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# *On the Pathwidth of Planar Graphs*

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## On the Pathwidth of Planar Graphs

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**Abstract:** Fomin and Thilikos in [5] conjectured that there is a constant  $c$  such that, for every 2-connected planar graph  $G$ ,  $\text{pw}(G^*) \leq 2\text{pw}(G) + c$  (the same question was asked simultaneously by Coudert, Huc and Sereni in [4]). By the results of Boedlander and Fomin [2] this holds for every outerplanar graph and actually is tight by Coudert, Huc and Sereni [4]. In [5], Fomin and Thilikos proved that there is a constant  $c$  such that the pathwidth of every 3-connected graph  $G$  satisfies:  $\text{pw}(G^*) \leq 6\text{pw}(G) + c$ . In this paper we improve this result by showing that the dual a 3-connected planar graph has pathwidth at most 3 times the pathwidth of the primal plus two. We prove also that the question can be answered positively for 4-connected planar graphs.

**Key-words:** planar graphs, pathwidth

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## A Propos de la Pathwidth des Graphes Planaires

**Résumé :** Fomin et Thilikos[5], après avoir démontré que la pathwidth de tout graphes planaires 3-connexe est au plus 6 fois celle de son dual à une constante près, ont conjecturé que pour tout graphe planaire biconnexe  $G$ ,  $\text{pw}(G^*) \leq 2\text{pw}(G) + \text{cte}$ . D'après Boedlander et Fomin [2] cela est vrai pour tout graphe outerplanaire. De plus cela est exact d'après Coudert, Huc and Sereni [4]. Dans cet article nous améliorons le résultat de Fomin et Thilikos en montrant que la pathwidth de tout graphe planaire 3-connexe est au plus 3 fois celle de son dual plus 2. Nous démontrons également que la conjecture est vrai pour tout graphe planaire dont le dual est 4-connexe.

**Mots-clés :** graphes planaires, pathwidth

## 1 Introduction

A *planar graph* is a graph that can be embedded in the plane without crossing edges. It is said to be *outerplanar* if it can be embedded in the plane without crossing edges and such that all its vertices are incident to the unbounded face. For any graph  $G$ , we denote by  $V(G)$  its vertex set and by  $E(G)$  its edge set. The *dual* of the planar graph  $G$ , denoted by  $G^*$ , is the graph obtained by putting one vertex for each face, and joining two vertices if and only if the corresponding faces are adjacent. Note that the dual of a planar graph can also be computed in linear time.

The notion of pathwidth was introduced by Robertson and Seymour [9]. A *path decomposition* of a graph  $G = (V, E)$  is a set system  $(X_1, \dots, X_r)$  of  $V$  such that

1.  $\bigcup_{i=1}^r X_i = V$ ;
2.  $\forall xy \in E, \exists i \in \{1, \dots, r\} : \{x, y\} \subset X_i$ ;
3. for all  $1 \leq i_0 < i_1 < i_2 \leq r$ ,  $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$ .

The *width* of the path decomposition  $(X_1, \dots, X_r)$  is  $\max_{1 \leq i \leq r} |X_i| - 1$ . The *pathwidth* of  $G$ , denoted by  $\text{pw}(G)$ , is the minimum width over its path decompositions.

Computing the pathwidth of graphs is an active research area, in which a lot of work has been done (For a survey see for instance [8]). Govindan et al. [6] gave an  $\mathcal{O}(n \log(n))$  time algorithm for approximating the pathwidth of outerplanar graphs with a multiplicative factor of 3. For biconnected outerplanar graphs, Bodlaender and Fomin [2] improved upon this result by giving a linear time algorithm which approximates the pathwidth of biconnected outerplanar graphs with a multiplicative factor 2. To do so, they exhibit a relationship between the pathwidth of an outerplanar graph and the pathwidth of its dual. More precisely, the following holds.

**Theorem 1 (Bodlaender and Fomin [2])** *Let  $G$  be a biconnected outerplanar graph without loops and multiple edges. Then  $\text{pw}(G^*) \leq \text{pw}(G) \leq 2\text{pw}(G^*) + 2$ .*

Since the weak dual of an outerplanar graph (which can be computed in linear time) is a tree and there exist linear time algorithms to compute the pathwidth of a tree [11], this yields the desired approximation.

