $\|y_T\|_{L^2(0, L)} + |\omega| \leq R/3$  and where  $R > 0$  small enough. Let  $(y_T, \omega) \in L^2(0, L) \times \mathbb{R}$  be such that  $\|y_T\|_{L^2(0, L)} + |\omega| \leq R/3$ . Let  $g, h \in H \cap \mathcal{B}_R$ . With (3.6), we have, for  $R > 0$  small enough,

$$\begin{aligned} \|\Pi(g)\|_{L^2(0, L)} &\leq \|y_T\|_{L^2(0, L)} + \|g + \omega e - F(G(g + \omega e))\|_{L^2(0, L)} \\ &\leq \frac{R}{3} + \frac{2R}{3} = R. \end{aligned} \quad (3.41)$$

Hence, for  $R > 0$  small enough,

$$\Pi(\mathcal{B}_R \cap H) \subset \mathcal{B}_R \cap H. \quad (3.42)$$

Let us now look at the contracting property of  $\Pi$ . Let  $\omega \in \mathbb{R}$ ,  $g \in H$  and  $h \in H$ . Let  $(\tilde{u}, \tilde{v}, \tilde{w}) = (u_+, v_+, w_+)$  if  $\omega \geq 0$  and  $(\tilde{u}, \tilde{v}, \tilde{w}) = (u_-, v_-, w_-)$  if  $\omega < 0$ . Let  $y, z, \tilde{y}, \tilde{z}$  be the mild solutions of the following problems

$$y_t + y_x + y_{xxx} + yy_x = 0, \quad (3.43)$$

$$y(t, 0) = y(t, L) = 0, \quad (3.44)$$

$$y_x(t, L) = \Gamma(g)(t) + |\omega|^{1/3}\tilde{u}(t) + |\omega|^{2/3}\tilde{v}(t) + |\omega|\tilde{w}(t), \quad (3.45)$$

$$y(0, x) = 0, \quad (3.46)$$

$$z_t + z_x + z_{xxx} + zz_x = 0, \quad (3.47)$$

$$z(t, 0) = z(t, L) = 0, \quad (3.48)$$

$$z_x(t, L) = \Gamma(h)(t) + \omega^{1/3}\tilde{u}(t) + \omega^{2/3}\tilde{v}(t) + \omega\tilde{w}(t), \quad (3.49)$$

$$z(0, x) = 0, \quad (3.50)$$

$$\tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \quad (3.51)$$

$$\tilde{y}(t, 0) = \tilde{y}(t, L) = 0, \quad (3.52)$$

$$\tilde{y}_x(t, L) = \Gamma(g)(t), \quad (3.53)$$

$$\tilde{y}(0, x) = 0, \quad (3.54)$$

$$\tilde{z}_t + \tilde{z}_x + \tilde{z}_{xxx} = 0, \quad (3.55)$$

$$\tilde{z}(t, 0) = \tilde{z}(t, L) = 0, \quad (3.56)$$

$$\tilde{z}_x(t, L) = \Gamma(h)(t), \quad (3.57)$$

$$\tilde{z}(0, x) = 0. \quad (3.58)$$

Let  $\phi = y - \tilde{y}$  and  $\psi = z - \tilde{z}$ . Let  $\gamma = \phi - \psi$ . By (3.43) to (3.58),  $\gamma$  is a mild solution of

$$\begin{aligned}\gamma_t + \gamma_x + \gamma_{xxx} + \gamma\gamma_x &= -(\gamma a)_x - b, \\ \gamma(t, 0) = \gamma(t, L) &= 0, \\ \gamma_x(t, L) &= 0, \\ \gamma(0, x) &= 0,\end{aligned}$$

with

$$\begin{aligned}a &= \psi + \tilde{y}, \\ b &= (\psi(\tilde{y} - \tilde{z}))_x + \tilde{y}\tilde{y}_x - \tilde{z}\tilde{z}_x.\end{aligned}$$

Let us notice that there exists  $C_7 = C_7(T) > 0$  such that

$$\begin{aligned}\|b\|_{L^1(L^2)} &\leq C_7(\|z\|_B + \|\tilde{y}\|_B + \|\tilde{z}\|_B)\|\tilde{y} - \tilde{z}\|_B, \\ \|(\gamma a)_x\|_{L^1(L^2)} &\leq C_7(\|z\|_B + \|\tilde{z}\|_B + \|\tilde{y}\|_B)\|\gamma\|_B.\end{aligned}$$

Then, using again Proposition 14 and Proposition 15 as in the the proof of (3.40), we get the existence of  $\epsilon_4 = \epsilon_4(T, L) > 0$  such that, for every  $(g, h, \omega) \in H \times H \times \mathbb{R}$  with

$$\|g\|_{L^2(0,L)} \leq \epsilon_4, \|h\|_{L^2(0,L)} \leq \epsilon_4, \text{ and } |\omega| \leq \epsilon_4,$$

one has

$$\|\gamma\|_B \leq \frac{1}{2}\|g - h\|_{L^2(0,L)}. \quad (3.59)$$

Note that (3.59) implies that

$$\|\Pi(g) - \Pi(h)\|_{L^2(0,L)} = \|\gamma(T, \cdot)\|_{L^2(0,L)} \leq \|\gamma\|_B \leq \frac{1}{2}\|g - h\|_{L^2(0,L)}. \quad (3.60)$$

Therefore, by (3.42) and (3.60), there exists  $\epsilon_5 = \epsilon_5(T, L) > 0$  such that, for  $R \leq \epsilon_5$ , for every  $(y_T, \omega) \in L^2(0, L) \times \mathbb{R}$  such that  $\|y_T\|_{L^2(0,L)} + |\omega| \leq R/3$ ,  $\Pi$  has a unique fixed point  $h(y_T, \omega)$  in  $\mathcal{B}_R \cap H$ . Note that the map  $h$  is continuous in a neighborhood of  $(0, 0) \in H \times \mathbb{R}$ .

We now apply the intermediate value theorem to the application

$$\begin{aligned}\tau : \mathbb{R} &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \rho(\omega e + h(y_T, \omega) + y_T - F(G(h(y_T, \omega) + \omega e))).\end{aligned}$$

By (3.6), there exists  $\epsilon_6 = \epsilon_6(T, L) > 0$  such that, if  $\|y_T\|_{L^2(0,L)} \leq \epsilon_6\sqrt{3L/8}$ , then  $\tau([- \epsilon_6, \epsilon_6]) \subset [- \epsilon_6, \epsilon_6]$ . Hence, if  $\|y_T\|_{L^2(0,L)} \leq \epsilon_6\sqrt{3L/8}$ , we deduce, by the intermediate value theorem, that  $\tau$  has at least a fixed point  $\omega_0$ . We have

$$F(G(h(y_T, \omega_0) + \omega_0 e)) = y_T,$$

which ends the proof of Theorem 2.

