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*Numerical analysis of junctions between thin shells, Part 1 : continuous problems*

Michel Bernadou , Annie Cubier

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————— THÈME 4 —————



*Rapport  
de recherche*



# Numerical analysis of junctions between thin shells, Part 1 : continuous problems

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Thème 4 — Simulation  
et optimisation  
de systèmes complexes  
Projet MODULEF

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**Abstract:** The junctions of beams, plates and shells are the basic components of any industrial structural construction. The numerical simulation of such junctions is a classical part of the commercial finite element codes. On the other hand it seems that there are very few mathematical studies of such junctions. In this paper, we propose a variational formulation of junctions between thin shells when the junction can be considered as an elastic or a rigid hinge. Then, we study the mathematical properties of these equations.

**Key-words:** Thin shells. Elastic junction. Rigid junction. Equilibrium equations. Variational formulations.

*(Résumé : tsvp)*

## **Analyse numérique de jonctions de coques minces, Partie 1 : Problèmes continus**

**Résumé :** Les jonctions de poutres, plaques et coques sont à la base de nombreuses constructions. La simulation numérique de telles jonctions peut être traitée par certains codes d'éléments finis. Cependant, il y a très peu d'analyse mathématique de ce type de problèmes. Dans cet article, nous proposons une formulation variationnelle de jonctions de coques minces dans le cas où la charnière a un comportement élastique ou rigide. Nous étudions alors les propriétés mathématiques de ces équations.

**Mots-clé :** Coques minces. Jonction élastique. Jonction rigide. Equations d'équilibre. Formulations variationnelles.

## 1 Introduction

Many industrial constructions use, as basic components, elastic beams, plates and shells. The numerical simulation of such assemblages needs a good approximation of each constitutive element as well as a good representation of their junctions. In engineering literature, there are many contributions on the best way to modelize and, particularly, to compute such constructions.

Thus, for the modelization of the mechanical engineering aspects, we refer for instance the reader to the chapter 6 of Baker-Kovalevsky-Rish (1981) which is devoted to multishell structures, to §§ 3.4 and 6.2 of Flügge (1973) which are concerned with polygonal domes and with different kinds of junctions between cylinders and spherical caps or ends, and to §§ 11.4 and 18.5 of Calladine (1983) for pressure-vessel junction problems. For finite element methods, we can recommend Bathe (1982) and Bathe-Ho(1981) and the references of these works.

By contrast there are very few mathematical studies in these directions. Nevertheless we can mention the works by Aufranc (1989) and Ciarlet (1990) which are mainly concerned with the problem of the junction between three-dimensional and two-dimensional linearly elastic structures while Le Dret (1991) consider asymptotic developments for junction between two plates.

In this series of papers, we restrict our attention to the numerical analysis of the junction between two shells. Our study follows the main lines of Bernadou-Fayolle-Léné (1989) and lies on the following assumptions :

- elastic, homogeneous, isotropic material,
- small deformations,
- deformation through the thickness satisfies the usual assumptions of Koiter (1966),
- the junction can be assimilated to a rigid or to an elastic hinge.

The contents of the Part 1 of this paper can be outline as follows : Section 2 discusses the mechanical modelling of the junction between thin shells in terms of partial differential equations. We start by recalling the general Koiter equations ; then, we introduce the modelization of the junction as an elastic or a rigid hinge and we conclude by giving some examples. Section 3 gives the variational formulations of these different junctions and the corresponding existence results. It is also proved that the solution of the elastic hinge problem converges to the solution of the rigid hinge problem when the elastic stiffness of the hinge becomes very large. The approximation by finite element methods will be analyzed in Part 2 of this paper.

## 2 Mechanical modelling

In this section, we briefly introduce the main notations and the basic equations that we subsequently use. We refer the reader to Koiter (1966) for more details concerning the equilibrium equations for one shell ; for convenience, we record the main topics in Paragraph 2.1.

### 2.1 Equilibrium equations for one shell

Let  $\Omega$  be a bounded open subset in a plane  $\mathcal{E}^2$ , with a sufficiently smooth boundary  $\partial\Omega$ . Then the middle surface  $\mathcal{S}$  of the shell is defined as the image of the set  $\bar{\Omega}$  by a mapping

$$\Phi : \bar{\Omega} \subset \mathcal{E}^2 \rightarrow \mathcal{E}^3,$$

where  $\mathcal{E}^3$  is the usual Euclidean space. Subsequently, we assume that  $\Phi \in (C^3(\bar{\Omega}))^3$  and that all points of  $\mathcal{S} = \Phi(\bar{\Omega})$  are regular, i.e., the two vectors  $\mathbf{a}_\alpha = \Phi_{,\alpha}$ ,  $\alpha = 1, 2$ , are linearly independent for all points  $\xi = (\xi^1, \xi^2) \in \bar{\Omega}$ . With the covariant basis  $(\mathbf{a}_\alpha)$  of the tangent plane, we associate the contravariant basis  $(\mathbf{a}^\alpha)$  which is defined through the relations  $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta_\beta^\alpha$  (no summation if  $\alpha = \beta$ ), where  $\delta_\beta^\alpha$  is the Kronecker's symbol. We also introduce the unit normal vector  $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2 / \|\mathbf{a}_1 \times \mathbf{a}_2\|$  and we set  $a = \det(a_{\alpha\beta})$ ,  $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ .

By using such local covariant and contravariant bases, the linear equilibrium equations can be written as (see Koiter (1966), equations (11.27) and (11.28)) :

$$[n^{\alpha\beta} + \frac{1}{2}b_\nu^\alpha m^{\beta\nu} - \frac{1}{2}b_\nu^\beta m^{\alpha\nu}] |_\beta + b_\nu^\alpha m^{\beta\nu} |_\beta + p^\alpha = 0 \quad \text{in } \mathcal{S}, \quad (2.1)$$

$$-m^{\alpha\beta} |_{\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} + p^3 = 0 \quad \text{in } \mathcal{S}, \quad (2.2)$$

with the following boundary conditions (see Koiter (1966), equations (11.29) and (11.31)) :

$$[n^{\alpha\beta} + \frac{3}{2}b_\nu^\alpha m^{\beta\nu} - \frac{1}{2}b_\nu^\beta m^{\alpha\nu}] n_\beta = N^\alpha + b_\beta^\alpha M^\beta \quad \text{on } \partial\mathcal{S}, \quad (2.3)$$

$$-m^{\alpha\beta} |_\alpha n_\beta - (m^{\alpha\beta} n_\alpha t_\beta)_{,s} = N^3 - (M^\alpha t_\alpha)_{,s} \quad \text{on } \partial\mathcal{S}, \quad (2.4)$$

$$m^{\alpha\beta} n_\alpha n_\beta = M^\alpha n_\alpha \quad \text{on } \partial\mathcal{S}. \quad (2.5)$$

In these equations, we have adopted the following notations :

$n^{\alpha\beta}, m^{\alpha\beta}$  = symmetric tensors of tangential stress resultants and stress couples,

$b_{\alpha\beta}, b_\alpha^\beta$  = covariant and mixed components of the second fundamental form of the middle surface  $\mathcal{S}$ ,

$(.) |_\alpha$  = covariant derivative with respect to  $\xi^\alpha$ ,

$\mathbf{p} = p^i \mathbf{a}_i$  = external loads referred to the middle surface  $\mathcal{S}$ ,

$$\mathbf{n} = n_\alpha \mathbf{a}^\alpha = \text{outward unit normal vector to the boundary } \partial\mathcal{S} \text{ in the tangent plane. This normal vector in the tangent plane should not be confused with the stress tensor,} \quad (2.6)$$

$$\mathbf{t} = \mathbf{a}_3 \times \mathbf{n} = t_\alpha \mathbf{a}^\alpha = \text{unit tangent vector to the boundary } \partial\mathcal{S}, \quad (2.7)$$

$$\mathbf{N} = N^i \mathbf{a}_i = \text{external edge loads per unit length of the boundary } \partial\mathcal{S}, \quad (2.8)$$

$$\mathbf{M} = M^\beta \mathbf{a}_\beta \times \mathbf{a}_3 = (M_n \mathbf{n} + M_t \mathbf{t}) \times \mathbf{a}_3 = M_t \mathbf{n} - M_n \mathbf{t} = \text{external edge moment loads per unit length of the boundary } \partial\mathcal{S}, \quad (2.9)$$

$ds$  = line element along the boundary  $\partial\mathcal{S}$ .

Now, by assuming that :

- i) the material of the shell is elastic, homogeneous and isotropic,
- ii) the strains are small everywhere in the shell,
- iii) the state of stress is approximatively plane and parallel to the middle surface,

it is proved in Koiter (1966) that

$$n^{\alpha\beta} = e E^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}, \quad m^{\alpha\beta} = \frac{e^3}{12} E^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}, \quad (2.10)$$

where  $E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} [a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu}]$ ,  $\gamma_{\lambda\mu}$  and  $\rho_{\lambda\mu}$  denote respectively the elastic moduli tensor for plane stresses, the middle surface strain tensor and the modified change of curvature tensor.