Coudert, Huc and Sereni in [4] improved this result by proving the following theorem:

**Theorem 2 (Coudert, Huc and Sereni [4])** *For every biconnected outerplanar graph  $G$ , we have  $\text{pw}(G^*) \leq \text{pw}(G) \leq 2 \text{pw}(G^*) - 1$  and all the values in the interval  $[\text{pw}(G^*), 2 \text{pw}(G^*) - 1]$  can be the pathwidth of  $G$ .*

Simultaneously Coudert, Huc and Sereni state the following question as an open problem in [4] and Fomin and Thilikos conjectured it in [5] :

**Conjecture 1 ([5],[4])** *Is there a constant  $c$  such that, for every 2-connected planar graph  $G$ ,  $\frac{1}{2}\text{pw}(G^*) - c \leq \text{pw}(G) \leq 2\text{pw}(G^*) + c$ ?*

It is worth noting that this conjecture is motivated by the following result about the treewidth, conjectured by Robertson and Seymour [10] and proved by Lapoire [7] using algebraic methods (notice that Bouchitté, Mazoit and Todinca [3] gave a shorter and combinatorial proof of this result).

**Theorem 3 ([7])** *For every planar graph  $G$ ,  $\text{tw}(G) \leq \text{tw}(G^*) + 1$ .*

Fomin and Thilikos made an even stronger conjecture :

**Conjecture 2 ([5])** *There is a constant  $c$  such that for every 2-connected planar graphs  $G$  of treewidth at least  $m$ ,  $\text{pw}(G^*) \leq \frac{m}{m-1}\text{pw}(G) + c$*

This conjecture does not hold. Indeed we can slightly modify the examples given in [4]. They are example of biconnected outerplanar graphs  $G$  such that  $\text{pw}(G) = 2\text{pw}(G^*) - 1$ . We modify this family of graph by plugging a  $3 \times 3$  grid on a face. This can be done without changing the pathwidth, whereas the treewidth increases from 2 to 3, so the equation is no longer satisfied.

The following theorem improves the previously known bound for 3-connected planar graphs.

**Theorem 4 (Main theorem)** *For every 3-connected planar graph  $G$  we have  $\text{pw}(G) \leq 3 \text{pw}(G^*) + 2$*

Actually our methods prove that the conjecture holds for every 4-connected planar graph.

## 2 Main Theorem

In this section we present the proof of our main theorem. We will use the following notations:

Given a graph  $G = (V(G), E(G))$  of maximum degree  $\Delta(G)$ , we will note  $V(G^*)$  either its face set or the vertex set of its dual and by  $f_G$ ,  $e_G$  and  $n_G$  respectively the number of faces, edges and vertices of  $G$ . Given a set  $A$ , by  $\mathcal{P}(A)$  we denote the family of all subsets of  $A$ .

**Definition 1** *Let  $G$  and  $H$  be two graphs. A connected map from  $G$  to  $H$  is a map  $\sigma : V(G) \rightarrow \mathcal{P}(V(H))$  from vertices of  $G$  to subsets of vertices of  $H$  satisfying the following two properties:*

1. *for every  $v \in V(G)$ ,  $\sigma(v)$  is connected.*
2. *for every adjacent vertices  $v, w \in V(G)$ ,  $\sigma(v) \cup \sigma(w)$  is also connected*

$\sigma$  is of degree at most  $k$  if it also satisfies

- $\forall w \in V(H)$  we have  $|\sigma^{-1}(w) := \{v \in V(G) | w \in \sigma(v)\}| \leq k$

**Lemma 1** *Let  $G$  and  $H$  be two graphs. for any connected map  $\sigma$  of degree at most  $k$  from  $G$  to  $H$ , we have:*

- $\text{pw}(G) \leq k \text{pw}(H) + k - 1$
- $\text{tw}(G) \leq k \text{tw}(H) + k - 1$

**Proof**

- Given a path-decomposition of  $H$  of width  $\ell$ , applying  $\sigma^{-1}$  on bags of our decomposition gives one of width  $k \cdot \ell + k - 1$  for  $G$ . This can be easily verified using the properties of  $\sigma$  listed above.
- Same proof gives the result for tree-width.

