Control of a maneuvering mobile robot by the transverse function approach : control design and simulation results

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N° 5155

Mars 2004

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5155--FR+ENG

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Thème 4 — Simulation et optimisation
de systèmes complexes
Projet ICARE

Rapport de recherche n° 5155 — Mars 2004 — 16 pages

Abstract: Based on the transverse function approach recently developed by two of the authors for the control of general nonlinear driftless systems, a control strategy for tracking an omnidirectional target with a unicycle-like robot is proposed. An original feature of the approach is the capacity to comply with a target which moves freely in the plane and performs motions which are not feasible by the nonholonomic robot. With respect to a previous publication on the subject, the proposed control solution involves two extensions in order to i) limit the control magnitude and the number of maneuvers when the initial tracking errors are large, and ii) adapt automatically the tracking precision depending on whether or not the target's motion corresponds to a feasible trajectory for the nonholonomic robot. Simulation results illustrate the practical usefulness of these extensions.

Key-words: Practical stabilization, target tracking, nonholonomic robot, transverse functions

This work is supported by the European Commission through the IST research program CyberCars.

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Automatisation des manœuvres d'un robot mobile en utilisant l'approche par fonction tranverse: Synthèse de la commande et simulations

Résumé : Une nouvelle stratégie de commande pour le suivi d'une cible omnidirectionnelle avec un robot unicycle est présentée. Basée sur l'utilisation de fonction transverse, notion récemment développée par deux des auteurs dans le cadre du contrôle des systèmes non-linéaires sans dérive, l'originalité de cette approche est que la cible n'est pas contrainte dans ces mouvements et peut donc suivre des trajectoires non réalisables par le robot non-holonôme. Par rapport à notre travail précédent sur le sujet [1, 2], la commande a été améliorée dans deux directions principales. D'une part, on a cherché à limiter l'amplitude du contrôle et le nombre de manœuvres lorsque l'erreur initiale de suivi est grande, et d'autre part une adaptation automatique de la précision du suivi est réalisée en fonction de la capacité du robot à suivre exactement ou non la trajectoire de la cible. Des résultats de simulation sont présentés afin d'illustrer l'intérêt pratique de cette approche.

Mots-clés : stabilisation pratique, suivi de cible, robot non-holonôme, fonction transverse

Contents

Introduction

As in [1, 2] we consider the problem of tracking a vehicle (called *target* from now on) with a nonholonomic unicycle-like robot. Unlike the robot, the target is not subjected to non-holonomic constraints. It can move freely in the plane and thus perform motions which may not be feasible for the robot. Such a situation occurs, for example, when the control problem consists in tracking a (virtual) frame —representing the target— located behind, and rigidly linked to, the body of a wheeled vehicle —a car, for instance. This would correspond to a classical car-platooning problem except that we would like to extend the operating domain of the control to the case when the leading vehicle moves backward and makes maneuvers. It is then not difficult to verify that such maneuvers usually yield trajectories of the target frame which are not feasible for a nonholonomic vehicle.

To our knowledge, the problem of tracking non-feasible trajectories has seldom been addressed in the control literature devoted to nonholonomic systems. This is much related to the fact that feedback control studies traditionally focus on asymptotic stabilization (see e.g. [6, 3]), whereas a non-feasible trajectory cannot, by definition, be asymptotically stabilized. In [5], a general framework, based on the use of so-called transverse functions, has been proposed for the design of feedback laws yielding practical stabilization of controllable driftless systems subjected to additive perturbations. This approach has been applied in [1, 2] to solve the target tracking problem here considered. In the present paper, the control methodology is further extended along two directions. The first issue is related to the determination of a vector field term which, in the control expression, ensures the stability of the controlled system and the property of practical stabilization. The extension here proposed aims at monitoring the transient behavior associated with large initial tracking errors in order to reduce the control effort and the number of robot maneuvers. The method relies on the constrained minimization of the control norm. The second issue involves the introduction of new transverse functions and additional design parameters which are used to automatically adjust the tracking precision depending on the nature of the target's motion. The underlying motivation is that small tracking errors are expected when the target's trajectory is feasible, whereas larger tracking errors are required to avoid fast oscillatory maneuvers otherwise.

The paper is organized as follows. Models for control design are introduced in Section 1. Control design issues are addressed in Section 2. Simulation results are reported in Section 3.

1 Modeling

Let us consider the three frames represented on Figure 1: $\mathcal{F}_0$ is a fixed frame, $\mathcal{F}$ is a frame attached to the unicycle, and $\mathcal{F}_r$ is a frame attached to the target. Let (x_m, y_m) denote the coordinates of $\overrightarrow{\mathrm{OP}}$ in $\mathcal{F}_0$, and α_m denote the angle between $\vec{\imath}_0$ and $\vec{\imath}$ (see Figure 1). The control inputs of the robot are the longitudinal velocity u_1 along the vector $\vec{\imath}$ of $\mathcal{F}$ and the angular velocity $u_2 = \dot{\alpha}_m$. With these notations, the well known kinematic equations of the unicycle are

$$\begin{cases} \dot{x}_m &= u_1 \cos \alpha_m \\ \dot{y}_m &= u_1 \sin \alpha_m \\ \dot{\alpha}_m &= u_2 \end{cases} \tag{1}$$

For stabilization purposes, we want to describe the kinematics of the robot with respect to the target. Let (x_r, y_r) denote the coordinates of $\overrightarrow{\mathrm{OP}}_r$ in $\mathcal{F}_r$, and α_r denote the angle between $\vec{\imath}_0$ and $\vec{\imath}_r$. The two components of the velocity of P_r, expressed in the basis of the target frame, are denoted as a and b, and the target angular velocity is denoted as c, i.e.

$$\begin{cases} \frac{d\overrightarrow{\mathrm{OP}}_r}{dt} &= a\vec{\imath}_r + b\vec{\jmath}_r \\ \dot{\alpha}_r &= c \end{cases} \tag{2}$$