Finally, the expressions of the components  $\gamma_{\lambda\mu}$  and  $\rho_{\lambda\mu}$  associated with a displacement vector  $\mathbf{v} = v_i \mathbf{a}^i$  are given by

$$\gamma_{\lambda\mu}(\mathbf{v}) = \frac{1}{2}(v_{\mu|\lambda} + v_{\lambda|\mu}) - b_{\lambda\mu} v_3, \quad (2.11)$$

$$\rho_{\lambda\mu}(\mathbf{v}) = v_{3|\lambda\mu} + b_{\mu|\lambda}^\nu v_\nu + \frac{1}{4} b_\lambda^\nu (3v_{\nu|\mu} - v_{\mu|\nu}) + \frac{1}{4} b_\mu^\nu (3v_{\nu|\lambda} - v_{\lambda|\nu}). \quad (2.12)$$

**Variational formulation** : subsequently, we assume that the shell is clamped along a part  $\partial\mathcal{S}_0 = \Phi(\gamma_0)$  of its boundary, with  $\text{meas}(\gamma_0) > 0$ , and that it is loaded along its complementary part  $\partial\mathcal{S}_1 = \Phi(\gamma_1)$ ,  $\gamma_1 = \partial\Omega - \gamma_0$  of its boundary by a distributed force  $\mathbf{N}$  and a distributed moment  $\mathbf{M}$ . From the equations (2.1) to



(2.5), (2.11) and (2.12), Green's formula allows to state the problem in the following form :

$$\left. \begin{aligned} & \text{For given } \mathbf{p} \in (L^2(\Omega))^3, \mathbf{N} \in (L^2(\partial\Omega_1))^3, \mathbf{M} \in (L^2(\partial\Omega_1))^3, \\ & \text{find } \mathbf{u} \in \mathbf{V} \text{ such that } a(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V} \end{aligned} \right\} \quad (2.13)$$

where :

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} eE^{\alpha\beta\lambda\mu} \{ \gamma_{\alpha\beta}(\mathbf{u}) \gamma_{\lambda\mu}(\mathbf{v}) + \frac{e^2}{12} \rho_{\alpha\beta}(\mathbf{u}) \rho_{\lambda\mu}(\mathbf{v}) \} \sqrt{ad} d\xi^1 d\xi^2, \\ \ell(\mathbf{v}) &= \int_{\Omega} \mathbf{p} \cdot \mathbf{v} \sqrt{ad} d\xi^1 d\xi^2 + \int_{\gamma_1} (\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})) d\gamma, \end{aligned} \quad (2.14)$$

and

$$\mathbf{V} = \{ \mathbf{v} \in (H^1(\Omega))^2 \times H^2(\Omega) ; \mathbf{v}|_{\gamma_0} = \mathbf{0}, v_{3,\nu}|_{\gamma_0} = 0 \},$$

where  $v_{,\nu}$  denotes the outward unit normal derivative to the boundary  $\gamma_0$ . Moreover, in (2.14) we have assumed that the boundary  $\gamma_1$  is parameterized by  $\eta \rightarrow \xi^\alpha = g^\alpha(\eta)$ , so that

$$d\gamma = \sqrt{a_{\alpha\lambda}(g^\alpha)'(g^\lambda)'} d\eta. \quad (2.15)$$

Let us detail some more the expression (2.14) : we have used the infinitesimal rotation vector  $\boldsymbol{\psi}$  whose expression as function of displacement  $\mathbf{v}$  is given by

$$\boldsymbol{\psi}(\mathbf{v}) = \varepsilon^{\lambda\beta} (v_{3,\beta} + b_\beta^\alpha v_\alpha) \mathbf{a}_\lambda + \frac{1}{2} \varepsilon^{\lambda\beta} v_{\beta|\lambda} \mathbf{a}_3, \quad (2.16)$$

with

$$\varepsilon^{\lambda\beta} = \frac{1}{\sqrt{a}} e^{\lambda\beta}, \quad e^{\lambda\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.17)$$

From (2.6), (2.7), (2.9) and (2.16), (2.17), we obtain

$$\mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v}) = (t^\beta M_t + n^\beta M_n) (v_{3,\beta} + b_\beta^\nu v_\nu), \quad (2.18)$$

so that

$$\ell(\mathbf{v}) = \int_{\Omega} \mathbf{p} \cdot \mathbf{v} \sqrt{ad} d\xi^1 d\xi^2 + \int_{\gamma_1} [ \mathbf{N} \cdot \mathbf{v} + (t^\beta M_t + n^\beta M_n) (v_{3,\beta} + b_\beta^\nu v_\nu) ] d\gamma.$$

**Theorem 2.1.1** (Bernadou-Ciarlet (1976), Bernadou-Ciarlet-Miara (1994)): *The problem (2.13) has one and only one solution.*

□

## 2.2 Junction between two thin shells

Consider two thin shells whose middle surfaces  $\mathcal{S}$  and  $\mathcal{S}$  joint along a common boundary  $\Gamma$ . Subsequently, we assume that these middle surfaces  $\mathcal{S}$  and  $\mathcal{S}$  are the images through bijections  $\Phi$  and  $\Phi$  of two bounded open subsets  $\Omega$  and  $\Omega$  of a plane  $\mathcal{E}^2$  with sufficiently smooth boundaries  $\partial\Omega$  and  $\partial\Omega$ . Moreover, we assume that :

- i) The middle surface  $\mathcal{S}$  is clamped along a part  $\Gamma_0$  of its boundary, with  $\text{meas}(\Gamma_0) > 0$ , joints with  $\mathcal{S}$  along  $\Gamma$ , and is loaded along  $\Gamma$  and along the complementary part  $\Gamma_1$  of the boundary. Briefly, we have

$$\partial\mathcal{S} = \Gamma_0 \cup \Gamma_1 \cup \Gamma. \tag{2.19}$$

Corresponding to this decomposition, we get through the application  $(\Phi)^{-1}$ :

$$\partial\Omega = \gamma_0 \cup \gamma_1 \cup \gamma. \tag{2.20}$$

- ii) The shell  $\mathcal{S}$  joints with  $\mathcal{S}$  along  $\Gamma$  and is loaded along  $\Gamma$  and along the complementary part  $\Gamma_1$  of its boundary so that

$$\partial\mathcal{S} = \Gamma_1 \cup \Gamma, \quad \partial\Omega = \gamma_1 \cup \gamma, \tag{2.21}$$

where  $\gamma = (\Phi)^{-1}(\Gamma)$ .

All these considerations are illustrated by Figure 1.

*Remark 2.2.1 :* The boundary conditions and the representations of shells  $\mathcal{S}$  and  $\mathcal{S}$  are purely indicative and can be extended to more general situations. In particular,

- i) we could consider the case of a shell  $\mathcal{S}$  which is also clamped along a part  $\Gamma_0$  of its boundary ;
- ii) we could consider more junctions using the same ideas.

□

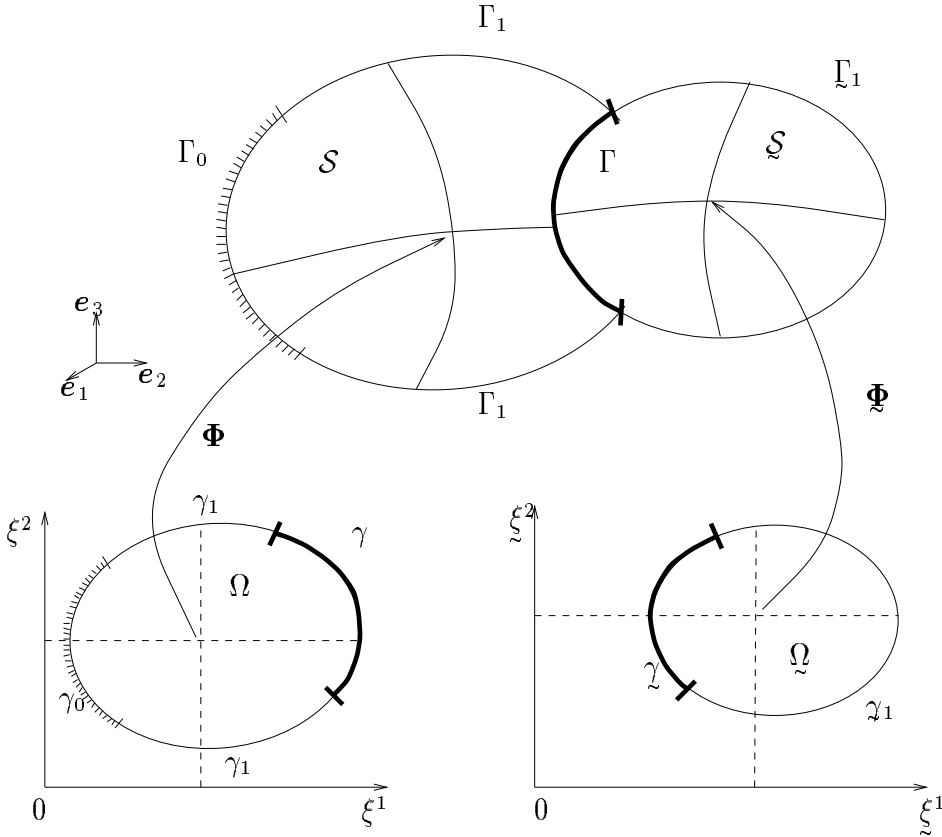
Subsequently, as a general rule, we note  $(\cdot)$  the quantities related to the shell  $\mathcal{S}$  while  $(\cdot)$  denotes the quantities related to the shell  $\mathcal{S}$ .