Let us remark that it follows from our proof of Theorem 2 that the following theorem, slightly more precise than Theorem 2, also holds.

**Theorem 13** *Let  $k$  be a positive integer and let  $T > 0$ . There exist  $r_1 > 0$  and  $C > 0$  such that, for every  $(y_0, y_T) \in L^2(0, 2k\pi)^2$  with  $\|y_0\|_{L^2(0, 2k\pi)} < r_1$  and  $\|y_T\|_{L^2(0, 2k\pi)} < r_1$ , there exist  $y \in C([0, T], L^2(0, 2k\pi)) \cap L^2(0, T, H^1(0, 2k\pi))$  satisfying, in the mild sense, (KdV) with  $L = 2k\pi$  such that*

$$y(0, \cdot) = y_0, \quad y(T, \cdot) = y_T,$$

$$\|y\|_B \leq C \left( \|\mathcal{P}_H(y_0)\|_{L^2(0, 2k\pi)} + \left| \int_0^{2k\pi} (1 - \cos(x))y_0 dx \right|^{1/3} + \|\mathcal{P}_H(y_1)\|_{L^2(0, 2k\pi)} + \left| \int_0^{2k\pi} (1 - \cos(x))y_1 dx \right|^{1/3} \right).$$

## A Appendix : Existence and uniqueness of solutions to Cauchy problems for KdV equations.

We first prove the existence of solutions to the Cauchy problem for nonlinear KdV equation (with small data)

**Proposition 14** *Let  $L > 0$  and  $T > 0$ . There exists  $\epsilon > 0$  and  $C > 0$  such that, for every  $f \in L^1(0, T, L^2(0, L))$ , every  $u \in L^2(0, T)$  and every  $y_0 \in L^2(0, L)$  such that*

$$\|f\|_{L^1(L^2)} + \|u\|_{L^2(0, T)} + \|y_0\|_{L^2(0, L)} \leq \epsilon,$$

*there exist at least a mild solution  $y$  of*

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = f, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = u(t), \\ y(0, x) = y_0(x). \end{cases} \quad (\text{A.1})$$

*which satisfies*

$$\|y\|_B \leq C(\|f\|_{L^1(L^2)} + \|u\|_{L^2(0, T)} + \|y_0\|_{L^2(0, L)}).$$

**Proof :** Let  $T > 0$ . For  $f \in L^1(0, T, L^2(0, L))$ ,  $u \in L^2(0, T)$  and  $y_0 \in L^2(0, L)$ , let us define

$$\begin{aligned} M_{f, u, y_0} : B &\longrightarrow B \\ y &\longmapsto \Psi(y_0, u, f - yy_x). \end{aligned}$$

A fixed point of  $M_{f, u, y_0}$  is a solution of (A.1). One easily gets the existence of  $C_8 = C_8(T) > 0$  such that, for every  $(y, z) \in B^2$ ,

$$\begin{aligned} \|yy_x\|_{L^1(L^2)} &\leq C_8\|y\|_B^2, \\ \|zz_x - yy_x\|_{L^1(L^2)} &\leq C_8(\|z\|_B + \|y\|_B)\|z - y\|_B. \end{aligned}$$

Hence, by continuity of  $\Psi$  (see Proposition 7), there exists a constant  $C_9 = C_9(T, L) > 0$  such that, for every  $f \in L^1(0, T, L^2(0, L))$ , every  $u \in L^2(0, T)$ , every  $y_0 \in L^2(0, L)$ , every  $y$  in  $B$  and every  $z$  in  $B$ ,

$$\begin{aligned} \|M_{f, u, y_0}(y)\|_B &\leq C_9(\|f\|_{L^1(L^2)} + \|u\|_{L^2(0, T)} + \|y_0\|_{L^2(0, L)} + \|y\|_B^2), \\ \|M_{f, u, y_0}(z) - M_{f, u, y_0}(y)\|_B &\leq C_9(\|z\|_B + \|y\|_B)\|z - y\|_B. \end{aligned}$$

From these two inequalities and the usual standard Banach fixed point theorem, one sees that Proposition 14 holds with

$$\epsilon := \frac{1}{9C_9^2}, \quad C := \frac{3C_9}{2}.$$

(Note that  $M_{f,u,y_0}(\tilde{B}) \subset \tilde{B}$  if  $\tilde{B} := \{y \in B; \|y\|_B \leq 1/(3C_9)\}$ ,  $\|M_{f,u,y_0}(z) - M_{f,u,y_0}(y)\|_B \leq (2/3)\|z - y\|_B$  for every  $(y, z) \in \tilde{B}^2$ , and  $\|y\|_{L^2(0,L)}(1 - C_9\|y\|_{L^2(0,L)}) \leq C_9(\|f\|_{L^1(L^2)} + \|u\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)})$  if  $y$  is a fixed point of  $M_{f,u,y_0}$ .)

We now prove the uniqueness of the mild solution of the Cauchy problem for our nonlinear KdV equation, together with estimates of this solution.

**Proposition 15** *Let  $T > 0$  and let  $L > 0$ . There exists  $C_{10} = C_{10}(T, L) > 0$  such that for every  $(y_0, z_0) \in L^2(0, L)^2$ , for every  $(u, v) \in L^2(0, T)^2$  and for every  $(f, g) \in L^1(0, T, L^2(0, L))^2$  for which there exist mild solutions  $y$  and  $z$  of*

$$y_t + y_x + y_{xxx} + yy_x = f, \quad (\text{A.2})$$

$$y(t, 0) = y(t, L) = 0, \quad (\text{A.3})$$

$$y_x(t, L) = u(t), \quad (\text{A.4})$$

$$y(0, x) = y_0(x), \quad (\text{A.5})$$

and of

$$z_t + z_x + z_{xxx} + zz_x = g, \quad (\text{A.6})$$

$$z(t, 0) = z(t, L) = 0, \quad (\text{A.7})$$

$$z_x(t, L) = v(t), \quad (\text{A.8})$$

$$z(0, x) = z_0(x), \quad (\text{A.9})$$

one has the following inequalities

$$\int_0^T \int_0^L (z_x(t, x) - y_x(t, x))^2 dx dt \leq \left( \int_0^L (z_0 - y_0)^2 dx + \|u - v\|_{L^2(0,T)}^2 + \|f - g\|_{L^1(L^2)}^2 \right) e^{C_{10}(1 + \|y\|_{L^2(H^1)}^2 + \|z\|_{L^2(H^1)}^2)}, \quad (\text{A.10})$$

$$\int_0^L (z(t, x) - y(t, x))^2 dx \leq \left( \int_0^L (z_0 - y_0)^2 dx + \|u - v\|_{L^2(0,T)}^2 + \|f - g\|_{L^1(L^2)}^2 \right) e^{C_{10}(1 + \|y\|_{L^2(H^1)}^2 + \|z\|_{L^2(H^1)}^2)}, \quad \forall t \in [0, T]. \quad (\text{A.11})$$