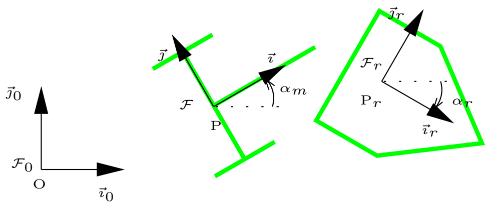


Figure 1: Configuration frames

The target velocity vector is $u_t = (a, b, c)^T$. With x and y denoting the coordinates of the position error vector between the robot and the target, expressed in the target frame $\mathcal{F}_r$, i.e. $\overrightarrow{\mathrm{P}_r\mathrm{P}} = x\vec{\imath}_r + y\vec{\jmath}_r$, and $\alpha = \alpha_m - \alpha_r$, one infers from (1) and (2)

$$\left\{\begin{array}{rcl} \dot{x} & = & u_1 \cos\alpha + cy - a \\ \dot{y} & = & u_1 \sin\alpha - cx - b \\ \dot{\alpha} & = & u_2 - c \end{array}\right. \tag{3}$$

System (3) represents the error system associated with the target tracking problem. It can also be written as

$$\dot{g} = u_1 b_1(g) + u_2 b_2 + b_0(g, u_t) \tag{4}$$

with $g = (x, y, \alpha)^T \in G := \mathbb{R}^2 \times \mathbb{T} \approx SE(2)$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, and

$$b_0(g, u_t) = (cy - a, -cx - b, -c)^T \tag{5}$$

$$b_1(g) = (\cos\alpha, \sin\alpha, 0)^T,\ b_2 = (0, 0, 1)^T \tag{6}$$

Note that $b_0 = 0$ if and only if the target is motionless.

2 Control design

The proposed control strategy is based on the *transverse function* approach [5] here applied to a unicycle-like robot, as in [1, 2]. Prior to presenting new developments of interest for our application, let us first review a few basic features of the approach (see the above mentioned references for more details).

2.1 Recalls

Let us temporarily focus on System (4) without the drift term b_0, i.e.

$$\dot{g} = u_1 b_1(g) + u_2 b_2 \tag{7}$$

A smooth function $f : \mathbb{T}^2 \longrightarrow G$ is called a *transverse function* for System (7) if, for any $(\theta, \beta) \in \mathbb{T}^2$, the matrix

$$H\left(f(\theta,\beta)\right) \triangleq \begin{pmatrix} b_1\left(f(\theta,\beta)\right) & b_2 & -\dfrac{\partial f}{\partial \theta}(\theta,\beta) \end{pmatrix} \tag{8}$$

is invertible. The dependence on the second variable β is not required in the original formulation of the approach, but it is here introduced for the sake of generalization. From now on, the variables θ and β should be interpreted as additional state components which are controlled via their time derivatives $\dot{\theta}$ and $\dot{\beta}$.

Recall that a Lie group G is a differentiable manifold equipped with a smooth group operation. The natural Lie group involved in the present application is $G = \mathbb{R}^2 \times \mathbb{T} \approx SE(2)$ endowed with the group operation $(g, g') \longmapsto gg'$ defined by

$$gg' = \begin{pmatrix} p + R(\alpha)p' \\ \alpha + \alpha' \end{pmatrix} \tag{9}$$

with $g = (p, \alpha)^T$, $g' = (p', \alpha')^T$, p and $p' \in \mathbb{R}^2$, and $R(\alpha)$ the rotation matrix of angle α. We denote by $e = (0, 0, 0)^T$ the unit element of this group. Note that we could equivalently define G as the group of homogeneous matrices in the plane, and the group operation as the product of these matrices. It is well known, and straightforward to verify, that System (7) defines a left-invariant control system on G, i.e., for any solution $g(.)$ of (7) with a control $(u_1(t), u_2(t))$ and any $g_0 \in G$, $g_0 g(.)$ is also a solution to (7) with the same control.

Let us finally define the variable z as the group product of g and $f^{-1}(\theta, \beta)$, with $f^{-1}(\theta, \beta)$ denoting the inverse of $f(\theta, \beta)$ on the group, i.e.

$$z := gf^{-1}(\theta, \beta) \tag{10}$$

The core of the control approach is the asymptotic stabilization of z to e by using the control variables u_1, u_2, and $\dot{\theta}$. The following proposition shows that this is a simple task, when f is a transverse function.

Proposition 1 *Let $f := (f_1, f_2, f_3)^T$ denote a transverse function, and $\bar{u}$ denote the "extended" control vector defined by $\bar{u} := (u_1, u_2, \dot{\theta})^T$. Then,*

i) along any solution $g(.)$ of (4)*, and any smooth time functions $\theta(.)$ and $\beta(.)$,*

$$\dot{z} = \bar{H}\bar{u} - w \tag{11}$$

with $\bar{H} = AH(f)$, $w = A\left(\frac{\partial f}{\partial \beta}\dot{\beta} - Bb_0(g, u_t)\right)$, and

$$A = \begin{pmatrix} R(\alpha - f_3) & R(\alpha - f_3)\begin{pmatrix} f_2 \\ -f_1 \end{pmatrix} \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} R(f_3 - \alpha) & 0 \\ 0 & 1 \end{pmatrix}$$

ii) given an arbitrary vector valued function $v \in C^0(G \times \mathbb{T}^2 \times \mathbb{R}; \mathbb{R}^3)$, the application of the following dynamic feedback law

$$\bar{u} = \bar{H}^{-1}(w + v) \tag{12}$$

applied to System (11) *yields $\dot{z} = v$.*

The proof of this result, easily obtained by differentiating the equality $zf = g$, is left to the reader. The part ii) of the above proposition tells us that any function v which ensures the asymptotic stability of the origin of the system $\dot{z} = v$ yields the convergence of the solutions of the controlled system (4,12) to $f(\mathbb{T}^2)$, *whatever* the "perturbation" term b_0 in (4) —and thus *whatever* the target's velocity. In other words the vector error g tends to $f(\mathbb{T}^2)$ when the control defined by (12) is used, provided that v asymptotically stabilizes the origin of the decoupled linear system $\dot{z} = v$. Moreover, the rate of convergence of z to zero is the same as the rate of convergence of g to $f(\mathbb{T}^2)$. To obtain exponential convergence, a simple possibility obviously consists in setting

$$v(z, \theta, \beta, t) = -Kz \tag{13}$$

with $(-K)$ denoting a Hurwitz stable matrix. In [1, 2], this possibility is further simplified by choosing K diagonal and positive definite. Now, since $\mathbb{T}^2$ is compact and f is smooth, the set $f(\mathbb{T}^2)$ is bounded. In particular, by choosing f such that this set is contained in a small neighborhood of zero, then g is ultimately close to zero.