Upon the boundaries  $\partial\mathcal{S}$  and  $\partial\mathcal{S}$ , we define two local direct orthonormal reference systems  $(\mathbf{n}, \mathbf{t}, \mathbf{a}_3)$  and  $(\mathbf{n}, \mathbf{t}, \mathbf{a}_3)$  which include as intrinsic vectors the outward unit normal vectors (see (2.6)), the unit tangent vectors (see (2.7)) and the vectors  $\mathbf{a}_3$  et  $\mathbf{a}_3$ .

*Remark 2.2.2 :* In a previous paper related to junction between plates, Bernadou-Fayolle-Léné (1989) introduced the angle  $\theta = (\mathbf{n}, \mathbf{n})_{\mathbf{t}}$  with respect to the reference system  $(\mathbf{n}, \mathbf{t}, \mathbf{a}_3)$ . This parameter  $\theta$  was constant ; for junctions between shells, we can similarly introduce the angle

$$\theta = (\mathbf{n}, \mathbf{n})_{\mathbf{t}}$$

with respect to the reference system  $(\mathbf{n}, \mathbf{t}, \mathbf{a}_3)$ , but in general, this angle is no longer constant.  $\square$



**Fig. 1** : Junction between two shells and its representation.

Whatever the behaviour of the hinge  $\Gamma$  is, the application of the action-reaction principle at any point of  $\Gamma$ , implies the transmission of the external efforts, i.e.,

$$\mathbf{N}(P) = \mathbf{N}_{\tilde{}}(P) \text{ and } \mathbf{M}(P) = \mathbf{M}_{\tilde{}}(P), \quad \forall P \in \Gamma. \quad (2.22)$$

Note that the relation  $\mathbf{M}(P) = \mathbf{M}_{\tilde{}}(P)$  in conjunction with (2.9) and its analog upon  $\mathcal{S}$  implies

$$M_t(P) = \underline{M}_{\tilde{t}}(P) = 0, \quad \forall P \in \Gamma, \quad (2.23)$$

as soon as  $\mathbf{n} \times \mathbf{n}_{\tilde{}} \neq \mathbf{0}$ . Then (2.9) and (2.23) imply :

$$\mathbf{M}(P) = \mathbf{M}_{\tilde{}}(P) = -M_n \mathbf{t}, \quad \forall P \in \Gamma. \quad (2.24)$$

Subsequently, we assume that relations (2.23) and (2.24) are still satisfied when  $\mathbf{n} \times \mathbf{n}_{\tilde{}} = \mathbf{0}$  (junction of class  $C^1$ ).

Then, we examine two types of hinge behaviour :

- i) a **rigid behaviour** which insures the continuity of the displacements and of the tangential rotations along the hinge for all points  $P$  of  $\Gamma$ , i.e.,

$$\left. \begin{aligned} \mathbf{u}(P) &= \mathbf{u}(P), \\ (\boldsymbol{\psi}(\mathbf{u}) \cdot \mathbf{t})(P) &= (\boldsymbol{\psi}(\mathbf{u}) \cdot \mathbf{t})(P) = [(\mathbf{t} \cdot \boldsymbol{\xi})(\boldsymbol{\psi}(\mathbf{u}) \cdot \boldsymbol{\xi})](P), \end{aligned} \right\} \quad (2.25)$$

where  $\boldsymbol{\psi}(\mathbf{u})$  (resp.  $\boldsymbol{\psi}(\mathbf{u})$ ) is defined by relation (2.16);

- ii) an **elastic behaviour** which only insures the continuity of the displacements for all points  $P$  of  $\Gamma$ , i.e.,

$$\left. \begin{aligned} \mathbf{u}(P) &= \mathbf{u}(P), \\ M_n(P) &= k[(\boldsymbol{\psi}(\mathbf{u}) - \boldsymbol{\psi}(\mathbf{u})) \cdot \mathbf{t}](P). \end{aligned} \right\} \quad (2.26)$$

Thus the second equation of relation (2.25) is replaced by the requirement : the tangential component of the moment  $\mathbf{M}$  (see (2.9)) is proportional to the jump of the tangential components of the rotations along the hinge  $\Gamma$ . The coefficient  $k$  measures the elastic stiffness along the hinge ; it is positive and it should be determined experimentally.

*Remark 2.2.3 :* Generally, the parameter  $k$  is dependent on the position along the hinge  $\Gamma$ .

On the other hand, the rigid behaviour can be interpreted as the limit case of the elastic behaviour of the hinge when stiffness coefficient  $k$  becomes very large. We come back to this remark in Paragraph 3.3.  $\square$

### 2.3 The equations of the junction problem

In this paragraph we summarize the equations of the junction problem. By taking into account the assumptions made in Paragraphs 2.1 and 2.2, these equations are given by :

$$\left. \begin{aligned} [n^{\alpha\beta} + \frac{1}{2}b_\nu^\alpha m^{\beta\nu} - \frac{1}{2}b_\nu^\beta m^{\alpha\nu}] |_\beta + b_\nu^\alpha m^{\beta\nu} |_\beta + p^\alpha &= 0 \quad \text{in } \mathcal{S}, \\ -m^{\alpha\beta} |_{\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta} + p^3 &= 0 \quad \text{in } \mathcal{S}, \end{aligned} \right\} \quad (2.27)$$

$$\left. \begin{aligned} [\underline{n}^{\alpha\beta} + \frac{1}{2}\underline{b}_\nu^\alpha \underline{m}^{\beta\nu} - \frac{1}{2}\underline{b}_\nu^\beta \underline{m}^{\alpha\nu}] |_\beta + \underline{b}_\nu^\alpha \underline{m}^{\beta\nu} |_\beta + \underline{p}^\alpha &= 0 \quad \text{in } \underline{\mathcal{S}}, \\ -\underline{m}^{\alpha\beta} |_{\alpha\beta} + \underline{b}_{\alpha\beta} \underline{n}^{\alpha\beta} + \underline{p}^3 &= 0 \quad \text{in } \underline{\mathcal{S}}. \end{aligned} \right\} \quad (2.28)$$

In relation (2.28), we note for simplicity ( $|_\beta$ ) the covariant derivatives with respect to  $\xi^\beta$ . The corresponding boundary conditions are

$$\mathbf{u} = \mathbf{0}, \quad u_{3,n} = 0 \quad \text{on } \Gamma_0, \quad (2.29)$$

$$\left\{ \begin{array}{l} [n^{\alpha\beta} + \frac{3}{2}b_\nu^\alpha m^{\beta\nu} - \frac{1}{2}b_\nu^\beta m^{\alpha\nu}]n_\beta = N^\alpha + b_\beta^\alpha M^\beta \quad \text{on } \Gamma_1 \cup \Gamma, \\ -m^{\alpha\beta} |_\alpha n_\beta - (m^{\alpha\beta} n_\alpha t_\beta)_{,s} = N^3 - (M^\alpha t_\alpha)_{,s} \quad \text{on } \Gamma_1 \cup \Gamma, \\ m^{\alpha\beta} n_\alpha n_\beta = M^\alpha n_\alpha \quad \text{on } \Gamma_1 \cup \Gamma ; \end{array} \right.$$

$$\left\{ \begin{array}{l} [\underline{n}^{\alpha\beta} + \frac{3}{2}\underline{b}_\nu^\alpha \underline{m}^{\beta\nu} - \frac{1}{2}\underline{b}_\nu^\beta \underline{m}^{\alpha\nu}]\underline{n}_\beta = \underline{N}^\alpha + \underline{b}_\beta^\alpha \underline{M}^\beta \quad \text{on } \underline{\Gamma}_1 \cup \Gamma, \\ -\underline{m}^{\alpha\beta} |_\alpha \underline{n}_\beta - (\underline{m}^{\alpha\beta} \underline{n}_\alpha \underline{t}_\beta)_{,\underline{s}} = \underline{N}^3 - (\underline{M}^\alpha \underline{t}_\alpha)_{,\underline{s}} \quad \text{on } \underline{\Gamma}_1 \cup \Gamma, \\ \underline{m}^{\alpha\beta} \underline{n}_\alpha \underline{n}_\beta = \underline{M}^\alpha \underline{n}_\alpha \quad \text{on } \underline{\Gamma}_1 \cup \Gamma ; \end{array} \right.$$

$$\left\{ \begin{array}{l} N(P) = \underline{N}(P), \quad \forall P \in \Gamma, \\ M(P) = \underline{M}(P) = -M_n(P)t, \quad \forall P \in \Gamma, \end{array} \right.$$

while the conditions of junction upon  $\Gamma$  depend on the type of hinge :

\* **rigid hinge** ( see (2.16) and (2.25))

$$\left. \begin{array}{l} \mathbf{u} = \underline{\mathbf{u}} \quad \text{on } \Gamma, \\ \boldsymbol{\psi}(\mathbf{u}) \cdot \mathbf{t} = (\mathbf{t} \cdot \underline{\mathbf{t}})\boldsymbol{\psi}(\underline{\mathbf{u}}) \cdot \underline{\mathbf{t}} \quad \text{on } \Gamma ; \end{array} \right\} \quad (2.30)$$

\* **elastic hinge** (see (2.16) and (2.26))

$$\left. \begin{array}{l} \mathbf{u} = \underline{\mathbf{u}} \quad \text{on } \Gamma, \\ k[\boldsymbol{\psi}(\mathbf{u}) \cdot \mathbf{t} - (\mathbf{t} \cdot \underline{\mathbf{t}})\boldsymbol{\psi}(\underline{\mathbf{u}}) \cdot \underline{\mathbf{t}}] = M_n \quad \text{on } \Gamma. \end{array} \right\} \quad (2.31)$$

## 2.4 Some examples

Let us consider three examples : two are extracted from practical engineering situations and the third from a classical benchmark.