**Proof :** Let

$$\Delta := z - y. \quad (\text{A.12})$$

Then  $\Delta$  is a mild solution of

$$\Delta_t + \Delta_x + \Delta_{xxx} = -y\Delta_x - z_x\Delta - (f - g), \quad (\text{A.13})$$

$$\Delta(t, 0) = \Delta(t, L) = 0, \quad (\text{A.14})$$

$$\Delta_x(t, L) = v(t) - u(t), \quad (\text{A.15})$$

$$\Delta(0, x) = z_0(x) - y_0(x). \quad (\text{A.16})$$

Formally, by integrating by parts in

$$\int_0^L 2x\Delta(\Delta_t + \Delta_x + \Delta_{xxx} + y\Delta_x + z_x\Delta + (f-g))dx = 0, \quad (\text{A.17})$$

using (A.14) and (A.15), we readily get

$$\begin{aligned} \frac{d}{dt} \int_0^L x\Delta^2 dx + 3 \int_0^L \Delta_x^2 = \int_0^L \Delta^2 dx + L(v(t) - u(t))^2 - 2 \int_0^L xy\Delta\Delta_x dx \\ + 2 \int_0^L z\Delta^2 dx + 4 \int_0^L xz\Delta\Delta_x dx - 2 \int_0^L x\Delta(f-g)dx. \end{aligned} \quad (\text{A.18})$$

(Note that the multiplier  $x\Delta$  for (A.13) has been introduced by Rosier in [12, p. 48].) Even if (A.17) is only formal, using standard approximation arguments, one easily sees that (A.18) always holds (in the sense of distributions).

By (A.3) and the continuous Sobolev embedding  $H_0^1(0, L) \subset C^0([0, T])$ , there exists  $C_{11} = C_{11}(L) > 0$  such that

$$2 \left| \int_0^L xy\Delta\Delta_x dx \right| \leq C_{11} \|y_x\|_{L^2(0,L)} \int_0^L |x\Delta\Delta_x| dx.$$

Thus,

$$2 \left| \int_0^L xy\Delta\Delta_x dx \right| \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + \frac{C_{11}^2}{2} \|y_x\|_{L^2(0,L)}^2 L \int_0^L x\Delta^2 dx. \quad (\text{A.19})$$

Similarly,

$$4 \left| \int_0^L xz\Delta\Delta_x dx \right| \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + 2C_{11}^2 \|z_x\|_{L^2(0,L)}^2 \int_0^L x\Delta^2 dx. \quad (\text{A.20})$$

We have the following lemma:

**Lemma 16** For every  $\phi \in H^1(0, L)$  with  $\phi(0) = 0$ , for every  $a \in [0, L]$ ,

$$\int_0^L \phi^2 dx \leq \frac{a^2}{2} \int_0^L \phi_x^2 dx + \frac{1}{a} \int_0^L x\phi^2 dx. \quad (\text{A.21})$$

Indeed

$$\begin{aligned} \int_0^L \phi^2 dx &= \int_0^a \phi^2 dx + \int_a^L \phi^2 dx \\ &\leq \int_0^a \left( \int_0^x \phi_x(s) ds \right)^2 dx + \frac{1}{a} \int_0^L x\phi^2 dx \\ &\leq \int_0^a \left( \int_0^L \phi_x^2(s) ds \right) x dx + \frac{1}{a} \int_0^L x\phi^2 dx \\ &\leq \frac{a^2}{2} \int_0^L \phi_x^2 dx + \frac{1}{a} \int_0^L x\phi^2 dx. \end{aligned}$$



Thanks to Lemma 16, there exists  $C_{12} > 0$  such that

$$\int_0^L \Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_{12} \int_0^L x \Delta^2 dx. \quad (\text{A.22})$$

Moreover, by (A.7) and the Sobolev embedding  $H_0^1(0, L) \subset C^0([0, T])$ , there exists  $C_{13} = C_{13}(L) > 0$  such that

$$2 \left| \int_0^L z \Delta^2 dx \right| \leq C_{13} \|z_x\|_{L^2(0, L)} \int_0^L \Delta^2 dx.$$

Hence, using (A.14) and Lemma 16 with  $a := \text{Min}\{C_{13}^{-1/2} \|z_x\|_{L^2(0, L)}^{-1/2}, L\}$ , there exists  $C_{14} = C_{14}(L) > 0$  such that

$$2 \int_0^L z \Delta^2 dx \leq \frac{1}{2} \int_0^L \Delta_x^2 dx + C_{14} (1 + \|z_x\|_{L^2(0, L)}^{3/2}) \int_0^L x \Delta^2 dx. \quad (\text{A.23})$$

Moreover,

$$2 \left| \int_0^L x \Delta(f - g) dx \right| \leq 2\sqrt{L} \|f - g\|_{L^2(0, L)} \left( \int_0^L x \Delta^2 dx \right)^{1/2}. \quad (\text{A.24})$$

Thus, using (A.22), (A.20), (A.22), (A.23), and (A.24), there exists  $C_{15} = C_{15}(L) > 0$  such that

$$\begin{aligned} \frac{d}{dt} \int_0^L x \Delta^2 dx + \int_0^L \Delta_x^2 \leq & L(v(t) - u(t))^2 + 2\sqrt{L} \|f - g\|_{L^2(0, L)} \left( \int_0^L x \Delta^2 dx \right)^{1/2} \\ & + C_{15} (1 + \|y_x\|_{L^2(0, L)}^2 + \|z_x\|_{L^2(0, L)}^2) \int_0^L x \Delta^2 dx. \end{aligned} \quad (\text{A.25})$$

In particular,

$$\begin{aligned} \frac{d}{dt} \int_0^L x \Delta^2 dx \leq & L(v(t) - u(t))^2 + 2\sqrt{L} \|f - g\|_{L^2(0, L)} \left( \int_0^L x \Delta^2 dx \right)^{1/2} \\ & + C_{15} (1 + \|y_x\|_{L^2(0, L)}^2 + \|z_x\|_{L^2(0, L)}^2) \int_0^L x \Delta^2 dx. \end{aligned} \quad (\text{A.26})$$

Let us assume for the moment being that the following lemma holds:

**Lemma 17** *Let  $T > 0$ . Let  $a, b$  and  $c$  be three nonnegative functions in  $L^1(0, T)$ . Let  $w \in C^0([0, T])$  be a non negative function such that, in the sense of distributions,*

$$\dot{w} \leq a(t) + b(t)\sqrt{w(t)} + c(t)w(t).$$

Then

$$w(t) \leq 3 \left( w(0) + \int_0^t a(s) ds + \left( \int_0^t b(s) ds \right)^2 \right) e^{\int_0^t c(s) ds}.$$

