A Donsker theorem to simulate one-dimensional processes with measurable coefficients

Pierre Etoré, Antoine Lejay

A Donsker theorem to simulate one-dimensional processes with measurable coefficients

Pierre ÉTORÉ & Antoine LEJAY

29th May 2006

Projet OMEGA, IECN, BP 239, F-54506 Vandœuvre-lès-Nancy CEDEX

Pierre.Etore@iecn.u-nancy.fr Antoine.Lejay@iecn.u-nancy.fr

Abstract

In this paper, we prove a Donker theorem for one-dimensional processes generated by an operator with measurable coefficients. We construct a random walk on any grid on the state space, using the transition probabilities of the approximated process, and the conditional average times it spends on each cell of the grid. Indeed we can compute these quantities by solving some suitable elliptic PDE problems.

Keywords

Monte Carlo methods, Donsker theorem, one-dimensional process, scale function, divergence form operators, Feynman-Kac formula, elliptic PDE problem

AMS Classification: 60J60, 65C

1 Introduction

In this paper we provide a scheme for simulating one-dimensional processes generated by Divergence Form Operators (DFO), with measurable coefficients. Those are operators of type

$$L = \frac{\rho}{2}\nabla\Big(a\nabla.\Big) + b\nabla. \tag{1.1}$$

A sufficient condition for L to generate a continuous Markov process is that a and ρ are uniformly elliptic and bounded and that b is bounded. Note that DFOs contain the case of non divergence form operators.

A lot of probabilistic numerical methods allow already to treat the case of smooth coefficients a, ρ and b (see [KP92] for example). However we wish to deal with irregular coefficients. Indeed DFOs appear in the modelization of diffusion phenomena, and the irregularity of the coefficient can reflect the irregularity of the media the particle is evolving in. This is interesting in a wide variety of physical situations, for example in fluid mechanics in porous media (see [RTW05]), in the modelization of the brain (see [Fau99]), and can also be used in finance (see [DDG05]).

Recently some authors have provided schemes in the case of coefficients having some discontinuities, in dimension $d = 1$. The case of dimension $d < 1$ appears to be a challenging problem.

In [Mar04] (see also [MT06]) M. Martinez has treated the case $\rho = 1$, and a and b having one point of discontinuity in some x_0. He uses the fact that the process X generated by L solves the following Stochastic Differential Equation with Local Time (SDELT),

$$X_t = X_0 + \int_0^t \sqrt{a\rho}(X_s)dW_s + \int_0^t \Big[\frac{\rho a'}{2} + b\Big](X_s)ds + \sum_{x\in\mathcal{I}} \frac{a(x+) - a(x-)}{a(x+) + a(x-)} L_t^x(X), \tag{1.2}$$

where $\mathcal{I}$ is the set of the points of discontinuity of a. Using a proper space transformation he gets rid of the local time in (1.2) and then uses an Euler scheme. To estimate the speed of convergence of his method, he needs a to be $\mathcal{C}^6$ outside the point of discontinuity. The initial condition has to be $\mathcal{C}^4$ almost everywhere and to satisfy other restrictive conditions.

In [LM06] (see also [Mar04]) A. Lejay and M. Martinez propose a different scheme. First they perform a piecewise constant approximation of the coefficients. Second they use another space transformation, to obtain a process that behaves locally like a Skew Brownian Motion (SBM) which is known to solve

$$Y_t = Y_0 + W_t + \beta L_t^y(Y),$$

with $\beta \in [-1, 1]$. Third they use an exact simulation method of the Skew Brownian Motion (SBM), based on the simulation of the exit times of a Brownian motion. In general the whole algorithm is slow and costly but allows to treat the case of coefficients a, ρ and b being right continuous with left limits (r.c.l.l.), and of class $\mathcal{C}^1$ except on countable set of points, without cluster point. Besides the initial condition can be taken in H^1, and the algorithm is well adapted to the case of coefficients being flat on large intervals outside their points of discontinuity.

In [ETO06] we have used the same kind of space transformation to obtain a process $Y = \Phi(X)$ that behaves like a SBM around each point of a regular grid $\Phi(\mathfrak{g}) = \{kh | k \in \mathbb{Z}\}$. We then perform an asymmetric random walk on $\Phi(\mathfrak{g})$ to approach the transformed process Y. The random walk is asymmetric because its transition probabilities can differ from $1/2$, but the time spent by the approximating process on each cell is always h^2. This corresponds to the average time spent by Y on each cell of $\Phi(\mathfrak{g})$. The obtained algorithm is easy to implement, fast, and allows to treat the same cases as in [LM06] except that the initial condition has to be taken in $\mathrm{W}^{1,\infty} \cap \mathrm{H}^1 \cap \mathcal{C}_0$. Notice that the convergence rate is of order $\mathcal{O}(h^{1/2})$, which is not surprising comparing to Donsker's theorem.

One of the limitations of this latter algorithm is that, in order to get a regular grid $\Phi(\mathfrak{g})$ for $Y = \Phi(X)$, we have to construct a grid $\mathfrak{g}$ for X that *depends on* the coefficients a, ρ and b. Besides we have to assume that the coefficients are of class $\mathcal{C}^1$ except on a countable set of points without cluster point. Indeed we deeply rely on the theory of SDELT which requires this assumption.

Here we do not use SDELTs and propose a more general algorithm that allows to get rid of both these limitations. We can take measurable coefficients and any grid on the real line. Roughly the idea is the following:

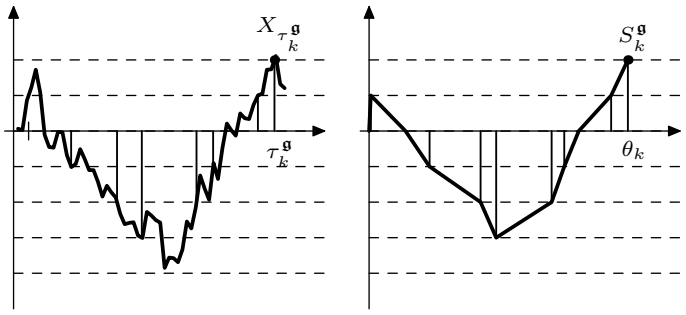


Figure 1: One path of X_t and the linear interpolation of the corresponding path of $S^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}$.

Let be $\mathfrak{g} = \{x_j\}_{j\in J}$ a set of points on the real line, with $x_j < x_{j+1}$, $\forall j \in J$. Let be X the process generated by L defined by (1.1).

At each point x_j in $\mathfrak{g}$, knowing X is in x_j, we ask ourselves two questions:

1) What is the probability $\pi(j, j+1)$ that the next point of $\mathfrak{g}$ the process X reaches is x_{j+1} and not x_{j-1}?

2) Knowing X has gone to x_{j+1} (respectively to x_{j-1}) what is the average time $T(j,+)$ (respectively $T(j,-)$) it has taken to go from x_j to x_{j+1} (respectively from x_j to x_{j-1})?

We will show (Sections 4 and 5) that it is possible to link the quantities $T(j,+)$ and $T(j,-)$ with the solutions of some elliptic problems involving the operator L.

Suppose we know all the $\pi(j, j+1)$'s, $T(j,+)$'s and $T(j,-)$'s and we want an approximation of X at time t. We set $\hat{t} = 0$, and perform the path on $\mathfrak{g}$ of a random walk $S^{\mathfrak{g}}$ with transition probabilities the $\pi(j, j+1)$'s. Each time $S^{\mathfrak{g}}$ passes from one point of $\mathfrak{g}$ to another one we add to $\hat{t}$ the corresponding $T(j,\pm)$. When $\hat{t}$ is greater than t we stop the algorithm and return the current position of $S^{\mathfrak{g}}$.

One can think of this construction as a generalization of the Donsker theorem. Assume the minimum and the maximum step size of the grid $\mathfrak{g}$ is related to some parameter h. One construct a random walk $(S^{\mathfrak{g}}_k)_k$ and an integer value function $K^{\mathfrak{g}}(t)$, such that $S^{\mathfrak{g}}$ reflects the successive positions of the trajectories of X on $\mathfrak{g}$, and $S^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}$ converges in distribution to X_t for each $t \geq 0$, as h tends to zero.

If $\tau^{\mathfrak{g}}_k$ denotes the first time after $\tau^{\mathfrak{g}}_{k-1}$ at which X_t passes through a point of $\mathfrak{g}$ different from $X_{\tau^{\mathfrak{g}}_{k-1}}$, and θ_k is the time constructed at which we consider $S^{\mathfrak{g}}_k$, then the two-dimensional path obtained by joining the (θ_k, S^h_k)'s by segments is an approximation of the path obtained by joining the $(\tau^{\mathfrak{g}}_k, X_{\tau_k})$'s by segments (see Figure 1).

If $a = \rho = 1$ and $b = 0$, then X is a Brownian motion and if $\mathfrak{g}$ is the uniform grid with step h, then $(S^{\mathfrak{g}}_k)_k$ is a simple random walk, $\theta_{k+1} - \theta_k = h^2$, and the convergence result we prove implies the Donsker theorem.

The scheme is very easy to implement if we first perform some suitable piecewise constant approximations of the coefficients. By doing this it is possible

to assume that the grid is constructed at a mesoscopic scale, and that the piecewise constant coefficients correspond to some effective coefficients of highly oscillating coefficients at a small scale, as the homogenization theory ([BLP78], [JKO94]) proves it. Numerical tests show a good behaviour of this algorithm.

Besides, compared to the previous ones, this scheme seems more adaptable to the multidimensional case. Indeed it relies strongly on the link between PDE theory and the one of Markov processes, and involves elliptic PDEs that could still be written in dimension 2 or 3.

Some Hypothesis and notations. We will denote by G the state space of the processes we will consider, which is here a connected interval of $\mathbb{R}$. The set G will be also the definition domain of the coefficients of the DFOs, and of the functions they can be applied.

For u in $\mathrm{L}^2(G)$ we denote by $\frac{du}{dx}$ the derivative of u in the distributionnal sense. We classically denote by $\mathrm{H}^1(G)$ the space of functions in $\mathrm{L}^2(G)$ such that $\frac{du}{dx} \in \mathrm{L}^2(G)$, and by $\mathrm{H}^1_0(G)$ the closure of $\mathcal{C}^\infty_c(G)$ in $\mathrm{H}^1(G)$ equipped with the norm $(\|u\|_2^2 + \|\frac{du}{dx}\|_2^2)^{1/2}$.

Outline of the paper. In Section 2 we recall some basic facts on DFOs and the associated processes. In Section 3 we present the convergence results we will need. In Section 4 we link the conditional moments of exit times we will need with the solutions of some suitable elliptic PDE problems. In Section 5 we present our scheme. In Section 6 we show this scheme converges and evaluate its rate of convergence. Section 7 deals with numerical considerations.

2 DFO and Markov processes

In this section we recall some facts on processes generated by a DFO. As these results are standard we do not much emphasize on the proofs.

2.1 The existence of a Markov process

For $0 < \lambda < \Lambda < \infty$ let us denote by $\mathfrak{Ell}(\lambda, \Lambda)$ the set of functions f on G that are measurable and such that

$$\forall x \in G, \ \ \lambda \leq f(x) \leq \Lambda.$$

We denote by $\mathfrak{B}(\Lambda)$ the set of functions f on G that are measurable and such that

$$\forall x \in G, \ \ |f(x)| \leq \Lambda.$$

For a, $\rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$, we will denote by $\mathfrak{L}(a, \rho, b)$ the DFO defined by

$$L = \frac{\rho}{2} \frac{d}{dx}\Big(a \frac{d}{dx}\Big) + b \frac{d}{dx},$$

and

$$D(L) = \{ \ f \in \mathrm{H}^1_0(G) \ , \ a\frac{df}{dx} \in \mathrm{H}^1(G)\}, \ \ \text{if } G \text{ is finite and for Dirichlet b.c.,}$$

$$D(L) = \{\ f \in \mathrm{H}^1(G)\ ,\ a\frac{df}{dx} \in \mathrm{H}^1(G)\},\ \text{ if } G \text{ is infinite or for Neumann b.c.}$$

For any measure $g(x)dx$ with a bounded density g we denote by $\mathrm{L}^2(G, g(x)dx)$ the set of the functions in $\mathrm{L}^2(G)$ endowed with the scalar product

$$(f,h) \longmapsto \int_G f(x)h(x)g(x)dx.$$

Note that when we simply write $\mathrm{L}^2(G)$ the involved measure is implicitely the Lebesgue one. With these notations we have the following theorem.