### ***Hyperbolic paraboloid roof for Hamburg Sechslingspforte swimming pool***

As a first example, let us recall that mentioned by Argyris-Lochner (1972) and Leonhardt-Schlaich (1970). The roof consists of two identical straight-edged shells  $ABCD$  and  $A'C'D$  (see Figure 2) which joint along the edge  $BD$  and which are symmetrical with respect to this edge. The middle surface of each shell has the form of a hyperbolic paraboloid with straight edges. Of course, in the real life situations, the structure is much more complicated since it includes stiffeners and other appliances.

### ***Junction between two cylindrical tubes***

Offshore platforms (see for example Alencar-Ferrante (1984)) are built by assembling a large number of tubular joints. The junction of two cylindrical tubes constitutes the basic node of such a structure (see Figures 3 and 4).

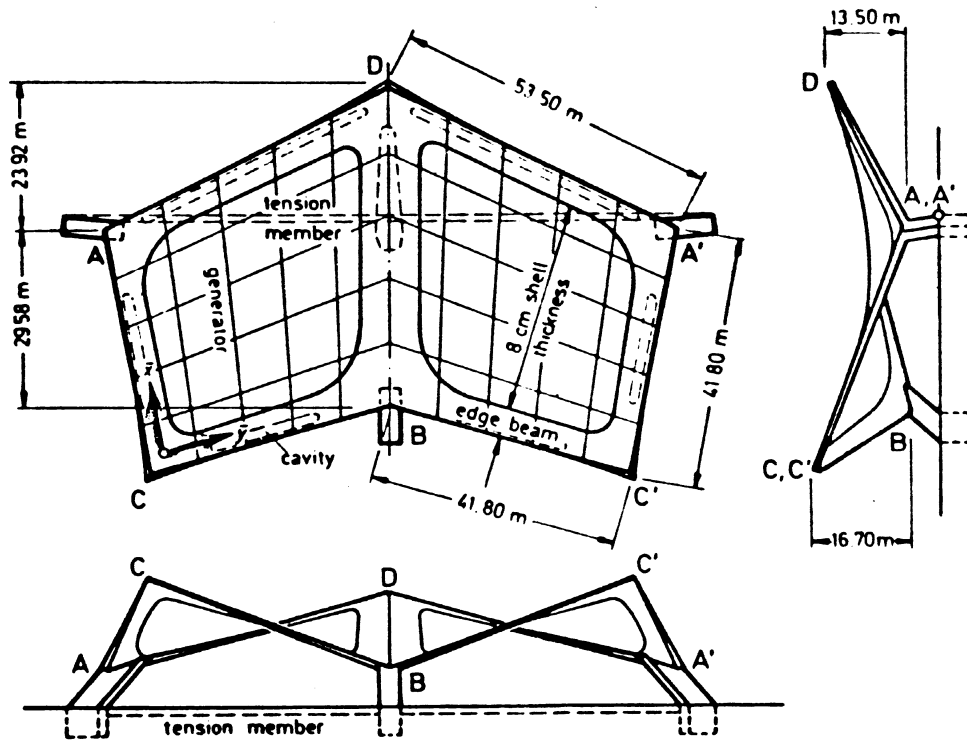
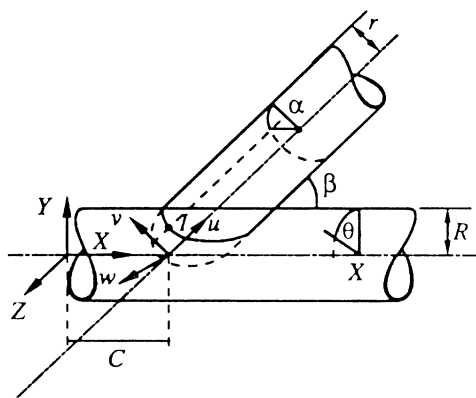


Fig. 2: Geometry of the hyperbolic paraboloidal shells.



Main tube

$$Y^2 + Z^2 = R^2 \Leftrightarrow \Phi = \begin{cases} X = X \\ Y = R \cos \theta \\ Z = R \sin \theta \end{cases}$$

Secondary tube

$$\Phi = \begin{cases} X = C + u \cos \beta - r \sin \beta \cos \alpha \\ Y = u \sin \beta + r \cos \beta \cos \alpha \\ Z = r \sin \alpha \end{cases}$$

Fig. 3 : Intersection lines.

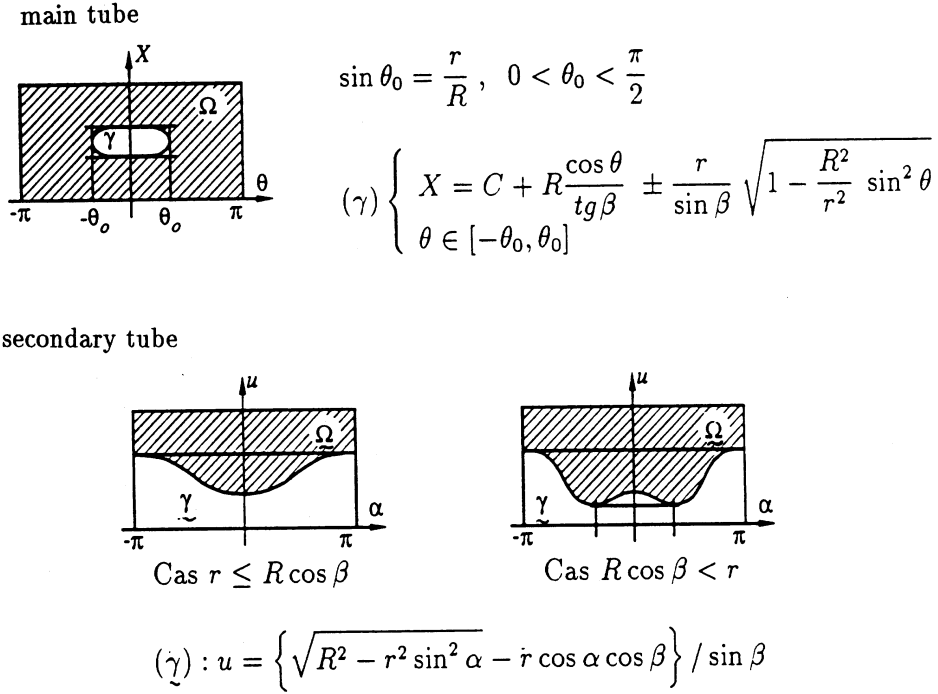


Fig. 4: Reference domains.

The next example is classical in engineering and it is very often used as a benchmark in finite element approximations.

### *Circular cylinder with spherical end cap under internal pressure*

This example has been studied by several authors, in particular Argyris-Lochner (1972), Baker-Kovalevsky-Rish (1981), Bernadou (1996), Calladine (1983), Cubier (1994), Flügge (1973). This example ( see Figure 5) is parameterized with the help of the angle  $\delta$  so that the junction is of order  $C^0$  ( discontinuity of slopes ) when  $\delta \neq \frac{\pi}{2}$  and of order  $C^1$  ( continuity of slopes and discontinuity of curvatures ) when  $\delta = \frac{\pi}{2}$ .

