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TUNING THE MESH OF A MIXED METHOD IN ψ, ω FOR NAVIER-STOKES EQUATIONS

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Tuning the mesh of a mixed method in ψ,ω for Navier-Stokes Equations

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Abstract :

Error estimates show that boundary layers do not require to refine the mesh for ψ but only for ω when the Reynolds number is large.
In this report we study a new mixed method for the Stokes and Navier-Stokes equations. The method uses two meshes, one very fine for the vorticity ω and a coarser one for the stream function ψ.
We prove estimates and study implementation problems.

Adaptation de Maillage pour les Equations de Navier-Stokes dans la formulation ψ,ω

Résumé :

Les estimations d'erreurs montrent que, quand le nombre de Reynolds est grand, on n'a pas besoin de raffiner le maillage pour ψ autant que pour ω. Dans ce rapport, nous étudions une nouvelle méthode mixte pour l'approximation des équations de Stokes et Navier-Stokes. Cette méthode utilise deux grilles, l'une très fine pour la vorticité ω, l'autre plus grossière pour la fonction de courant ψ.
Nous faisons une analyse d'erreur et étudions les problèmes d'implémentation.

Introduction

Throughout the paper, Ω denotes an open domain of $\mathbb{R}^2$ with boundary Γ.
For the sake of simplicity but without loss of generality, we assume Ω simply connected.
The aim of the paper is to give a new algorithm for the numerical approximation of the solutions (ψ,ω) of the Navier-Stokes equations for incompressible fluids, in their stream-vorticity formulation :

$$(1)\qquad \begin{cases} -\Delta\psi = \omega & \text{in } \Omega \\ \dfrac{\partial\omega}{\partial t} - \nu\Delta\omega + u.\nabla\omega = f & \text{in } \Omega \\ u = \nabla \times \psi & \text{in } \Omega \\ \psi = g_o & \text{on } \Gamma \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_1 \\ \omega = g_1 & \text{on } \Gamma_2 \end{cases},$$

where Γ_1 and Γ_2 are two subsets of Γ such that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.
Hereafter, f belongs to $L^2(\Omega)$.
When the Reynolds number is large, the viscosity effects can be neglected in large regions of Ω, but they cannot be neglected in particular near Γ_1; indeed, near Γ_1, a boundary layer matches the solution of the Euler equations with the boundary conditions. Therefore, if the flow is approached by means of finite elements, it is necessary to refine the mesh near Γ_1, in order to catch the very rapid variations of the vorticity ω. However, one can guess that the variations of the stream function ψ are slower, since $\frac{\partial\psi}{\partial n} = 0$. Then it seems numerically efficient to compute ψ on a coarser mesh than ω. The goal of this paper is to describe a variational formulation of (1) yielding a mixed finite element method to approach ψ and ω with different grids.

Let us first recall some results on the mixed method for the stream vorticity formulation of the Navier-Stokes equations. For the sake of simplicity, we shall focus on the case where $\Gamma_2 = \emptyset$ and where the Stokes equations are considered :

$$(2)\qquad \begin{cases} -\Delta\psi = \omega & \text{in } \Omega \\ -\nu\Delta\omega = f & \text{in } \Omega \\ \psi = g_o & \text{on } \Gamma \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma \end{cases}.$$

Ciarlet-Raviart [1] give a saddle point variational formulation for (2) : Find $(\psi,\omega,\mu^*) \in H_o^1(\Omega) \times L^2(\Omega) \times H^1(\Omega)$ such that

$$(3) \qquad \begin{cases} \forall(\varphi,v) \in H_o^1(\Omega) \times L^2(\Omega), \nu \displaystyle\int_\Omega \omega v + \int_\Omega \nabla\varphi.\nabla\mu^* - \int_\Omega v\mu^* = \int_\Omega f\varphi \\ \gamma_o(\psi) = g_o \\ \forall\mu \in H^1(\Omega), \quad \displaystyle\int_\Omega \nabla\psi.\nabla\mu - \int_\Omega w\mu = 0 \end{cases},$$

where γ_o stands for the trace of H^1 functions on Γ.
In (3), the weak formulation of $-\Delta\psi = \omega; \frac{\partial\psi}{\partial n} = 0$ is viewed as a constraint. Ciarlet-Raviart show the existence and uniqueness of (ψ,ω,μ^*) satisfying (3), and give a mixed approximation for (3). They choose $X_h \subset H^1(\Omega), Y_h \subset L^2(\Omega), M_h \subset H^1(\Omega)$, so that $X_h \subset Y_h \subset M_h$. These authors prove that, if $g_o \in X_h \backslash X_h^o$, the following problem has a unique solution :
Find $(\psi_h,\omega_h,\mu_h^*) \in X_h \times Y_h \times M_h$ so that $\psi_h - g_o \in X_h^o$, and

$$(4) \qquad \begin{cases} \forall(\varphi_h,v_h) \in X_h^o \times Y_h, \nu \displaystyle\int_\Omega \omega_h v_h + \int_\Omega \nabla\varphi_h.\nabla\mu_h^* - \int_\Omega v_h\mu_h^* = \int_\Omega f\varphi_h \\ \forall\mu_h \in M_h, \qquad \displaystyle\int_\Omega \nabla\psi_h.\nabla\mu_h - \int_\Omega \omega_h\mu_h = 0 \end{cases}.$$

Furthermore, these authors obtain the following error bound :
If ψ solution of (2) belongs to $H_o^2(\Omega) \times H^{k+2}(\Omega)$, for some integer $k \geq 2$, if $Y_h = M_h$, and if X_h and M_h are P^k finite element spaces, related to uniformly regular triangulations :

$$(5) \qquad \|\Delta\psi - \omega_h\|_o + \|\psi - \psi_h\|_1 \leq K[\|\psi\|_{k+1} + \|\omega\|_k]h^{k-1} \quad .$$

Scholz [7] improves this error estimate, and shows that the convergence is also obtained for a P^1 approximation of (3), if ψ belongs to $H^3(\Omega)$.
In the right member of (5), (and also in the improved error estimate), we notice that in the case of a boundary layer for example, the error due to the approximation of ψ by functions of X_h seems smaller than the other errors. Hence, it seems to be a waste to discretize ψ with a mesh as fine as for ω.
At this point, it is useful to recall the results of Glowinski and Pironneau [2]. These authors consider the special case where $X_h = M_h$, and they design an algorithm, based on a boundary operator, which enables one to compute ω_h and ψ_h in (4) without having to compute μ_h^*.

We also would like to avoid computing μ_h^*, but we cannot use the method of [2], since the mesh for ψ_h is desired coarser. We can now specify the aim of the paper : it is to give a variational formulation for (2) and then for (1), in order to approach ω and ψ on different grids, by means of a boundary operator, and without computing any Lagrange multiplier like μ_h^*.

The first part of the paper is devoted to a formulation of the Stokes equations (2) and to a related mixed discretization. It includes an error analysis. The second part of the paper deals with an analogue method for the generalized Stokes equations. An error analysis is also sketched. The third part concerns the implementation of the algorithm for the approximation of the whole Navier-Stokes equations, and contains heuristic considerations in order to choose the sizes of the meshes for ω and ψ, when the flow develops a laminar boundary layer.

Numerical experiments for this algorithm are not carried out so far, but will be performed in a forecoming work.

I. A saddle point variational formulation for the Stokes problem.

I.1. The continuous problem

I.1.a. A well posed constrained problem

In order to derive a variational formulation for (2), we introduce the set :

$$V = \{v \in L^2(\Omega), \Delta v \in L^2(\Omega)\} \quad . \tag{6}$$

V is a Hilbert space with the following scalar product :

$$(u, v)_V = \int_\Omega uv + \int_\Omega \Delta u \Delta v \quad . \tag{7}$$

We also introduce the following closed subset of V :

$$W = \{v \in L^2(\Omega); \Delta v = 0\} \quad . \tag{8}$$

In (2), by an harmonic shift of g_o, we may assume that

$$g_o = 0 \quad . \tag{9}$$

Let us also shift the inhomogeneous data f by solving first : Find $\omega_o \in H_o^1(\Omega)$, so that :

$$(10) \qquad \forall v \in H_o^1(\Omega), \quad \nu \int_\Omega \nabla\omega_o.\nabla v = \int_\Omega fv \quad .$$

Then consider the variational problem posed on $W \times H_o^1(\Omega)$: Find $(\bar{\omega}, \psi) \in W \times H_o^1(\Omega)$,

$$(11) \qquad \begin{cases} \forall v \in W, \quad \int_\Omega \bar{\omega} v = -\int_\Omega \omega_o v \\ \forall \varphi \in H_o^1(\Omega), \quad \int_\Omega \nabla\psi.\nabla\varphi = \int_\Omega (\bar{\omega} + \omega_o)\varphi \end{cases} .$$

Remark 1

Note that the variational problem (11) is posed on an unusual space, namely $W \times H_o^1(\Omega)$. □

Remark 2

In formulation (10), (11) the problems on ω and ψ can be solved separately. This is specific to equations (2). When the problem of discretizing ω is overcome, it will be easy to take different meshes for ω and ψ. □

Lemma 1 *Problem (11) has a unique solution.* □

Proof

If $\bar{\omega}$ exists and is unique, then the Lax-Milgram lemma yields the existence and uniqueness for (11).
But $\bar{\omega}$ is the L^2 projection of $-\omega_o$ onto W which exists and is unique because W is closed for the L^2 topology. □

When Ω is smooth enough, we can show that ψ is the solution of a more classical biharmonic problem :