From (A.16)), (A.26) and Lemma 17, we get, for every  $t \in [0, T]$ ,

$$\int_0^L x \Delta^2(t, x) dx \leq 3 \left( \int_0^L x (z_0 - y_0)^2 dx + L \|u - v\|_{L^2(0, T)}^2 + 4L \|f - g\|_{L^1(L^2)}^2 \right) e^{C_{15}(T + \|y\|_{L^2(H^1)}^2 + \|z\|_{L^2(H^1)}^2)}. \quad (\text{A.27})$$

Using (A.12), (A.25) and (A.27), we get the existence of  $C_{16} = C_{16}(T, L) > 0$  such that

$$\int_0^T \int_0^L (z_x(t, x) - y_x(t, x))^2 dx dt \leq \left( \int_0^L (z_0 - y_0)^2 dx + \|u - v\|_{L^2(0, T)}^2 + \|f - g\|_{L^1(L^2)}^2 \right) e^{C_{16}(1 + \|y\|_{L^2(H^1)}^2 + \|z\|_{L^2(H^1)}^2)}, \quad (\text{A.28})$$

which gives (A.10). Finally, in order to get (A.11), we multiply (A.13) by  $\Delta$ , integrate on  $[0, L]$ . Using (A.14), (A.15) and integrations by parts, we get

$$\frac{1}{2} \frac{d}{dt} \int_0^L \Delta^2 dx + \frac{1}{2} \Delta_x^2(t, 0) = \frac{1}{2} (v(t) - u(t))^2 - \int_0^L (y \Delta_x - 2z \Delta_x + f - g) \Delta dx. \quad (\text{A.29})$$

Moreover

$$- \int_0^L (y \Delta_x - 2z \Delta_x) \Delta dx \leq \int_0^L \Delta_x^2 dx + \int_0^L \left( \frac{1}{2} y^2 + 2z^2 \right) \Delta^2 dx. \quad (\text{A.30})$$

By (A.3), (A.7), (A.29), (A.30), and the continuous Sobolev embedding  $H_0^1(0, L) \subset C^0([0, L])$  there exists  $C_{17} = C_{17}(L) > 0$  such that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L \Delta^2 dx \leq \frac{1}{2} (v(t) - u(t))^2 + \int_0^L \Delta_x^2 dx + \|f(t, \cdot) - g(t, \cdot)\|_{L^2(0, L)} \left( \int_0^L \Delta^2 dx \right)^{1/2} \\ + C_{17} (\|y_x\|_{L^2(0, L)}^2 + \|z_x\|_{L^2(0, L)}^2) \int_0^L \Delta^2 dx, \end{aligned}$$

which, with (A.16), (A.28) and Lemma 17, gives (A.11) for  $C_{10} = C_{10}(T, L) > 0$  large enough.

It remains to prove Lemma 17. Considering

$$\tilde{w}(t) := w(t) e^{-\int_0^t c(s) ds},$$

we easily see that, without loss of generality, we may assume that  $c = 0$ . Moreover, still without loss of generality, we may also assume that there exists  $\epsilon > 0$  such that, for almost every  $s$  in  $[0, T]$ ,  $a(s) \geq \epsilon$ . Let

$$\tilde{w}(t) := \text{Max} \left( w(t) - \int_0^t a(s) ds, 0 \right).$$

Then, in the sense of distributions,

$$\frac{d}{dt} \left( \sqrt{\tilde{w} + \int_0^t a(s) ds} \right) \leq \frac{b(t)}{2} + \frac{a(t)}{2\sqrt{\int_0^t a(s) ds}}.$$

Integrating this inequality on  $[0, t]$ , we get

$$\sqrt{\tilde{w}(t) + \int_0^t a(s)ds} \leq \sqrt{w(0)} + \frac{1}{2} \int_0^t b(s)ds + \sqrt{\int_0^t a(s)ds},$$

which gives

$$w(t) \leq 3 \left( w(0) + \frac{1}{4} \left( \int_0^t b(s)ds \right)^2 + \int_0^t a(s)ds \right),$$

and ends the proof of Lemma 17.

## B Motion failure in the $\pm(1 - \cos(x))$ directions for a 2<sup>nd</sup> order power series expansion.

Throughout all this section, we again assume that (2.5) holds. We now denote by  $L^2(0, L)$  the space of the measurable complex valued functions such that  $\int_0^L |f|^2 dx < \infty$ . We use the similar convention for  $C^\infty([0, L])$ ,  $L^2(0, T)$ ,  $H$ ,  $\mathcal{D}(A)$ , mild solutions etc. We also still denote by  $\mathcal{P}_H$  the orthogonal projection on  $H \subset L^2(0, L)$  for the hermitian product on  $L^2(0, L)$ . The main result of this section is the following one.

**Proposition 18** *Let  $z$  be a mild solution of*

$$z_t + z_x + z_{xxx} = 0, \tag{B.1}$$

$$z(t, 0) = z(t, L) = 0, \tag{B.2}$$

$$z(0, x) = z(T, x) = 0, \tag{B.3}$$

and  $y$  be a mild solution of

$$y_t + y_x + y_{xxx} = 0, \tag{B.4}$$

$$y(t, 0) = y(t, L) = 0, \tag{B.5}$$

such that

$$\int_0^L y(t, x)(1 - \cos(x))dx = 0, \forall t \in [0, T]. \tag{B.6}$$

Then

$$\int_0^T \int_0^L yz \sin(x) dx dt = 0. \tag{B.7}$$

Before giving the proof of this proposition, let us first mention a corollary of this proposition.

**Corollary 19** *Let  $T, L, y$  and  $z$  be as in Proposition 18. Let  $w \in B$  be a mild solution of*

$$w_t + w_x + w_{xxx} = -(yz)_x, \tag{B.8}$$

$$w(t, 0) = w(t, L) = 0, \tag{B.9}$$

such that

$$w(0, x) = 0, \quad (\text{B.10})$$

$$\mathcal{P}_H(w(T, \cdot)) = 0. \quad (\text{B.11})$$

Then

$$w(T, x) = 0. \quad (\text{B.12})$$

Indeed, from (B.8) and (B.9), we get

$$\frac{d}{dt} \int_0^L (1 - \cos(x))w dx = \int_0^L yz \sin(x) dx,$$

which, with (B.7) and (B.10), gives

$$\int_0^L (1 - \cos(x))w(T, x) dx = 0. \quad (\text{B.13})$$

Finally (B.12) follows from (B.11) and (B.13).

Let us now prove Proposition 18 in two special cases, from which we will then deduce the general case.