*Equation of the cylinder :*

This cylinder is parameterized through  $\xi^1 = x_3 \in [0, L]$  and through  $\xi^2 \in [0, 2\pi]$ , so that the position vector is given by :

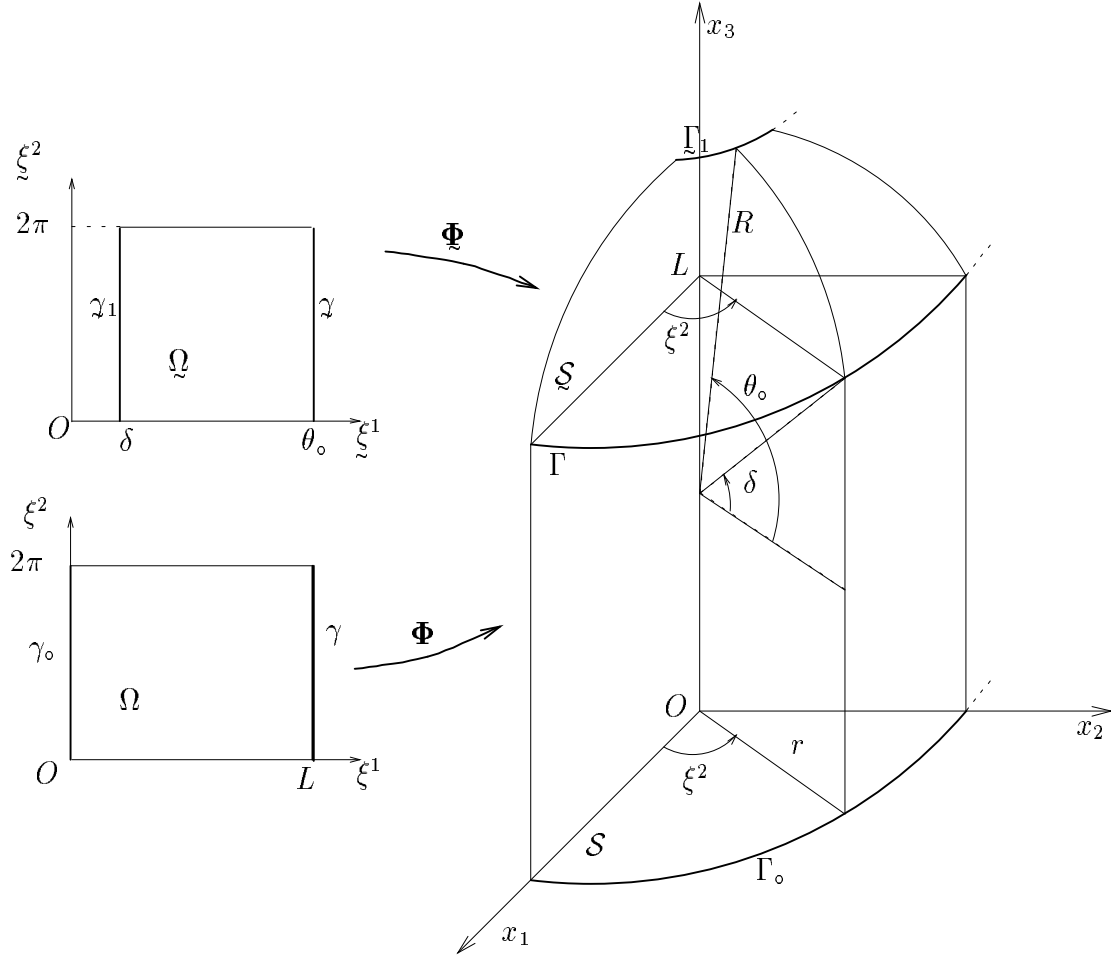
$$\mathbf{OP} = \left( x_1 = r \cos(\xi^2), x_2 = r \sin(\xi^2), x_3 = \xi^1 \right)$$

*Equation of the spherical end cap :*

This end cap is parameterized through  $\xi^1 = \delta + \frac{\xi^1 - L}{R}$  with  $\xi^1 \in [L, L + R(\theta_0 - \delta)]$ ,

$0 < \delta < \theta_0 \leq \frac{\pi}{2}$  and through  $\xi^2 = \xi^2 \in [0, 2\pi]$ , so that the position vector is given by

$$\mathbf{OP} = (x_1 = R \cos(\xi^1) \cos(\xi^2), x_2 = R \cos(\xi^1) \sin(\xi^2), x_3 = R \sin(\xi^1) + L - R \sin \delta)$$



**Fig. 5** : Junction between a circular cylinder and a spherical end cap and their representation

### 3 Variational formulations and existence results

From the equations stated in Paragraph 2.3, we derive the corresponding variational formulations in suitable spaces and then, we prove existence and uniqueness results. We conclude by proving that the solutions  $(\mathbf{u}^k, \mathbf{u}^k)$  of the elastic hinge problem are converging to the solution  $(\mathbf{u}_{rig}, \mathbf{u}_{rig})$  of the rigid hinge problem when coefficient  $k$  becomes very large.



### 3.1 Case of an elastic hinge

Let us introduce the first shell  $\mathcal{S}$  independently of a possible junction with another shell  $\mathcal{G}$ . We use the following form of Green's theorem (see Green-Zerna (1968, p. 39)) :

$$\int_{\mathcal{S}} v^\alpha |_\alpha d\mathcal{S} = \int_{\partial\mathcal{S}} n_\alpha v^\alpha ds,$$

where  $\mathbf{n} = n_\alpha \mathbf{a}^\alpha$  denotes the outward unit normal vector to  $\partial\mathcal{S}$  (see (2.6)). Then, by multiplying the equilibrium equations (2.1) and (2.2) by appropriate test functions  $v_\alpha$  and  $v_3$ , by integrating by parts upon the middle surface  $\mathcal{S}$  and by taking into account the general boundary conditions (2.3) to (2.5) and the symmetry of  $n^{\alpha\beta}$  and  $m^{\alpha\beta}$ , we obtain with (2.6) to (2.9), (2.11) and (2.12) :

$$\left. \begin{aligned} \int_{\mathcal{S}} \{n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v})\} d\mathcal{S} &= \int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} d\mathcal{S} \\ + \int_{\partial\mathcal{S}} [N \cdot \mathbf{v} + b^{\alpha\beta} (M_n n_\beta + M_t t_\beta) v_\alpha + (M_t v_{3,t} + M_n v_{3,n})] ds. \end{aligned} \right\} \quad (3.1)$$

Subsequently, it will be convenient to use the infinitesimal rotation vector  $\boldsymbol{\psi}$  whose expression as function of displacement  $\mathbf{v}$  is given by (2.16). Thus, from (2.6) (2.9), we obtain

$$\mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v}) = b^{\alpha\beta} (M_n n_\beta + M_t t_\beta) v_\alpha + (M_t v_{3,t} + M_n v_{3,n}),$$

so that the expression (3.1) can be rewritten

$$\int_{\mathcal{S}} \{n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v})\} d\mathcal{S} = \int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} d\mathcal{S} + \int_{\partial\mathcal{S}} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] ds. \quad (3.2)$$

Exactly in the same way, we would get for the second shell  $\mathcal{G}$

$$\int_{\mathcal{G}} \{n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v})\} d\mathcal{G} = \int_{\mathcal{G}} \mathbf{p} \cdot \mathbf{v} d\mathcal{G} + \int_{\partial\mathcal{G}} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] d\mathcal{G}. \quad (3.3)$$

Now, we assume that both shell  $\mathcal{S}$  and  $\mathcal{G}$  are such that :

- i) the shell  $\mathcal{S}$  is clamped along  $\Gamma_0$ , i.e., the conditions (2.29) are satisfied ;
- ii) the shells  $\mathcal{S}$  and  $\mathcal{G}$  joint along the common side  $\Gamma$  so that conditions (2.30) or (2.31) are satisfied, depending of the type of hinge into consideration.

With notations (2.19) and (2.21), we obtain by adding equations (3.2), (3.3) :

$$\left. \begin{aligned} \int_{\mathcal{S}} \{n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v})\} d\mathcal{S} + \int_{\mathcal{G}} \{n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v})\} d\mathcal{G} = \\ \int_{\mathcal{S}} \mathbf{p} \cdot \mathbf{v} d\mathcal{S} + \int_{\mathcal{G}} \mathbf{p} \cdot \mathbf{v} d\mathcal{G} + \int_{\Gamma_1} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] ds + \int_{\Gamma_1} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] ds \\ + \int_{\Gamma} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] ds + \int_{\Gamma} [N \cdot \mathbf{v} + \mathbf{M} \cdot \boldsymbol{\psi}(\mathbf{v})] ds. \end{aligned} \right\} \quad (3.4)$$

With relations (2.22) (2.24), we can rewrite the integrals upon  $\Gamma$  by observing that

$$ds = -ds, \quad N = \underline{N}, \quad M = \underline{M} = -M_n \mathbf{t} \text{ on } \Gamma,$$

so that

$$\left. \begin{aligned} \int_{\Gamma} [N \cdot \mathbf{v} + M \cdot \boldsymbol{\psi}(\mathbf{v})] ds + \int_{\Gamma} [\underline{N} \cdot \mathbf{v} + \underline{M} \cdot \boldsymbol{\psi}(\mathbf{v})] d\underline{s} = \\ \int_{\Gamma} [N \cdot (\mathbf{v} - \mathbf{v}) - M_n \{ \boldsymbol{\psi}(\mathbf{v}) \cdot \mathbf{t} - (\mathbf{t} \cdot \underline{\mathbf{t}}) \boldsymbol{\psi}(\mathbf{v}) \cdot \underline{\mathbf{t}} \}] ds. \end{aligned} \right\} \quad (3.5)$$