Proposition 1 *If Γ is $C^{1,1}$ or if Ω is polygonal and convex, then ψ solution of (10) (11), is also the solution of*

$$(12) \qquad \begin{cases} \psi \in H_o^2(\Omega) \\ \nu \int_\Omega \Delta\psi\Delta\varphi = \int_\Omega f\varphi, \quad \forall \varphi \in H_o^2(\Omega) \end{cases} ,$$

which is the usual variational formulation for (2). □

Proof

The solution ψ of (11) belongs to $H^2(\Omega)$ because of well known regularity results (cf. [3]).
Then $\omega = \omega_o + \bar{\omega} = -\Delta\psi$.
To prove that $\frac{\partial\psi}{\partial n} = 0$, we use the fact that when Ω is smooth enough, $\forall\rho \in H^{\frac{1}{2}}(\Gamma)$, there exists $\tilde{\rho} \in H^1(\Omega)$ with $\rho = \tilde{\rho}$ on Γ and :

$$(13) \qquad \forall\varphi \in H^1_o(\Omega), \int_\Omega \nabla\tilde{\rho}.\nabla\varphi = 0 \quad .$$

Hence :

$$(14) \qquad \int_\Omega \Delta\psi\tilde{\rho} = -\int_\Omega \nabla\psi.\nabla\tilde{\rho} + \int_\Gamma \frac{\partial\psi}{\partial n}\tilde{\rho} = \int_\Gamma \frac{\partial\psi}{\partial n}\tilde{\rho} \quad .$$

On the other hand,

$$(15) \qquad \int_\Omega \Delta\psi\tilde{\rho} = \int_\Omega -(\omega + \bar{\omega}_o)\tilde{\rho} = 0 \quad .$$

In (15) the first equality comes from the second equation of (11) and the fact that $\psi \in H^2(\Omega)$, and the second equality comes from the first equation of (11). So we have proved that
$\forall\rho \in H^{\frac{1}{2}}(\Gamma), \int_\Omega \frac{\partial\psi}{\partial n}\rho = 0$; hence $\frac{\partial\psi}{\partial n} = 0$.
Finally, for every $\varphi \in H^2_o(\Omega)$,

$$\nu\int_\Omega \Delta\psi\Delta\varphi = -\nu\int_\Omega (\omega_o + \bar{\omega})\Delta\varphi \quad .$$

Since $\varphi \in H^2_o(\Omega)$,

$$\int_\Omega \bar{\omega}\Delta\varphi = \int_\Omega \Delta\bar{\omega}\varphi = 0 \quad .$$

and

$$-\nu\int_\Omega \omega_o\Delta\varphi = \nu\int_\Omega \nabla\omega_o.\nabla\varphi = \int_\Omega f\varphi \quad .$$

Proposition 1 is proved. □

We are now going to replace problem (11) with a problem posed on $H^1(\Omega) \times H^1_o(\Omega)$.

I.1.b.An ill posed constrained problem

The set W is not easy to discretize. The aim of this section is to formulate a new problem, in order to replace W with a new set W', which has a variational definition. We introduce the set :

$$W' = \left\{ v \in H^1(\Omega) : \forall u \in H^1_o(\Omega), \int_\Omega \nabla v . \nabla u = 0 \right\} \quad . \tag{16}$$

We can now introduce a new problem :
Find $(\bar{\omega}, \psi) \in W' \times H^1_o(\Omega)$ such that

$$\begin{cases} \forall v \in W', \quad \displaystyle\int_\Omega \bar{\omega} v = -\int_\Omega \omega_o v \\ \forall \varphi \in H^1_o(\Omega), \quad \displaystyle\int_\Omega \nabla \psi . \nabla \varphi = \int_\Omega (\bar{\omega} + \omega_o)\varphi \end{cases} \quad . \tag{17}$$

Remark 3

Let us emphasize that (17) may have no solution. However, we shall see later that the discrete version of (17) has a unique solution. The problem (17) or its discrete version is more convenient than problem (11) because the space W' enjoys a variational definition which is not the case for W. □

Proposition 2 *If $(\bar{\omega}, \psi)$ is a solution of (17), then it is the only solution and $(\bar{\omega}, \psi)$ is the solution of (11). Conversely, if the solution ψ of (11) belongs to $H^3(\Omega)$, then $(-\Delta\psi - \omega_o, \psi)$ is the solution of (17).* □

Proof

If $\bar{\omega}$ satisfies the first equality of (17), it also satisfies the first equality of (11), because W' is dense in W for the L^2 norm.
To show this density result, let us recall that $H^{\frac{1}{2}}(\Gamma)$ is isomorphic to W' and $H^{-\frac{1}{2}}(\Gamma)$ is isomorphic to W. (The latter result can be found in [4]). Using these isomorphisms, the density of $H^{\frac{1}{2}}(\Gamma)$ in $H^{-\frac{1}{2}}(\Gamma)$ yields the density result.
So if $(\bar{\omega}, \psi)$ satisfies (17), it satisfies (11). The uniqueness for (11) yields the uniqueness for (17).
If ψ solution of (11) belongs to $H^3(\Omega)$, it is of course solution of (17), because $-\Delta\psi - \omega_o \in H^1(\Omega)$. □

I.1.c. A saddle point problem for (17)

Problem (17) can be viewed as a constrained problem where the constraint is : $\omega \in W'$. It can be changed into an unconstrained problem, at the cost of adding a Lagrange multiplier. Problem (17) is equivalent to the following saddle point problem :
Find $(\bar{\omega}, \psi, \mu^*) \in H^1(\Omega) \times H_o^1(\Omega) \times H_o^1(\Omega)$

$$(18) \qquad \begin{cases} \forall v \in H^1(\Omega), \int_\Omega \bar{\omega} v + \int_\Omega \nabla v . \nabla \mu^* = - \int_\Omega \omega_o v \\ \forall \mu \in H_o^1(\Omega), \int_\Omega \nabla \bar{\omega} . \nabla \mu = 0 \\ \forall \varphi \in H_o^1(\Omega), \int_\Omega \nabla \psi . \nabla \varphi = \int_\Omega (\omega_o + \bar{\omega}) \varphi \end{cases}$$

Remark 4
We can add to remark 3 that we can obtain (18) because of the variational definition of W'. Such a formulation cannot be obtained from (11). □

Remark 5
It is easy to notice that if ψ, solution of (11) belongs to $H^3(\Omega)$, then $(-\Delta\psi - \omega_o, \psi, -\psi)$ is the solution of (18). This remark will prove useful for the error estimate. □

I.1.d. A boundary formulation for (17)

As in [2], we would like to introduce a boundary operator to change (17) into a problem posed on $H_o^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$.
Consider the following isomorphism between $H^{\frac{1}{2}}(\Gamma)$ and W' :

$$\lambda \rightarrow \bar{\omega}(\lambda) \qquad ,$$

where $\bar{\omega}(\lambda)$ is the unique solution of the following problem :

$$(19) \qquad \begin{cases} \gamma_o(\bar{\omega}(\lambda)) = \lambda \\ \forall v \in H_o^1(\Omega), \int_\Omega \nabla \bar{\omega}(\lambda) . \nabla v = 0 \end{cases}$$

The following lemma is used to transform (17) into a problem posed on $H_o^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$.

Lemma 2 *Problem (18) is equivalent to :*
Find $(\psi, \lambda) \in H_o^1(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ *such that*

$$(20) \qquad \begin{cases} \forall \mu \in H^{\frac{1}{2}}(\Gamma), \int_\Omega \bar{\omega}(\lambda)\bar{\omega}(\mu) = -\int_\Omega \omega_o \bar{\omega}(\mu) \\ \forall \varphi \in H_o^1(\Omega), \int_\Omega \nabla\psi.\nabla\varphi = \int_\Omega (\omega_o + \bar{\omega}(\lambda))\varphi \end{cases}$$

□

Proof The proof is straightforward. □

We can now define the following symmetric bilinear form on $H^{\frac{1}{2}}(\Gamma)$:

$$(21) \qquad a(\lambda, \mu) = \int_\Omega \bar{\omega}(\lambda)\bar{\omega}(\mu) \quad .$$

Since a is continuous, (21) defines a linear mapping A from $H^{\frac{1}{2}}(\Gamma)$ into $(H^{\frac{1}{2}}(\Gamma))'$ by

$$(22) \qquad < A(\lambda), \mu >= a(\lambda, \mu), \qquad \forall \lambda, \mu \in H^{\frac{1}{2}}(\Gamma) \quad .$$

One can also define $B \in (H^{\frac{1}{2}}(\Gamma))'$ by

$$(23) \qquad < B, \mu >= -\int_\Omega \omega_o \bar{\omega}(\mu) \quad .$$

Problem (20) reads :
Find $(\lambda, \psi) \in H^{\frac{1}{2}}(\Gamma) \times H_o^1(\Gamma)$

$$(24) \qquad \begin{cases} A(\lambda) = B \\ \forall \varphi \in H_o^1(\Omega), \int_\Omega \nabla\psi.\nabla\varphi = \int_\Omega (\omega_o + \bar{\omega}(\lambda))\varphi \end{cases}$$

I.2. Approximation by a mixed finite element method

Throughout this section, Ω is a polygonal domain.
Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of Ω so that $\mathcal{T}_1 \subset \mathcal{T}_2$. $\mathcal{T}_1$ is the coarser mesh and will be used to approach ψ.