Theorem 2.1 *Let be $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be $T > 0$.*

i) The operator $\mathfrak{L}(a, \rho, b)$ is the infinitesimal generator of a strongly continuous semigroup of contraction $(S_t)_{0\leq t\leq T}$ on $\mathrm{L}^2(G, \rho^{-1}(x)dx)$.

ii) Moreover $(S_t)_{0\leq t\leq T}$ is a Feller semigroup. Thus $\mathfrak{L}(a, \rho, b)$ is the infinitesimal generator of a Markov process $(X_t, 0 \leq t \leq T)$.

iii) The process $(X_t, 0 \leq t \leq T)$ has continuous trajectories.

Proof. See [Lej00], Chapter 0. □

Sometimes it is more convenient to have on operator of the form $\mathfrak{L}(a, \rho, 0)$ than one of the form $\mathfrak{L}(a, \rho, b)$ (see subsection 2.2). It is possible to pass from one form to the other.

Proposition 2.1 *Let be $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be $(L, D(L)) = \mathfrak{L}(a, \rho, b)$. Let be $[l, r] \subset G$. For $x_0 \in [l, r]$, let be Ψ defined by*

$$\Psi(x) = 2\int_{x_0}^{x} \frac{b(y)}{a(y)\rho(y)}dy, \quad \forall x \in [l, r]. \tag{2.1}$$

Consider on $[l, r]$ the operator $(\bar{L}, D(\bar{L})) = \mathfrak{L}(ae^{\psi}, \rho e^{-\Psi}, 0)$. The restriction on $[l, r]$ of $(L, D(L))$ is equal to $(\bar{L}, D(\bar{L}))$.

Proof. As b is in $\mathfrak{B}(\Lambda)$ it is possible to show that ae^{Ψ} and $\rho e^{-\Psi}$ are in $\mathfrak{Ell}(\lambda', \Lambda')$ on $[l, r]$, with λ' and Λ' depending on λ, Λ and $l - r$.

As the functions $e^{\pm\Psi}$ are bounded and have bounded classical derivatives they are in $\mathrm{H}^1([l, r])$. So the properties of the product of functions of class H^1 (see [Bre83] Corollary VIII.9) allow to assert that, on $[l, r]$, $D(L) = D(\bar{L})$, and that

$$\forall f \in D(L), \quad \frac{\rho e^{-\Psi}}{2}\frac{d}{dx}\Big(ae^{\Psi}\frac{df}{dx}\Big) = \frac{\rho}{2}\frac{d}{dx}\Big(a\frac{df}{dx}\Big) + b\frac{df}{dx}.$$

□

2.2 Scale function and speed measure

We recall here the definitions of the scale function and the speed measure of a process X on an interval G of the real line. We follow [Bre68], Chapter 16.

Definition 2.1 *The scale function of a Markov process X on G is the strictly increasing function s, unique up to a linear transformation, such that for all $[l,r] \subset G$, and for all $x \in [l,r]$,*

$$\mathbb{P}^x[X_\tau = r] = \frac{s(x) - s(l)}{s(r) - s(l)},$$

where $\tau = \inf\{t \geq 0 | X_t \in \{l,r\}\}$.

The scale function of the Brownian motion is the identity function. A process X is said to be on its *natural scale* if its scale function is the identity function. When X is not on its natural scale it is easy to check that $s(X)$ is.

Definition 2.2 *Let be X a Markov process on G that is on its natural scale. The speed measure of X is the unique measure m on G such that, for all $[l,r] \subset G$, and for all $x \in [l,r]$, with $\tau = \inf\{t \geq 0 | X_t \in \{l,r\}\}$, we have,*

$$\mathbb{E}^x[\tau] = \int_l^r G_{[l,r]}(x,z) m(dz), \tag{2.2}$$

where,

$$G_{[l,r]}(x,z) = \begin{cases} \frac{2(x-l)(r-z)}{r-l}, & \forall x,z \in [l,r],\ x \leq z, \\ \frac{2(z-l)(r-x)}{r-l}, & \forall x,z \in [l,r],\ x \geq z, \\ 0, & \textit{otherwise.} \end{cases}$$

When X is not on its natural scale, there is still a notion of speed measure. (see Problem 16.7 in [Bre68]).

Lemma 2.1 *Let be X a Markov process on G and s its scale function. Assume the speed measure of $s(X)$ is $V(x)dx$. We have for all $[l,r] \subset G$, and for all $x \in [l,r]$, with $\tau = \inf\{t \geq 0 | X_t \in \{l,r\}\}$,*

$$\mathbb{E}^x[\tau] = \int_l^r G_{[l,r]}(x,y) V(s(y)) s'(y) dy,$$

where,

$$G_{[l,r]}(x,y) = \begin{cases} \frac{2(s(x)-s(l))(s(r)-s(y))}{s(r)-s(l)}, & \forall x,y \in [l,r],\ x \leq y, \\ \frac{2(s(y)-s(l))(s(r)-s(x))}{s(r)-s(l)}, & \forall x,y \in [l,r],\ x \geq y, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.3}$$

We call $m(dx) = V(s(x))s'(x)dx$ the speed measure of X.

Proof. We have $\mathbb{E}^x[\tau] = \mathbb{E}^{s(x)}[\tilde{\tau}]$, with $\tilde{\tau} = \inf\{t \geq 0 | s(X_t) \in \{s(l), s(r)\}\}$. As $s(X)$ is on its natural scale it suffices to use in (2.2) the change of variable $z = s(y)$ to complete the proof. □

In dimension one to give the scale function and the speed measure of a Markov process suffices to fully characterize it.

Given a Markov process generated by a DFO we can directly determine its scale function and speed measure.

Proposition 2.2 *Let be $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$. Let be X generated by $\mathfrak{L}(a, \rho, 0)$. Let be $x_0 \in G$. The scale function and speed measure of X are given by,*

$$s(x) = \int_{x_0}^{x} \frac{dy}{a(y)}, \tag{2.4}$$

$$\text{and } m(dx) = \frac{dx}{\rho(x)}. \tag{2.5}$$

Proof. See [Lej00] Chapter 5, where it is made use of convergence results proved in [FW94].

□

Remark 2.1 Note that as the scale function s is unique up to a linear transform, the choice of the reference point x_0 in (2.4) is arbitrary. We may choose any point in G.

2.3 Using estimates on the kernel

We will need the following lemma, proved by D.W. Stroock in [Str88].

Lemma 2.2 *Let be a, $\rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be $T > 0$. Let be X generated by $\mathfrak{L}(a, \rho, b)$. There exists a constant A depending on λ, Λ and T, such that, for all $(t, x) \in [0, T] \times G$ and any $d > 0$,*

$$\mathbb{P}^x\Big[\sup_{s\in[0,t]} |X_s - x| \geq d\Big] \leq A \exp\Big(-\frac{Ad^2}{t}\Big).$$

Proof. In [Str88] the lemma is proved for $\rho = 1$ and $b = 0$. However the proof is based on Aronson's estimates that remain valid with any $b \in \mathfrak{B}(\Lambda)$ provided that we are at some finite horizon T. Besides Aronson's estimates concern the transition probability density of X with respect to the Lebesgue measure in the case $\rho = 1$. In the case $\rho \neq 1$, we have the same estimates for the density with respect ot the measure $\rho^{-1}(x)dx$. As ρ^{-1} is in $\mathfrak{Ell}(1/\Lambda, 1/\lambda)$, we finally get the same result. □

3 Convergence results

We first assert very general convergence results concerning DFOs. They lie on the theory of G-convergence, about which we can refer the reader to [ZKOT79] and [ZKO81]. We first assert a result concerning elliptic problems.

Theorem 3.1 *Assume G is finite. Let be $a \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be (a^n) a sequence of $\mathfrak{Ell}(\lambda, \Lambda)$ and (b^n) a sequence of $\mathfrak{B}(\Lambda)$ such that*

$$\frac{1}{a^n} \xrightarrow[n\to\infty]{\mathrm{L}^2(G)} \frac{1}{a}, \quad \text{and} \quad \frac{b^n}{a^n} \xrightarrow[n\to\infty]{\mathrm{L}^2(G)} \frac{b}{a}. \tag{3.1}$$

Let be $(L, D(L)) = \mathfrak{L}(a, 1, b)$ *and for each* n *let be* $(L^n, D(L^n)) = \mathfrak{L}(a^n, 1, b^n)$. *Let be* (f^n) *a sequence converging in* H^{-1} *to some* f. *Let be* u *the solution to the problem* $Lu = -f$ *submitted to uniform Dirichlet boundary conditions, and for each* n *let be* u^n *the solution to* $L^n u^n = -f^n$.

Then the sequence (u^n) *converges weakly in* $\mathrm{H}_0^1(G)$ *to* u.

Proof. This a direct consequence of Theorems 5 and 17 in [ZKOT79]. □

Remark 3.1 If G is infinite or we want to treat Neumann b.c. the conclusion of Theorem 3.1 still holds, with a weak convergence in $\mathrm{H}^1(G)$, but under the assumption the weak convergences (3.1) hold in $\mathrm{L}^2_{loc}(G)$.

In [ZKO81] the authors applied the G-convergence theory to parabolic problems. In [Roz96] the author used this to prove some results on the convergence in law of processes generated by DFOs.

Theorem 3.2 *Let be* $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$ *and* $b \in \mathfrak{B}(\Lambda)$. *Let* (a^n) *and* (ρ^n) *two sequences of* $\mathfrak{Ell}(\lambda, \Lambda)$ *and* (b^n) *a sequence of* $\mathfrak{B}(\Lambda)$ *such that*

$$\frac{1}{a^n} \xrightarrow[n\to\infty]{\mathrm{L}^2_{loc}(G)} \frac{1}{a}, \quad \frac{1}{\rho^n} \xrightarrow[n\to\infty]{\mathrm{L}^2_{loc}(G)} \frac{1}{\rho} \text{ and } \frac{b^n}{a^n\rho^n} \xrightarrow[n\to\infty]{\mathrm{L}^2_{loc}(G)} \frac{b}{a\rho}. \tag{3.2}$$

Let us denote by S *and* X *respectively the semigroup and the process generated by* $\mathfrak{L}(a, \rho, b)$ *and by* (S^n) *and* (X^n) *the sequences of semigroups and processes generated by the sequence of operators* $\mathfrak{L}(a^n, \rho^n, b^n)$.

Then for any $T > 0$ *and any* $f \in \mathrm{L}^2(G)$ *we have :*

i) The function $S_t^n f(x)$ *converges weakly in* $\mathrm{L}^2(0, T; \mathrm{E})$ *to* $S_t f(x)$, *where* $\mathrm{E} = \mathrm{H}_0^1(G)$ *in the case of Dirichlet b.c. and* $\mathrm{E} = \mathrm{H}^1(G)$ *in the case of an infinite* G *or for Neumann b.c.*

ii) The continuous version of $S_t^n f(x)$ *converges uniformly on each compact of* $(0, T) \times G$ *to the continuous version of* $S_t f(x)$.

iii)

$$(X_t^n,\ t \geq 0) \xrightarrow[n\to\infty]{} (X_t,\ t \geq 0) \quad \textit{in law.}$$

Proof. See [ZKO81] and [Roz96]. See also [LM06]. □

We will finally need the following result concerning the convergence of the exit times of a familly of processes.

Proposition 3.1 *Let be* $[l, r] \subset G$. *Let be* (X^n) *that converges in law to* X. *Let be* $\tau = \inf\{t \geq 0 \,|\, X_t \in \{l, r\}\}$ *and for each* n, $\tau^n = \inf\{t \geq 0 \,|\, X_t^n \in \{l, r\}\}$. *Then for all* $T > 0$ *and for any starting point* x *we have,*

$$\mathbb{P}^x \circ (X^n, \tau^n)^{-1} \xrightarrow[n\to\infty]{} \mathbb{P}^x \circ (X, \tau)^{-1},$$

with respect to the topology of $\mathcal{C}([0, T], \mathbb{R}) \times \mathbb{R}$. *In particular* $X^n_{\cdot\wedge\tau^n}$ *converges in law to* $X_{\cdot\wedge\tau}$ *under each* $\mathbb{P}^x$.

Proof. See in [LM06] the end of the proof of Proposition 3. See also [Lej00] Chapter 0, Section 6. These proofs use the fact that any starting point x is regular for X. □

4 Conditional moments of exit times as solutions of elliptic problems

To build our algorithm and prove its convergence we will need values and estimates of conditional moments of exit times of the approximated process X from an interval $[l, r] \subset G$. In order to get them we link in this section such conditional moments with the solutions of some suitable elliptic problems.

When no special mention is made, we mean by the solution u of an elliptic PDE involving an operator $(L, D(L))$, the continuous version of its solution. Indeed that is necessary if we want to identify $u(x)$ with the expectation starting from x of some functionnal of the process generated by $(L, D(L))$.