By using the mapping  $\Phi$  and  $\underline{\Phi}$ , we are able to specify the integrals of relation (3.4) upon the reference domains  $\Omega$  and  $\underline{\Omega}$  and upon their boundaries  $\partial\Omega$  and  $\partial\underline{\Omega}$  except for the integral formulated on  $\Gamma$ . Indeed, relation (3.5) contains terms defined on  $\mathcal{S}$  or  $\underline{\mathcal{S}}$  which cannot be directly expressed only on  $\gamma$  or  $\underline{\gamma}$ . Thus, we continue to express (3.5) on  $\Gamma$ ; from relations (3.4) and (3.5) we get :

$$\left. \begin{aligned} \int_{\Omega} \{ n^{\alpha\beta} \gamma_{\alpha\beta}(\mathbf{v}) + m^{\alpha\beta} \rho_{\alpha\beta}(\mathbf{v}) \} \sqrt{ad} d\xi^1 d\xi^2 + \int_{\underline{\Omega}} \{ \underline{n}^{\alpha\beta} \underline{\gamma}_{\alpha\beta}(\mathbf{v}) + \underline{m}^{\alpha\beta} \underline{\rho}_{\alpha\beta}(\mathbf{v}) \} \sqrt{\underline{a}} d\underline{\xi}^1 d\underline{\xi}^2 = \\ \int_{\Omega} \mathbf{p} \cdot \mathbf{v} \sqrt{ad} d\xi^1 d\xi^2 + \int_{\underline{\Omega}} \underline{\mathbf{p}} \cdot \underline{\mathbf{v}} \sqrt{\underline{a}} d\underline{\xi}^1 d\underline{\xi}^2 + \int_{\gamma_1} [N \cdot \mathbf{v} + M \cdot \boldsymbol{\psi}(\mathbf{v})] d\gamma + \\ \int_{\underline{\gamma}_1} [\underline{N} \cdot \mathbf{v} + \underline{M} \cdot \boldsymbol{\psi}(\mathbf{v})] d\underline{\gamma} + \int_{\Gamma} N \cdot (\mathbf{v} - \mathbf{v}) - M_n \{ \boldsymbol{\psi}(\mathbf{v}) \cdot \mathbf{t} - (\mathbf{t} \cdot \underline{\mathbf{t}}) \boldsymbol{\psi}(\mathbf{v}) \cdot \underline{\mathbf{t}} \} ds, \end{aligned} \right\} \quad (3.6)$$

where  $d\gamma$  (and similarly for  $d\underline{\gamma}$ ) is defined by relation (2.15) while  $ds$  is the line element along the boundary  $\Gamma$ .

Let us emphasize that the expression (3.6) is valid for rigid hinges as well as for elastic hinges. Now let us specialize equation (3.6) to the case of an elastic hinge. In order to set up the variational formulation of the problem we have to :

- i) substitute relations (2.10) ( and similar relations for the shell  $\underline{\mathcal{S}}$  ) into the equation (3.6);
- ii) define a suitable admissible displacement space.

In this way, we set :

$$\left. \begin{aligned} \mathbf{V} = V_1 \times V_1 \times V_2, \\ V_1 = \{ v \in H^1(\Omega), v = 0 \text{ on } \gamma_0 \}, \quad V_2 = \{ v \in H^2(\Omega), v = v_{,\nu} = 0 \text{ on } \gamma_0 \} \end{aligned} \right\} \quad (3.7)$$

and

$$\mathbf{V} = H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega), \quad (3.8)$$

where  $v_{,\nu}$  denotes the outward unit normal derivative to the boundary  $\gamma_0$ . Then, the space of kinematically admissible displacement field for an elastic junction problem is defined by :

$$W_{el} = \{ (\mathbf{w} ; \underline{\mathbf{w}}) \in \mathbf{V} \times \underline{\mathbf{V}} ; \mathbf{w} = \underline{\mathbf{w}} \text{ at the corresponding points of } \gamma \text{ and } \underline{\gamma} \}; \quad (3.9)$$

iii) substitute condition (2.31)<sub>2</sub> into relation (3.6).

Hence, the variational formulation of the problem can be stated as follows :

$$\left. \begin{aligned} & \text{Find } (\mathbf{u}^k ; \mathbf{u}^k) \in W_{el} \text{ such that} \\ & a[(\mathbf{u}^k ; \mathbf{u}^k), (\mathbf{v} ; \mathbf{v})] + kb[(\mathbf{u}^k ; \mathbf{u}^k), (\mathbf{v} ; \mathbf{v})] = \ell(\mathbf{v} ; \mathbf{v}), \quad \forall (\mathbf{v} ; \mathbf{v}) \in W_{el}, \end{aligned} \right\} \quad (3.10)$$

where

$$\left. \begin{aligned} a[(\mathbf{u} ; \mathbf{u}), (\mathbf{v} ; \mathbf{v})] &= \int_{\Omega} e E^{\alpha\beta\lambda\mu} [\gamma_{\lambda\mu}(\mathbf{u}) \gamma_{\alpha\beta}(\mathbf{v}) + \frac{e^2}{12} \rho_{\lambda\mu}(\mathbf{u}) \rho_{\alpha\beta}(\mathbf{v})] \sqrt{ad} d\xi^1 d\xi^2 \\ &+ \int_{\Omega} \underline{e} \underline{E}^{\alpha\beta\lambda\mu} [\underline{\gamma}_{\lambda\mu}(\mathbf{u}) \underline{\gamma}_{\alpha\beta}(\mathbf{v}) + \frac{\underline{e}^2}{12} \underline{\rho}_{\lambda\mu}(\mathbf{u}) \underline{\rho}_{\alpha\beta}(\mathbf{v})] \sqrt{\underline{a}} d\underline{\xi}^1 d\underline{\xi}^2, \end{aligned} \right\} \quad (3.11)$$

$$b[(\mathbf{u} ; \mathbf{u}), (\mathbf{v} ; \mathbf{v})] = \int_{\Gamma} \{ \psi(\mathbf{u}) \cdot \mathbf{t} - (\mathbf{t} \cdot \underline{\mathbf{t}}) \psi(\mathbf{u}) \cdot \underline{\mathbf{t}} \} \{ \psi(\mathbf{v}) \cdot \mathbf{t} - (\mathbf{t} \cdot \underline{\mathbf{t}}) \psi(\mathbf{v}) \cdot \underline{\mathbf{t}} \} ds \quad (3.12)$$

$$\left. \begin{aligned} \ell(\mathbf{v} ; \mathbf{v}) &= \int_{\Omega} \mathbf{p} \cdot \mathbf{v} \sqrt{ad} d\xi^1 d\xi^2 + \int_{\Omega} \underline{\mathbf{p}} \cdot \underline{\mathbf{v}} \sqrt{\underline{a}} d\underline{\xi}^1 d\underline{\xi}^2 + \\ &\int_{\gamma_1} [\mathbf{N} \cdot \mathbf{v} + \mathbf{M} \cdot \psi(\mathbf{v})] d\gamma + \int_{\underline{\gamma}_1} [\underline{\mathbf{N}} \cdot \underline{\mathbf{v}} + \underline{\mathbf{M}} \cdot \psi(\mathbf{v})] d\underline{\gamma}. \end{aligned} \right\} \quad (3.13)$$

Thus, we obtain the following theorem :

**Theorem 3.1.1** *Assume that the geometrical data are sufficiently smooth and that :*

$$\left\{ \begin{aligned} & \mathbf{p} \in (L^2(\Omega))^3, \quad \mathbf{N} \in (L^2(\partial\Omega))^3, \quad \mathbf{M} \in (L^2(\partial\Omega))^3, \\ & \underline{\mathbf{p}} \in (L^2(\underline{\Omega}))^3, \quad \underline{\mathbf{N}} \in (L^2(\partial\underline{\Omega}))^3, \quad \underline{\mathbf{M}} \in (L^2(\partial\underline{\Omega}))^3, \\ & k = \text{constant} > 0, \quad E > 0, \quad \underline{E} > 0, \quad 0 < \nu < \frac{1}{2}, \quad 0 < \underline{\nu} < \frac{1}{2}. \end{aligned} \right.$$

*Then, the problem (3.10) has one and only one solution.*

**Proof :** We only give the main lines of this proof. It takes four steps :

*Step 1 : The space  $W_{el}$  defined by relation (3.9) is a closed subspace of the space*

$$E = (H^1(\Omega))^2 \times H^2(\Omega) \times (H^1(\underline{\Omega}))^2 \times H^2(\underline{\Omega})$$

*equipped with the norm*

$$\|(\mathbf{v} ; \mathbf{v})\|_E = \{ \|v_1\|_{1,\Omega}^2 + \|v_2\|_{1,\Omega}^2 + \|v_3\|_{2,\Omega}^2 + \|\underline{v}_1\|_{1,\underline{\Omega}}^2 + \|\underline{v}_2\|_{1,\underline{\Omega}}^2 + \|\underline{v}_3\|_{2,\underline{\Omega}}^2 \}^{1/2}. \quad (3.14)$$

Indeed, let  $(\mathbf{v}_n, \underline{\mathbf{v}}_n)$  be a sequence of functions in the space  $W_{el}$  which converges to an element  $(\mathbf{v} ; \mathbf{v}) \in E$ . By using the continuity of the trace operator  $\text{tr} : v \in$

$H^1(\Omega) \rightarrow \text{tr} v \in L^2(\partial\Omega)$  (resp.  $\text{tr} : \mathfrak{v} \in H^1(\tilde{\Omega}) \rightarrow \text{tr} v \in L^2(\partial\tilde{\Omega})$ ) it follows that  $\text{tr}(\mathbf{v}_n, \mathfrak{v}_n)$  converges to  $\text{tr}(\mathbf{v}; \mathfrak{v})$  in the space  $(L^2(\gamma))^3 \times (L^2(\tilde{\gamma}))^3$ . Then the sequence  $\text{tr}(\mathbf{v}_n, \mathfrak{v}_n)$  contains a subsequence which converges almost everywhere to  $\text{tr}(\mathbf{v}; \mathfrak{v})$  and thus  $\text{tr}(\mathbf{v}) = \text{tr}(\mathfrak{v})$  a.e on  $\gamma$  and  $\tilde{\gamma}$ . This implies  $(\mathbf{v}; \mathfrak{v}) \in W_{el}$ .