2.2 Transient dynamics adjustment

As explained above, the feedback control law used in [1, 2] is

$$\bar{u}_{zl} = \bar{H}^{-1}\left(w - Kz\right) \tag{14}$$

with K a positive definite matrix, and the index zl reminding us that this control yields a linear system in the z variable. When no further requirement is made on the controlled system, choosing the control term v as in (13) makes sense, since it combines simplicity and fast (exponential) convergence. However, there are many other possible choices. Also, nothing proves that, during the transient phase when $|z(t)|$ converges to zero from an initially large value $z(0)$, the robot's trajectory in the cartesian space resulting from this choice has interesting properties. As a matter of fact, simulations show that this choice tends to produce an unnecessary large number of maneuvers. In order to improve on this aspect, we propose that v be instead chosen so as to minimize the norm of a function of the control $\bar{u}$, under the constraint of having $|z|$ still converge exponentially to zero. More precisely, we consider the following optimization problem

$$\begin{cases} \min_{\bar{u}} J := \frac{1}{2}\left\|\bar{u} - \dot{\beta}H^{-1}\frac{\partial f}{\partial \beta}\right\|_{\delta}^{2} \\ z^T(\bar{H}\bar{u} - w) = -z^T K z \end{cases} \tag{15}$$

with $\|.\|_\delta$ defined by $\|x\|_\delta^2 = \delta_1(x_1^2 + x_2^2) + \delta_2 x_3^2$, $\delta_1, \delta_2 > 0$, and K a positive definite matrix. Note that, by (11), the constraint in (15) is equivalent to $z^T\dot{z} = -z^T K z$. The solution to this problem is easily derived by applying the standard method of Lagrange multipliers — see Appendix A for details.

Lemma 1 *The solution to the optimization problem* (15) *is*

$$\begin{pmatrix} u \\ \dot{\theta} \end{pmatrix}^{\star} = \bar{u}^{\star} = \begin{pmatrix} -\frac{1}{\delta_1 \eta}(z^T K z + z^T \tilde{w})\bar{H}_1^T z \\ -\frac{1}{\delta_2 \eta}(z^T K z + z^T \tilde{w})\bar{H}_2^T z \end{pmatrix} + \dot{\beta}H^{-1}\frac{\partial f}{\partial \beta} \tag{16}$$

with

$$\tilde{w} := ABb_0(g, u_t) \quad , \quad \eta = z^T\left(\delta_1^{-1}\bar{H}_1\bar{H}_1^T + \delta_2^{-1}\bar{H}_2\bar{H}_2^T\right)z \tag{17}$$

and $\bar{H}_1 \in \mathbb{R}^{3\times 2}, \bar{H}_2 \in \mathbb{R}^{3\times 1}$ *the matrices defined by* $\bar{H} = (\bar{H}_1, \bar{H}_2)$.

When $\tilde{w} = 0$, the feedback law (16) can be defined at $z = 0$ by continuity. Otherwise, this feedback is not defined at $z = 0$. To circumvent this difficulty, and retain the qualities of the above optimal solution when $|z|$ is large, we propose to combine the feedback laws (14) and (16) as specified in the following proposition.

Proposition 2 *Let*

$$\bar{u} = \left(1 - \frac{\eta}{\eta + \psi}\right)\bar{u}_{zl} + \frac{\eta}{\eta + \psi}\bar{u}^{\star} \tag{18}$$

with $\bar{u}_{zl}$, $\bar{u}^\star$, *and* η *defined by* (14), (16), *and* (17) *respectively, and* ψ *a positive constant number used for regularization purposes. Then, the feedback control law* (18) *is well defined everywhere, and the origin* $z = 0$ *of the closed-loop system* (11)–(18) *is exponentially stable (because* $z^T\dot{z} = -z^T K z$ *along any trajectory of this system).*

The proof of Proposition 2 is given in Appendix B.

2.3 Monitoring of the tracking precision

The choice of the transverse function f is obviously of central importance. In [1, 2] the following functions, depending on a single variable θ, and two positive parameters $(\varepsilon_1, \varepsilon_2)$, were used:

$$f(\theta) = (\varepsilon_1 \sin\theta, \frac{\varepsilon_1\varepsilon_2}{4}\sin 2\theta, \varepsilon_2\cos\theta)^T \tag{19}$$

with $\varepsilon_1, \varepsilon_2 > 0$. Note that by setting $\bar{f}(\theta,\beta) = f(\theta+\beta)$ the above function is formally transformed into a transverse function which depends on two variables, as specified at the beginning of Section 2.1. Therefore, in order to use the function (19) in the control laws derived earlier, one only has to set $\dot{\beta} = 0$ in the control expressions. Note also that the "size" of f, which determines the ultimate bound of the tracking errors, is directly related to ε_1 and ε_2.

The next proposition points out another set of transverse functions which depend on the variable β in a more useful way, and also on a third parameter γ.

Lemma 2 *For any $\varepsilon_1, \varepsilon_2 > 0$, and any $\gamma \in \mathbb{R}$, $f : \mathbb{T}^2 \longrightarrow G$ defined by*

$$f = \left(\bar{f}_1, \bar{f}_2, \arctan \bar{f}_3\right)^T \tag{20}$$

with

$$\begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \bar{f}_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_1\left(\sin(\theta+\beta) - \gamma\sin\beta\right) \\ \frac{\varepsilon_1\varepsilon_2}{2}\left(\left(\sin(\theta+\beta) - \gamma\sin\beta\right)\left(\cos(\theta+\beta) - \gamma\cos\beta\right) - \gamma\sin\theta\right) \\ \varepsilon_2\left(\cos(\theta+\beta) - \gamma\cos\beta\right) \end{pmatrix} \tag{21}$$

is a transverse function.

The proof of Lemma 2 can be found in Appendix C.

The variable β and the parameter γ represent extra degrees of freedom which can be used to "monitor" the size of $f(\theta,\beta)$ and, in doing so, achieve complementary control objectives. This possibility is illustrated by the following proposition — see Appendix D.

Proposition 3 *Assume that the dynamic feedback law $\bar{u}_{zl}$ given by (14), or $\bar{u}$ given by (18), is applied to the control system (4), with the transverse function f defined by (20,21). Assume also that the target velocity vector $u_t = (a,b,c)^T$ is bounded, then:*

1) For $\gamma = 1$ and $\dot{\beta} = k\tan(\theta/2)$ $(k > 0)$ with $\theta(0) = 0$, $g = 0$ is an asymptotically stable equilibrium point if $b_0 = 0$ (i.e. if the target does not move).