In the sequel the intervals $[l, r]$ will be the cells of the grid on G we use to build our algorithm. As the size of these cells will tend to zero, we make in this section the reasonable assumption that

$$r - l \leq 2. \tag{4.1}$$

We mainly want to prove the following proposition.

Proposition 4.1 *Let be $a, \rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be the DFO $(L, D(L)) = \mathfrak{L}(a, \rho, b)$. Let be X the Markov process generated by $(L, D(L))$. Let be $[l, r] \subset G$. Let be $\tau = \inf\{t \geq 0 \,|\, X_t \in \{l, r\}\}$. For all $x \in [l, r]$ we have that $\mathbb{E}^x\big[\tau \, \mathbf{1}_{\{X_\tau = r\}}\big] = v_1(x)$, where v_1 is solution of,*

$$\mathcal{P}_1([l,r],+) \begin{cases} Lv_1 = -v_0 & in \quad (l, r) \\ v_1 = 0 & on \quad \{l, r\}, \end{cases}$$

where v_0 is the solution of

$$\mathcal{P}_0([l,r],+) \begin{cases} Lv_0 = 0 & in \quad (l, r) \\ v_0(l) = 0, \\ v_0(r) = 1. \end{cases}$$

In the same manner $\mathbb{E}^x\big[\tau \, \mathbf{1}_{\{X_\tau = l\}}\big] = v_1(x)$, where v_1 is the solution of $\mathcal{P}_1([l, r], -)$, which is written as $\mathcal{P}_1([l, r], +)$ but with v_0 the solution of

$$\mathcal{P}_0([l,r],-) \begin{cases} Lv_0 = 0 & in \quad (l, r) \\ v_0(l) = 1, \\ v_0(r) = 0. \end{cases}$$

To prove Proposition 4.1 we need a few lemmas.

Lemma 4.1 *Let be a and ρ in $\mathfrak{Ell}(\lambda,\Lambda)\cap\mathcal{C}^\infty(G)$ and b in $\mathfrak{B}(\Lambda)\cap\mathcal{C}^\infty(G)$. Let be $(L,D(L))=\mathfrak{L}(a,\rho,b)$. Let be X the Markov process generated by $(L,D(L))$. Let be $[l,r]\subset G$ and $\tau=\inf\{t\geq 0\,|\,X_t\in\{l,r\}\}$. For all $x\in[l,r]$ we have $\mathbb{E}^x\big[\,\tau\,\mathbf{1}_{\{X_\tau=r\}}\big]=v_1(x)$, where v_1 is solution of $\mathcal{P}_1([l,r],+)$ as defined in Proposition 4.1.*

Proof. As a, ρ and b are $\mathcal{C}^\infty$, L can be written,

$$L=\frac{\rho\,a}{2}\frac{d^2}{dx^2}+\Big[\frac{\rho\,a'}{2}+b\Big]\frac{d}{dx}.$$

Thus X solves,

$$dX_t=\sqrt{a\rho}(X_t)dW_t+\Big[\frac{\rho\,a'}{2}+b\Big](X_t)dt.$$

As a, ρ and b are $\mathcal{C}^\infty$, the solution v_0 of $\mathcal{P}_0([l,r],+)$ as described in Proposition 4.1 is of class $\mathcal{C}^2$.

We first notice that because of the boundary conditions in $\mathcal{P}_0([l,r],+)$ we have $\tau v_0(X_\tau)=\tau\mathbf{1}_{\{X_\tau=r\}}$.

Second, for $0\leq t\leq\tau$, applying the Itô rule to $tv_0(X_t)$, and using $Lv_0=0$ we get

$$\begin{aligned} tv_0(X_t) &= \textstyle\int_0^t v_0(X_s)ds+M_t+\int_0^t sLv_0(X_s)ds \\ &= \textstyle\int_0^t v_0(X_s)ds+M_t, \end{aligned}$$

where $M_t=\int_0^t sv_0'(X_s)\sqrt{a\rho}(X_s)dW_s$ is a local martingale. As v_0' is bounded on $[l,r]$ and τ is a stopping time, $M_{t\wedge\tau}$ is a martingale, so, taking the expectation,

$$\mathbb{E}^x[\tau v_0(X_\tau)]=\mathbb{E}^x\Big[\int_0^\tau v_0(X_s)ds\Big].$$

Third, because of the smoothness of the coefficients, the solution v_1 of $\mathcal{P}_1([l,r],+)$ is $\mathcal{C}^2$ too. So using the Feynman-Kac formula we get

$$\mathbb{E}^x[\tau\mathbf{1}_{\{X_\tau=r\}}]=\mathbb{E}^x\Big[\int_0^\tau v_0(X_s)ds\Big]=v_1(x)\quad\forall x\in[l,r].$$

□

Lemma 4.2 *Let be a and ρ in $\mathfrak{Ell}(\lambda,\Lambda)\cap\mathcal{C}^\infty(G)$ and b in $\mathfrak{B}(\Lambda)\cap\mathcal{C}^\infty(G)$. Let be $(L,D(L))=\mathfrak{L}(a,\rho,b)$. Let be X the Markov process generated by $(L,D(L))$. Let be $[l,r]\subset G$ and $\tau=\inf\{t\geq 0\,|\,X_t\in\{l,r\}\}$. For all $x\in[l,r]$ we have $\mathbb{E}^x\big[\,\tau^2\,\mathbf{1}_{\{X_\tau=r\}}\big]=v_2(x)$, where v_2 is solution of,*

$$\mathcal{P}_2([l,r],+)\begin{cases} Lv_2=-2\,v_1 & \text{in}\quad (l,r)\\ \\ v_2=0 & \text{on}\quad \{l,r\},\end{cases}$$

where v_1 is the solution of $\mathcal{P}_1([l,r],+)$.

In the same manner $\mathbb{E}^x\big[\,\tau^2\,\mathbf{1}_{\{X_\tau=l\}}\big]=v_2(x)$, where v_2 is the solution of $\mathcal{P}_2([l,r],-)$, which is written as $\mathcal{P}_2([l,r],+)$ but with v_1 the solution of $\mathcal{P}_1([l,r],-)$.

Proof. We treat the case of $\mathbb{E}^x[\,\tau^2\,\mathbf{1}_{\{X_\tau=r\}}]$ and consider the problems $\mathcal{P}_i([l,r],+)$ for $i=0,1,2$.

Because a, ρ and b of class $\mathcal{C}^\infty$ the functions v_0, v_1 and v_2 are of class $\mathcal{C}^2$.

We first notice that we have $\tau^2\,\mathbf{1}_{\{X_\tau=r\}}=\tau^2 v_0(X_\tau)$.

Second for all $0\le t\le\tau$, the Itô rule applied to $t^2v_0(X_t)$ leads to

$$\begin{aligned} t^2v_0(X_t) &= 2\int_0^t sv_0(X_s)ds + M_t + \int_0^t s^2Lv_0(X_s)ds \\ &= -2\int_0^t sLv_1(X_s)ds + M_t, \end{aligned} \tag{4.2}$$

with M a local martingale with bracket $\langle M\rangle_t=\int_0^t s^4[(v_0')^2a\rho](X_s)ds$.

Third, applying the Itô rule to $tv_1(X_t)$ for all $0\le t\le\tau$,

$$tv_1(X_t)=\int_0^t v_1(X_s)ds+N_t+\int_0^t sLv_1(X_s)ds,$$

with N another local martingale with bracket $\langle N\rangle_t=\int_0^t s^2[(v_1')^2a\rho](X_s)ds$. Taking in account (4.2) and the boundary conditions in $\mathcal{P}_1([l,r],+)$ we have,

$$\begin{aligned} \tau^2v_0(X_\tau) &= M_\tau+2\Big(\int_0^\tau v_1(X_s)ds+N_\tau-\tau v_1(X_\tau)\Big) \\ &= \int_0^\tau 2v_1(X_s)ds+2N_\tau+M_\tau. \end{aligned}$$

As v_0' and v_1' are bounded on $[l,r]$ and τ is a stopping time, $M_{t\wedge\tau}$ and $N_{t\wedge\tau}$ are martingales. So finally, taking expectation and using again the Feynman-Kac formula,

$$\mathbb{E}^x[\tau^2\,\mathbf{1}_{\{X_\tau=r\}}]=\mathbb{E}^x\Big[\int_0^\tau 2v_1(X_s)ds\Big]=v_2(x),\quad\forall x\in[l,r].$$

□

Lemma 4.3 *Let be a and ρ in $\mathfrak{EU}(\lambda,\Lambda)\cap\mathcal{C}^\infty(G)$ and b in $\mathfrak{B}(\Lambda)\cap\mathcal{C}^\infty(G)$. Let be $(L,D(L))=\mathfrak{L}(a,\rho,b)$. Let be $[l,r]\subset G$. For all function f continuous and bounded, the solution of*

$$(\mathcal{P})\begin{cases} Lu=-f & \text{in} \quad (l,r) \\ \\ u=0 & \text{on} \quad \{l,r\}, \end{cases}$$

verifies $\|u\|_\infty\le C\,\|f\|_\infty\,(r-l)^2$, where C is a constant depending only on λ and Λ.

Proof. Note first that $(L,D(L))=\mathfrak{L}(a\,e^{\Psi},\rho\,e^{-\Psi},0)$ with Ψ defined by $\Psi(x)=2\int_l^x\frac{b(y)}{a(y)\rho(y)}dy$.

Let be f continuous and bounded and u the solution of $(\mathcal{P})$. It is of class $\mathcal{C}^2$ and by Feynman-Kac we have $u(x)=\mathbb{E}^x[\int_0^\tau f(X_s)ds]$. But according to Problem 10 in Section 16.6 in [Bre68] and Proposition 2.2 we have that $\mathbb{E}^x[\int_0^\tau f(X_s)ds]=\int_l^r G_{[l,r]}(x,y)f(y)\exp(\Psi(y))\rho^{-1}(y)dy$, where $G_{[l,r]}$ is defined by (2.3) with $s(x)=\int_l^x\exp(-\Psi(y))/a(y)\,dy$. Thus,

$$u(x)=\int_l^r G_{[l,r]}(x,y)f(y)\exp(\Psi(y))\rho^{-1}(y)dy, \tag{4.3}$$

and we have an explicit expression of $G_{[l,r]}$. After some easy computations and using condition (4.1) we get

$$\forall x \in [l,r], \quad \exp(\pm\Psi(x)) \leq \kappa, \tag{4.4}$$

where $\kappa = \max(1, e^{4\Lambda/\lambda^2})$ und thus,

$$\forall x,y \in [l,r] \;\; G_{[l,r]} \leq 2\frac{\kappa}{\lambda}(r-l) \quad \text{and} \quad \forall y \in [l,r],\; \exp(\Psi(y))\rho^{-1}(y) \leq \frac{\kappa}{\lambda},$$

Using this in (4.3) we complete the proof.

□

Lemma 4.4 *For $a, \rho \in \mathfrak{Ell}(\lambda,\Lambda)$ and $b \in \mathfrak{B}(\Lambda)$, let be the DFO $(L, D(L)) = \mathfrak{L}(a,\rho,b)$. The solution of $\mathcal{P}_0([l,r],+)$ is given by*

$$v_0(x) = \frac{\int_l^x e^{-\Psi(y)}/a(y)dy}{\int_l^r e^{-\Psi(y)}/a(y)dy}.$$

Proof. Straightforward: this function solves $\mathcal{P}_0([l,r],+)$ which has a unique solution. □

Remark 4.1 The function v_0 is the scale function of the process generated by $\mathfrak{L}(a,\rho,b)$ (see Propositions 2.1 and 2.2). In fact the scale function is build by solving such elliptic problems as $\mathcal{P}_0([l,r],+)$.

Lemma 4.5 *Let be a and ρ in $\mathfrak{Ell}(\lambda,\Lambda) \cap \mathcal{C}^\infty(G)$ and b in $\mathfrak{B}(\Lambda) \cap \mathcal{C}^\infty(G)$. Let be $(L, D(L)) = \mathfrak{L}(a,\rho,b)$. Let be $[l,r] \subset G$. Let be v_1 and v_2 respectively the solutions of $\mathcal{P}_1([l,r],+)$ and $\mathcal{P}_2([l,r],+)$ involving $(L, D(L))$. The following estimates hold,*

$$\|v_1\|_\infty \leq C(r-l)^2, \tag{4.5}$$

$$\|v_2\|_\infty \leq K\,(r-l)^4, \tag{4.6}$$

where C and K are positive constants depending only on λ and Λ.