Let  $\lambda \in \mathbb{C} \setminus \sigma(A)$ . Then there exists one and only one  $y_\lambda \in C^\infty([0, L])$  such that

$$\lambda y_\lambda + y_{\lambda x} + y_{\lambda x x} = 0, \quad (\text{B.14})$$

$$y_\lambda(0) = y_\lambda(L) = 0, \quad (\text{B.15})$$

$$y_{\lambda x}(L) = 1. \quad (\text{B.16})$$

Let

$$Y_\lambda(t, x) := e^{\lambda t} y_\lambda(x). \quad (\text{B.17})$$

Then the following lemma holds

**Lemma 20** For  $\lambda \in \mathbb{C} \setminus \sigma(A)$ , Theorem 18 holds if

$$y(t, x) := Y_\lambda(t, x). \quad (\text{B.18})$$

(Note that (B.14), (B.15), (B.17) and (B.18) imply (B.4) and (B.5).) Let us prove this lemma. Since  $\lambda \in \mathbb{C} \setminus \sigma(A)$ , there exists one and only one  $\phi \in C^\infty([0, L])$  such that

$$\lambda \phi + \phi_x + \phi_{x x} = y_\lambda \sin(x), \quad (\text{B.19})$$

$$\phi(0) = \phi(L) = 0, \quad (\text{B.20})$$

$$\phi_x(L) = 0. \quad (\text{B.21})$$

After some integrations by parts, we get, using (B.1), (B.2), (B.19), (B.20), (B.17) and (B.18),

$$\int_0^L yz \sin(x) dx = \frac{d}{dt} \int_0^L z \phi e^{\lambda t} - e^{\lambda t} [z_x \phi_x]_0^L$$

which, with by (B.3) and (B.21), gives (B.7) if

$$\phi_x(0) = 0. \quad (\text{B.22})$$

We multiply (B.19) by  $y_\lambda(L-x)$ . We get, with (2.5),

$$\int_0^L \lambda \phi y_\lambda(L-x) + \int_0^L (\phi_x + \phi_{xxx}) y_\lambda(L-x) dx = \int_0^L y_\lambda(L-x) y_\lambda(x) \sin(x) dx = 0. \quad (\text{B.23})$$

Performing integrations by parts in (B.23), we get, using (B.14), (B.15) and (B.20)

$$\phi_x(0) y_{\lambda x}(L) = 0,$$

which, with (B.16), gives (B.22) and ends the proof of Lemma 20.

Let now  $\mu \in \sigma(\tilde{A})$  and  $\xi \in C^\infty([0, L])$  be such that

$$\mu \xi + \xi_x + \xi_{xxx} = 0, \quad (\text{B.24})$$

$$\xi(0) = \xi(L) = 0, \quad (\text{B.25})$$

$$\xi_x(L) = \xi_x(0). \quad (\text{B.26})$$

If  $\mu = 0$ , we assume that

$$\int_0^L (1 - \cos(x)) \xi dx = 0. \quad (\text{B.27})$$

Note that (B.27) is implied by (B.24) and (B.25) if  $\mu \neq 0$ . One has the following lemma.

**Lemma 21** *Proposition 18 holds for*

$$y(t, x) := e^{\mu t} \xi(x). \quad (\text{B.28})$$

(Again (B.24) to (B.28) imply (B.4) to (B.6).) Let us prove this lemma. Let us first treat the case where  $\mu \neq 0$ . Then, since, by [12, Remark 3.6 (ii)],

$$\sigma(A) \cap \sigma(\tilde{A}) = \{0\}, \quad (\text{B.29})$$

$\mu \notin \sigma(A)$ . Hence

$$(\xi_x(L) = 0) \Rightarrow (\xi = 0) \quad (\text{B.30})$$

and there exists one and only one  $\phi \in C^\infty([0, L])$  such that

$$\begin{aligned} \mu \phi + \phi_x + \phi_{xxx} &= \xi \sin(x), \\ \phi(0) &= \phi(L) = 0, \\ \phi_x(L) &= 0. \end{aligned}$$

We then proceed as in the proof of Lemma 20 and get the desired property.

Let us now turn to the case where  $\mu = 0$ . Then (B.24) to (B.27) imply the existence of  $c \in \mathbb{C}$  such that

$$\xi(x) = c \sin(x). \quad (\text{B.31})$$

We now define  $\phi \in C^\infty([0, L])$  by

$$\phi(x) := \frac{1}{6} \left( x - x \cos(x) + \frac{1}{2} \sin(2x) - \sin(x) \right).$$

We have

$$\begin{aligned}\phi_x + \phi_{xxx} &= \sin^2(x) - \frac{1}{3}(1 - \cos(x)), \\ \phi(0) = \phi(L) = \phi_x(0) = \phi_x(L) &= 0,\end{aligned}$$

which, with (B.1) and (B.2), leads to

$$\int_0^L z \sin^2(x) dx = \frac{1}{3} \int_0^L (1 - \cos(x)) z dx + \frac{d}{dt} \int_0^L z \phi. \quad (\text{B.32})$$

But, from (B.1) and (B.2), we get

$$\frac{d}{dt} \int_0^L (1 - \cos(x)) z dx = 0,$$

which, with (B.3), gives

$$\int_0^L (1 - \cos(x)) z dx = 0. \quad (\text{B.33})$$

Equality (B.7) follows from (B.3), (B.28), (B.31), (B.32) and (B.33). This ends the proof of Lemma 21.

Let  $\mathcal{E}$  be the subspace of  $L^2(0, T) \times L^2(0, L)$  spanned by the following pairs

- $(Y_{\lambda_x}(\cdot, L), Y_\lambda(0, \cdot))$  with  $\lambda \in \mathbb{C} \setminus \sigma(A)$ ,
- $(e^{\mu t} \xi_x(L), \xi)$  with  $\mu \in \sigma(\tilde{A})$  and  $\xi \in C^\infty([0, L])$  satisfying (B.24), (B.25), (B.26) and (B.27).

Let us point out that, as in section 2, we still have (2.52). Therefore  $\mathcal{E} \subset L^2(0, T) \times H$ . We have

**Lemma 22** *For every  $T > 0$ ,  $\mathcal{E}$  is dense in  $L^2(0, T) \times H$ .*

Let us prove this Lemma. Let  $v \in L^2(0, T)$  and  $h \in L^2(0, L)$  such that  $(v, h) \in \mathcal{E}^\perp$ . Thus,  $\forall \lambda \in \mathbb{C} \setminus \sigma(A)$ ,

$$\int_0^T \bar{v}(t) e^{\lambda t} dt + \int_0^L \bar{h}(x) y_\lambda(x) dx = 0. \quad (\text{B.34})$$

Let us give an estimate on  $y_\lambda$  in order to prove that  $v = 0$ . We multiply (B.14) by  $\bar{y}_\lambda$ , take the real part and integrate on  $[0, L]$ . Then, using integrations by parts together with (B.15) and (B.16), we get

$$2\text{Re}(\lambda) \int_0^L |y_\lambda|^2 dx = 1 - |y_{\lambda_x}(0)|^2. \quad (\text{B.35})$$

From this equality we get that

$$\|y_\lambda\|_{L^2(0, L)} \leq \frac{1}{2\text{Re}(\lambda)} \text{ if } \text{Re}(\lambda) > 0. \quad (\text{B.36})$$

The same computations show that, if  $w \in \mathcal{D}(A)$  satisfies  $Aw = \lambda w$  with  $\lambda \in \mathbb{C}$  then  $2\text{Re}(\lambda) \int_0^L |w|^2 dx = -|w_x(0)|^2$ . Therefore  $\sigma(A) \subset \{\lambda \in \mathbb{C}; \text{Re}(\lambda) \leq 0\}$ . Hence by (B.34) and (B.36) the holomorphic function  $\lambda \in \mathbb{C} \rightarrow \int_0^T \bar{v}(t) e^{\lambda t} dt$  is bounded and converges to 0 as  $\text{Re}(\lambda)$  tends to  $+\infty$ . Therefore this holomorphic function is identically equal to 0, which implies that  $v = 0$ .