2) For $\gamma \in (0,1)$, and

$$\dot{\beta} = \frac{k}{1+\gamma^2-2\gamma\cos\theta}\tan\frac{\theta}{2} + \frac{2}{\varepsilon_1\varepsilon_2}\frac{\bar{f}_3(c\bar{f}_2 - a) + c\bar{f}_1}{1+\gamma^2-2\gamma\cos\theta} \tag{22}$$

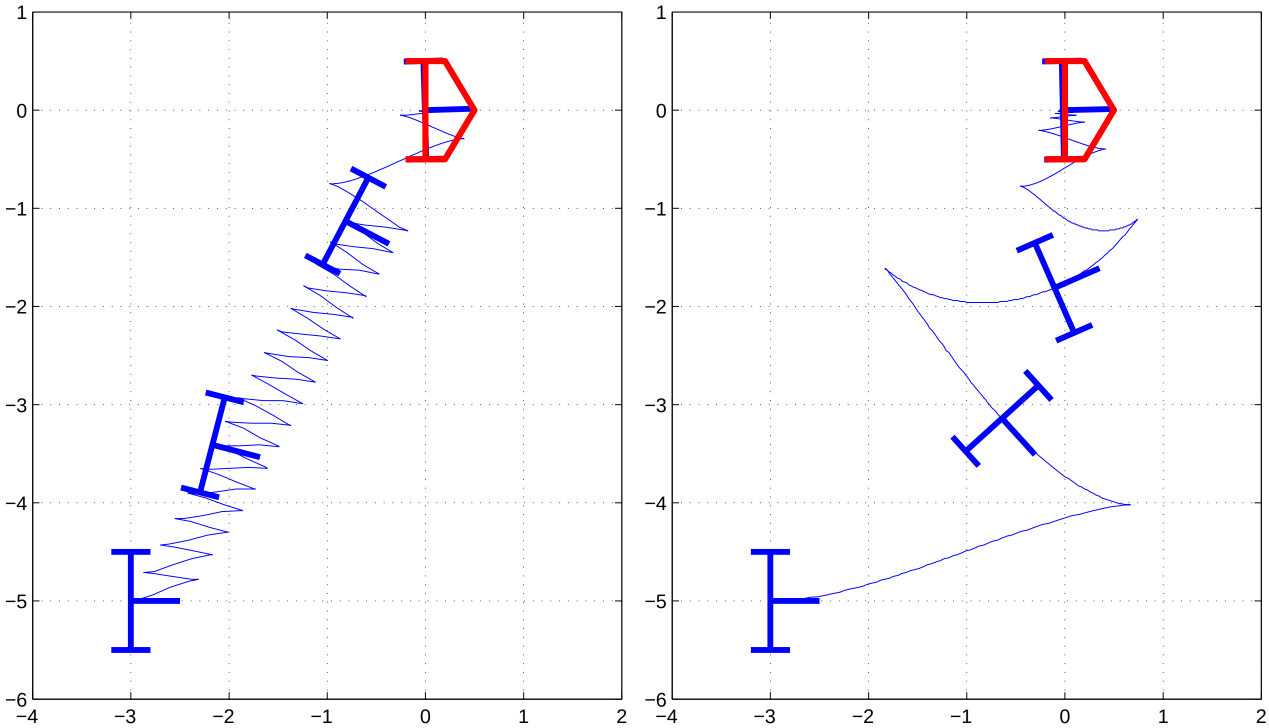
if the component b of the target velocity vector in (2) is equal to zero, then $\theta(t)$ converges to zero exponentially. In this case, the convergence of $g(t)$ to $f(\mathbb{T}^2)$ implies that the norm of the tracking error g is ultimately bounded by $(1-\gamma)\|(\varepsilon_1, \varepsilon_2, \frac{\varepsilon_1\varepsilon_2}{4})\|$.

3) In all cases, the convergence of $g(t)$ to $f(\mathbb{T}^2)$ ensures that the norm of the tracking error g is ultimately bounded by $2\|(\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)\|$.

The first part of the proposition indicates that asymptotic stabilization of the target frame can be achieved when the target is motionless, whereas the second part points out the controls's capacity to automatically augment the tracking precision when the target's trajectory is feasible for the robot. Recall from (2) that it is feasible when $b = 0$. By choosing γ close to one, small tracking errors are obtained in this case. When the trajectory is not feasible $(b \neq 0)$, it is intuitively preferable to reduce the tracking precision so as to lower the frequency of the maneuvers and, subsequently, the overall control effort. In all cases, as pointed out in the third part of the proposition, the tracking errors are ultimately bounded by a number whose size is adjustable via the choice of the parameters ε_1 and ε_2.

3 Simulation results

Figure 2 shows trajectories of the robot's point P when the target is motionless and the initial tracking errors are large. The control applied to the robot is given by (14) in the case of Figure 2(a), and (18) in the case of Figure 2(b). The matrix K used in both cases is equal to $0.5\mathrm{I}_3$, with I_3 the identity matrix. The parameters of the control (18) are as follows: $\delta_1 = 1$, $\delta_2 = 10$, $\psi = 0.01$. In both cases the transverse function (20,21) is used with ε_1, $\varepsilon_2 = 0.3$, and $\gamma = 0.9$. The time derivative of β is given by (22) with $k = 0.2$. One can observe that, as anticipated, the control (18) produces much