Proof. We have $\|v_0\|_\infty \leq 1$. So (4.5) follows from Lemma 4.3. Then (4.6) follows from (4.5) and the same lemma. □

Proof of Proposition 4.1. We prove the link between $\mathbb{E}^x[\,\tau\,\mathbf{1}_{\{X_\tau=r\}}]$ and the solution of $\mathcal{P}_1([l,r],+)$. The link between $\mathbb{E}^x[\,\tau\,\mathbf{1}_{\{X_\tau=l\}}]$ and $\mathcal{P}_1([l,r],-)$ is proven in the same manner.

Step 1. We build two sequences (a^k) and (ρ^k) in $\mathfrak{Ell}(\lambda,\Lambda) \cap \mathcal{C}^\infty(G)$, and a sequence (b^k) in $\mathfrak{B} \cap \mathcal{C}^\infty(G)$, such that

$$a^k \xrightarrow[k\to\infty]{} a, \qquad \rho^k \xrightarrow[k\to\infty]{} \rho \;\text{ and }\; b^k \xrightarrow[k\to\infty]{} b \quad \text{a.e.} \tag{4.7}$$

Let be $L^k := \mathfrak{L}(a^k,\rho^k,b^k)$. Recall that $(L, D(L)) = \mathfrak{L}(a\,e^{\Psi}, \rho\,e^{-\Psi}, 0)$ and $(L^k, D(L^k)) = \mathfrak{L}(a^k\,e^{\Psi^k}, \rho\,e^{-\Psi^k}, 0)$ with $\Psi(x) = \int_l^x \frac{b(y)}{a(y)\rho(y)}dy$ and $\Psi^k(x) = \int_l^x \frac{b^k(y)}{a^k(y)\rho^k(y)}dy$.

We call v_0^k the solution of the problem $\mathcal{P}_0^k(x,+)$ which is written like $\mathcal{P}_0([l,r],+)$ but with L replaced by L^k. We call v_1^k the solution of the problem $\mathcal{P}_1^k([l,r],+)$ which is written like $\mathcal{P}_1([l,r],+)$ but with L replaced by L^k and v_0 replaced by v_0^k. For each $k \in \mathbb{N}$ let be X^k the process generated by $(L^k, D(L^k))$ and $\tau^k = \inf\{t \geq 0 \,|\, X_t^k \in \{l,r\}\}$.

For each $k \in \mathbb{N}$ we can apply Lemma 4.1 and get that

$$\mathbb{E}^x\big[\,\tau^k\,\mathbf{1}_{\{X^k_{\tau^k}=r\}}\big] = v_1^k(x) \quad \forall x \in [l,r]. \tag{4.8}$$

Step 2. We show that $v_1^k(x) \to v_1(x)$ as $k \to \infty$, for all $x \in [l,r]$, where the v_1^k's and v_1 are continuous versions of the solutions of the involved PDEs.

The generalized solutions v_1^k and v_1 are elements of $\mathrm{H}_0^1([l,r])$ that respectively verify,

$$\Big(\frac{1}{2}\frac{d}{dx}\Big(a^k\frac{d}{dx}\Big) + \frac{b^k}{\rho^k}\frac{d}{dx}\Big)v_1^k = -\frac{v_0^k}{\rho^k} \quad \text{and} \quad \Big(\frac{1}{2}\frac{d}{dx}\Big(a\frac{d}{dx}\Big) + \frac{b}{\rho}\frac{d}{dx}\Big)v_1 = -\frac{v_0}{\rho}.$$

By Lemma 4.4,

$$v_0^k(x) = \frac{\int_l^x e^{-\Psi^k(y)}/a^k(y)dy}{\int_l^r e^{-\Psi^k(y)}/a^k(y)dy}, \quad \text{and} \quad v_0(x) = \frac{\int_l^x e^{-\Psi(y)}/a(y)dy}{\int_l^r e^{-\Psi(y)}/a(y)dy}.$$

Thus, by dominated convergence, we first get that v_0^k converges pointwise to v_0. Again by dominated convergence we can get that v_0^k/ρ^k tends to v_0/ρ in $\mathrm{L}^2([l,r])$, and thus in $\mathrm{H}^{-1}([l,r])$. Note that,

$$\forall k \in \mathbb{N}, \qquad \big\|v_0^k\big\|_\infty \leq 1 \quad \text{and} \quad \Big\|\frac{v_0^k}{\rho^k}\Big\|_\infty \leq \frac{1}{\lambda}. \tag{4.9}$$

Again by dominated convergence it is obvious that,

$$\frac{1}{a^k} \xrightarrow[k\to\infty]{} \frac{1}{a}, \quad \text{and} \quad \frac{b^k}{a^k\rho^k} \xrightarrow[k\to\infty]{} \frac{b}{a\rho},$$

weakly in $\mathrm{L}^2([l,r])$. Thus Theorem 3.1 ensures that v_1^k converges to v_1 weakly in $\mathrm{H}_0^1([l,r])$ and using a compact injection argument we get the strong convergence in $\mathrm{L}^2([l,r])$. (Note that, because of the rather strong convergence we have assumed on the coefficients, we could have shown that $v_1^k \to v_1$ in $\mathrm{L}^2([l,r])$, by making some computations on the norms instead of using Theorem 3.1).

We consider now the continuous versions of the v_1^k's and v_1. By Lemma 4.5,

$$\forall k \in \mathbb{N}, \qquad \big\|v_1^k\big\|_\infty \leq C(r-l)^2 \tag{4.10}$$

with C depending on λ, Λ, but uniform in k. Taking in account (4.9) and (4.10) Theorem 8.22 in [GT83], allows to assert there are positive constants $\widetilde{C}$ and α, uniform in k, such that,

$$\forall k \in \mathbb{N},\ \forall [y-h, y+h] \subset (l,r), \qquad \mathrm{osc}(v_1^k, [y-h, y+h]) \leq \widetilde{C}h^\alpha.$$

Then, by the Arzela-Ascoli theorem, a subsequence $(v_1^{k'})$ converges uniformly on each compact of $[l,r]$. But its limit is necessarily v_1, because of the convergence in $\mathrm{L}^2([l,r])$ of v_1^k to v_1. Thus v_1^k converges uniformly on each compact to v_1, and then pointwise.

Step 3. We show that $\mathbb{E}^x[\,\tau^k\,\mathbf{1}_{\{X^k_{\tau^k}=r\}}] \to \mathbb{E}^x[\,\tau\,\mathbf{1}_{\{X_\tau=r\}}]$ as $k\to\infty$, for all $x\in[l,r]$.

Considering (4.7) it is clear that (3.2) holds. So combining Theorem 3.2 and Proposition 3.1 we have that $\tau^k\,\mathbf{1}_{\{X^k_{\tau^k}=r\}}$ converges in law to $\tau\,\mathbf{1}_{\{X_\tau=r\}}$.

But by Lemma 4.2 and (4.6) in Lemma 4.5,

$$\forall k\in\mathbb{N},\quad \mathbb{E}^x\Big[\big(\tau^k\,\mathbf{1}_{\{X^k_{\tau^k}=r\}}\big)^2\Big]\le K(r-l)^4,$$

with K depending on λ and Λ but not on k. This ensures that the $\tau^k\,\mathbf{1}_{\{X^k_{\tau^k}=r\}}$'s are uniformly integrable (see in [Bil68] the discussion between Theorems 5.3 and 5.4).

So Step 3 is proven.

Taking in account Steps 1, 2 and 3 we complete the proof.

□

We assert now some estimates on v_1 that will be useful to prove our Donsker theorem.

Proposition 4.2 *Let be* $(L,D(L))=\mathfrak{L}(a,\rho,b)$ *with* $a,\rho\in\mathfrak{Ell}(\lambda,\Lambda)$ *and* $b\in\mathfrak{B}(\Lambda)$. *The solution* v_1 *of* $\mathcal{P}_1([l,r],+)$ *satisfies:*

i)

$$\|v_1\|_\infty\le C(r-l)^2, \tag{4.11}$$

where C *is a positive constant depending only on* λ *and* Λ.

ii) for all $x\in(l,r)$ *and with* $\delta:=\min(r-x,x-l)$,

$$v_1(x)\ge C'\frac{\delta^4}{(r-l)^2}, \tag{4.12}$$

where C' *is a positive constant depending only on* λ *and* Λ.

Similar estimates hold for the solution of $\mathcal{P}_1([l,r],-)$.

Proof. In the proof of Proposition 4.1 we have seen in Step 2 that for all $x\in[l,r]$, $v_1(x)=\lim_{k\to\infty}v_1^k(x)$, with the v_1^k's defined in Step 1.

i) In this proof we have also seen that the v_1^k's satisfy (4.10). Passing to the limit we get $|v_1(x)|\le C(r-l)^2$ for all $x\in[l,r]$ and (4.11) is proven.

ii) Let be $x\in(l,r)$ fixed. As in the proof of Lemma 4.3 we have for each v_1^k the representation

$$\int_l^r G^k_{[l,r]}(x,y)v_0^k(y)\exp(\Psi^k(y))(\rho^k)^{-1}(y)dy,$$

with the adequate $G^k_{[l,r]}$ and Ψ^k. Thanks to (4.4), $\exp(\pm\Psi^k) \geq 1/\kappa$ for all $k \in \mathbb{N}$, and, for all $k \in \mathbb{N}$, and all $y \in [l,r]$,

$$\begin{gathered}
\exp(\Psi^k(y))(\rho^k)^{-1}(y) \geq \tfrac{1}{\Lambda\kappa}, \\
v_0^k(y) \geq \tfrac{\lambda}{\Lambda\kappa^2}.\tfrac{y-l}{r-l}, \\
G^k_{[l,r]}(x,y) \geq 2\tfrac{\lambda}{\Lambda^2\kappa^3}\tfrac{h}{r-l}(y-l) \ \text{ if } y \leq x, \\
\text{and} \quad G^k_{[l,r]}(x,y) \geq 2\tfrac{\lambda}{\Lambda^2\kappa^3}\tfrac{h}{r-l}(r-y) \ \text{ if } y \geq x.
\end{gathered} \tag{4.13}$$

We finally get $v_1^k(x) \geq (\lambda^2/\Lambda^4\kappa^6)(\delta^4/(r-l)^2)$ for all k, and passing to the limit we prove (4.12).

□

Remark 4.2 Note that the regularization arguments in the proofs of Propositions 4.1 and 4.2 allow to assert that, in the general case of measurable functions a, ρ and b, the Green function associated to the measure $\exp(\Psi(y))\rho^{-1}(y)dy$ for elliptic problems on $[l,r]$ involving $(L, D(L))$ is $G_{[l,r]}$ (with Ψ and $G_{[l,r]}$ like in the proof of Lemma 4.3).

Indeed for each f let be u the solution of $(\mathcal{P})$, and a sequence (u^k) constructed to approach u as in Step 1 in the proof of Proposition 4.1. For each k we have $u^k(x) = \int_l^r G^k_{[l,r]}(x,y)f(y)\exp(\Psi^k(y))(\rho^k)^{-1}(y)dy$. For each $x \in [l,r]$ using dominated convergence and passing to the limit we can identify $u(x)$ and $\int_l^r G_{[l,r]}(x,y)f(y)(y)\exp(\Psi(y))\rho^{-1}(y)dy$.

This could have been used to prove Proposition 4.2.

We finally assert an estimate concerning the second order moment.

Proposition 4.3 *Let be $a, \rho \in \mathfrak{Ell}(\lambda,\Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be the DFO $(L, D(L)) = \mathfrak{L}(a,\rho,b)$. Let be X the Markov process generated by $(L, D(L))$. Let be $[l,r] \subset G$. Let be $\tau = \inf\{t \geq 0 \,|\, X_t \in \{l,r\}\}$. For all $x \in [l,r]$,*

$$\mathbb{E}^x\big[\tau^2 \mathbf{1}_{\{X_\tau = r\}}\big] \leq K\,(r-l)^4, \tag{4.14}$$

where K is a positive constant depending on λ and Λ.

An estimate similar to (4.14) holds for $\mathbb{E}^x\big[\tau^2 \mathbf{1}_{\{X_\tau = l\}}\big]$.