We introduce the finite element spaces $X_H \subset H^1(\Omega)$ and $Y_h \subset H^1(\Omega)$, defined by :

$$(25) \qquad X_H = \{\varphi_H \in C^o(\bar{\Omega}); \forall T \in \mathcal{T}_1, \varphi_H|_T \in P^k\} \quad ,$$

$$Y_h = \{v_h \in C^o(\bar{\Omega}); \forall T \in \mathcal{T}_2, v_h|_T \in P^k\} \quad . \tag{26}$$

Finally we define X_H^o and Y_h^o by

$$X_H^o = \{\varphi_H \in X_H; \varphi_H = 0 \quad \Gamma\} \quad , \tag{27}$$

$$Y_h^o = \{v_h \in Y_h; v_h = 0 \quad \Gamma\} \quad . \tag{28}$$

1.2.a. Discrete analogue to (17)

We define ω_{oh}, the Y_h^o conforming approximation of ω_o, defined by (10) ; ω_{oh} is the unique solution of :
Find $\omega_{oh} \in Y_h^o$ such that

$$\forall v_h \in Y_h^o, \quad \nu \int_\Omega \nabla\omega_{oh}.\nabla v_h = \int_\Omega f v_h \quad . \tag{29}$$

Just as for the continuous formulation, we introduce the space W_h' :

$$W_h' = \{v_h \in Y_h : \forall u_h \in Y_h^o \int_\Omega \nabla v_h.\nabla u_h = 0\} \quad . \tag{30}$$

The discretized problem for (17) is :
Find $(\bar{\omega}_h, \psi_H) \in W_h' \times X_H^o$, such that :

$$\begin{cases} \forall v_h \in W_h', \int_\Omega \bar{\omega}_h v_h = -\int_\Omega \omega_{oh} v_h \\ \forall \varphi_H \in X_H^o, \quad \int_\Omega \nabla\psi_H.\nabla\varphi_H = \int_\Omega (\bar{\omega}_h + \omega_{oh})\varphi_H \end{cases} \tag{31}$$

Notice that (31) is a mixed approximation of (17), since $W_h' \not\subset W'$. Here, unlike to (17), the following lemma is true :

Lemma 3 *Problem (31) has a unique solution.* □

Proof W_h' is a vectorial subspace of Y_h, closed for the discrete L^2 norm, since Y_h has a finite dimension. Hence, the L^2 projection of $-\omega_{oh}$ onto W_h' is uniquely defined. The proof is completed. □

Remark 6

The fact that $X_H \subset Y_h$ implies that $\int_\Omega (\omega_{oh} + \bar{\omega}_h)\varphi_H$ can be computed exactly. Thus (31) does not introduce any interpolation errors. □

Remark 7

Problem (31) is not directly implementable, because the space W_h' is not explicitly known. □

I.2.b. Discrete analogue to (20)

Just as for the continuous problem, one can define an isomorphism between $Y_h \backslash Y_h^o$ and W_h', $\lambda_h \rightarrow \bar{\omega}_h(\lambda_h)$ by :
$\forall \lambda_h \in Y_h \backslash Y_h^o, \bar{\omega}_h(\lambda_h)$ is the solution of :

$$\text{(32)} \qquad \begin{cases} \bar{\omega}_h(\lambda_h) - \lambda_h \in Y_h^o \\ \forall v_h \in Y_h^o, \int_\Omega \nabla \bar{\omega}_h(\lambda_h).\nabla v_h = 0 \end{cases}$$

Let a_h be the symmetric, positive definite bilinear form :

$$\text{(33)} \qquad a_h(\lambda_h, \mu_h) = \int_\Omega \bar{\omega}_h(\lambda_h)\bar{\omega}_h(\mu_h) \quad ,$$

and b_h the vector of $Y_h \backslash Y_h^o$ defined by :

$$\text{(34)} \qquad \forall \mu_h \in Y_h \backslash Y_h^o, (b_h, \mu_h) = -\int_\Omega \omega_{oh}\bar{\omega}_h(\mu_h) \quad .$$

It is straightforward to show the following lemma, which provides a method for solving (31).

Lemma 4 *Problem (31) is equivalent to :*
Find $(\lambda_h, \psi_H) \in (Y_h \backslash Y_h^o) \times X_H^o$ *such that*

$$\text{(35)} \qquad \begin{cases} \forall \mu_h \in Y_h \backslash Y_h^o, \quad a_h(\lambda_h, \mu_h) = (b_h, \mu_h) \\ \forall \varphi_H \in X_H^o, \quad \int_\Omega \nabla \psi_H.\nabla \varphi_H = \int_\Omega [\omega_{oh} + \bar{\omega}_h(\lambda_h)]\varphi_H \end{cases}$$

□

I.2.c. Implementation

An algorithm based on (35) would be :

. Compute $\bar{\omega}_h(\mu_i)$ for a basis (μ_i) of $Y_h \backslash Y_h^o$.

. Compute the bilinear form a_h by means of the functions $\bar{\omega}_h(\mu_i)$.

. Solve problem (29) to compute b_h and then solve (35).

Of course, this algorithm is expensive for a single Stokes problem. It will be efficient for a Navier-Stokes approximation, since one has to solve many Stokes problems, and the computations of $\bar{\omega}_h(\mu_i)$ and a_h are done once for all.

I.2.d. Error bounds

Throughout this paragraph, we suppose that Ω is polygonal convex, that the integrals $\int_\Omega f v_h$ are computed exactly, and that ψ, solution of (12) belongs to $H^3(\Omega)$.

We recall that, if ψ, solution of (12) belongs to $H^3(\Omega)$, then $(-\Delta\psi - \omega_o, \psi, -\psi)$ is the solution of (18).

We are going to perform an error analysis for P^1 and P^2 finite elements. The same analysis would be true for P^k finite elements, for $k > 2$. The following theorem gives the error estimates for the discretization described above :

Theorem 1 *Assume that Ω is a convex polygonal domain and that the triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ are triangulations of sizes $O(H)$ and $O(h)$ respectively, with a regularity bounded independently of H and h.*
Let k be equal to 1 or 2. If ψ solution of (12) belongs to $H^{k+2}(\Omega)$, then if X_H and Y_h are constructed with P^k finite elements, the following error estimate is true for the approximation (ω_h, ψ_H) :

$$\|\omega_o - \omega_{oh}\|_o \leq Ch^2\|\omega_o\|_2 \leq C\frac{h^2}{\nu}\|f\|_0 \quad . \tag{36}$$

$$\begin{aligned} \forall \varepsilon < 1/2, \quad & \|\bar{\omega} - \bar{\omega}_h\|_o + \|\psi - \psi_H\|_1 \\ & \leq C[h^k\|\omega\|_k + h^2\frac{1}{\nu}\|f\|_0 + c(\varepsilon)h^{k-1/2-\varepsilon}\|\psi\|_{k+2} + H^k\|\psi\|_{k+1}]. \end{aligned} \tag{37}$$

where C is a constant independent of h, H, ε, ν and $c(\varepsilon)$ depends only on ε. □

Proof

Let us estimate $\|\omega_o - \omega_{oh}\|_o$. Since ω_{oh} is a conforming P^k finite element approximation of ω_o, this estimate is classical (cf. [5] or [6]) ; to obtain (36), first estimate $\|\omega_o - \omega_{oh}\|_1$ by means of standard finite element theory, then use the Aubin-Nitsche lemma, to estimate $\|\omega_o - \omega_{oh}\|_o$.
Now, (37) is harder to derive.

Remark 5 and (18) yield :

$$\forall v_h \in W_h', \quad \int_\Omega \bar{\omega} v_h - \int_\Omega \nabla\psi.\nabla v_h = -\int_\Omega \omega_o v_h \qquad . \tag{38}$$

On the other hand, (31) gives

$$\forall v_h \in W_h', \quad \int_\Omega \bar{\omega}_h v_h = -\int_\Omega \omega_{oh} v_h \qquad . \tag{39}$$

Because of the definition of W_h', (39) can also be written

$$\forall \mu_h \in Y_h^o, \quad \int_\Omega \bar{\omega}_h v_h - \int_\Omega \nabla v_h.\nabla \mu_h = -\int_\Omega \omega_{oh} v_h \qquad . \tag{40}$$

In (38) and (40) let us replace v_h with $v_h - \bar{\omega}_h$, and let us substract (40) to (38). We obtain :

$$\begin{aligned} \forall v_h \in W_h', \forall \mu_h \in Y_h^o \int_\Omega (\bar{\omega} - \bar{\omega}_h).(v_h - \bar{\omega}_h) - \int_\Omega \nabla(\psi - \mu_h).\nabla(v_h - \bar{\omega}_h) \\ = -\int_\Omega (\omega_o - \omega_{oh}).(v_h - \bar{\omega}_h) \qquad . \end{aligned} \tag{41}$$

Therefore, since $\omega = \omega_o + \bar{\omega}$,

$$\begin{aligned} &\forall v_h \in W_h', \forall \mu_h \in Y_h^o, \\ &\|v_h - \bar{\omega}_h\|_o^2 = -\int_\Omega (\omega - \omega_{oh} - v_h)(v_h - \bar{\omega}_h) + \int_\Omega \nabla(\psi - \mu_h).\nabla(v_h - \bar{\omega}_h). \end{aligned} \tag{42}$$

which implies :

$$\begin{aligned} &\forall v_h \in W_h', \forall \mu_h \in Y_h^o, \\ &\|v_h - \bar{\omega}_h\|_o^2 \leq \|\omega - \omega_{oh} - v_h\|_o \|v_h - \bar{\omega}_h)\|_o + \left| \int_\Omega \nabla(\psi - \mu_h).\nabla(v_h - \bar{\omega}_h) \right|. \end{aligned} \tag{43}$$

To find out an error estimate from (43), we need the following lemma, which will be proved later on :

Lemma 5 *The following estimate is true :*

$$\inf_{v_h \in W_h'} \|\omega - \omega_{oh} - v_h\|_o \leq Ch^k \|\omega\|_k \quad . \tag{44}$$

□

This lemma deals with the first part of the second member of (43). Let us see what we obtain by naively processing the second part of the second member in (43) : we notice that there exists a constant C, so that (see [5])

$$\forall v_h \in W_h', \quad |v_h - \bar{\omega}_h|_1 < \frac{C}{h} \|v_h - \bar{\omega}_h\|_o \quad . \tag{45}$$