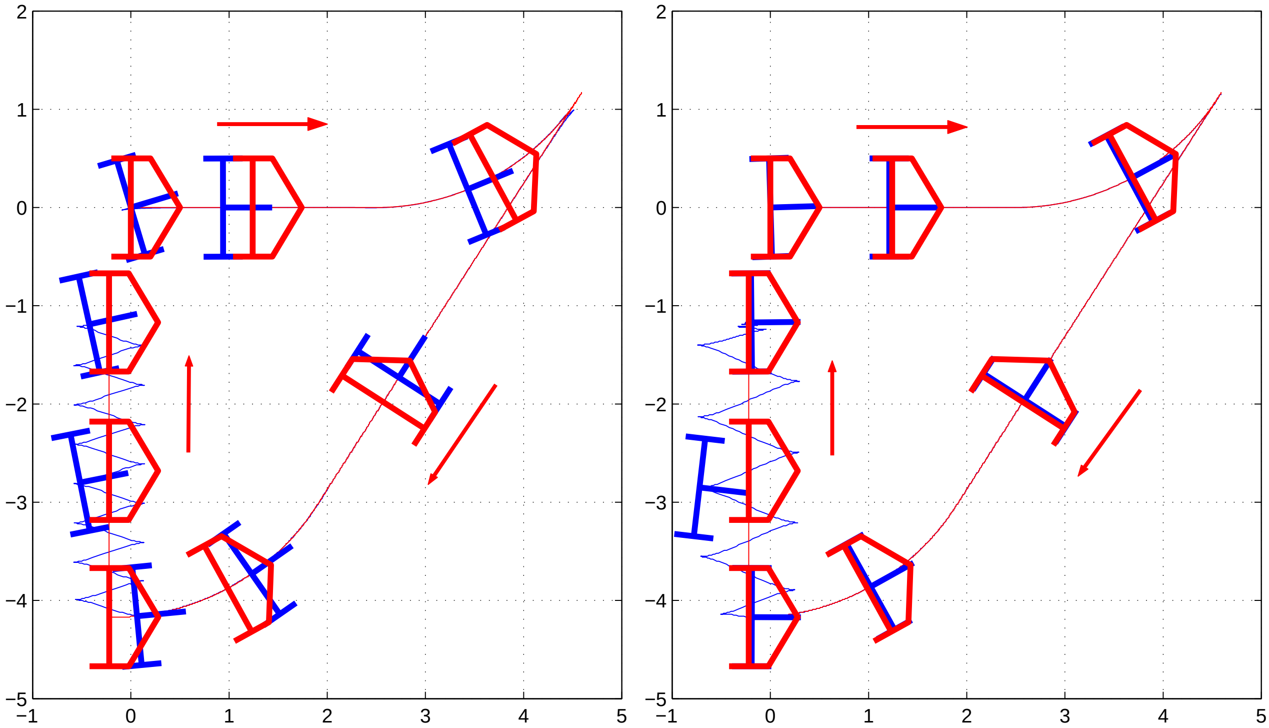
(a) Control law (12)

(b) Control law (18)

Figure 2: Cartesian motion: robot and target

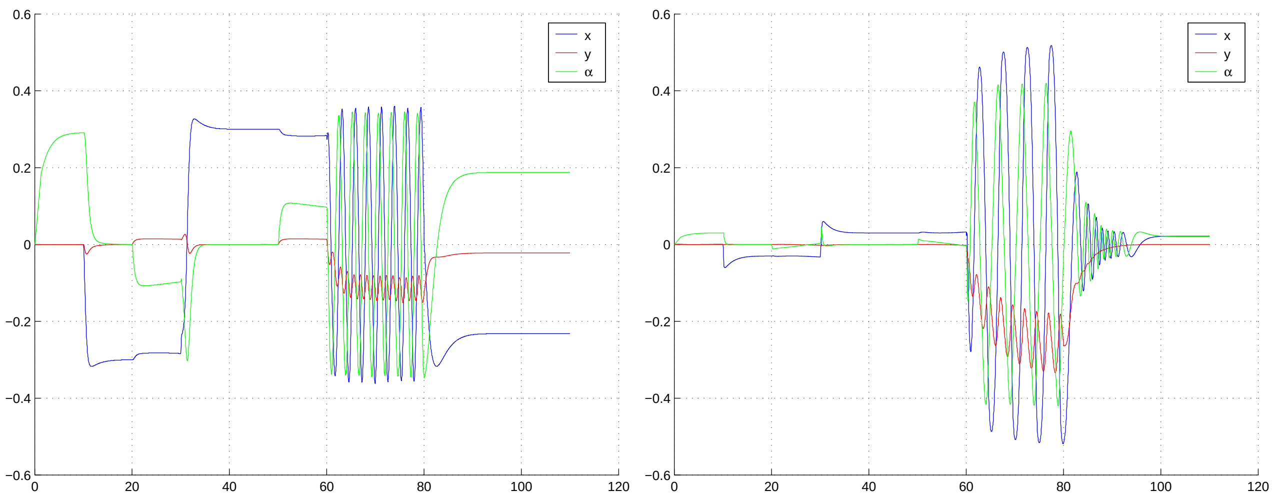
fewer maneuvers than the control (14). Once z has converged to zero, using one control or the other is indifferent, because they yield the same zero dynamics.

Figure 3 illustrates what happens in the long range (i.e. on the zero dynamics $z = 0$), and show how the monitoring mechanism described in Section 2.3, associated with the second variable β in the transverse function, modifies the tracking precision depending on the target's motion. The transverse function (19), which depends on a single variable, is used for the simulation results of Figure 3(a), with no monitoring of the tracking precision in this case. The transverse function defined by (20,21), and the adaptation mechanism as specified by (22), are used for the simulation results of Figure 3(b). All parameters are the same as before. In these simulations the target starts moving along a trajectory which is feasible for the unicycle (i.e. $b = 0$), with a positive longitudinal velocity at first (forward motion, $a > 0$), then with a negative one (backward motion, $a < 0$). During this part of the simulation, the tracking precision is much better for the right figure, due to the action of the adaptation mechanism. This is confirmed by Figure 4 which shows the components of the tracking error g (Compare Figure 4(a) and 4(b) for $t \in [0, 60]s$). During the last part of the simulations, the target performs a lateral motion ($b > 0$, $a = c = 0$) which is not feasible for the robot. One can



(a) Control law using the transverse function (19) (b) Control law using the transverse function (20,21)

Figure 3: Cartesian motion: robot and target



(a) Control law using the transverse function (19) (b) Control law using the transverse function (20,21)

Figure 4: Tracking error g between the robot and the target

then observe the maneuvers that the robot has to perform to track the target. Whereas the tracking precision has not changed in the case of the control simulated in the first figure, it has automatically been lowered in the case of the control simulated in the second figure so as to reduce the frequency of the robot's maneuvers — Compare Figure 4(a) and 4(b) for $t \in [60, 80]s$.