*Step 2 : The application  $(\mathbf{v}; \mathfrak{v}) \in W_{el} \rightarrow \|(\mathbf{v}; \mathfrak{v})\|_{W_{el}}$  is a norm on  $W_{el}$  where*

$$\|(\mathbf{v}; \mathfrak{v})\|_{W_{el}} = \{a[(\mathbf{v}; \mathfrak{v}), (\mathbf{v}; \mathfrak{v})] + kb[(\mathbf{v}; \mathfrak{v}), (\mathbf{v}; \mathfrak{v})]\}^{1/2}. \quad (3.15)$$

Clearly, this application is a semi-norm on  $W_{el}$ . It is a norm since :

- i)  $\int_{\Omega} eE^{\alpha\beta\lambda\mu}[\gamma_{\lambda\mu}(\mathbf{v})\gamma_{\alpha\beta}(\mathfrak{v}) + \frac{e^2}{12}\rho_{\lambda\mu}(\mathbf{v})\rho_{\alpha\beta}(\mathfrak{v})]\sqrt{a}d\xi^1d\xi^2 = 0, \quad \mathbf{v} \in \mathbf{V}$ , implies  $\mathbf{v} = \mathbf{0}$  in  $\Omega$   
( see Bernadou-Ciarlet (1976) or Bernadou-Ciarlet-Miara(1994) );
- ii) the condition  $\mathbf{v} = \mathfrak{v}$  at the corresponding points of  $\gamma$  and  $\tilde{\gamma}$  implies  $\mathfrak{v} = \mathbf{0}$  on  $\tilde{\gamma}$  which is equivalent to  $\mathfrak{v} = \mathbf{0}$  on  $\Gamma$ . Then the condition  $b[(\mathbf{v}; \mathfrak{v}), (\mathbf{v}; \mathfrak{v})] = 0$  gives in addition  $\mathfrak{v}_{3,\mathfrak{v}} = 0$  on  $\Gamma$ . These conditions, carried out along  $\tilde{\gamma}$  through the mapping  $\Phi$ , are equivalent to clamped conditions along this boundary. Then, as in i) we obtain  $\mathfrak{v} = \mathbf{0}$  in  $\Omega$ .

*Step 3 : Upon the space  $W_{el}$ , the norms  $\|(\mathbf{v}; \mathfrak{v})\|_E$  and  $\|(\mathbf{v}; \mathfrak{v})\|_{W_{el}}$  are equivalent.*

- i) There exists a constant  $C > 0$  such that

$$\|(\mathbf{v}; \mathfrak{v})\|_{W_{el}} \leq C\|(\mathbf{v}; \mathfrak{v})\|_E, \quad \forall (\mathbf{v}; \mathfrak{v}) \in W_{el}.$$

By assuming that the geometrical data are sufficiently regular, we get

$$a[(\mathbf{v}; \mathfrak{v}), (\mathbf{v}; \mathfrak{v})] \leq C\|(\mathbf{v}; \mathfrak{v})\|_E^2.$$

For any function  $f : \Gamma \rightarrow \mathbb{R}$ , we have

$$\|f\|_{0,\Gamma} \leq C\|f \circ \Phi\|_{0,\gamma},$$

where the mapping  $\Phi$  is such that  $\Phi(\gamma) = \Gamma$ . By using this inequality in conjunction with the continuity of the trace operators from  $H^1(\Omega)$  into  $L^2(\partial\Omega)$  and from  $H^1(\tilde{\Omega})$  into  $L^2(\partial\tilde{\Omega})$ , we obtain

$$b[(\mathbf{v}; \mathfrak{v}), (\mathbf{v}; \mathfrak{v})] \leq C\|(\mathbf{v}; \mathfrak{v})\|_E^2.$$

- ii) Conversely, there exists a constant  $C > 0$  such that

$$\|(\mathbf{v}; \mathfrak{v})\|_E \leq C\|(\mathbf{v}; \mathfrak{v})\|_{W_{el}}, \quad \forall (\mathbf{v}; \mathfrak{v}) \in W_{el}. \quad (3.16)$$

This proof is divided into four points :

- There exists a constant  $C > 0$  such that

$$C\|(\mathbf{v}; \mathfrak{v})\|_E^2 \leq \|(\mathbf{v}; \mathfrak{v})\|_{W_{el}}^2 + \|\mathfrak{x}_1\|_{0,\Omega}^2 + \|\mathfrak{x}_2\|_{0,\Omega}^2 + \|\mathfrak{x}_3\|_{1,\Omega}^2, \quad \forall (\mathbf{v}; \mathfrak{v}) \in W_{el}. \quad (3.17)$$

Indeed, by using Bernadou-Ciarlet (1976, Theorem 6.1.3), there exists a constant  $C_1 > 0$  such that for any element  $\mathbf{v} \in V$ , we have

$$C_1 \|\mathbf{v}\|_{(H^1(\Omega))^2 \times H^2(\Omega)}^2 \leq \int_{\Omega} e E^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}(\mathbf{v}) \gamma_{\lambda\mu}(\mathbf{v}) + \frac{e^2}{12} \rho_{\alpha\beta}(\mathbf{v}) \rho_{\lambda\mu}(\mathbf{v})] \sqrt{ad} \xi^1 d\xi^2. \quad (3.18)$$

By analogy with the proof of Bernadou-Ciarlet (1976, Theorem 6.1.3), we prove the existence of a constant  $C_2 > 0$  such that for any element  $\mathbf{v} \in \mathcal{V}$  we have

$$\left. \begin{aligned} C_2 \|\mathbf{v}\|_{(H^1(\Omega))^2 \times H^2(\Omega)}^2 &\leq \|\mathbf{v}_1\|_{0,\Omega}^2 + \|\mathbf{v}_2\|_{0,\Omega}^2 + \|\mathbf{v}_3\|_{1,\Omega}^2 \\ &+ \int_{\Omega} \xi \tilde{E}^{\alpha\beta\lambda\mu} [\tilde{\gamma}_{\alpha\beta}(\mathbf{v}) \tilde{\gamma}_{\lambda\mu}(\mathbf{v}) + \frac{\xi^2}{12} \tilde{\rho}_{\alpha\beta}(\mathbf{v}) \tilde{\rho}_{\lambda\mu}(\mathbf{v})] \sqrt{\tilde{a}\tilde{d}} d\xi^1 d\xi^2. \end{aligned} \right\} \quad (3.19)$$

Finally, for any element  $(\mathbf{v} ; \mathbf{v}) \in W_{el}$ , we have

$$0 \leq b[(\mathbf{v} ; \mathbf{v}), (\mathbf{v} ; \mathbf{v})]. \quad (3.20)$$

To obtain (3.17), it remains to add inequalities (3.18) (3.19) and (3.20).

- The application  $(\mathbf{v} ; \mathbf{v}) \in W_{el} \rightarrow \|(\mathbf{v} ; \mathbf{v})\|_{W_{el}}$  is weakly lower semi-continuous for the topology induced by the norm (3.14). This arises from properties of convexity and strong continuity of this application.

- Any sequence  $(\mathbf{v}_n, \mathbf{v}_n) \in W_{el}$  satisfying

$$\|(\mathbf{v}_n, \mathbf{v}_n)\|_E = 1, \quad \forall n, \quad (3.21)$$

$$\|(\mathbf{v}_n, \mathbf{v}_n)\|_{W_{el}} < \frac{1}{n} \quad (3.22)$$

converges to  $(\mathbf{0}, \mathbf{0})$ , weakly in the space  $E$ , strongly in the space  $(L^2(\Omega))^2 \times H^1(\Omega) \times (L^2(\underline{\Omega}))^2 \times H^1(\underline{\Omega})$ .

The space  $E$  is reflexive so that the assumption (3.21) and the Eberlein-Schmulyan theorem (see Yosida (1968)) involve the existence of a subsequence  $(\mathbf{v}_n, \mathbf{v}_n)$ , which is weakly convergent in space  $E$  to an element  $(\mathbf{v} ; \mathbf{v}) \in E$ . From the compactness of the injection of  $H^{m+1}$  into  $H^m$ ,  $m = 0$  or  $1$ , there exists again an extracted subsequence, still denoted  $(\mathbf{v}_n, \mathbf{v}_n)$  which is strongly convergent in  $(L^2(\Omega))^2 \times H^1(\Omega) \times (L^2(\underline{\Omega}))^2 \times H^1(\underline{\Omega})$  to  $(\mathbf{w} ; \mathbf{w}) \in (L^2(\Omega))^2 \times H^1(\Omega) \times (L^2(\underline{\Omega}))^2 \times H^1(\underline{\Omega})$ . Since the limit of a weakly convergent sequence is unique, we obtain  $(\mathbf{v} ; \mathbf{v}) = (\mathbf{w} ; \mathbf{w})$ . Finally, the property proved in the previous point and (3.21) implies that  $(\mathbf{v} ; \mathbf{v}) = (\mathbf{w} ; \mathbf{w}) = (\mathbf{0}, \mathbf{0})$ .