On the other hand, since $\psi \in H^3(\Omega) \cap H_o^2(\Omega)$,

$$\inf_{\mu_h \in Y_h^\circ} |\psi - \mu_h|_1 \leq Ch^k \|\psi\|_{k+1} \tag{46}$$

Because of (44), (45) and (46), (43) yields :

$$\inf_{v_h \in W_h'} \|v_h - \bar{\omega}_h\|_o \leq Ch^k \|\omega\|_k + Ch^{k-1} \|\psi\|_{k+1} \quad .$$

This estimate does not yield the convergence of the approximation in the case $k = 1$. A more refined estimate is necessary. To obtain it, we need a lemma due to Scholz [7], [8]:

Lemma 6 (Scholz) *Suppose that Ω and T_2 satisfy the assumptions of theorem 1. Let k be an integer and p be a real number such that $1 \leq k$ and $2 \leq p \leq \infty$. Suppose that Y_h is related to P^k finite elements. If $P_h\varphi$ denotes the projection of φ onto Y_h^0, there exists a constant $C > 0$ such that, for all functions $\varphi \in W^{k+1,p}(\Omega) \cap H_0^1(\Omega)$,*

$$sup_{u_h \in Y_h} \quad \frac{1}{\|u_h\|_0} \left| \int_\Omega \nabla(\varphi - P_h\varphi).\nabla u_h \right| \leq Ch^{k-1/2-1/p} \|\varphi\|_{k+1,p} \quad ,$$

where $\|\varphi\|_{k+1,p}$ is the norm of φ in $W^{k+1,p}(\Omega)$. □

Sobolev imbeddings yield that

$$\forall \varepsilon < 1/2, \quad \begin{cases} \psi \in W^{k+1,1/\varepsilon}(\Omega) \\ \|\psi\|_{k+1,1/\varepsilon} \leq c(\varepsilon) \|\psi\|_{k+2} \end{cases} .$$

Hence,

$$\left| \int_\Omega \nabla(\psi - P_h\psi).\nabla(v_h - \bar{\omega}_h) \right| \leq c(\varepsilon)\|\psi\|_{k+2}\|v_h - \omega_h\|_0 \quad .$$

Therefore, replacing μ_h with $P_h\psi$ in (43) yields

$$(47) \qquad \forall \varepsilon < 1/2, \qquad \|\bar{\omega} - \bar{\omega}_h\|_o \leq C[h^k\|\omega\|_k + c(\varepsilon)h^{k-1/2-\varepsilon}\|\psi\|_{k+2}] \quad .$$

Now, let us find an estimate for $\|\psi - \psi_H\|_1$.

Remark 6 tells us that there are no interpolation errors from the term :

$$\int_\Omega (\bar{\omega}_h + \omega_{oh})\varphi_H, \qquad \text{in} \quad (31).$$

Thus estimating $\|\psi - \psi_H\|_1$ is straightforward. From (31), we find that

$$\|\psi - \psi_H\|_1 \leq C\left[\|\omega_{oh} + \omega_h - \omega\|_o + Inf_{\varphi_H \in X_H^o}\|\psi - \varphi_H\|_1\right].$$

which together with (36) and (47) yields

$$(48) \qquad \|\psi - \psi_H\|_1 \leq C\left[\frac{1}{\nu}\|f\|_0 h^2 + \|\omega\|_k h^k + c(\varepsilon)h^{k-1/2-\varepsilon}\|\psi\|_{k+2} + H^k\|\psi\|_{k+1}\right].$$

Theorem 1 is proved. □

Let us now prove lemma 5.

Proof of Lemma 5

Let us choose v_h such that

$$(49) \qquad \begin{cases} \forall u_h \in Y_h^o, \int_\Omega \nabla v_h.\nabla u_h = 0 \\ v_h = \Pi_h\omega \quad \text{on } \Gamma \end{cases} ,$$

where $\Pi_h\omega$ is the $L^2(\Gamma)$ projection of ω onto $Y_h \backslash Y_h^o$.
The standard error estimates for the Dirichlet problem for $-\Delta$ with P^k elements yield

$$(50) \qquad \|\omega - \omega_{oh} - v_h\|_1 \leq Ch^{k-1}\|\omega\|_k \quad .$$

To obtain (44), the Aubin-Nitsche trick is used :

$$\|\omega - \omega_{oh} - v_h\|_o = Sup_{g \in L^2(\Omega)} \frac{\int_\Omega (\omega - \omega_{oh} - v_h) g}{\|g\|_o} \quad . \tag{51}$$

Let u be the solution of the following problem.

$$\begin{aligned} &\text{Find } u \in H_o^1(\Omega), \\ &\forall z \in H_o^1(\Omega), \int_\Omega \nabla u . \nabla z = \int_\Omega gz \quad . \end{aligned} \tag{52}$$

and let u_h be the solution of (52) with P^k elements.
Find $u_h \in Y_h^o$, so that

$$\forall z_h \in Y_h^o, \int_\Omega \nabla u_h . \nabla z_h = \int_\Omega g z_h \quad . \tag{53}$$

Then,

$$\|u - u_h\|_1 \leq Ch\|u\|_2 \leq Ch\|g\|_o \quad ,$$

because Ω is polygonal convex.
Taking $\omega - \omega_{oh} - v_h$ as a test function in (52), and using the H^2 regularity of u yield :

$$\int_\Omega g(\omega - \omega_{oh} - v_h) = \int_\Omega \nabla u . \nabla(\omega - \omega_{oh} - v_h) - \int_\Gamma \frac{\partial u}{\partial n}(\omega - \omega_{oh} - v_h) \quad .$$

On the other hand

$$\int_\Omega \nabla u_h . \nabla(\omega - \omega_{oh} - v_h) = 0 \quad .$$

Hence,

$$\int_\Omega g(\omega - \omega_{oh} - v_h) = \int_\Omega \nabla(u - u_h) . \nabla(\omega - \omega_{oh} - v_h) - \int_\Gamma \frac{\partial u}{\partial n}(\omega - \omega_{oh} - v_h)$$

$$\int_\Omega g(\omega - \omega_{oh} - v_h) \leq |u - u_h|_1 |\omega - \omega_{oh} - v_h|_1 + \|\omega - \omega_{oh} - v_h\|_{-\frac{1}{2},\Gamma} \|\frac{\partial u}{\partial n}\|_{\frac{1}{2},\Gamma} \tag{54}$$

But, since $\omega - v_h \in L^2(\Gamma)$,

$$\|\omega - \omega_{oh} - v_h\|_{-\frac{1}{2}\Gamma} = \|\omega - v_h\|_{-\frac{1}{2}\Gamma} = sup_{z \in H^{\frac{1}{2}}(\Gamma)} \frac{\int_\Gamma (\omega - v_h) z}{\|z\|_{\frac{1}{2},\Gamma}}.$$

Since v_h is the L^2 projection of ω onto $Y_h \backslash Y_h^o$,

$$\forall z_h \in Y_h, \int_\Gamma (\omega - v_h) z = \int_\Gamma (\omega - v_h)(z - z_h) \quad .$$

Hence

$$\|\omega - \omega_{oh} - v_h\|_{-\frac{1}{2}\Gamma} \leq sup_{z \in H^{\frac{1}{2}}(\Gamma)} inf_{z_h \in Y_h \backslash Y_h^o} \frac{\|\omega - v_h\|_{0,\Gamma} \|z - z_h\|_{0,\Gamma}}{\|z\|_{\frac{1}{2},\Gamma}}.$$

Standard approximation results yield :
There exists C so that :

$$\forall z \in H^{\frac{1}{2}}(\Gamma), inf_{z_h \in Y_h \backslash Y_h^o} \|z - z_h\|_{0,\Gamma} \leq C h^{\frac{1}{2}} \|z\|_{\frac{1}{2},\Gamma} \quad .$$

On the other hand, since $\omega \in H^k(\Omega)$,

$$\|\omega - v_h\|_{0,\Gamma} \leq C h^{k-\frac{1}{2}} \|\omega\|_{k-\frac{1}{2},\Gamma} \leq C h^{k-\frac{1}{2}} \|\omega\|_k \quad .$$

Hence

$$\|\omega - \omega_{oh} - v_h\|_{-\frac{1}{2},\Gamma} \leq C h^k \|\omega\|_k \quad .$$

Therefore, (54) yields

$$\begin{aligned} \int_\Omega g(\omega - \omega_{oh} - v_h) &\leq C h \|u\|_2 h^{k-1} \|\omega\|_k + C \|u\|_2 h^k \|\omega\|_k \\ &\leq C h^k \|\omega\|_k \|g\|_o \quad . \end{aligned}$$

The proof is completed because of (51). □

Error estimates (36) and (37) seem to show that it is not necessary to approach ψ on a grid as sharp as ω, because $h^{k-\frac{1}{2}-\epsilon}\|\psi\|_{k+2}$ and $H^k\|\psi\|_{k+1}$ play the same role in (37). We shall go back to this matter in III.1.
For higher order approximations, we have the following theorem :

Theorem 2 *Assume that ψ, solution of (12), belongs to $H^{k+2}(\Omega)$, for $k \geq 2$, and ω_o solution of (10) belongs to $H^k(\Omega)$, then if X_H and Y_h are constructed with P^k conforming finite element, then the following estimate is true :*

$$\forall \varepsilon < 1/2,$$
$$\|\psi - \psi_h\|_1 + \|\omega - \omega_h\|_o \leq C \left[h^k [\|\omega_o\|_k + \|\Delta\psi\|_k] + c(\varepsilon) h^{k-1/2-\varepsilon} \|\psi\|_{k+2} + H^k \|\psi\|_{k+1} \right].$$

I.3. Non homogeneous problem

If $\frac{\partial \psi}{\partial n} = g_1 \neq 0$ in (2), we have to replace formulation (11) by :
Find $(\bar{\omega}, \psi)$ in $W' \times H_o^1(\Omega)$:

$$(55) \qquad \begin{cases} \forall v \in W', \int_\Omega \bar{\omega} v = -\int_\Omega \omega_o v - \int_\Gamma g_1 v \\ \forall \varphi \in H_o^1(\Omega), \int_\Omega \nabla\psi . \nabla\varphi = \int_\Omega (\bar{\omega} + \omega_o)\varphi \end{cases}$$

The discretized problem is deduced from (55), and the error estimate is the same.