Conclusions

We have presented extensions of the transverse function control approach applied to the problem of tracking an omnidirectional target frame with a nonholonomic unicycle-like mobile robot, with simulations illustrating their practical usefulness. The proposed control solutions have also been tested on our experimental benchmark —a description of which can be found in [7]. Experimenting on a physical system involves complementary issues (control discretization, control saturation, state reconstruction, target velocity estimation, etc.) which have been addressed in [1, 2]. The extension of the proposed control approach to car-like mobile robots will be the subject of future articles.

A Proof of Lemma 1

We consider the following optimization problem

$$\begin{cases} \min_{\bar{u}} J := \frac{1}{2}\left\|\bar{u} - \dot{\beta}H^{-1}\frac{\partial f}{\partial \beta}\right\|_{\delta}^{2} \\ z^T(\bar{H}\bar{u} - w) = -z^T K z \end{cases} \tag{23}$$

with $\|.\|_\delta$ defined by $\|x\|_\delta^2 = \delta_1(x_1^2 + x_2^2) + \delta_2 x_3^2$, $\delta_1, \delta_2 > 0$, and K a positive definite matrix. Define the Lagrangien

$$L := J + \lambda\left(z^T\left(\begin{pmatrix}\bar{H}_1 & \bar{H}_2\end{pmatrix}\left(\bar{u} - \dot{\beta}H^{-1}\frac{\partial f}{\partial \beta}\right) + \tilde{w}\right) + z^T K z\right) \tag{24}$$

with λ the Lagrange parameter, $\bar{H}_1 \in \mathbb{R}^{3\times 2}, \bar{H}_2 \in \mathbb{R}^{3\times 1}$ the matrices defined by $\bar{H} = (\bar{H}_1, \bar{H}_2)$, and $\tilde{w} := ABb_0(g, u_t)$. The solution to the optimization problem (23) satisfies the following equation:

$$\frac{\partial L}{\partial u} = \delta_1\left(u - \left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)\dot{\beta}H^{-1}\frac{\partial f}{\partial \beta}\right)^T + \lambda z^T\bar{H}_1 = 0 \quad \Rightarrow \quad u^\star = -\frac{\lambda}{\delta_1}\bar{H}_1^T z + \left(\begin{smallmatrix}1&0&0\\0&1&0\end{smallmatrix}\right)\dot{\beta}H^{-1}\frac{\partial f}{\partial \beta} \tag{25a}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \delta_2(\dot{\theta} - \left(\begin{smallmatrix}0&0&1\end{smallmatrix}\right)\dot{\beta}H^{-1}\frac{\partial f}{\partial \beta} + \lambda z^T\bar{H}_2 = 0 \quad \Rightarrow \quad \dot{\theta}^\star = -\frac{\lambda}{\delta_2}\bar{H}_2^T z + \left(\begin{smallmatrix}0&0&1\end{smallmatrix}\right)\dot{\beta}H^{-1}\frac{\partial f}{\partial \beta} \tag{25b}$$

with $u = (u_1, u_2)^T$. By definition, $(u^\star, \dot{\theta}^\star)$ satisfies the equality

$$z^T(\bar{H}_1 u^\star + \bar{H}_2\dot{\theta}^\star - w) = -z^T K z \tag{26}$$

and we deduce from (25) and (26) that

$$\lambda = \frac{z^T K z + z^T\tilde{w}}{z^T(\delta_1^{-1}\bar{H}_1\bar{H}_1^T + \delta_2^{-1}\bar{H}_2\bar{H}_2^T)z} \tag{27}$$

Lemma 1 follows from Equations (25) and (27).

B Proof of Proposition 2

Let us show that the control $\bar{u}$ is defined everywhere. First of all $\bar{u}_{zl}$ is well defined since A is clearly invertible and, by the definition of a transverse function, H is also invertible so that $\bar{H}^{-1}$ is well defined. Now $\eta + \psi \geq \psi > 0$ and $\left(1 - \frac{\eta}{\eta+\psi}\right)\bar{u}_{zl}$ is therefore well defined. It follows from (16) that

$$\frac{\eta}{\eta+\psi}\bar{u}^\star = \begin{pmatrix} -\frac{1}{\delta_1(\eta+\psi)}(z^T K z + z^T\tilde{w})\bar{H}_1^T z \\ -\frac{1}{\delta_2(\eta+\psi)}(z^T K z + z^T\tilde{w})\bar{H}_2^T z \end{pmatrix} + \frac{\eta}{\eta+\psi}\dot{\beta}H^{-1}\frac{\partial f}{\partial \beta} \tag{28}$$

is also well defined everywhere. Therefore $\bar{u}$, defined by (18) is well defined.

Now we show that the origin $z = 0$ of the closed-loop system (11)–(18) is exponentially stable. Let

$$V(z) := \frac{1}{2}z^T z \tag{29}$$

Along the trajectories of the closed-loop system (11)–(18) its time-derivate $\dot{V}$ is given by:

$$\dot{V} = z^T\dot{z} = z^T(\bar{H}\bar{u} - w) = z^T\left(\bar{H}\left(\left(1 - \frac{\eta}{\eta+\psi}\right)\bar{u}_{zl} + \frac{\eta}{\eta+\psi}\bar{u}^\star\right) - w\right) \tag{30}$$

$$= \left(1 - \frac{\eta}{\eta+\psi}\right)z^T\left(\bar{H}\bar{u}_{zl} - w\right) + \frac{\eta}{\eta+\psi}z^T\left(\bar{H}\bar{u}^\star - w\right) \tag{31}$$

From the definition (14) of $\bar{u}_{zl}$

$$z^T\left(\bar{H}\bar{u}_{zl} - w\right) = -z^T K z \tag{32}$$

and since $\bar{u}^\star$ is a solution to the optimization problem (15),

$$z^T\left(\bar{H}\bar{u}^\star - w\right) = -z^T K z \tag{33}$$

It follows from (31–33) that

$$\begin{aligned}
\dot{V} &= z^T(\bar{H}\bar{u} - w) && (34)\\
&= -\left(1 - \frac{\eta}{\eta+\psi}\right) z^T K z - \frac{\eta}{\eta+\psi} z^T K z && (35)\\
\dot{V} &= -z^T K z && (36)
\end{aligned}$$

We deduce from (29) and (36) that V is a Lyapunov function for the closed-loop system, and that the origin $z = 0$ of this system is exponentially stable.