Proof. As in the proof of Proposition 4.1, we can build a sequence (X^k) of processes generated by smooth coefficients, such that $(\tau^k)^2 \mathbf{1}_{\{X^k_{\tau^k} = r\}}$ converges in law to $\tau^2 \mathbf{1}_{\{X_\tau = r\}}$. But, as in Step 3 in the proof of Proposition 4.1, Lemmas 4.2 and 4.5, ensure that for all $x \in [l,r]$,

$$\forall k \in \mathbb{N}, \quad \mathbb{E}^x\big[(\tau^k)^2 \mathbf{1}_{\{X^k_{\tau^k} = r\}}\big] \leq K\,(r-l)^4,$$

with K depending on λ and Λ. So Theorem 5.3 in [Bil68] leads to,

$$\forall x \in [l,r], \quad \mathbb{E}^x\big[\tau^2 \mathbf{1}_{\{X_\tau = r\}}\big] \leq \liminf_k \mathbb{E}^x\big[(\tau^k)^2 \mathbf{1}_{\{X^k_{\tau^k} = r\}}\big] \leq K\,(r-l)^4.$$

□

5 The algorithm

Let be a, $\rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be X the process generated by $(L, D(L)) = \mathfrak{L}(a, \rho, b)$.

We know present the algorithm, that takes as input arguments (except a, ρ, b) the starting point x and the time t at which we want an approximation of X.

STEP 1. Construction of the grid and initialization.

We first give us a grid $\mathfrak{g} = \{x_j\}_{j \in J}$ on G, with $x_j \leq x_{j+1}$ for all $j \in J$.

We assume that $\mathfrak{g}$ has a minimum cell size h and a maximum cell size H. As h and H will tend to zero we assume that $H \leq 1$. This corresponds to the condition (4.1) in Section 4.

For such a grid $\mathfrak{g}$ we define $R := H/h$ and we call the couple (h, R) the *characteristic* of $\mathfrak{g}$.

We then initialize the algorithm by setting $\hat{t} \leftarrow 0$ and $S \leftarrow x_j$ where x_j is the point of $\mathfrak{g}$ that is the closest to x.

STEP 2. Computation of the transition probabilities and the conditional expectations of exit times.

Let be $(\tau_p^{\mathfrak{g}})_{p \in \mathbb{N}}$ the sequence of random times defined by,

$$\begin{aligned} &\tau_0^{\mathfrak{g}} = 0, \\ &\tau_p^{\mathfrak{g}} = \inf\big\{t \geq \tau_{p-1}^{\mathfrak{g}} \mid X_t \in \mathfrak{g} \setminus \{X_{\tau_{p-1}^{\mathfrak{g}}}\}\big\}. \end{aligned} \tag{5.1}$$

The second step consists in computing for each $x_j \in \mathfrak{g}$:

i) the transition probabilities

$$\pi(j, j+1) := \mathbb{P}\big[X_{\tau_p^{\mathfrak{g}}} = x_{j+1} \mid X_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big],$$

ii) the conditionnal expectations

$$\begin{aligned} T(j,+) &:= \mathbb{E}[\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}} \mid X_{p-1} = x_j; X_p = x_{j+1}] \\ &= \mathbb{E}[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X_{\tau_p^{\mathfrak{g}}} = x_{j+1}\}} | X_{\tau_{p-1}^{\mathfrak{g}}} = x_j]/\pi(j, j+1), \end{aligned}$$

and

$$\begin{aligned} T(j,-) &:= \mathbb{E}[\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}} \mid X_{p-1} = x_j; X_p = x_{j-1}] \\ &= \mathbb{E}[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X_{\tau_p^{\mathfrak{g}}} = x_{j-1}\}} | X_{\tau_{p-1}^{\mathfrak{g}}} = x_j]/(1 - \pi(j, j+1)). \end{aligned}$$

This can be done thanks to the following proposition.

Proposition 5.1 *Let be $x_j \in \mathfrak{g}$. We have for all $p \in \mathbb{N}^*$:*

i)

$$\mathbb{P}\big[X_{\tau_p^{\mathfrak{g}}} = x_{j+1} \mid X_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big] \;=\; \frac{\int_{x_{j-1}}^{x_j} e^{-\Psi(y)}/a(y)dy}{\int_{x_{j-1}}^{x_{j+1}} e^{-\Psi(y)}/a(y)dy},$$

where,

$$\Psi(x) = 2\int_{x_{j-1}}^{x} \frac{b(y)}{a(y)\rho(y)}dy, \quad \forall x \in [x_{j-1}, x_{j+1}].$$

ii)

$$\mathbb{E}\big[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X_{\tau_p^{\mathfrak{g}}}=x_{j+1}\}} | X_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big] = v_1(x_i),$$

where v_1 is the solution of $\mathcal{P}_1([x_{j-1}, x_{j+1}], +)$ as defined in Proposition 4.1, and

$$\mathbb{E}\big[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X_{\tau_p^{\mathfrak{g}}}=x_{j-1}\}} | X_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big] = v_1(x_i),$$

where v_1 is the solution of $\mathcal{P}_1([x_{j-1}, x_{j+1}], -)$ as defined in Proposition 4.1.

Proof. By the strong Markov property we have for all $p \in \mathbb{N}^*$,

$$\mathbb{P}[X_{\tau_p^{\mathfrak{g}}} = x_{j+1} \,|\, X_{\tau_{p-1}^{\mathfrak{g}}} = x_j] = \mathbb{P}^{x_j}[X_{\tau_1^{\mathfrak{g}}} = x_{j+1}]$$

and

$$\mathbb{E}[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X_{\tau_p^{\mathfrak{g}}}=x_{j+1}\}} | X_{\tau_{p-1}^{\mathfrak{g}}} = x_j] = \mathbb{E}^{x_j}[\tau_1^{\mathfrak{g}}\mathbf{1}_{\{X_{\tau_1^{\mathfrak{g}}}=x_{j+1}\}}].$$

Point i) is a consequence of Proposition 2.1 and 2.2 and Definition 2.1. We apply Proposition 4.1 to get ii).

□

Remark 5.1 Note that $\mathbb{P}[X_{\tau_p^{\mathfrak{g}}} = x_{j+1} \,|\, X_{\tau_{p-1}^{\mathfrak{g}}} = x_j] = v_0(x_j)$ where v_0 solves $\mathcal{P}_0([x_{j-1}, x_{j+1}], +)$.

It is clear that if we define $(S_p^{\mathfrak{g}})_{p\in\mathbb{N}} = (X_{\tau_p^{\mathfrak{g}}})_{p\in\mathbb{N}}$ we get a random walk on the grid $\mathfrak{g}$ with transition probabilities $\mathbb{P}[S_p^{\mathfrak{g}} = x_{j+1} \,|\, S_{p-1}^{\mathfrak{g}} = x_j] = \pi(j, j+1)$.

STEP 3. Simulation of the approximating process.

We compute an approximated value of X_t, denoted by $\widehat{X}_t^{\mathfrak{g}}$, in the following manner:

Main loop. We have $S = x_j$ for some $x_j \in \mathfrak{g}$.
Simulate $Ber(\pi(j, j+1))$.

If success occurs
Then set $S \leftarrow x_{j+1}$ and $\hat{t} \leftarrow \hat{t} + T(j, +)$.
Else
Then set $S \leftarrow x_{j-1}$ and $\hat{t} \leftarrow \hat{t} + T(j, -)$.

If $\hat{t} < t$
Then go on in main loop.
Else $\hat{t} \geq t$
Then stop and **return** $\widehat{X}_t^{\mathfrak{g}} = S$.

In the next section we prove that this algorithm converges and evaluate at which speed.

6 Speed of convergence of the algorithm

In this section we will prove the following theorem, that gives us the rate of strong convergence of our scheme.

Theorem 6.1 *Let be a, $\rho \in \mathfrak{Ell}(\lambda, \Lambda)$ and $b \in \mathfrak{B}(\Lambda)$. Let be X the process generated by $(L, D(L)) = \mathfrak{L}(a, \rho, b)$. Let be $0 < T < \infty$.*

For any grid $\mathfrak{g}$ on G, let be $(\tau_p^{\mathfrak{g}})$ the sequence of stopping times described in Section 5 at Step 2. Let also be

$$K^{\mathfrak{g}}(t) = \inf\big\{k \in \mathbb{N} \,|\, \sum_{p=1}^{k} \mathbb{E}[\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}} \,|\, X_{\tau_{p-1}^{\mathfrak{g}}}, X_{\tau_p^{\mathfrak{g}}}] \geq t\big\}.$$

We have that $X_{\tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}}$ has the same law as the approximating process $\widehat{X}_t^{\mathfrak{g}}$ described in Section 5.

Besides for all $\gamma \in (0, 1/2)$ there exists a constant K_0 depending on γ, T, λ and Λ such that for any grid $\mathfrak{g}$ on G of characteristic (h, R),

$$\forall t \in [0,T],\, \forall x \in G \;\; \mathbb{E}^x\big|X_{\tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}} - X_t\big| \leq \; K_0 C_0(R)\, h^{\gamma},$$

where $C_0(.)$ is a positive and increasing function of R.

To prove Theorem 6.1 we need a few Lemmas. For all grid $\mathfrak{g}$ we call $S^{\mathfrak{g}}$ the random walk on $\mathfrak{g}$ defined by $S_p^{\mathfrak{g}} = X_{\tau_p^{\mathfrak{g}}}$ for all $p \in \mathbb{N}$.

Lemma 6.1 *For all $\gamma \in (0, 1/2)$ and all $1 < p < 4/3$ there exists a constant C_p^{γ} depending on γ, p, λ, Λ and T, such that,*

$$\mathbb{E}\Big(\sup_{s \neq t,\; s,t \in [0,T]} \frac{|X_t - X_s|}{|t-s|^{\gamma}}\Big)^p \;\leq\; C_p^{\gamma} < \infty.$$

To prove this lemma we will use Lemma 2.2 and the technique used by one of the authors in [Lej05] to prove his Lemma 3.

Proof. *Step 1.* We prove that for any $\alpha > 2$ there is a constant B depending on λ, Λ and T such that for all $0 \leq s \leq t \leq T$,

$$\sup_{x \in G} \mathbb{E}^x\big[\sup_{r \in [s,t]} |X_r - X_s|^{\alpha}\big] \leq B|t-s|^{\alpha/2}.$$

With the Markov property,

$$\sup_{x \in G} \mathbb{E}^x\big[\sup_{r \in [s,t]} |X_r - X_s|^{\alpha}\big] = \sup_{x \in G} \mathbb{E}^x\big[\sup_{r \in [0,t-s]} |X_r - x|^{\alpha}\big].$$

Let be $\alpha > 2$ and $t \in [0,T]$ fixed. For $x \in G$ and $d > 0$ let be the exit time $\tau = \inf\{r \geq 0 | X_r \notin (x-d, x+d)\}$. Using $(a+b)^{\alpha} \leq 2^{\alpha-1}(a^{\alpha} + b^{\alpha})$ and the strong Markov property we get,

$$\sup_{x\in G}\mathbb{E}^x[\sup_{r\in[0,t]}|X_r-x|^\alpha]\le d^\alpha+\sup_{x\in G}\mathbb{E}^x[\sup_{r\in[0,t]}|X_r-x|^\alpha\mathbf{1}_{\{t>\tau\}}]$$

$$\le d^\alpha+\sup_{x\in G}\mathbb{E}^x[\sup_{r\in[\tau,t]}|X_r-x|^\alpha\mathbf{1}_{\{t>\tau\}}]$$

$$\le d^\alpha+2^{\alpha-1}(d^\alpha+\sup_{x\in G}\mathbb{E}^x[\sup_{r\in[\tau,t]}|X_r-X_\tau|^\alpha\mathbf{1}_{\{t>\tau\}}]$$

$$\le(2^{\alpha-1}+1)d^\alpha+2^{\alpha-1}\sup_{x\in G}\mathbb{E}^x[\sup_{r\in[0,t]}|X_r-x|^\alpha]\sup_{x\in G}\mathbb{P}^x[t>\tau].$$

Thanks to Lemma 2.2 we have $\sup_{x\in G}\mathbb{P}^x[t>\tau]\le A\exp(-Ad^2/t)$. Hence Step 1 is proven if we take d equal to $\mu\sqrt{t}$ with μ large enough so that we have $2^{\alpha-1}A\exp(-A\mu^2)\le 1/2$, and if we finally replace t by $t-s$.

Step 2. Let be $\gamma\in(0,1/2)$. There exists $\alpha>2$ such that $0<\gamma<1/2-1/\alpha$. Combining Step 1 and the Kolmogorov-Čentsov theorem (see [RY91]),

$$\mathbb{E}\Big(\sup_{s\neq t,\ s,t\in[0,T]}\frac{|X_t-X_s|}{|t-s|^\gamma}\Big)^\alpha\ \le\ C_\alpha^\gamma<\infty,$$

with C_α^γ depending on γ, α, λ, Λ and T.