II. THE GENERALIZED STOKES PROBLEM

A method frequently used to approach numerically the nonlinear convection term in the Navier-Stokes equations for incompressible fluids is the Galerkin characteristics method, also known as the modified method of characteristics ([9]). This method will be briefly described later. For our purpose here, we only need to know that Navier-Stokes equations, discretized by the Galerkin characteristic method require the solution of the following generalized Stokes problem at each time step.

$$(56) \qquad \begin{cases} \alpha\omega^{n+1} - \nu\Delta\omega^{n+1} = f^{n+1} & \text{in } \Omega \\ -\Delta\psi^{n+1} = \omega^{n+1} & \text{in } \Omega \\ \psi^{n+1} = 0 \quad \text{on } \Gamma \\ \dfrac{\partial \psi^{n+1}}{\partial n} = 0 \quad \text{on } \Gamma \end{cases}$$

It differs from (2) by a zero order term in the equation satisfied by ω, which comes from the discretization of $\frac{\partial w}{\partial t}$.
We are going to find a variational formulation for (56) just as in (I). In III, we shall make use of both this formulation and the Galerkin characteristics method in order to solve the

full Navier-Stokes equations. The discretization of the generalized Stokes problem is very similar to what was described for (2), so not all the details will be spell out.

II.1. Analogue of (I) for problem (56). The continuous problem

Let us consider $\omega_{\alpha,\nu}$ defined by :

$$\omega_{\alpha,\nu} = \left\{v \in L^2(\Omega); \nu\Delta v = \alpha v\right\} \quad . \tag{57}$$

It is easy to check that $\omega_{\alpha,\nu}$ is a subspace of V, closed for the L^2 topology, and isomorphic to $H^{-\frac{1}{2}}(\Gamma)$.
We shall also consider $\mathcal{H}$, the following subspace of $V \times H_o^1(\Omega)$:

$$\mathcal{H} = \left\{(v,\varphi) \in \omega_{\alpha,\nu} \times H_o^1(\Omega) : \forall \mu \in H_o^1(\Omega), \int_\Omega \nabla\varphi.\nabla\mu = \int_\Omega v\mu\right\} \quad . \tag{58}$$

$\mathcal{H}$ is closed for the $L^2 \times H_o^1$ topology.

As we did in (10), let us consider ω_o, the unique solution of :

$$\begin{cases} \text{Find } \omega_o \in H_o^1(\Omega), \\ \forall v \in H_o^1(\Omega), \nu \displaystyle\int_\Omega \nabla\omega_o.\nabla v + \alpha \int_\Omega \omega_o v = \int_\Omega fv \end{cases} \quad . \tag{59}$$

Here we also need to solve a problem for ψ :

$$\begin{cases} \text{Find } \psi_o \in H_o^1(\Omega), \\ \forall \varphi \in H_o^1(\Omega), \displaystyle\int_\Omega \nabla\psi_o.\nabla\varphi = \int_\Omega \omega_o\varphi \end{cases} \quad . \tag{60}$$

We can now consider the analogue of (11) : Find $(\bar{\omega},\bar{\psi}) \in \mathcal{H}$, such that

$$\begin{aligned} \forall (v,\varphi) \in \mathcal{H},\ \nu \int_\Omega \bar{\omega} v + \alpha \int_\Omega \nabla\bar{\psi}.\nabla\varphi &= -\nu \int_\Omega \omega_o v - \alpha \int_\Omega \psi_o v \\ &= -\nu \int_\Omega \omega_o v - \alpha \int_\Omega \nabla\psi_o.\nabla\varphi. \end{aligned} \tag{61}$$

Problem (61) has a unique solution, $(\bar{\omega},\bar{\psi})$ which is the projection of $(-\omega_o,-\psi_o)$ onto $\mathcal{H}$, closed for the $L^2 \times H_o^1$ topology.

We can also introduce an ill posed problem just as (17). We define $W'_{\alpha,\nu}$ and $\mathcal{H}'$ by :

$$(62)\qquad \begin{aligned} W'_{\alpha,\nu} &= \left\{ v \in H^1(\Omega) : \forall u \in H^1_o(\Omega), \nu \int_\Omega \nabla v.\nabla u + \alpha \int_\Omega vu = 0 \right\}. \\ \mathcal{H}' &= \left\{ (v,\varphi) \in W'_{\alpha,\nu} \times H^1_o(\Omega) : \forall \mu \in H^1_o(\Omega), \int_\Omega \nabla\varphi.\nabla\mu = \int_\Omega v\mu \right\}. \end{aligned}$$

The ill posed constrained problem analogue to (17) is :

$$(63)\qquad \begin{cases} \text{Find } (\bar\omega,\bar\psi) \in \mathcal{H}', \\ \forall (v,\varphi) \in \mathcal{H}', \qquad \nu \displaystyle\int_\Omega \bar\omega v + \alpha \int_\Omega \nabla\bar\psi.\nabla\varphi = -\nu \int_\Omega \omega_o v - \alpha \int_\Omega \psi_o v. \end{cases}$$

It is possible to prove the following lemma, similar to proposition 2 :

Lemma 7 *If $(\bar\omega,\bar\psi)$ is solution of (63), then it is the only solution, and it is the solution of (61).*
Furthermore, if ψ, solution of :

$$(64)\qquad \psi \in H^2_o(\Omega),\ \forall \varphi \in H^2_o(\Omega),\ \nu \int_\Omega \Delta\psi\Delta\varphi + \alpha \int_\Omega \nabla\psi.\nabla\varphi = \int_\Omega f\varphi$$

belongs to H^3, then $(\Delta\psi - \omega_o, \psi - \psi_o)$ is the solution of (63). □

It is also possible to find an unconstrained problem for (63), at the cost of adding two Lagrange multipliers :
Find $(\bar\omega,\bar\psi) \in \mathcal{H}, (\mu^*,\eta^*) \in H^1_o(\Omega) \times H^1_o(\Omega)$ so that

$$(65)\qquad \begin{cases} \forall (v,\varphi) \in H^1(\Omega) \times H^1_o(\Omega), \nu \displaystyle\int_\Omega \bar\omega v + \alpha \int_\Omega \nabla\bar\psi.\nabla\varphi + \nu \int_\Omega \nabla v.\nabla\mu^* + \alpha \int_\Omega v\mu^* \\ \quad + \displaystyle\int_\Omega \nabla\varphi.\nabla\eta^* - \int_\Omega v\eta^* = -\nu \int_\Omega \omega_o v - \alpha \int_\Omega \psi_o v \\ \forall \mu \in H^1_o(\Omega), \nu \displaystyle\int_\Omega \nabla\bar\omega.\nabla\mu + \alpha \int_\Omega \bar\omega\mu = 0 \\ \forall \eta \in H^1_o(\Omega), \displaystyle\int_\Omega \nabla\bar\psi.\nabla\eta - \int_\Omega \bar\omega\eta = 0 \end{cases}$$

Remark 8

It is easy to notice that if ψ solution of (64) belongs to $H^3(\Omega)$, then

$$\mu^* = -\psi \text{ and } \eta^* = -\alpha(\psi - \psi_o).$$

□

II.2. Approximation of (56) with a mixed finite element method

The spaces X_H and Y_h are defined by (25) and (26). The functions ω_o and ψ_o are approached by conforming finite elements :
Find $(\omega_{oh}, \psi_{oH}) \in Y_h^o \times X_H^o$, such that

$$(66) \qquad \begin{cases} \forall v_h \in Y_h^o, & \nu \int_\Omega \nabla\omega_{oh}.\nabla v_h + \alpha \int_\Omega \omega_{oh} v_h = \int_\Omega f v_h, \\ \forall \varphi_H \in X_H^o, & \int_\Omega \nabla\psi_{oH}.\nabla\varphi_H = \int_\Omega \omega_{oh}\varphi_H. \end{cases}$$

To find the discrete analogue to (63), we introduce the set :

$$(67) \qquad \mathcal{H}'_{h,H} = \left\{ (v_h, \varphi_H) \in Y_h \times X_H^o, \middle| \begin{array}{l} \forall \mu_h \in Y_h^o, \nu \int_\Omega \nabla v_h.\nabla\mu_h + \alpha \int_\Omega v_h\mu_h = 0 \\ \forall \eta_H \in X_H^o, \int_\Omega \nabla\varphi_H.\nabla\eta_H - \int_\Omega v_h\eta_H = 0 \end{array} \right\}.$$

The discrete analogue to (63) is : Find $(\bar{\omega}_h, \bar{\psi}_H) \in \mathcal{H}'_{h,H}$, such that

$$(68) \qquad \forall (v_h, \varphi_H) \in \mathcal{H}'_{h,H}, \nu \int_\Omega \bar{\omega}_h v_h + \alpha \int_\Omega \nabla\bar{\psi}_H.\nabla\varphi_H = -\nu \int_\Omega \omega_{oh} v_h - \alpha \int_\Omega \psi_{oH} v_h.$$

We can prove just as for lemma 3 that problem (68) has a unique solution.