C Proof of Lemma 2

By definition, the transversality condition corresponds to the inversibility of the matrix

$$H(\theta,\beta) \triangleq \begin{pmatrix} b_1(f(\theta,\beta)) & b_2 & -\dfrac{\partial f}{\partial \theta}(\theta,\beta) \end{pmatrix}$$

for any $(\theta,\beta) \in \mathbb{T}^2$. We deduce from (6), (20) and (21) that

$$H(\theta,\beta) = \begin{pmatrix} \frac{1}{\sqrt{1+\bar{f}_3^2}} & 0 & -\varepsilon_1 \cos(\theta+\beta) \\ \frac{\bar{f}_3}{\sqrt{1+\bar{f}_3^2}} & 0 & -\frac{\varepsilon_1\varepsilon_2(-\gamma\cos\theta + \cos(\theta+\beta)(\cos(\theta+\beta)-\gamma\cos\beta) - \sin(\theta+\beta)(\sin(\theta+\beta)-\gamma\sin\beta))}{2} \\ 0 & 1 & \varepsilon_2 \frac{\sin(\theta+\beta)}{1+\bar{f}_3^2} \end{pmatrix}$$

A simple calculation yields

$$\det H(\theta,\beta) = -\frac{\varepsilon_1\varepsilon_2}{2\sqrt{1+\bar{f}_3^2}}$$

One concludes that for $\varepsilon_1, \varepsilon_2 > 0$, $H(\theta,\beta)$ is invertible and the lemma follows.

D Proof of Proposition 3

Let us first remark that the convergence of $g(t)$ to the set $f(\mathbb{T}^2)$, independently of the choice of $\dot{\beta}$ and independently of the target velocity b_0, is a direct consequence of Proposition 2.

For the proof of Property 1, we refer to the proof of [4, Theo. 1]. For the second property, we assume $b_0 \not\equiv 0$, and $\gamma \in (0,1)$ and we define $\dot{\theta}_{zl} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \bar{u}_{zl}$. We give the proof when the feedback law $\bar{u}$ given by (18) is used. The proof with the feeback law u_{zl} is similar. Equation (18) gives

$$\dot{\theta} = \left(1 - \frac{\eta}{\eta+\psi}\right)\dot{\theta}_{zl} + \frac{\eta}{\eta+\psi}\dot{\theta}^\star \tag{37}$$

By using (8), with (20,21), the third row of the matrix $H(\sigma)^{-1}$, with $\sigma = (\theta,\beta)$, is

$$\frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix} -\bar{f}_3 & 1 & 0 \end{pmatrix}$$

Therefore, from (14), (16), and (18),

$$\dot{\theta} = \left(1 - \frac{\eta}{\eta+\psi}\right) \frac{2}{\varepsilon_1\varepsilon_2} \begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix} A^{-1}(w - Kz) + \frac{\eta}{\eta+\psi}\left(-\frac{1}{\delta_2\eta}(z^TKz + z^T\tilde{w})\bar{H}_2^T z + \frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\frac{\partial f}{\partial\beta}\dot{\beta}\right) \quad (38)$$

Since

$$w = A\left(\frac{\partial f}{\partial\beta}\dot{\beta} - Bb_0(g, u_t)\right) \quad (39)$$

Equation (38) yields

$$\dot{\theta} = \left(1 - \frac{\eta}{\eta+\psi}\right) \frac{2}{\varepsilon_1\varepsilon_2} \begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\left(\frac{\partial f}{\partial\beta}\dot{\beta} - \left(Bb_0(g,u_t) + A^{-1}Kz\right)\right) + \frac{\eta}{\eta+\psi}\left(-\frac{1}{\delta_2\eta}(z^TKz + z^T\tilde{w})\bar{H}_2^T z + \frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\frac{\partial f}{\partial\beta}\dot{\beta}\right) \quad (40)$$

$$\dot{\theta} = \frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\frac{\partial f}{\partial\beta}\dot{\beta} - \frac{2}{\varepsilon_1\varepsilon_2}\left(1 - \frac{\eta}{\eta+\psi}\right)\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\left(Bb_0(g,u_t) + A^{-1}Kz\right) - \frac{1}{\delta_2}\frac{1}{\eta+\psi}\left((z^TKz + z^T\tilde{w})\bar{H}_2^T z\right) \quad (41)$$

$$\dot{\theta} = -\left(1 + \gamma^2 - 2\gamma\cos\theta\right)\dot{\beta} - \frac{2}{\varepsilon_1\varepsilon_2}\left(1 - \frac{\eta}{\eta+\psi}\right)\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\left(Bb_0(g,u_t) + A^{-1}Kz\right) - \frac{1}{\delta_2}\frac{1}{\eta+\psi}\left(z^TKz + z^TABb_0(g,u_t)\right)\bar{H}_2^T z \quad (42)$$

From the definition (22) of $\dot{\beta}$, and (5) of b_0, one can show that

$$\dot{\beta} = \frac{k\tan\left(\frac{\theta}{2}\right) - \frac{2}{\varepsilon_1\varepsilon_2}b - \frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}b_0(f,u_t)}{1 + \gamma^2 - 2\gamma\cos\theta} \quad (43)$$

and we infer from (42) that

$$\dot{\theta} = -k\tan\frac{\theta}{2} + \frac{2}{\varepsilon_1\varepsilon_2}b + R(z,\sigma) \quad (44)$$

with

$$R(z,\sigma) = \frac{2}{\varepsilon_1\varepsilon_2}\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}b_0(f,u_t) - \frac{2}{\varepsilon_1\varepsilon_2}\left(1 - \frac{\eta}{\eta+\psi}\right)\begin{pmatrix}-\bar{f}_3 & 1 & 0\end{pmatrix}\left(Bb_0(g,u_t) + A^{-1}Kz\right) - \frac{1}{\delta_2}\frac{1}{\eta+\psi}\left(z^TKz + z^TABb_0(g,u_t)\right)\bar{H}_2^T z \quad (45)$$