- The inequality (3.16) is true.

Otherwise there exists a sequence  $(\mathbf{v}_n, \mathbf{z}_n) \in W_{el}$  satisfying relations (3.21) and (3.22). Then, substituting  $(\mathbf{v}_n, \mathbf{z}_n)$  for  $(\mathbf{v}; \mathbf{z})$  into relation (3.17) we get :

$$0 < C \leq \frac{1}{n^2} + \|\mathcal{L}_{1n}\|_{0,\Omega}^2 + \|\mathcal{L}_{2n}\|_{0,\Omega}^2 + \|\mathcal{L}_{3n}\|_{1,\Omega}^2,$$

which involves the contradiction when  $n \rightarrow +\infty$ . Thus, the inequality (3.16) is true.

*Step 4 : The problem (3.10) has one and only one solution.*

The inequality (3.16) proves the  $W_{el}$ -ellipticity of the bilinear form which appears in the first hand member of the variational equation (3.10). It remains to add the obvious properties of  $W_{el}$ - continuity of the bilinear and linear forms (3.11) to (3.13) ; and to apply the Lax-Milgram lemma.  $\square$

*Remark 3.1.1 :* The assumption  $k = \text{constant} > 0$  is not essential. All the proof can be extended to the case of a coefficient  $k$  which is a smooth function of the arc length  $x$  along  $\Gamma$  with  $k(x) > 0$ .  $\square$

### 3.2 Case of a rigid hinge.

The junction conditions (2.30) of the hinge can be expressed on the boundaries  $\gamma$  and  $\tilde{\gamma}$ , so that the space of kinematically admissible displacement fields is now :

$$\left. \begin{aligned} W_{rig} = \{(\mathbf{w}; \boldsymbol{\psi}) \in \mathbf{V} \times \mathcal{V} ; \mathbf{w} = \boldsymbol{\psi} \text{ and} \\ n^\beta(w_{3,\beta} + b_\beta^\alpha w_\alpha) = (\mathbf{t} \cdot \tilde{\mathbf{t}}) \tilde{n}^\beta(\boldsymbol{\psi}_{3,\beta} + \tilde{b}_\beta^\alpha \boldsymbol{\psi}_\alpha) \text{ at the corresponding points of } \gamma \text{ and } \tilde{\gamma} \} \end{aligned} \right\} \quad (3.23)$$

The variational formulation of the junction between two shells with a rigid hinge can be obtained by analogy with the case of an elastic hinge as follows :

$$\left. \begin{aligned} \text{Find } (\mathbf{u}_{rig}; \boldsymbol{\psi}_{rig}) \in W_{rig} \text{ such that} \\ a[(\mathbf{u}_{rig}; \boldsymbol{\psi}_{rig}), (\mathbf{v}; \boldsymbol{\psi})] = \ell(\mathbf{v}; \boldsymbol{\psi}), \quad \forall (\mathbf{v}; \boldsymbol{\psi}) \in W_{rig} \end{aligned} \right\} \quad (3.24)$$

where  $a[.,.]$  and  $\ell(.)$  are the bilinear and linear forms defined by relations (3.11) and (3.13). We prove :

**Theorem 3.2.1 :** *Under the assumptions of Theorem 3.1.1, the problem (3.24) has one and only one solution.*

**Proof :** This proof is similar to that of Theorem 3.1.1. It suffices to note that the application  $(\mathbf{v}; \boldsymbol{\psi}) \in W_{rig} \rightarrow \|(\mathbf{v}; \boldsymbol{\psi})\|_{W_{rig}}$  is a norm on  $W_{rig}$ , where we set

$$\|(\mathbf{v}; \boldsymbol{\psi})\|_{W_{rig}} = \{a[(\mathbf{v}; \boldsymbol{\psi}), (\mathbf{v}; \boldsymbol{\psi})]\}^{1/2}.$$

Moreover, condition  $\mathbf{v} = \mathbf{v} = \mathbf{0}$  at the corresponding points of  $\gamma$  and  $\underline{\gamma}$  derives from step 2 while  $\underline{v}_{3,\underline{v}} = 0$  on  $\underline{\gamma}$  derives from the last condition of the definition (3.23). Similar arguments would prove that the norms  $\|(\mathbf{v} ; \mathbf{v})\|_E$  and  $\|(\mathbf{v} ; \mathbf{v})\|_{W_{rig}}$  are equivalent on the space  $W_{rig}$ , and the conclusion would arise once again from the Lax-Milgram lemma.  $\square$

### 3.3 Study of the behaviour of an elastic hinge when $k \rightarrow \infty$ .

**Theorem 3.3.1** *Let  $(\mathbf{u}^k, \underline{\mathbf{u}}^k)$  be the solution of the elastic hinge problem (3.10) and let  $(\mathbf{u}_{rig} ; \underline{\mathbf{u}}_{rig})$  be the solution of the rigid hinge problem (3.24). Then*

$$\lim_{k \rightarrow +\infty} \|(\mathbf{u}^k, \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig} ; \underline{\mathbf{u}}_{rig})\|_{W_{el}} = 0 \quad (3.25)$$

**Proof :** It takes four steps

*Step 1 : Weak convergence of  $(\mathbf{u}^k, \underline{\mathbf{u}}^k)$  in  $W_{el}$ .*

By considering the equation (3.10) and by using the definition (3.15) and the continuity of the linear form  $\ell(\cdot)$ , we immediately obtain :

$$\min(k, 1) \|(\mathbf{u}^k, \underline{\mathbf{u}}^k)\|_{W_{el}} \leq \|\ell\|,$$

where  $\|\ell\|$  is independent of  $k$ . Since for any  $k \geq 1$ , the sequence  $(\mathbf{u}^k, \underline{\mathbf{u}}^k)$  is bounded in  $W_{el}$ , there exists a subsequence, still denoted  $(\mathbf{u}^k, \underline{\mathbf{u}}^k)$ , which is weakly convergent in  $W_{el}$  to a limit  $(\mathbf{u}^*, \underline{\mathbf{u}}^*)$ .

*Step 2 : The limit  $(\mathbf{u}^*, \underline{\mathbf{u}}^*) \in W_{rig}$ .*

For any  $k \geq 1$ , the equation (3.10) implies

$$b[(\mathbf{u}^k, \underline{\mathbf{u}}^k); (\mathbf{u}^k, \underline{\mathbf{u}}^k)] \leq \frac{\|\ell\|^2}{k}.$$

Since the application  $(\mathbf{w} ; \underline{\mathbf{w}}) \in E \rightarrow b[(\mathbf{w} ; \underline{\mathbf{w}}); (\mathbf{w} ; \underline{\mathbf{w}})]$  is convex and continuous, it is weakly lower semi-continuous. Then, from Step 1, we obtain at the limit ( $k \rightarrow +\infty$ )

$$\text{tr}[n^\beta(u_{3,\beta}^* + b_\beta^\alpha u_\alpha^*)] = \text{tr}[\underline{n}^\beta(\underline{\mathbf{t}} \cdot \underline{\mathbf{t}})(\underline{v}_{3,\beta}^* + \underline{b}_\beta^\alpha \underline{v}_\alpha^*)] \quad \text{on } \Gamma,$$

so that  $(\mathbf{u}^*, \underline{\mathbf{u}}^*) \in W_{rig}$ .

*Step 3 : In fact  $(\mathbf{u}^*, \underline{\mathbf{u}}^*) = (\mathbf{u}_{rig} ; \underline{\mathbf{u}}_{rig})$*

It suffices to write the equation (3.10) for any  $(\mathbf{v} ; \underline{\mathbf{v}}) \in W_{rig} \subset W_{el}$  so that at the limit when  $k \rightarrow +\infty$ ,  $(\mathbf{u}^*, \underline{\mathbf{u}}^*)$  is solution of equation (3.24). Hence the uniqueness of the solution of equation (3.24) gives the result.

Step 4 : Strong convergence (3.25).

As soon as  $k \geq 1$ , we obtain with (3.15)

$$\begin{aligned}
 0 &\leq \|(\mathbf{u}^k; \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig})\|_{W_{el}}^2 \\
 &= a[(\mathbf{u}^k; \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig}), (\mathbf{u}^k; \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig})] \\
 &\quad + kb[(\mathbf{u}^k; \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig}), (\mathbf{u}^k; \underline{\mathbf{u}}^k) - (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig})] \\
 &= \ell(\mathbf{u}^k; \underline{\mathbf{u}}^k) + \ell(\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig}) - 2a[(\mathbf{u}^k; \underline{\mathbf{u}}^k), (\mathbf{u}_{rig}; \underline{\mathbf{u}}_{rig})],
 \end{aligned}$$

so that we get (3.25) for  $k \rightarrow +\infty$ . Moreover, since the limit is unique, this result is independent of the subsequence into consideration.  $\square$

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