Remark 9

The approximation (68) is a mixed approximation, since $\mathcal{H}'_{h,H} \not\subset \mathcal{H}'$. □

Remark 10

Discretization (66) does not induce any interpolation error, since $\int_\Omega \omega_{oh}\varphi_H$ is computed exactly. □

The discretized problem (68) is not directly implementable, because $\mathcal{H}'_{h,H}$ is not explicitly known. To implement it, one need to introduce a boundary problem similar as (35) which we shall discuss in section II.4.

II.3. An error estimate for problem (56)

Let us estimate the error when problem (56) is approached by problems (66)-(68). However, we shall not specify how the constants which appear in the estimates depend on α and

ν since this calculation is very technical. Hence, this estimate will not indicate how to choose h, H, α for a Navier-Stokes problem. For the following, we shall replace f with f_h in (66). The error estimate will then depend on $\|f - f_h\|_o$.

Theorem 3 Error estimate for the approximation of (56)

Assume that Ω is polygonal convex. Assume that Y_h and X_H are related to P^k finite elements ($k = 1$ or 2), and that the triangulations are uniformly regular in h and H. Assume that ψ the solution of (64) belongs to $H^{k+2}(\Omega)$, then for all $\varepsilon < 1/2$, there exist constants $C(\alpha, \nu)$ and $c(\varepsilon)$ so that :

(69)

$$\|\omega - \omega_h\|_o + \|\psi - \psi_H\|_1 \leq C(\alpha, \nu, \varepsilon) \begin{bmatrix} h^{k-1/2-\varepsilon}\|\psi\|_{k+2} + H^k\|\psi\|_{k+1} + h^k\|\omega\|_k + \\ H^2\|f\|_{-1} + h^2\|f\|_o + \|f - f_h\|_o \end{bmatrix} .$$

□

The proof of theorem 3 is in appendix 1.

Remark 11

In fact, there should be a different constant depending on α, ν and ε for each norm appearing in the second member of (69). A sharp error estimate cannot be obtained if these constants are not computed. □

II.4. A boundary formulation for (56)

$\mathcal{H}'_{h,H}$ is isomorphic to $Y_h \backslash Y_h^o$. The isomorphism is :

$$\lambda_h \in Y_h \backslash Y_h^o \rightarrow (\bar{\omega}_h(\lambda_h), \bar{\psi}_H(\lambda_h)) \in \mathcal{H}'_{h,H} \quad ,$$

where $\bar{\omega}_h(\lambda_h)$ satisfies :

(70)
$$\begin{cases} \bar{\omega}_h(\lambda_h) - \lambda_h \in Y_h^o \\ \forall v_h \in Y_h^o, \nu \int_\Omega \nabla\bar{\omega}_h(\lambda_h).\nabla v_h + \alpha \int_\Omega \bar{\omega}_h(\lambda_h) v_h = 0 \end{cases} .$$

Just as for the Stokes problem (2), we can define $a_{h,H}$ the symmetric positive definite form on $Y_h \backslash Y_h^o$ given by :

(71)
$$a_{h,H}(\lambda_h, \mu_h) = \nu \int_\Omega \bar{\omega}_h(\lambda_h)\bar{\omega}_h(\mu_h) + \alpha \int_\Omega \nabla\bar{\psi}_H(\lambda_h).\nabla\bar{\psi}_H(\mu_h) \quad .$$

Just as for the Stokes problem (2), solving (68) is equivalent to solving :

$$(72) \qquad a_{h,H}(\lambda_h, \mu_h) = (b_{h,H}, \mu_h) = -\nu \int_\Omega \omega_{oh} \bar{\omega}_h(\mu_h) - \alpha \int_\Omega \psi_{oH} \bar{\omega}_h(\mu_h)$$

Let

$$(\bar{\omega}_h(\mu_i), \bar{\psi}_H(\mu_i))_{1 \le i \le I}$$

be a basis of $\mathcal{H}'_{h,H}$. This basis bas been computed once for all, and stored in memory.

One can then compute the matrix associated with $a_{h,H}$. If this is done, solving (68) can be achieved by :

1) Computing $-\nu \int_\Omega \omega_{oh} \bar{\omega}_h(\mu_i) - \alpha \int_\Omega \nabla \psi_{oH} . \nabla \bar{\psi}_H(\mu_i)$ for each $1 \le i \le I$.

2) Finding λ_h, by solving a small linear system with I unknowns.

3) Computing $\bar{\omega}_h(\lambda_h), \bar{\psi}_H(\lambda_h)$ from λ_h.

III THE NAVIER-STOKES EQUATIONS

When $\nu \to 0$ the velocity near the boundary $u = \{\psi_y, -\psi_x\}$ may have a thin region of strong gradient : a boundary layer.
Typically the dependency of the velocity upon ν is of the form (see Landau-Lifschitz[10], for example) $u = g(x/\sqrt{\nu})$ where g is a function of order 1 and x is the distance from the wall. Thus $\psi = \sqrt{\nu}\ \tilde{\psi}(x/\sqrt{\nu})$ and $\omega = \tilde{\omega}(x/\sqrt{\nu})/\sqrt{\nu}$. These expressions are valid within the boundary layer which is a thin region around the walls of thickness $\sqrt{\nu}$ and also in the wake of an aerodynamic profile.

III-1 From Stokes to Navier-Stokes

It has been shown by Rappaz et al [11] that the error analysis for the Stokes equations

$$-\nu \Delta u + \nabla p = f, \quad \nabla . u = 0$$

carries on to the Stationnary Navier-Stokes equations

$$-\nu \Delta u + \nabla p = -u \nabla u, \quad \nabla . u = 0 \qquad ,$$

without difficulty with f replaced by $u \nabla u$. This is because the fixed point algorithm

$$-\nu \Delta u^{n+1} + \nabla p^{n+1} = -u^n \nabla u^n, \quad \nabla . u^{n+1} = 0 \qquad .$$

converges to an isolated solution of the Navier-Stokes equation if it is properly initialized.

In the formulation with the stream function ψ and the vorticity ω in two dimensions $u\nabla u$ is replaced by $u\nabla\omega$ because the fixed point algorithm is

$$-\nu\Delta\omega^{n+1} = -u^n\nabla\omega^n \quad ,$$

$$-\Delta\psi^{n+1} = \omega^{n+1} \quad .$$

In view of (36)(37), the error estimates for a numerical method based on a dicretization of the above biharmonic problem by the method studied in Section II would be that of the Stokes problem with $f = -u\nabla\omega$:

$$\forall\varepsilon < 1/2, \quad \|\omega^{n+1} - \omega_h^{n+1}\|_o + \|\psi^{n+1} - \psi_H^{n+1}\|_1$$
$$\leq C[h^k\|\omega^n\|_k + h^2\frac{1}{\nu}\|u^n\nabla\omega^n\|_0 + c(\varepsilon)h^{k-1/2-\epsilon}\|\psi^n\|_{k+2} + H^k\|\psi^n\|_{k+1}].$$

But the algorithm converges so it is reasonable to expect that $||\psi^n|| \approx ||\psi||$ and $||\omega^n|| \approx ||\omega||$.
Now if there are no other singularity than the boundary layer we have:

$$||\psi||_3 \approx ||\omega||_1 \approx (\int_\Omega |\frac{\partial^3\psi}{\partial x^3}|^2)^{\frac{1}{2}} \approx (\int_0^{\sqrt{\nu}} |\sqrt{\nu}\frac{\partial^3\tilde{\psi}}{\partial x^3}(\frac{x}{\sqrt{\nu}})|^2)^{\frac{1}{2}} \approx \nu^{-\frac{3}{4}}(\int_0^1 |\frac{\partial^3\tilde{\psi}}{\partial y^3}(y)|^2)^{\frac{1}{2}},$$

$$||\psi||_4 \approx ||\omega||_2 \approx (\int_\Omega |\frac{\partial^2\omega}{\partial x^2}|^2)^{\frac{1}{2}} \approx (\int_0^{\sqrt{\nu}} |\sqrt{\nu}\frac{\partial^2\tilde{\omega}}{\partial x^2}(\frac{x}{\sqrt{\nu}})|^2)^{\frac{1}{2}} \approx \nu^{-\frac{5}{4}}(\int_0^1 |\frac{\partial^2\tilde{\omega}}{\partial y^2}(y)|^2)^{\frac{1}{2}},$$

$$\frac{1}{\nu}||u\nabla\omega||_0 \approx ||\omega||_2 \approx \nu^{-\frac{5}{4}}.$$

Therefore if an error of order η is desired, we must have :

$$h^k\|\omega^n\|_k + h^2\frac{1}{\nu}\|u^n\nabla\omega^n\|_0 + c(\varepsilon)h^{k-1/2-\epsilon}\|\psi^n\|_{k+2} + H^k\|\psi^n\|_{k+1}$$
$$\approx (h^k + c(\varepsilon)h^{k-1/2-\epsilon})\nu^{\frac{-2k-1}{4}} + h^2\nu^{-\frac{5}{4}} + H^k\nu^{-\frac{2k-1}{4}} << \eta,$$

i.e.
if k=1, $h << \eta^2\nu^{\frac{3}{2}}$ and $H << \eta\nu^{\frac{1}{4}}$,
if k=2, $h << \eta^{\frac{2}{3}}\nu^{\frac{5}{6}}$ and $H << \eta^{\frac{1}{2}}\nu^{\frac{3}{8}}$.

Fundamental remark

We note that the gain is very substancial because, in the case $k = 1$ for instance, this analysis gives $H = O(h^{1/6})$. We can also notice that the gain is all the larger that the order of approximation is smaller. □

III.2 The characteristic-Galerkin method.