Assuming that the target velocities $(a,b,c)^T$ are bounded, we infer from (45) that there exists $\eta_1 > 0$ such that

$$\|R(z,\sigma)\| \leq \eta_1\|z\| \quad (46)$$

Since, by assumption, $b \equiv 0$, it follows from (44) that

$$\dot{\theta} = -k \tan \frac{\theta}{2} + R(z, \sigma) \tag{47}$$

Let

$$V \triangleq \tan^2 \frac{\theta}{2} \tag{48}$$

The time-derivate of V is

$$\begin{aligned} \dot{V} &= \tan \frac{\theta}{2} \left(1 + \tan^2 \frac{\theta}{2}\right) \dot{\theta} && (49) \\ &= \tan \frac{\theta}{2} \left(1 + \tan^2 \frac{\theta}{2}\right) \left(-k \tan \frac{\theta}{2} + R(z, \sigma)\right) && (50) \\ \dot{V} &= -kV\,(1+V) + \sqrt{V}\,(1+V)\,R(z, \sigma) && (51) \end{aligned}$$

Using (46) and Young's inequality which implies that for any α_i such that $0 < \alpha_i < 2$,

$$V^{\alpha_i} \eta_1 \|z\| \leq \frac{k}{4} V^2 + \left(\frac{4\alpha_i}{k}\right)^{\frac{\alpha_i}{2-\alpha_i}} \frac{2-\alpha_i}{2} \left(\eta_1 \|z\|\right)^{\frac{2}{2-\alpha_i}} \tag{52}$$

we obtain

$$\dot{V} \leq -kV \left(1 + \frac{V}{2}\right) + \sum_{i=1}^{2} c_{\beta_i} \|z\|^{\beta_i} \tag{53}$$

with β_i, c_{β_i} some positive constants such that $1 < \beta_i \leq 4$.

By Proposition 2, $z = 0$ is exponentially stable for the closed-loop system (11)–(18), Therefore there exist some constants $k_z, \gamma_z > 0$ such that

$$\forall t > 0, \quad \|z(t)\| \leq k_z \|z(0)\| e^{-\gamma_z t} \tag{54}$$

This yields

$$\dot{V} \leq -kV \left(1 + \frac{V}{2}\right) + \sum_{i=1}^{2} c_{\beta_i} \|z\|^{\beta_i} \leq -kV + \eta_2 e^{-\gamma_z t} \tag{55}$$

for some positive constant η_2. The exponential convergence of θ to zero follows from this inequality.

In this case, the convergence of $g(t)$ to $f(\mathbb{T}^2)$ implies that the norm of the tracking error g is ultimately bounded by $\max_{\beta \in \mathbb{T}} f(0, \beta)$. Equation (20) yields

$$f(0, \beta) = \begin{pmatrix} \varepsilon_1\,(1-\gamma) \sin \beta \\ \frac{\varepsilon_1 \varepsilon_2}{2}\,(1-\gamma)^2 \sin \beta \cos \beta \\ \arctan\left(\varepsilon_2\,(1-\gamma) \cos \beta\right) \end{pmatrix} \tag{56}$$

Since each coordinate of $f(0, \beta)$ can be independently bounded as follows

$$\forall \beta \in \mathbb{T} \quad \text{and} \quad \forall \gamma \in (0,1), \qquad \left\{ \begin{array}{rcc} \bar{f}_1(0, \beta) & \leq & (1-\gamma)\varepsilon_1 \\ \bar{f}_2(0, \beta) & \leq & \frac{\varepsilon_1 \varepsilon_2}{4}\,(1-\gamma)^2 \leq (1-\gamma)\,\frac{\varepsilon_1 \varepsilon_2}{4} \\ \arctan\left(\bar{f}_3(0, \beta)\right) & \leq & \arctan\left(\varepsilon_2\,(1-\gamma)\right) \leq (1-\gamma)\,\varepsilon_2 \end{array} \right. \tag{57}$$

one concludes that the norm of the tracking error g is ultimately bounded by $(1-\gamma)\|(\varepsilon_1, \varepsilon_2, \frac{\varepsilon_1 \varepsilon_2}{4})\|$.

Finally, let us consider the Property 3. Since, by Proposition 2, $g(t)$ converges to $f(\mathbb{T}^2)$, the ultimate bound is given by $\max_{(\theta,\beta) \in \mathbb{T}^2} \|f(\theta, \beta)\|$. For any $(\theta, \beta) \in \mathbb{T}^2$, and for any $\gamma \in (0,1)$, each coordinate of f can be independently bounded

$$\forall (\theta, \beta) \in \mathbb{T}^2, \left\{ \begin{array}{rcl} \bar{f}_1(\theta, \beta) & \leq & 2\varepsilon_1 \\ \bar{f}_2(\theta, \beta) & \leq & 2\varepsilon_1 \varepsilon_2 \\ \arctan\left(\bar{f}_3\,(\theta, \beta)\right) & \leq & \arctan(2\varepsilon_2) \leq 2\varepsilon_2 \end{array} \right. \tag{58}$$

This yields $\max_{(\theta,\beta) \in \mathbb{T}^2} \|f(\theta, \beta)\| \leq 2\|\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2\|$.

References

[1] Guillaume Artus, Pascal Morin, and Claude Samson. Tracking of an omnidirectional target with a nonholonomic mobile robot. In *The 11th International Conference on Advanced Robotics*, volume 3, pages 1468–1473, July 2003.

[2] Guillaume Artus, Pascal Morin, and Claude Samson. Tracking of an omnidirectional target with a nonholonomic mobile robot: control design and experimental results. Research Report 4849, INRIA, June 2003.

[3] A. de Luca, G. Oriolo, and C. Samson. Feedback control of a nonholonomic car-like robot. In J.-P. Laumond, editor, *Robot motion planning and control*, volume 229 of *LNCIS*. Springer Verlag, 1998.

[4] Pascal Morin and Claude Samson. Practical and asymptotic stabilization of the 3-d chained system by the transverse function approach. In *IEEE Conference on Decision and Control*, pages 4700–4704, December 2003.

[5] Pascal Morin and Claude Samson. Practical stabilization of driftless systems on lie groups: The transverse function approach. *IEEE Transactions on Automatic Control*, 48(9):1496–1508, September 2003.

[6] Claude Samson. Control of chained systems. Application to path following and time-varying point-stabilization. *IEEE Transactions on Automatic Control*, 40:64–77, 1995.

[7] D.P. Tsakiris, K. Kapellos, C. Samson, P. Rives, and J.J. Borrelly. Experiments in real-time vision-based point stabilization of a nonholonomic mobile manipulator. In *5th International Symposium on Experimental Robotics*, 1997.

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http://www.inria.fr
ISSN 0249-6399