As $\alpha>2$ it then suffices to use the Jensen inequality to complete the proof. □

Lemma 6.2 *In the context of Theorem 6.1, the $(\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}})$'s are independent conditionally to the paths of $S^{\mathfrak{g}}$.*

Proof. Tedious but straightforward. □

Lemma 6.3 *There are constants M, M' and M'' depending on λ and Λ such that, for any grid $\mathfrak{g}$ on G of characteristic (h,R), in the context of Theorem 6.1,*

i) $\forall p\in\mathbb{N},\ \forall x_j\in\mathfrak{g},\ \ \dfrac{M'}{R^2}\,h^2\le\mathbb{E}[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}|X^{\mathfrak{g}}_{\tau_{p-1}}=x_j;X^{\mathfrak{g}}_{\tau_p^{\mathfrak{g}}}=x_{j\pm1}]\le M\,R^3\,h^2.$

ii) $\forall p\in\mathbb{N},\ \forall x_j\in\mathfrak{g},\ \ \mathbb{V}ar[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}|X^{\mathfrak{g}}_{\tau_{p-1}}=x_j;X^{\mathfrak{g}}_{\tau_p^{\mathfrak{g}}}=x_{j\pm1}]\le M''\,R^4\,h^4.$

Proof. Remember that for all $x_j\in\mathfrak{g}$, $h\le x_{j+1}-x_j\le Rh$. To prove Point i) it then suffices to use Propositions 4.2 and 5.1, Remark 5.1 and the second line of (4.13). Point ii) is a direct consequence of (4.14) in Proposition 4.3. □

Lemma 6.4 *In the context of Theorem 6.1 there exists a constant K depending on T, λ and Λ such that for any grid $\mathfrak{g}$ of characteristic (h,R),*

$$\mathbb{E}|\tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}-t|^2\ \le\ KC_1(R)\,h^2,$$

where $C_1(.)$ is a positive and increasing function of R.

Proof. Let be $\mathfrak{g}$ a grid of characteristic (h, R). Using $(a+b)^2 \leq 2(a^2+b^2)$ we have,

$$\begin{aligned}\mathbb{E}|\tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)} - t|^2 \quad &\leq \quad 2\,\mathbb{E}\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}] - t\Big|^2 \\ &\qquad + 2\,\mathbb{E}\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}] - \tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}\Big|^2. \end{aligned} \tag{6.1}$$

By definition of $K^{\mathfrak{g}}(t)$ and the i) of Lemma 6.3 it is obvious that

$$\begin{aligned}&\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}] - t\Big| \\ \leq \quad &\max_{x_j \in \mathfrak{g}} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}} = x_j, X_{\tau^{\mathfrak{g}}_p} = x_{j\pm 1}] \\ \leq \quad &MR^3h^2 \quad \text{a.s.},\end{aligned}$$

and we thus get an estimate of the first term on the right in (6.1):

$$\mathbb{E}\Big|\sum_{p=1}^{K^{\mathfrak{g}}(t)} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}] - t\Big|^2 \;\leq\; M^2R^6h^4. \tag{6.2}$$

We will now estimate the second term. We have,

$$\begin{aligned}&\mathbb{E}\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}] - \tau^{\mathfrak{g}}_{K^{\mathfrak{g}}(t)}\Big|^2 \\ = \quad &\mathbb{E}\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \big(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} - \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}]\big)\Big|^2.\end{aligned} \tag{6.3}$$

For some $k \in \mathbb{N}$ there are paths of $S^{\mathfrak{g}}$ such that $K^{\mathfrak{g}}(t) = k$ and $\mathbb{P}[\text{path}] \neq 0$. So we can write

$$\begin{aligned}&\mathbb{E}\Big|\textstyle\sum_{p=1}^{K^{\mathfrak{g}}(t)} \big(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} - \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}]\big)\Big|^2 \\ = \textstyle\sum_{k\in\mathbb{N}} &\mathbb{E}\Big[\,\mathbf{1}_{\{K^{\mathfrak{g}}(t)=k\}}\Big|\textstyle\sum_{p=1}^{k} \big(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} - \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}]\big)\Big|^2\Big].\end{aligned} \tag{6.4}$$

But for each k,

$$\begin{aligned}&\mathbb{E}\Big[\,\mathbf{1}_{\{K^{\mathfrak{g}}(t)=k\}}\Big|\textstyle\sum_{p=1}^{k} \big(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} - \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}]\big)\Big|^2\Big] \\ = \textstyle\sum_{\text{paths s.t. } K^{\mathfrak{g}}(t)=k} &\mathbb{E}\Big[\,\mathbf{1}_{\{\text{path}\}}\big|\textstyle\sum_{p=1}^{k} \big(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} - \mathbb{E}[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1} \,|\, X_{\tau^{\mathfrak{g}}_{p-1}}, X_{\tau^{\mathfrak{g}}_p}]\big)\big|^2\Big] \\ = \textstyle\sum_{\text{paths s.t. } K^{\mathfrak{g}}(t)=k} &\mathbb{E}\Big[\,\mathbf{1}_{\{\text{path}\}}\mathbb{V}ar\big[\,\textstyle\sum_{p=1}^{k}(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1}) \,|\, \text{path}\,\big](\text{path})\Big].\end{aligned}$$

We use now Lemma 6.2, which allows us to assert that $\mathbb{V}ar[\sum_{p=1}^{k}(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1}) \,|\, \text{path}\,] = \sum_{p=1}^{k} \mathbb{V}ar[\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1}|\text{path}]$, and the ii) of Lemma 6.3. Thus,

$$\mathbb{E}\Big[\,\mathbf{1}_{\{K^{\mathfrak{g}}(t)=k\}}\big|\sum_{p=1}^{k}\big(\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}-\mathbb{E}[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}\,|\,X_{\tau_{p-1}^{\mathfrak{g}}},X_{\tau_p^{\mathfrak{g}}}]\big)\big|^2\Big]$$

$$\leq\ \sum_{\text{paths s.t. }K^{\mathfrak{g}}(t)=k} k\,M''R^4h^4\mathbb{E}[\,\mathbf{1}_{\{\text{path}\}}\,] \tag{6.5}$$

$$=\ M''R^4h^4\ k\,\mathbb{P}[K^{\mathfrak{g}}(t)=k].$$

Taking in account (6.3), (6.4) and (6.5) we then have,

$$\mathbb{E}\Big|\sum_{p=1}^{K^{\mathfrak{g}}(t)}\mathbb{E}[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}\,|\,X_{\tau_{p-1}^{\mathfrak{g}}},X_{\tau_p^{\mathfrak{g}}}]-\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}\Big|^2\leq M''R^4h^4\,\mathbb{E}[\,K^{\mathfrak{g}}(t)\,]. \tag{6.6}$$

It remains to evaluate $\mathbb{E}[\,K^{\mathfrak{g}}(t)\,]$. But, by definition of $K^{\mathfrak{g}}(t)$ and the i) of Lemma 6.3, for all $t\in[0,T]$

$$\begin{aligned}K^{\mathfrak{g}}(t)\ &\leq\ \Big\lfloor\frac{t}{\min\limits_{x_j\in\mathfrak{g}}\mathbb{E}[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}|X_{\tau_{p-1}^{\mathfrak{g}}}=x_j,X_{\tau_p^{\mathfrak{g}}}=x_{j\pm1}]}\Big\rfloor+1\\ &\leq\ \frac{T}{M'}\frac{R^2}{h^2}+1\quad\text{a.s.}\end{aligned}$$

Using this and (6.6) we get

$$\mathbb{E}\Big|\sum_{p=1}^{K^{\mathfrak{g}}(t)}\mathbb{E}[\tau_p^{\mathfrak{g}}-\tau_{p-1}^{\mathfrak{g}}\,|\,X_{\tau_{p-1}^{\mathfrak{g}}},X_{\tau_p^{\mathfrak{g}}}]-\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}\Big|^2\leq\frac{TM''}{M'}R^6h^2+M''R^4h^4. \tag{6.7}$$

Taking in account (6.2) and (6.7) and the fact that $h\leq1$ (see (4.1) and Step 1 in Section 5) we complete the proof. □

Proof of Theorem 6.1. That $X_{\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}}$ has the same law as $\widehat{X}_t^{\mathfrak{g}}$ is clear.

Let be $\gamma\in(0,1/2)$ and $q=2/\gamma$. Let be p the conjugate of q. We have $1<p<4/3$. By the Hölder inequality we can write for any grid $\mathfrak{g}$ of characteristic (h,R), any $t\in[0,T]$,

$$\mathbb{E}\big|X_{\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}}-X_t\big|\leq\Big(\mathbb{E}\big[\sup_{s\neq t}\frac{|X_t-X_s|}{|t-s|^\gamma}\big]^p\Big)^{1/p}\cdot\Big(\mathbb{E}|\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}-t|^2\Big)^{1/q},$$

and by Lemma 6.1 and Lemma 6.4 we get

$$\mathbb{E}\big|X_{\tau_{K^{\mathfrak{g}}(t)}^{\mathfrak{g}}}-X_t\big|\leq C(\gamma)\,(KC_1(R)\,h^2)^{1/q}=C(\gamma)\,K^{\gamma/2}\big(C_1(R)\big)^{\gamma/2}\,h^\gamma,$$

where the constant $C(\gamma)\,K^{\gamma/2}$ depends on γ, T, λ and Λ but not on the grid. The proof is completed. □

7 Simulations

One can wonder how to practically make the computations in STEP 2 of the algorithm. In fact we can first construct piecewise approximations $\widetilde{a}$, $\widetilde{\rho}$, $\widetilde{b}$ of the given coefficients a, ρ and b. We can then approach the process generated by $\mathfrak{L}(\widetilde{a},\widetilde{\rho},\widetilde{b})$, by another one for which the computations in STEP 2 are easy to make.

7.1 The choice of the piecewise constant coefficients

In many practical situations G is bounded and the coefficients a, ρ and b can actually be considered as being r.c.l.l., and of class $\mathcal{C}^1$ except on a set $\mathcal{I}$ of points of finite cardinal.

Then we can choose the grid $\mathfrak{g}$ of characteristic (h,R) such that $\mathcal{I}\subset\mathfrak{g}$ and define r.c.l.l. piecewise constant coefficients $\widetilde{a}$, $\widetilde{\rho}$ and $\widetilde{b}$, by $\widetilde{a}(x)=a(x)$, $\widetilde{\rho}(x)=\rho(x)$ and $\widetilde{b}(x)=b(x)$ for all $x\in\mathfrak{g}$.

Using the mean value theorem it is then easy to show that the coefficients $\widetilde{a}$, $\widetilde{\rho}$ and $\widetilde{b}$ tend pointwise respectively to a, ρ and b as h tends to zero. By dominated convergence and Theorem 3.2, the process generated by $\mathfrak{L}(\widetilde{a},\widetilde{\rho},\widetilde{b})$ then converges in law to the one generated by $\mathfrak{L}(a,\rho,b)$ (using a similar technique to the one in the proof of Proposition 7.2 below, we could even show the weak error introduced by this approximation is of order $\mathcal{O}(h)$, that is less than the one of our scheme).

On the other hand, our algorithm also allows us to perform some change of scale, and then to catch directly some small scale behavior of the process, by considering that the grid is constructed at a mesoscope scale. The homogenization theory can then be applied to deal with the local behavior of the coefficients. The article [OZ05] perform the construction of a Markov chain that approximate a diffusion in dimension 2 or 3 in presence of locally irregular coefficients, but they need to solve a finite volume problem. In our simpler case, we construct the random walk by taking into account only local informations.

To be more precise, in some practical situation, one can have very irregular but locally periodic coefficients, with rapid oscillations. In this situation the theory of G-convergence can help us the get homogenized coefficients that will be piecewise constant (see [ZKOT79], see also [BLP78] on homogenization).

To fix ideas let be $\mathcal{I}=\{x_i\}_{i\in I}$ a set of points in G and let us consider $\hat{a}\in\mathfrak{Ell}(\lambda,\Lambda)$ that is periodic on each interval $[x_i,x_{i+1}]$. Let be the coefficient a defined by $a(x)=\hat{a}(x/\varepsilon_i)$, $\forall x\in[x_i,x_{i+1}]$, with a very small coefficient ε_i on each interval $[x_i,x_{i+1}]$. The coefficient a is rapidly oscillating. Suppose we want to simulate the paths of the process X generated by $\mathfrak{L}(a,1,0)$. It can be shown that, on each $[x_i,x_{i+1}]$,

$$\frac{1}{\hat{a}(./\varepsilon_i)}\xrightarrow[\varepsilon_i\downarrow 0]{\mathrm{L}^2}\int_{x_i}^{x_{i+1}}\frac{dy}{\hat{a}(y)}.$$

So we define a coefficient $\widetilde{a}$, constant on each $[x_i,x_{i+1})$, by,

$$\widetilde{a}(x_i)=\Big(\int_{x_i}^{x_{i+1}}\frac{dy}{\hat{a}(y)}\Big)^{-1},$$

in order that, on each $[x_i,x_{i+1}]$, $1/\hat{a}(./\varepsilon_i)$ converges weakly in L^2 to $(1/\widetilde{a})$.