The characteristic-Galerkin method, also known as the modified method of characteristics [9] [12] [13] [14] is an efficient discretization of the convection terms which replaces (1) by a sequence of generalized Stokes problems.
Let us recall the method briefly. Consider

$$\frac{dX}{d\tau} = u(X,\tau) \qquad X_{|\tau=t} = x \qquad , \tag{73}$$

and a time step k ; φ^n will denote φ at $t = nk$; $X(x,t;\tau)$ is the position at time τ of the fluid particle which was at position x at time t. Denote

$$X^n(x) = X(x,(n+1)k;nk) \qquad . \tag{74}$$

Now it is not hard to show that

$$\left(\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega\right)_{|x,t=(n+1)k} = \frac{1}{k}[\omega^{n+1}(x) - \omega^n(X^n(x))] + O(k) \qquad ; \tag{75}$$

so a reasonable semi-discretization of (1) is

$$\begin{cases} -\Delta\psi^{n+1} = \omega^{n+1} \\ \dfrac{1}{k}\omega^{n+1} - \nu\,\Delta\omega^{n+1} = f + \dfrac{1}{k}\omega^n \circ X^n \\ \psi_{|\Gamma} = g_0 \quad , \quad \dfrac{\partial\psi}{\partial n}_{|\Gamma_1} = 0 \quad , \quad \omega_{|\Gamma_2} = g_1 \end{cases} \qquad . \tag{76}$$

where X^n is computed by (73) with $u = \nabla\times\psi^n$. Thus, after discretization, the algorithm discussed earlier is as follows.

Initialization.

Construct a triangulation $\mathcal{T}_1$ of Ω of size H ; subdivide it to obtain a triangulation $\mathcal{T}_2$ of size h. Denote X_H, Y_h the spaces of conforming finite element of degree 1 or 2.

Denote $\{\mu_i\}_{i=1}^I$ the functions of Y_h which are equal to 1 at nodes i of Γ_1 and 0 at all other nodes, respectively. Solve for all $i = 1, ..., I$

$$(77) \qquad \begin{cases} \overline{\omega}_k(\mu_i) - \mu_i \in Y_h^0 \\ \forall v_h \in Y_h^0 \quad , \quad \nu \int_\Omega \nabla \overline{\omega}_h(\mu_i) \cdot \nabla v_h + \frac{1}{k} \int_\Omega \overline{\omega}_h(\nu_i) v_h = 0 \end{cases}$$

Solve

$$(78) \qquad \begin{cases} \overline{\psi}_H(\mu_i) \in X_H^0 \\ \forall \varphi_H \in X_H^0 \qquad \int_\Omega \nabla \overline{\psi}_H(\mu_i) \cdot \nabla \varphi_H = \int_\Omega \overline{\omega}_h(\mu_i) \varphi_H \end{cases}$$

Store $\{\overline{\omega}_h(\mu_i), \overline{\psi}_H(\mu_i)\}$; compute and store

$$(79) \qquad A_{ij}^{h,H} = \nu \int_\Omega \overline{\omega}_h(\mu_i) \overline{\omega}(\mu_j) + \frac{1}{k} \int_\Omega \overline{\psi}_H(\mu_i) \cdot \nabla \overline{\psi}_H(\mu_j) \quad .$$

For each time step.

Solve

$$(80) \qquad \begin{cases} \omega_{0h}^{n+1} - \omega_{1h} \in Y_h^0 \\ \forall v_h \in Y_h^0 \quad , \quad \nu \int_\Omega \nabla \omega_{0h}^{n+1} \cdot \nabla v_h + \frac{1}{k} \int_\Omega \omega_{0h}^{n+1} v_h = \int_\Omega f v_h + \frac{1}{h} \int_\Omega (\omega_h^n \circ X_H^n) v_h \end{cases}$$

where X_H^n is an approximation of X^n.

Solve

$$(81) \qquad \begin{cases} \psi_{0H}^{n+1} - g_0 \in X_H^0 \\ \forall \varphi_H \in X_H^0 \quad , \quad \int_\Omega \nabla \psi_{0H}^{n+1} \cdot \nabla \varphi_H = \int_\Omega \omega_{0H}^{n+1} \varphi_H \end{cases}$$

Compute

$$(82) \qquad b_i^{h,H} = -\nu \int_\Omega \omega_{0h}^{n+1} \overline{\omega}_h(\mu_i) - \frac{1}{k} \int_\Omega \nabla \psi_{0H}^{n+1} \cdot \nabla \overline{\psi}_H(\mu_i) \quad ,$$

and solve

$$A^{h,H} \lambda = b^{h,H} \quad .$$

Set

$$(84) \qquad \omega_h^{n+1} = \omega_{0h}^{n+1} + \sum_{i=1,...,I} \lambda_i \overline{\omega}_h(\mu_i) \quad ,$$

$$\psi_H^{n+1} = \psi_{0H}^{n+1} + \sum_{i=1,\dots,I} \lambda_i \overline{\psi}_h(\mu_i) \quad . \tag{85}$$

Notice that it is not essential to require $\omega_{0h}^{n+1}|_\Gamma = \omega_{1h}$ in (80) and $\psi_{0H}^{n+1}|_\Gamma = g_0$ in (81); these constraints could be relaxed and imposed at the level of (84) (85) by superposition. In some cases this remark makes room for a substantial speed up of the numerical method.

We have seen earlier that for a steady or quasi-steady flow around an airfoil for example, the right choice is $h \ll \nu^{3/2}; H \ll \nu^{1/4}$ in the boundary layer and in the wake. Thus the computing time will be dominated by the computation of ω_{0h}^{n+1}.

III.3. Speeding up the method.

III.3.1. The χ-method.

To speed up the computation of ω_{0h}^{n+1} we must use the parabolic character of boundary layers.

It is known since Prandtl (see Rosenhead [15]) that $\partial^2\omega/\partial s^2 \ll \partial^2\omega/\partial n^2$ in wakes and boundary layers when s is in the direction of u and n normal to it. Thus in (76) and (80) $-\nu\Delta\omega^{n+1}$ could be replaced by $-\nu\,\partial^2\omega/\partial n^2$ in these regions. Another way of saying the same thing is to say that if (76) or (80) were solved by a block Gauss-Seidel relaxation sweep in the direction of the flow, a few iterations would be enough. Some preliminary tests with this idea have been done in (Derrico-Pironneau [16]); these show not only that it is efficient but also that it is better than direct methods which tend to be very ill-conditionned with elements of large aspect ratio.

However there are two difficulties:

- the thickness and position of boundary layers and wakes are known only crudely.
- the method requires a sophisticated renumbering ofthe nodes in directions normal to the flow. The second difficulty has been solved in [17].

For the first difficulty, we shall use the χ-method. Given a small positive number δ we denote by $\chi_\delta(x)$ the function which is zero when $|x| < \delta - \delta^2$, x when $|x| > \delta$ and the linear interpolation in between ($\chi_\delta(x) = \delta - 1 + x/\delta$ when $\delta - \delta^2 \le |x| \le \delta$).

In the χ-method the first two equations of (76) are replaced by

$$\begin{cases} -\Delta\psi^{n+1} = \chi_\delta(\omega^{n+1}) \\ \dfrac{1}{k}\omega^{n+1} - \nu\displaystyle\sum_i \frac{\partial}{\partial x_i}\chi_{\delta c_i}\Big(\frac{\partial\omega^{n+1}}{\partial x_i}\Big) = f + \frac{1}{k}\omega^n \circ X^n \end{cases} , \tag{86}$$

where $c_i = (u_i^n)^2 / |\frac{\partial \omega^n}{\partial X_i}|$.
Convergence when $\delta \to 0$ for this class of methods is studied in [18] [19]. The basic idea is to notice that

$$|\chi_j(\omega) - \omega| \leq \delta \quad , \tag{87}$$

therefore (86) is a δ-perturbation of (76).
From the numerical point of view the equation on ω^{n+1} in (86) is much easier to solve than the original one in (76) because it has the parabolic character of Prandtl's equations in the boundary layers.

III.3.2. Another basis $\overline{\omega}(\mu^i)$.

Beside the storage problem for $\overline{\omega}(\mu^i)$; the computation of $\int_\Omega \omega^0 \overline{\omega}_h(\mu^i)$ is expensive because $\overline{\omega}_h(\mu^i)$ does not have a small compact support in Ω.
Again we observe that it is not essential to have $\overline{\omega}_h(\mu^i) = \mu^i$ on the boundary. Other basis of $Y_h \backslash Y_h^0$ can be chosen.
Let x^i denote the i-th boundary node and let

$$\overline{\omega}^i = \xi(|x - x^i|) \quad \text{where} \quad -\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\xi + \frac{\xi}{k\nu} = 0 \quad .$$

Obviously

$$-\nu \Delta \overline{\omega}^i + \frac{\overline{\omega}^i}{k} = 0 \quad .$$

and $|\overline{\omega}^i|_i$ is a basis of $Y - Y_h^0$. Now $\overline{\omega}^i(x)$ decays rapidly when $|x - x^i|$ increases so

$$\int_\Omega \omega_{0h}^{n+1} \overline{\omega}^i \simeq \int_{V(x^i)} \omega_{0h}^{n+1} \overline{\omega}^i \quad ,$$

where $V(X^i)$ is a neighborhood of x^i and where δ' is the distance betwwen $\partial V(x^i)$ and x^i. Notice that it is no longer necessary to store $\overline{\omega}^i$, nevertheless since it may be time consuming to recompute the values of ξ we may also store the values $\omega^i(x^j)$, for all nodes x^j of $V(X^i)$.

IV. CONCLUSIONS.

Adaptative mesh refinement for the Navier-Stokes equations is a powerful tool for an efficient management of the computer resources at high Reynolds number. We have shown

on one formulation of the equations, vorticity-stream function in 2 dimensions, and one discretization method, the mixed finite element method, that it is not necessary to refine the mesh uniformely for both variables; only the vorticity needs to be refined in regions of strong gradients like boundary layers and wakes. However it was necessary to modify the standard mixed method at the cost of an increase in the memory requirement. Error estimates show that a reasonable choice for a P^1 approximation is $h(\omega) \ll \nu^{3/2}, H(\psi) \ll \nu^{1/4}$.