Let us now prove that  $h \in H^\perp$ . We have

$$\int_0^T \bar{h} \xi dx = 0 \quad (\text{B.37})$$

for every  $\xi \in C^\infty([0, L])$  satisfying (B.24) for some  $\mu \in \sigma(\tilde{A})$ , (B.25), (B.26) and (B.27). But, since  $i\tilde{A}$  is selfadjoint with compact resolvent, it follows from the spectral decomposition of such operators that the vector space spanned by such  $\xi$  is dense in  $H$ . Hence, by (B.37),  $h \in H^\perp$ .

Let us now end the proof of Proposition 18 by a density argument on  $y$ . Let  $z$  be as in the hypotheses of Proposition 18. Let

$$\begin{aligned} F : L^2(0, T) \times H &\rightarrow \mathbb{C} \\ (u, \phi) &\mapsto \int_0^T \int_0^L y z dx dt, \end{aligned}$$

where  $y$  is the mild solution of

$$y_t + y_x + y_{xxx} = 0, \quad y(t, 0) = y(t, L) = 0, \quad y_x(t, L) = u(t), \quad y(0, x) = \phi(x).$$

By Lemma 20 and Lemma 21, this linear map  $F$  vanishes on  $\mathcal{E}$ . By Proposition 7,  $F$  is continuous. Hence  $F$  vanishes on the closure of  $\mathcal{E}$ , which, by Lemma 22, is equal to  $L^2(0, T) \times H$ . This ends the proof of Proposition 18.

## C Proof of Lemma 11.

Let  $b \in \mathbb{R} \setminus [-1/\sqrt{3}, 1/\sqrt{3}]$  be such that  $-2\lambda = 2ib(4b^2 - 1)$ . We have

$$y_{-2\lambda} = \eta e^{(-\sqrt{3b^2-1}-ib)x} + (1 - \eta) e^{(\sqrt{3b^2-1}-ib)x} - e^{2ibx}$$

with

$$\eta := \frac{e^{2ibL} - e^{(\sqrt{3b^2-1}-ib)L}}{e^{(-\sqrt{3b^2-1}-ib)L} - e^{(\sqrt{3b^2-1}-ib)L}}.$$

After some integrations par parts, we get, using (2.49) for  $-2\lambda$  instead of  $\lambda$ , (2.67) and (2.75),

$$\int_0^L (2\lambda(\delta_\lambda - \psi_\lambda) + (\delta_\lambda - \psi_\lambda)_x + (\delta_{\lambda xx} - \psi_{\lambda xxx})) y_{-2\lambda} dx = -(\delta_{\lambda_x}(L) - \psi_{\lambda_x}(L))(y_{-2\lambda_x}(L) - y_{-2\lambda_x}(0)).$$

Thus, by (2.66) and (2.74),

$$\int_0^L (z_\lambda \sin(x) - y_\lambda \phi_{\lambda_x}) y_{-2\lambda} dx = (\delta_{\lambda_x}(L) - \psi_{\lambda_x}(L))(y_{-2\lambda_x}(L) - y_{-2\lambda_x}(0)).$$

Then, in order to prove Lemma 11, one just needs to check that

$$a \rightarrow \int_0^L (z_\lambda \sin(x) - y_\lambda \phi_{\lambda_x}) y_{-2\lambda} dx \text{ is not identically 0 on } \Sigma. \quad (\text{C.38})$$

Straightforward computations give

$$\begin{aligned}
\phi_\lambda(x) = & -\frac{1}{12a(2a+1)(2a-1)} \left[ \kappa \left[ (a+1-i\sqrt{3a^2-1})e^{(-\sqrt{3a^2-1}-ia+i)x} \right. \right. \\
& \left. \left. + (a-1-i\sqrt{3a^2-1})e^{(-\sqrt{3a^2-1}-ia-i)x} \right] \right. \\
& \left. + (1-\kappa) \left[ (a+1+i\sqrt{3a^2-1})e^{(\sqrt{3a^2-1}-ia+i)x} + (a-1+i\sqrt{3a^2-1})e^{(\sqrt{3a^2-1}-ia-i)x} \right] \right. \\
& \left. + \left[ (2a-1)e^{(2ia+i)x} + (2a+1)e^{(2ia-i)x} \right] + K_1 e^{2iax} + K_2 e^{(-\sqrt{3a^2-1}-ia)x} + K_3 e^{(\sqrt{3a^2-1}-ia)x}, \right. \\
& \left. \right. \tag{C.39}
\end{aligned}$$

with

$$\begin{aligned}
K_1 := & \frac{1}{3(2a+1)(2a-1)}, \\
K_2 := & \left[ (6ia^2 - i + 2a\sqrt{3a^2-1})e^{-2iaL} + (-6ia^2 + i - 2a\sqrt{3a^2-1})e^{L(-\sqrt{3a^2-1}+ia)} + \right. \\
& (6ia^2 - 2i - 2a\sqrt{3a^2-1})e^{4iaL} + (-i - 4a\sqrt{3a^2-1})e^{2L(\sqrt{3a^2-1}-ia)} \\
& \left. + (3i - 6ia^2 + 6a\sqrt{3a^2-1})e^{L(\sqrt{3a^2-1}+ia)} \right] \\
& \left/ \left[ 6a(2a+1)(2a-1)(e^{L(-\sqrt{3a^2-1}-ia)} - e^{L(\sqrt{3a^2-1}-ia)})(-3ia + \sqrt{3a^2-1})e^{-L(\sqrt{3a^2-1}+ia)} \right. \right. \\
& \left. \left. - 2\sqrt{3a^2-1}e^{2iaL} + (3ia + \sqrt{3a^2-1})e^{L(\sqrt{3a^2-1}-ia)} \right] \right], \\
K_3 := & \left[ (-i + 4a\sqrt{3a^2-1})e^{L(-\sqrt{3a^2-1}-ia)} + (6ia^2 - i - 2a\sqrt{3a^2-1})e^{L(\sqrt{3a^2-1}-ia)} \right. \\
& \left. + (-6ia^2 + 2i - 2a\sqrt{3a^2-1})e^{2iaL} \right] \left[ e^{L(-\sqrt{3a^2-1}-ia)} - e^{2iaL} \right] \\
& \left/ \left[ 6a(2a+1)(2a-1)(e^{L(-\sqrt{3a^2-1}-ia)} - e^{L(\sqrt{3a^2-1}-ia)})(-3ia + \sqrt{3a^2-1})e^{-L(\sqrt{3a^2-1}+ia)} \right. \right. \\
& \left. \left. - 2\sqrt{3a^2-1}e^{2iaL} + (3ia + \sqrt{3a^2-1})e^{L(\sqrt{3a^2-1}-ia)} \right] \right].
\end{aligned}$$