As the ε_i are very small, by theorem 3.2 we can consider that the process $\widetilde{X}$ generated by $\mathfrak{L}(\widetilde{a}, 1, 0)$ is a good weak approximation of X. In the same manner we could homogenize the coefficients ρ and b, if needed.

7.2 Approximation of the process generated by piecewise constant coefficients

Assume that G is bounded. Let be $\mathfrak{g}$ a grid on G of characteristic (h, R). Let be $\widetilde{a}, \widetilde{\rho} \in \mathfrak{Ell}(\lambda, \Lambda)$ and $\widetilde{b} \in \mathfrak{B}(\Lambda)$ that are right continuous with left limit, and are constant on each cell $[x_j, x_{j+1})$ of $\mathfrak{g}$. Let be $\widetilde{X}$ the process generated by $\mathfrak{L}(\widetilde{a}, \widetilde{\rho}, \widetilde{b})$. Let be $x_0 \in G$ and

$$\Psi(x) = \int_{x_0}^{x} \frac{\widetilde{b}(y)}{\widetilde{a}(y)\widetilde{\rho}(y)} dy, \quad \forall x \in G.$$

Let be $\Psi^{\mathfrak{g}}$ right continuous with left limit, constant on each cell $[x_j, x_{j+1})$ of $\mathfrak{g}$, and defined by $\Psi^{\mathfrak{g}}(x_j) = \Psi(x_j)$ for all $x_j \in \mathfrak{g}$.

Let be $X^{\mathfrak{g}}$ the process generated by $\mathfrak{L}(\widetilde{a}e^{\Psi^{\mathfrak{g}}}, \widetilde{\rho}e^{-\Psi^{\mathfrak{g}}}, 0)$. In this context we have the two following propositions.

Proposition 7.1 *Let be $x_j \in \mathfrak{g}$. Let be $l = x_{j+1} - x_j$, $l' = x_j - x_{j-1}$ and $c = \exp\big(2\widetilde{b}(x_{j-1})l'/[\widetilde{a}(x_{j-1})\widetilde{\rho}(x_{j-1})]\big)$. For all $p \in \mathbb{N}^*$:*

i)

$$\mathbb{P}\big[X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_p} = x_{j+1} \,|\, X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_{p-1}} = x_j\big] \;=\; \frac{\widetilde{a}(x_j)l'}{\widetilde{a}(x_j)l' + \widetilde{a}(x_{j-1})l/c}.$$

ii)

$$\mathbb{E}\big[(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1})\mathbf{1}_{\{X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_p} = x_{j+1}\}} | X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_{p-1}} = x_j\big]$$

$$= \frac{1}{(\widetilde{a}(x_{j-1})l + \widetilde{a}(x_j)l'c)^2}\Big[\tfrac{1}{3}\tfrac{\widetilde{a}(x_{j-1})}{\widetilde{\rho}(x_j)} c\, l^3 l' + \tfrac{\widetilde{a}(x_j)}{\widetilde{\rho}(x_j)} c^2\, l^2 (l')^2 + \tfrac{2}{3}\tfrac{\widetilde{a}(x_j)}{\widetilde{\rho}(x_{j-1})} c\, l (l')^3\Big],$$

and

$$\mathbb{E}\big[(\tau^{\mathfrak{g}}_p - \tau^{\mathfrak{g}}_{p-1})\mathbf{1}_{\{X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_p} = x_{j-1}\}} | X^{\mathfrak{g}}_{\tau^{\mathfrak{g}}_{p-1}} = x_j\big]$$

$$= \frac{1}{(\widetilde{a}(x_{j-1})l + \widetilde{a}(x_j)l'c)^2}\Big[\tfrac{2}{3}\tfrac{\widetilde{a}(x_j)}{\widetilde{\rho}(x_j)} c\, l^3 l' + \tfrac{\widetilde{a}(x_{j-1})}{\widetilde{\rho}(x_{j-1})}\, l^2 (l')^2 + \tfrac{1}{3}\tfrac{\widetilde{a}(x_j)}{\widetilde{\rho}(x_{j-1})} c\, l (l')^3\Big],$$

Proof. To prove the point i) it suffices to compute

$$\frac{s(x_j) - s(x_{j-1})}{s(x_{j+1}) - s(x_{j-1})},$$

with the scale function s defined by (2.4) with a replaced by $\widetilde{a}e^{\Psi^{\mathfrak{g}}}$.

To prove the point ii) let us consider the process $\bar{X}$ generated by $(\bar{L}, D(\bar{L})) := \mathfrak{L}(\bar{a}, \bar{\rho}, 0)$ with

$$\bar{a}(x) = \begin{cases} \widetilde{a}(x_{j-1})e^{\Psi^{\mathfrak{g}}(x_{j-1})} & \text{if } x < 0 \\ \\ \widetilde{a}(x_j)e^{\Psi^{\mathfrak{g}}(x_j)} & \text{if } x \geq 0, \end{cases}$$

and

$$\bar{\rho}(x) = \begin{cases} \widetilde{\rho}(x_{j-1})e^{-\Psi^{\mathfrak{g}}(x_{j-1})} & \text{if } x < 0 \\ \\ \widetilde{\rho}(x_j)e^{-\psi^{\mathfrak{g}}(x_j)} & \text{if } x \geq 0, \end{cases}$$

and set $\tau = \inf\{t \geq 0 \,|\, \bar{X}_t \in \{-l', l\}\}$. By the definitions of $X^{\mathfrak{g}}$ and $\bar{X}$ and the strong Markov property it is clear that,

$$\mathbb{E}^0\big[\,\tau \mathbf{1}_{\{X_\tau = l\}}\,\big] = \mathbb{E}\big[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X^{\mathfrak{g}}_{\tau_p^{\mathfrak{g}}} = x_{j+1}\}} | X^{\mathfrak{g}}_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big].$$

But proposition 4.1 says that $\mathbb{E}^0[\,\tau \mathbf{1}_{\{X_\tau = l\}}\,] = v_1(0)$ where v_1 is a function in $D(\bar{L}) \cap \mathcal{C}$ that solves:

$$\begin{cases} \frac{\widetilde{\rho}(x_{j-1})\widetilde{a}(x_{j-1})}{2} v_1''(x) = -v_0(x) \quad \text{for } x \in [-l', 0) \\ \\ \frac{\widetilde{\rho}(x_j)\widetilde{a}(x_j)}{2} v_1''(x) = -v_0(x) \quad \text{for } x \in [0, l] \\ \\ v_1(0-) = v_1(0+) \\ \\ \widetilde{a}(x_{j-1})e^{\Psi^{\mathfrak{g}}(x_{j-1})} v_1'(0-) = \widetilde{a}(x_j)e^{\Psi^{\mathfrak{g}}(x_j)} v_1'(0+) \\ \\ v_1'(-l') = v_1'(l) = 0, \end{cases}$$

where v_0 is itself in $D(\bar{L}) \cap \mathcal{C}$ and solves

$$\begin{cases} \frac{\widetilde{\rho}(x_{j-1})\widetilde{a}(x_{j-1})}{2} v_0''(x) = 0 \quad \text{for } x \in [-l', 0) \\ \\ \frac{\widetilde{\rho}(x_j)\widetilde{a}(x_j)}{2} v_0''(x) = 0 \quad \text{for } x \in [0, l] \\ \\ v_0(0-) = v_0(0+) \\ \\ \widetilde{a}(x_{j-1})e^{\Psi^{\mathfrak{g}}(x_{j-1})} v_0'(0-) = \widetilde{a}(x_j)e^{\Psi^{\mathfrak{g}}(x_j)} v_0'(0+) \\ \\ v_0'(-l') = 0 \\ \\ v_0'(l) = 1. \end{cases}$$

Note that in both systems the third line traduces the continuity of the solution, while the fourth line indicates that it belongs to $D(\bar{L})$.

So we compute first v_0 and get a function of the form $Ax + B$ on $[-l', 0)$ and $Bx + C$ on $[0, l]$. Then we compute v_1, and finaly $v_1(0)$ that gives us $\mathbb{E}\big[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X^{\mathfrak{g}}_{\tau_p^{\mathfrak{g}}} = x_{j+1}\}} | X^{\mathfrak{g}}_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big]$.

We proceed in the same manner for computing

$$\mathbb{E}\big[(\tau_p^{\mathfrak{g}} - \tau_{p-1}^{\mathfrak{g}})\mathbf{1}_{\{X^{\mathfrak{g}}_{\tau_p^{\mathfrak{g}}} = x_{j-1}\}} | X^{\mathfrak{g}}_{\tau_{p-1}^{\mathfrak{g}}} = x_j\big].$$

□

Proposition 7.2 *Let be f in $\mathrm{H}_0^1(G) \cap \mathcal{C}_0(G)$. There exists a constant C depending on T, λ, Λ, G, $\|f\|_\infty$ and $\|df/dx\|_2$, such that,*

$$\sup_{(t,x)\in[0,T]\times G} |\mathbb{E}^x f(\widetilde{X}_t) - \mathbb{E}^x f(X_t^{\mathfrak{g}})| \leq C\, Rh.$$

Proof. We can consider that $\widetilde{X}$ is generated by $\mathfrak{L}(\widetilde{a}e^{\Psi}, \widetilde{\rho}e^{-\Psi}, 0)$. The functions $\mathbb{E}^x f(\widetilde{X}_t)$ and $\mathbb{E}^x f(X_t^{\mathfrak{g}})$ are continuous versions of the solutions to parabolic PDEs with initial condition f involving respectively the operators $\mathfrak{L}(\widetilde{a}e^{\Psi}, \widetilde{\rho}e^{-\Psi}, 0)$ and $\mathfrak{L}(\widetilde{a}e^{\Psi^{\mathfrak{g}}}, \widetilde{\rho}e^{-\Psi^{\mathfrak{g}}}, 0)$. As the points of discontinuity of $\widetilde{a}e^{\Psi}$ and $\widetilde{\rho}e^{-\Psi}$ are included in those of $\widetilde{a}e^{\Psi^{\mathfrak{g}}}$ and $\widetilde{\rho}e^{-\Psi}$, it can be shown (see Proposition 6.2 in [ETO06] for instance), that there is a constant $\tilde{C}$ depending on T, λ, Λ, G, $\|f\|_\infty$ and $\|df/dx\|_2$, such that,

$$\sup_{(t,x)\in[0,T]\times G} |\mathbb{E}^x f(\widetilde{X}_t) - \mathbb{E}^x f(X_t^{\mathfrak{g}})| \leq \tilde{C}\Big(\left\|\widetilde{a}e^{\Psi} - \widetilde{a}e^{\Psi^{\mathfrak{g}}}\right\|_\infty^2 + \left\|\widetilde{\rho}e^{-\Psi} - \widetilde{\rho}e^{-\Psi^{\mathfrak{g}}}\right\|_\infty \Big).$$

A simple application of the mean value theorem on each cell of the grid then leads to the desired result.

□

Proposition 7.1 means that it is easy to apply our scheme to $X^{\mathfrak{g}}$ because we have for this process explicit expressions of the quantities involved in Step 2 of the algorithm.

Proposition 7.2 means that the weak error we make by approximating $\widetilde{X}$ by $X^{\mathfrak{g}}$ is less than the approximation error of the scheme.

We used this approach in the numerical experiments presented in the next subsection.

7.3 Numerical experiments

Example 1. We take $b = 0$ and a and ρ to be

$$a(x) = \begin{cases} 1 & \text{if } x < -1/2 \\ 2 & \text{if } x \in [-1/2, 1/2) \\ 1 & \text{if } x \geq 1/2, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} 1 & \text{if } x < -1/2 \\ 1/2 & \text{if } x \in [-1/2, 1/2) \\ 1 & \text{if } x \geq 1/2. \end{cases}$$

We take a domain $G = [-5, 5]$ and a grid $\mathfrak{g} = \{k0.02 \in G | k \in \mathbb{Z}\}$. We consider the process X generated by $\mathfrak{L}(a, \rho, 0)$. As $a\rho = 1$ everywhere, the average time spent by X on a cell of $\mathfrak{g}$, knowing it has gone right or left, is the same as for the standard Brownian motion, that is to say 0.02^2.

The jumps of a in $-1/2$ and $1/2$ make that, heuristically, the process has a probability $2/3$ to go on the right in $-1/2$, while in $1/2$ this probability is $1/3$. Where the coefficient a is flat, the process X behaves like the Brownian motion and has a probability $1/2$ to go on the right.