Finally we have noticed that this tool does more than a speed up by a factor two because the partial differential equation on ω degenerates in the boundary layers and wakes and it is much easier to solve than the one on ψ, even though the number of degrees of freedom is much larger.

This analysis is theoretical but it will be followed by a numerical simulation and until this is done and the implementation details are solved it will of course be hard to convince the reader that it is worthwhile the complexity of the programming.

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Appendix 1 : Proof of theorem 3

Here, in order to shorten the notations, $(u,v)_o$ stands for the $L^2(\Omega)$ scalar product.
Let us first estimate the error on ω_o. Since ω_o is approached by conforming finite elements,

$$\|\omega_o - \omega_{oh}\|_1 \leq C(\alpha,\nu)[h\|\omega_o\|_2 + \|f - f_h\|_0] \quad .$$

Aubin Nitsche trick yields :

$$\begin{aligned} |\omega_o - \omega_{oh}\|_o &\leq C(\alpha,\nu)[h^2\|\omega_o\|_2 + \|f - f_h\|_0] \\ &\leq C(\alpha,\nu)[h^2\|f\|_0 + \|f - f_h\|_0] \quad . \end{aligned} \tag{A.1}$$

Let us now give estimates for the error for ψ_o. They will be used for estimating $\|\bar{\omega} - \bar{\omega}_h\|_o$.
Conforming finite element approximation results yield :

$$\begin{aligned} \|\psi_o - \psi_{oH}\|_1 &\leq CH\|\psi_o\|_2 + \|\omega_o - \omega_{oh}\|_o \\ &\leq C(\alpha,\nu)[H\|f\|_{-1} + \|\omega_o - \omega_{oh}\|_o] \quad . \end{aligned}$$

Aubin Nitsche lemma gives :

$$\|\psi_o - \psi_{oH}\|_o \leq C(\alpha,\nu)[H^2\|f\|_{-1} + \|\omega_o - \omega_{oh}\|_o] \quad . \tag{A.2}$$

The main part of the proof deals with the error estimate for $\bar{\omega}$.
Let us take $(v_h - \bar{\omega}_h, \varphi_H - \bar{\psi}_H) \in \mathcal{H}'_{h,H}$ as a test function in (65).

Remark 8 yields : $\forall \mu_h \in Y_h^o, \forall \eta_H \in X_H^o$,

$$\begin{aligned} &\nu(\bar{\omega}, v_h - \bar{\omega}_h)_o + \alpha(\nabla\bar{\psi}, \nabla\varphi_H - \nabla\bar{\psi}_H)_o \\ &+ \nu(-\nabla\psi - \nabla\mu_h, \nabla v_h - \nabla\bar{\omega}_h)_o + \alpha(-\psi - \mu_h, v_h - \bar{\omega}_h)_o \\ &+ \alpha(-\nabla\bar{\psi} - \nabla\eta_H, \nabla\varphi_H - \nabla\bar{\psi}_H)_o + \alpha(\bar{\psi} + \eta_H, \varphi_H - \bar{\psi}_H)_o = \\ &- \nu(\omega_o, v_h - \bar{\omega}_h)_o - \alpha(\nabla\psi_o, \nabla\varphi_H - \nabla\bar{\psi}_H)_o \quad . \end{aligned} \tag{A.3}$$

Taking $(v_h - \bar{\omega}_h, \varphi_H - \bar{\psi}_H)$ as a test function in (68) yields :

$$\begin{aligned} &\nu(\bar{\omega}_h, v_h - \bar{\omega}_h)_o + \alpha(\nabla\bar{\psi}_H, \nabla\varphi_H - \nabla\bar{\psi}_H)_o = \\ &- \nu(\omega_{oh}, v_h - \bar{\omega}_h)_o + \alpha(\nabla\psi_{oH}, \nabla\varphi_H - \nabla\bar{\psi}_H)_o \quad . \end{aligned} \tag{A.4}$$

Substracting (A.4) from (A.3), one gets : $\forall \mu_h \in Y_h^o, \forall \eta_H \in X_H^o$,

$$
\begin{aligned}
& \nu(\omega - \omega_{oh} - \bar{\omega}_h, v_h - \bar{\omega}_h)_o + \alpha(\nabla\psi - \nabla\psi_{oH} - \nabla\psi_H, \nabla\varphi_H - \nabla\bar{\psi}_H)_o \\
& \nu(-\nabla\psi - \nabla\mu_h, \nabla v_h - \nabla\bar{\omega}_h)_o + \alpha(-\psi - \mu_h, v_h - \bar{\omega}_h)_o \\
& + \alpha(-\nabla\bar{\psi} - \nabla\eta_H, \nabla\varphi_H - \nabla\bar{\psi}_H) + \alpha(\bar{\psi} - \eta_H, \varphi_H - \bar{\psi}_H)_o \\
& = 0 \quad .
\end{aligned} \tag{A.5}
$$

Hence, using (60) and (66),

$$
\nu\|v_h - \bar{\omega}_h\|_o + \alpha|\varphi_H - \bar{\psi}_H|_1 = a + b + c + d \quad , \tag{A.6}
$$

with

$$
\begin{aligned}
a &= \nu(\omega - \omega_{oh} - \bar{\omega}_h, \bar{\omega}_h - v_h)_o + (\omega - \omega_{oh} - \bar{\omega}_h, \bar{\psi}_H - \varphi_H)_o \quad , \\
b &= \nu(\nabla\psi - \nabla\mu_h, \nabla v_h - \nabla\bar{\omega}_h)_o + \alpha(\psi - \mu_h, v_h - \bar{\omega}_h)_o \quad , \\
c &= \alpha(\nabla\psi - \nabla\psi_{oH} - \nabla\eta_H, \nabla\varphi_H - \nabla\bar{\psi}_H)_o - \alpha(\psi - \psi_{oH} - \eta_H, \varphi_H - \bar{\psi}_H)_o \quad , \\
d &= -\alpha(\omega_o - \omega_{oh}, \varphi_H - \bar{\psi}_H)_o + \alpha(\psi_o - \psi_{oH}, \varphi_H - \bar{\psi}_H)_o \quad .
\end{aligned}
$$

The idea is now to restrict ourselves to $\eta_H = \varphi_H$.

Let us call $R = \left[\nu\|v_h - \bar{\omega}_h\|_o^2 + \alpha\|\varphi_H - \bar{\psi}_H\|_o^2\right]^{\frac{1}{2}}$.
The following inequalities are true :

$$
\begin{aligned}
& a \leq C(\alpha, \nu)\|\omega - \omega_{oh} - v_h\|_o R \quad , \\
& \forall \epsilon < 1/2, \quad b \leq C(\alpha, \nu, \epsilon) h^{k-1/2-\epsilon}\|\psi\|_{k+2} R \quad , \quad \text{(Sholz lemma)} \\
& c \leq C(\alpha, \nu)\left[\|\omega - \omega_{oh} - v_h\|_o + \|\psi - \psi_{oH} - \varphi_H\|_o\right] R \quad , \\
& d \leq C(\alpha, \nu)\left[\|\omega_o - \omega_{oh}\|_o + \|\psi_o - \psi_{oH}\|_o\right] R \quad .
\end{aligned}
$$

Hence,

$$
R \leq C(\alpha, \nu, \epsilon)\left[\begin{array}{l} \|\omega - \omega_{oh} - v_h\|_o + \|\psi - \psi_{oh} - \varphi_h\|_o + h^{k-1/2-\epsilon}\|\psi\|_{k+2} \\ + \|\omega_o - \omega_{oh}\|_o + \|\psi_o - \psi_{oH}\|_o \end{array}\right] . \tag{A.7}
$$

On the other hand

$$
\|\bar{\omega} - \bar{\omega}_h\|_o \leq C(\alpha, \nu)\left[\|\omega - \omega_{oh} - v_h\|_o + \|\omega_o - \omega_{oh}\|_o + R\right] \quad .
$$

Therefore

$$(A.8) \qquad \|\bar{\omega} - \bar{\omega}_h\|_o \leq C(\alpha, \nu, \varepsilon)[A + B + D] \quad .$$

with

$$A = Inf_{(v_h, \varphi_H) \in \mathcal{H}'_{h,H}} [\|\omega - \omega_{oh} - v_h\|_o + \|\psi - \psi_{oH} - \varphi_H\|_o] \quad ,$$

$$B = \|\omega_o - \omega_{oh}\|_o + \|\psi_o - \psi_{oH}\|_o \quad ,$$

$$D = h^{k-1/2-\varepsilon}\|\psi\|_{k+2} \quad .$$

The estimate for A is obtained just as lemma 5, by means of an Aubin-Nitsche trick :

$$A \leq C(\alpha, \nu) \left[\|\omega\|_k h^k + \|f - f_h\|_o + H^2\|\psi\|_2\right] \quad .$$

Hence

$$\|\bar{\omega} - \bar{\omega}_h\|_o \leq C(\alpha, \nu, \varepsilon) \left[\begin{array}{l} h^{k-1/2-\varepsilon}\|\psi\|_{k+2} + H^2\|\psi\|_2 + h^k\|\omega\|_k + H^2\|f\|_{-1} + h^2\|f\|_o \\ + \|f - f_h\|_o \end{array}\right] .$$

The last thing to do is to study the error on ψ. This is a standard result :

$$\|\psi - \psi_H\|_1 \leq C\left[H^k\|\psi\|_{k+1} + \|\omega_o - \omega_{oh}\|_o + \|\bar{\omega} - \bar{\omega}_h\|_o\right] \quad .$$

Theorem 3 is proved. □

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