We also have

$$\begin{aligned}
z_\lambda(x) = & k_1(x) + \left( -k_1(0) + \frac{k_1(L) - k_1(0)}{e^{(\sqrt{3b^2-1}+ib)L} - e^{(-\sqrt{3b^2-1}+ib)L}} \right) e^{(-\sqrt{3b^2-1}+ib)x} \\
& - \frac{k_1(L) - k_1(0)}{e^{(\sqrt{3b^2-1}+ib)L} - e^{(-\sqrt{3b^2-1}+ib)L}} e^{(\sqrt{3b^2-1}+ib)x} \\
& + M_\lambda(-e^{2ibx} + \bar{\eta}e^{(-\sqrt{3b^2-1}+ib)x} + (1-\bar{\eta})e^{(\sqrt{3b^2-1}+ib)x}), \tag{C.40}
\end{aligned}$$



where  $M_\lambda \in \mathbb{C}$  is a constant and

$$\begin{aligned}
k_1(x) := & -\frac{\kappa^2 i(-\sqrt{3a^2-1}-ia)}{2(24a^3-9a+3i\sqrt{3a^2-1})} e^{(-2\sqrt{3a^2-1}-2ia)x} - \frac{(1-\kappa)^2 i(\sqrt{3a^2-1}-ia)}{2(24a^3-9a-3i\sqrt{3a^2-1})} e^{(2\sqrt{3a^2-1}-2ia)x} \\
& + \frac{\kappa(1-\kappa)}{12a^2-3} e^{-2iax} + \frac{e^{4iax}}{24a^2} - \frac{\kappa i(-\sqrt{3a^2-1}+ia)}{2(12a^3-3a)} e^{(-\sqrt{3a^2-1}+ia)x} \\
& - \frac{i(1-\kappa)(\sqrt{3a^2-1}+ia)}{2(12a^3-3a)} e^{(\sqrt{3a^2-1}+ia)x}.
\end{aligned} \tag{C.41}$$

Since

$$\int_0^L (-e^{2ibx} + \bar{\eta} e^{(-\sqrt{3b^2-1}+ib)x} + (1-\bar{\eta}) e^{(\sqrt{3b^2-1}+ib)x}) y_{-2\lambda} \sin(x) dx = \int_0^L \bar{y}_{-2\lambda} y_{-2\lambda} \sin(x) dx = 0, \tag{C.42}$$

one does not need to know  $M_\lambda$  to compute the integral in (C.38). Let

$$Q := D \int_0^L (z_\lambda \sin(x) - y_\lambda \phi_{\lambda x}) y_{-2\lambda} dx,$$

with

$$\begin{aligned}
D = & \left( e^{L(-\sqrt{3a^2-1}-ia)} - e^{L(\sqrt{3a^2-1}-ia)} \right) \left( e^{L(\sqrt{3b^2-1}+ib)} - e^{L(-\sqrt{3b^2-1}+ib)} \right) \\
& \left( (-3ia + \sqrt{3a^2-1}) e^{-L(\sqrt{3a^2-1}+ia)} - 2\sqrt{3a^2-1} e^{2iaL} + (3ia + \sqrt{3a^2-1}) e^{L(\sqrt{3a^2-1}+ia)} \right)
\end{aligned}$$

Then  $Q$  is a finite sum of terms of the form  $P_c(a) e^{(c_1\sqrt{3a^2-1}+c_2ia+c_3\sqrt{3b^2-1}+c_4ib)L}$  where  $P_c(a)$  is a rational function of  $a$ ,  $b$ ,  $\sqrt{3a^2-1}$  and  $\sqrt{3b^2-1}$  and  $c := (c_1, c_2, c_3, c_4) \in \mathbb{Z}^4$ . (There are 110 such terms.) It is easy to prove that (C.38) is identically 0 (if and) only if every  $P_c$  is identically 0.

We look at  $P_{(3,1,2,2)}$ . After lengthy but straightforward computations, we get

$$\begin{aligned}
P_{(3,1,2,2)}(a) = & -(3ia + \sqrt{3a^2-1}) (f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9) \\
& + \frac{6ia^2 - i - 2a\sqrt{3a^2-1}}{6a(2a+1)(2a-1)} (f_{10} + f_{11}),
\end{aligned} \tag{C.43}$$

with

$$\begin{aligned}
f_1 = & \frac{(\sqrt{3a^2-1}-ia)i}{48a^3-18a-6i\sqrt{3a^2-1}} \left( \frac{1}{(2\sqrt{3a^2-1}-2ia+\sqrt{3b^2-1}-ib)^2+1} \right. \\
& \left. - \frac{1}{(2\sqrt{3a^2-1}-2ia+2ib)^2+1} \right), \\
f_2 = & \frac{-1}{24a^2} \left( \frac{1}{(4ia+\sqrt{3b^2-1}-ib)^2+1} - \frac{1}{(4ia+2ib)^2+1} \right),
\end{aligned}$$

$$f_3 = \frac{(\sqrt{3a^2-1}+ia)i}{24a^3-6a} \left( \frac{1}{(\sqrt{3a^2-1}+ia+\sqrt{3b^2-1}-ib)^2+1} - \frac{1}{(\sqrt{3a^2-1}+ia+2ib)^2+1} \right),$$

$$f_4 = \frac{1}{12a(2a+1)(2a-1)} \left[ \left( \frac{a+1+i\sqrt{3a^2-1}}{2\sqrt{3a^2-1}-2ia+i+\sqrt{3b^2-1}-ib} + \frac{a-1+i\sqrt{3a^2-1}}{2\sqrt{3a^2-1}-2ia-i+\sqrt{3b^2-1}-ib} \right) - \left( \frac{a+1+i\sqrt{3a^2-1}}{2\sqrt{3a^2-1}-2ia+i+2ib} + \frac{a-1+i\sqrt{3a^2-1}}{2\sqrt{3a^2-1}-2ia-i+2ib} \right) \right],$$