Remark 7.1 We could describe the behavior of X in terms of SDELT. Indeed X solves,

$$X_t = X_0 + W_t + \frac{1}{3}L_t^{-1/2}(X) - \frac{1}{3}L_t^{1/2}(X),$$

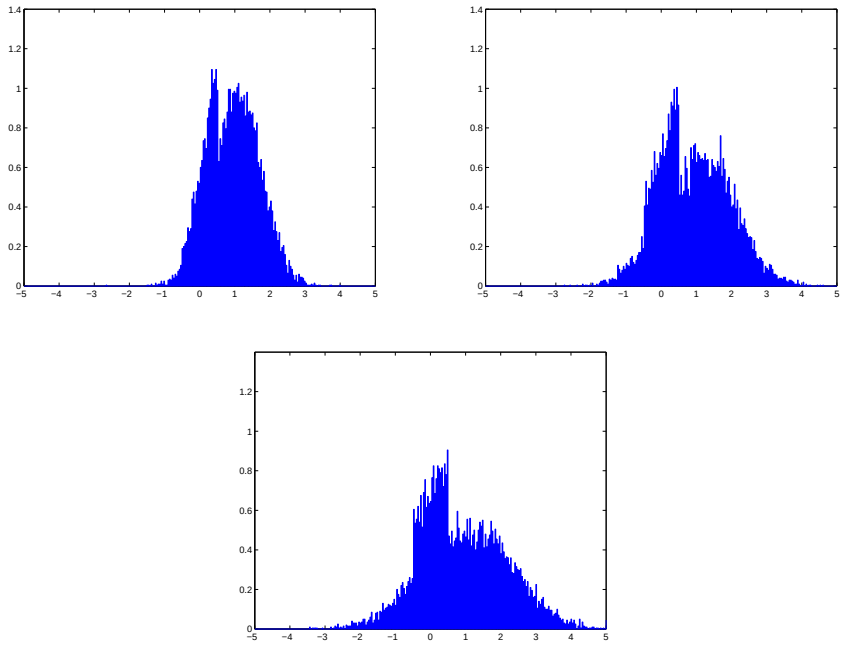


Figure 2: Approximation of $p(t, x, y)$ for $x = 1.0$ at times $t = 0.5$, $t = 1.0$ and $t = 1.5$ (with N=10000 particules).

where $L_t^x(X)$ is the symmetric local time of X in x (see [LM06]). We see that X behaves like the Brownian motion W except in $-1/2$ and $1/2$. Such a process X is called Doubly Skew Brownian motion.

We simulate 10000 paths of $\widehat{X}^{\mathfrak{g}}$ starting from $x = 1.0$ at times $t = 0.5$, $t = 1.0$ and $t = 1.5$. Compared to a gaussian density centered in $x = 1.0$ the histograms we get show a concentration of particles in the interval $[-1/2, 1/2]$ (Figure 2).

Example 2. We take $G = [-5, 5]$ and the grid $\mathfrak{g} = \{k0.05 \in G | k \in \mathbb{Z}\}$. We simulate 10000 paths of $\widehat{X}^g$ starting from $x = 0.0$ and plot an histogram of the positions at time $t = 1.0$. In Figure 2, X is generated by $\mathfrak{L}(a, \rho, 0)$ with $\rho = 1$ and

$$a(x) = \begin{cases} 1 & \text{if } x < 0 \\ 5 & \text{if } x \geq 0. \end{cases}$$

In Figure 3, $a = 1$ and

$$\rho(x) = \begin{cases} 1 & \text{if } x < 0 \\ 5 & \text{if } x \geq 0. \end{cases}$$

Comparing Figures 2 and 3 shows that the process tends to go where a is the highest and ρ the lowest.

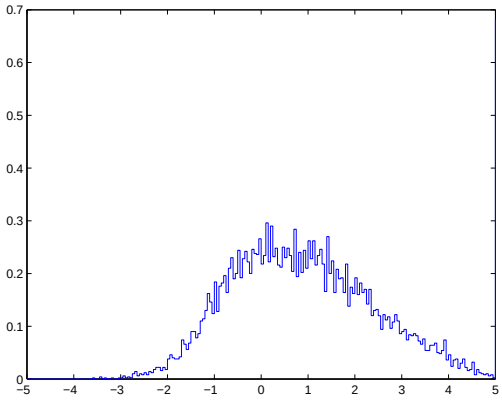


Figure 3: Histogram of the positions at time $t = 1.0$ of 10000 paths starting from $x = 0.0$ of X with $\rho = 1$ and a with value 1 on $\mathbb{R}^+$ and 5 on $\mathbb{R}^+$.

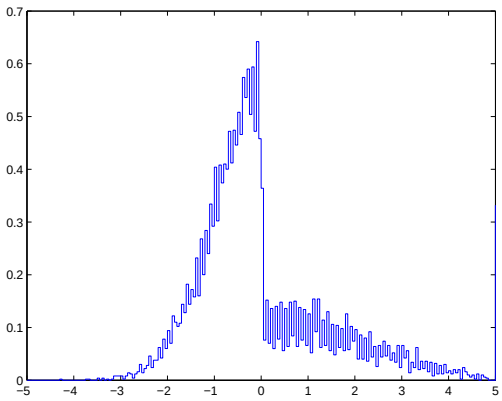
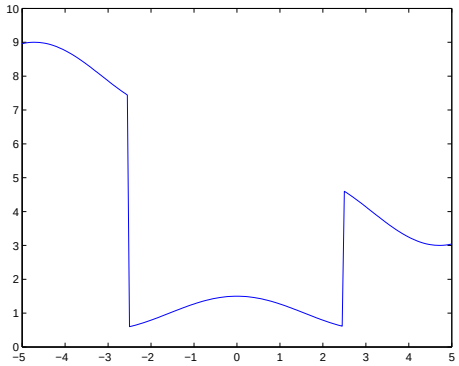


Figure 4: Histogram of the positions at time $t = 1.0$ of 10000 paths starting from $x = 0.0$ of X with $a = 1$ and ρ with value 1 on $\mathbb{R}^+$ and 5 on $\mathbb{R}^+$.

Figure 5: graph of the coefficient a.



Example 3. We take $\rho = 1$, $b = 2$ and an irregular a whose graph is ploted on Figure 5. We know that $\mathbb{E}^x f(X_t)$ is the solution $u(t,x)$ of the parabolic problem $\partial u/\partial t = Lu$, with initial condition f and $(L, D(L)) = \mathfrak{L}(a,\rho,b)$. So the quantity $\frac{1}{N}\sum_{n=1}^{N} f\big(\widehat{X}_t^{\mathfrak{g},n}\big)$ should approach $u(t,x)$.

We use the routine `pdepe` of MATLAB to compute with a deterministic method an approximation $\bar{u}(t,x)$ of $u(t,x)$, at times $t = 0.5$ and $t = 1.0$, with the initial condition

$$f(x) = \cos\big(\frac{\pi}{10}x\big).$$

Note that we have imposed uniform Dirichlet boundary conditions to the parabolic problem.

At each time $t \in \{0.5,\ 1.0\}$ we simulate $N = 8000$ paths of $\widehat{X}^{\mathfrak{g}}$ starting from $x \in \{-4,-3,...,3,4\}$ and compute for each x the quantity

$$\tilde{u}(t,x) = \frac{1}{N}\sum_{n=1}^{N} f\big(\widehat{X}_t^{\mathfrak{g},n}\big).$$

We then plot on the same figure the graphs of $\bar{u}(t,.)$ and $\tilde{u}(t,.)$. Figure 6 shows a good adequation between the probabilistic and the deterministic approximations of $u(t,x)$.

Acknowledgements: This research was supported by the GdR MOMAS.

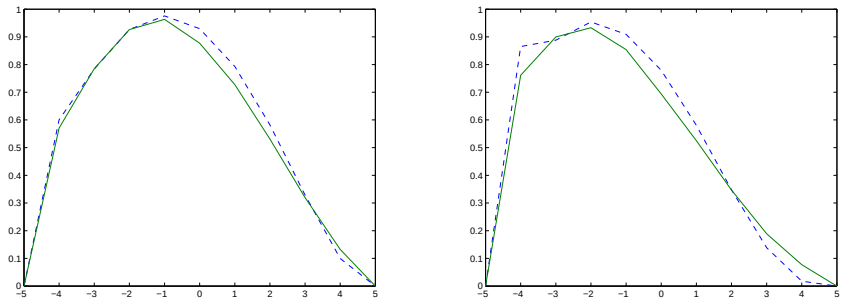


Figure 6: Graph of $\tilde{u}(t,.)$ together with the one of $\bar{u}(t,.)$ (represented by the dashed line), for $t = 0.5$ and $t = 1.0$.

References

[BLP78] A. Bensoussan, J.L. Lions and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*. North-Holland, Amsterdam, 1978.

[Bil68] P. Billingsley. *Convergence of Probabilities Measures*. John Wiley & Sons, 1968.

[Bre68] L. Breiman. *Probability*. Addison-Wesley Series in Statistics, 1968.

[Bre83] H. Brezis. *Analyse fonctionnelle*. Masson, 1983.

[DDG05] M. Decamps, A. De Schepper and M. Goovaerts. *Applications of δ-function pertubation to the pricing of derivative securities*. Physica A 342:3-4 (2004), 677-692.

[ETO06] P. Étoré. *On random walk simulation of one-dimensional diffusion processes with discontinuous coefficients*. Electronic Journal of Probability, vol. 11 (2006), 249-275.

[Fau99] O. Faugeras, F. Clément, R. Deriche, R. Keriven, T. Papadopoulo, J. Roberts, T. Viéville, F. Devernay, J. Gomes, G. Hermosillo, P. Kornprobst and D. Lingrand. *The inverse EEG and MEG problems: The adjoint state approach I: The continuous case*. INRIA research report RR-3673, 1999.

[FW94] M. Freidlin and A.D. Wentzell. *Necessary and Sufficient Conditions for Weak Convergence of One-Dimensional Markov Processes*. Festschrift dedicated to 70^{th} Birthday of Professor E.B. Dynkin, Birkhäuser (1994), 95-109.

[GT83] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin, 1983.

[JKO94] V.V. Jikov, S.M. Kozlov and O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer, 1994.

[KP92] P.E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations.* Springer, Berlin, 1992.

[Lej00] A. Lejay. *Méthodes probabilistes pour l'homogénéisation des opérateurs sous forme divergence: cas linéaires et semi-linéaires.* PhD thesis, Université de Provence, Marseille, France, 2000.

[Lej05] A. Lejay. *Stochastic Differential Equations Driven by Processes Generated by Divergence Form Operators I: A Wong-Zakai Theorem.* Preprint, Institut Élie Cartan, Nancy, France, 2005.

[LM06] A. Lejay and M. Martinez. *A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients.* The Annals of Applied Probability 16:1 (2006), 107-139.

[Mar04] M. Martinez. *Interprétations probabilistes d'opérateurs sous forme divergence et analyse de méthodes numériques probabilistes associées.* PhD thesis, Université de Provence, Marseille, France, 2004.

[MT06] M. Martinez and D. Talay. *Discrétisation d'équations différentielles stochastiques unidimensionnelles à générateur sous forme divergence avec coefficient discontinu.* C.R. Acad. Sci. Paris 342:1 (2006), 51-56.

[OZ05] H. Owhadi and L. Zhang. *Metric based upscaling.* Accepted for publication in Communications on Pure and Applied Mathematics (2005).

[RTW05] J.M. Ramirez, E.A. Thomann, E.C. Waymire, R. Haggerty and B. Wood. *A generalized Taylor-Aris formula and Skew Diffusion.* Preprint, 2005.

[RY91] D. Revuz and M. Yor. *Continuous Martingale and Brownian Motion.* Springer, Heidelberg, 1991.

[Roz96] A. Rozkosz. *Stochastic Representation of Diffusions Corresponding to Divergence Form Operators.* Stochastics Stochastics Reports 57 (1996), 129-157.

[Str88] D. W. Stroock. *Diffusion semigroups corresponding to uniformly elliptic divergence form operators.* Springer, Lecture Notes in Mathematics, Seminaire de Probabilités XXII 1321 (1988), 316-347.

[ZKOT79] V.V. Zhikov, S.M. Kozlov, O.A. Oleinik and K. T'en Ngoan. *Averaging and G-convergence of Differential Operators.* Russian Math. Survey 34:5 (1979), 69-147.

[ZKO81] V.V. Zhikov, S.M. Kozlov and O.A. Oleinik. *G-convergence of Parabolic Operators.* Russian Math. Survey 36:1 (1981), 9-60.