$$f_5 = -\frac{1}{12a(2a+1)(2a-1)} \left[ \left( \frac{a+1+i\sqrt{3a^2-1}}{\sqrt{3a^2-1}+ia+i+\sqrt{3b^2-1}-ib} + \frac{a-1+i\sqrt{3a^2-1}}{\sqrt{3a^2-1}+ia-i+\sqrt{3b^2-1}-ib} \right) - \left( \frac{a+1+i\sqrt{3a^2-1}}{\sqrt{3a^2-1}+ia+i+2ib} + \frac{a-1+i\sqrt{3a^2-1}}{\sqrt{3a^2-1}+ia-i+2ib} \right) \right],$$

$$f_6 = \frac{i}{12a} \left[ \left( \frac{1}{\sqrt{3a^2-1}+ia+i+\sqrt{3b^2-1}-ib} + \frac{1}{\sqrt{3a^2-1}+ia-i+\sqrt{3b^2-1}-ib} \right) - \left( \frac{1}{\sqrt{3a^2-1}+ia+i+2ib} + \frac{1}{\sqrt{3a^2-1}+ia-i+2ib} \right) \right],$$

$$f_7 = \frac{-i}{12a} \left[ \left( \frac{1}{4ia+i+\sqrt{3b^2-1}-ib} + \frac{1}{4ia-i+\sqrt{3b^2-1}-ib} \right) - \left( \frac{1}{4ia+i+2ib} + \frac{1}{4ia-i+2ib} \right) \right],$$

$$f_8 = \frac{-2ia}{3(2a+1)(2a-1)} \left[ \frac{1}{\sqrt{3a^2-1}+ia+\sqrt{3b^2-1}-ib} - \frac{1}{\sqrt{3a^2-1}+ia+2ib} \right],$$

$$f_9 = \frac{2ia}{3(2a+1)(2a-1)} \left[ \frac{1}{4ia+\sqrt{3b^2-1}-ib} - \frac{1}{4ia+2ib} \right],$$

$$f_{10} = -(\sqrt{3a^2-1}-ia) \left[ \frac{1}{2\sqrt{3a^2-1}-2ia+\sqrt{3b^2-1}-ib} - \frac{1}{2\sqrt{3a^2-1}-2ia+2ib} \right],$$

$$f_{11} = (\sqrt{3a^2 - 1} - ia) \left[ \frac{1}{\sqrt{3a^2 - 1} + ia + \sqrt{3b^2 - 1} - ib} - \frac{1}{\sqrt{3a^2 - 1} + ia + 2ib} \right],$$

As  $a \rightarrow +\infty$ , one gets that  $P_{(3,1,2,2)}(a) \sim p/a$  with

$$p := \frac{(\sqrt{3} - i)(6i - 2\sqrt{3})}{24} \left( -\frac{1}{2\sqrt{3} - 2i + 2^{1/3}\sqrt{3} + i2^{1/3}} + \frac{1}{2\sqrt{3} - 2i - i2^{4/3}} + \frac{1}{\sqrt{3} + i + 2^{1/3}\sqrt{3} + i2^{1/3}} - \frac{1}{\sqrt{3} + i - i2^{4/3}} \right) \neq 0.$$

In particular  $P_{(3,1,2,2)}$  is not identically equal to 0.

## References

- [1] A. Agrachev, *Newton diagrams and tangent cones to attainable sets*, in: *Analysis of Controlled Dynamical Systems (Lyon 1990)*, (B. Bonnard et al., eds.), Progr. Systems Control Theory 8, Birkhäuser, Boston, 1991, pp. 1-12.
- [2] R.M. Bianchini and G. Stefani, *Controllability along a trajectory: a variational approach*, SIAM J. Control Optim. 31 (1993), pp. 900-927.
- [3] J. Bona and R. Winther, *The Korteweg-de Vries equation posed in a quarter-plane*, SIAM J. Math. Anal., 14 (1983), pp. 1056-1106.
- [4] J.-M. Coron, *Contrôlabilité exacte frontière de l'équation d'Euler des fluides parfaits incompressibles bidimensionnels*, C.R. Acad. Sci. Paris, 317 (1993), pp. 271-276.
- [5] J.-M. Coron, *On the controllability of 2-D incompressible perfect fluids*, J. Math. Pures & Appliquées, 75 (1996), pp. 155-188.
- [6] J.-M. Coron, *Local controllability of a 1-D tank containing a fluid modeled by the shallow water equations*, ESAIM: COCV, 8 (2002), pp. 513-554.
- [7] J.-M. Coron and E. Trélat, *Global steady-state controllability of 1-D semilinear heat equations*, preprint, Université Paris-Sud, 2003, submitted for publication in SIAM J. Control and Opt.
- [8] E. Crépeau, *Exact controllability of the Korteweg-de Vries equation around a non trivial stationary solution*, Int. J. Control, 74 (2001), pp. 1096-1106.
- [9] H. Hermes, *Controlled stability*, Anali di Matematica pura ed applicata, 54 (1977), pp. 103-119.
- [10] M. Kawski, *High-order small time local controllability*, in: *Nonlinear Controllability and Optimal Control*, (H.J. Sussmann, ed.), Monogr. Textbooks Pure Appl. Math. 113, Dekker, New York, 1990, pp. 431-467.
- [11] S. Micu and J.H. Ortega, *On the controllability of a linear coupled system of Korteweg-de Vries equations*, Proceedings of the Fifth International Conference on mathematical and numerical aspects of wave propagation, Santiago de Compostela, Spain, July 2000, SIAM Proceedings Series, Philadelphia (2000) pp. 1020-1024.
- [12] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM: COCV, 2 (1997), pp. 33-55.

- 
- [13] L. Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equations on a bounded domain*, ESAIM: Proceedings, 4 (1998), pp. 255-267.
- [14] L. Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line*, SIAM J. Control Optim., 39 (2000), pp. 331-351.
- [15] L. Rosier, *Control of the surface of a fluid by a wavemaker*, submitted to ESAIM: COCV.
- [16] D.L. Russell and B-Y. Yang, *Exact controllability and stabilizability of the Korteweg-de Vries equation*, Trans. Amer. Math. Soc., 348 (1996), pp. 3643-3672.
- [17] H.J. Sussmann, *Lie brackets and local controllability : a sufficient condition for scalar-input systems*, SIAM J. Control Optim., 21 (1983), pp. 686-713.
- [18] H.J. Sussmann, *A general theorem on local controllability*, SIAM J. Control Optim., 25 (1987), pp. 158-194.
- [19] G.B. Whitham, *Linear and nonlinear waves*, John Wiley, New York, 1974.
- [20] A.I. Tret'yak, *On odd-order necessary conditions for optimality in a time-optimal control problem for systems linear in the control*, Math. USSR Sbornik, 79 (1991), pp. 47-63.



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