□

From now on we suppose  $G$  to be a 3 vertex connected planar graph. We aim to find a low degree connected map from  $G^*$  to  $G$ .

A *Face-To-Edge assignment* is a system of distinct representatives for faces of  $G$ . In other words, a Face-To-Edge assignment is a function  $\tau$  such that we associate to a given face  $F$  of  $G$  an edge  $\tau(F) = (v, w) \in E(F) \subset E(G)$ , in such a way that two different faces are associated to different edges. Given a Face-To-Edge assignment, the map  $\sigma : G^* \rightarrow \mathcal{P}(V(G))$  associates to every vertex  $F$  of  $G^*$  (face of  $G$ ) the subset  $V(F) \setminus V(\tau(F))$ .

**Proposition 1** *The so defined map  $\sigma$  is connected.*

**Proof** Two faces  $F_1, F_2$  sharing an edge  $e$  can't be both associated to  $e$  since they are associated to different edges. Consequently  $\sigma(F_1) \cup \sigma(F_2)$  is connected.  $\square$

Given a Face-To-Edge assignment, let  $H$  be the subgraph of  $G$  consisting of non selected edges; i.e.  $H = G \setminus \{\tau(F) | F \in V(G^*)\}$ . Using Euler's Formula ( $f_S + n_S = e_S + 2$ ) we know that  $H$  contains exactly  $n - 2$  edges. We have

**Proposition 2**  $\forall v \in V(G), |\sigma^{-1}(v)| = \text{deg}_H(v)$

**Proof** A selected edge (an edge of  $G \setminus H$ ) should be associated to one of the two faces containing it. Given a vertex  $v$  of  $G$  of degree  $d$ , it appears exactly in  $d$  faces. Suppose  $r$  edges incident to  $v$  are selected, so they should be associated to exactly  $r$  faces incident with  $v$ .  $v$  doesn't appear in  $|\sigma(v)|$  of these faces, and appears in  $|\sigma(v)|$  of other faces incident to  $v$ . So  $|\sigma^{-1}(v)| = d - r = \text{deg}_H(v)$   $\square$

**Corollary 1**  $\sigma$  is of degree at most  $\Delta(H)$ .

Remark that the average degree in  $H$  is always  $\leq 2$ .

**Definition 2** We call  $H \subset G$  a nice subgraph if it has  $n - 2$  edges and such that we can find a Face-To-Edge assignment  $\tau$  with  $\tau(F) \in E(G) \setminus E(H)$ .

**Definition 3** Given a graph  $G$ , we call adjacency graph, the bipartite graph  $A$  on vertex set  $(V(G^*) \cup E(G \setminus H))$ , with an edge between a vertex of  $V(G^*)$  (i.e. a face  $F$  of  $G$ ) and an edge of  $G \setminus H$  if this edge belongs to  $F$ .

**Corollary 2** Let  $H$  be a nice subgraph of  $G$  of max-degree  $\Delta$ . Then we have  $\text{pw}(G^*) \leq \Delta \text{pw}(G) + \Delta - 1$

We will need the following theorem of Barnette [1]:

**Theorem 5** (Barnette) Every 3-connected planar graph has a spanning tree of max-degree 3.

**Corollary 3** Every 3-connected planar graph  $G$  contains a nice subgraph of max-degree 3

**Proof** By Barnette's theorem there exists a sub-forest  $H$  of  $G$  of max-degree 3 containing  $n - 2$  edges (i.e. is a spanning tree minus an edge). We want to prove that such a subgraph is nice by applying Hall's matching theorem to the adjacency graph  $A$  between faces of  $G$  and edges of  $G \setminus H$ . Given a set of faces  $\{F_1, \dots, F_i\}$  we should prove that in  $A$  the corresponding set has at least  $i$  neighbors. Considering the planar graph  $S$  obtained by taking the union of  $F_i$ , we have:

- $f_S \geq i + 1$  (because  $G^*$  is connected)
- $f_S + n_S = e_S + 2$  (Euler's formula)

We conclude  $e_S - (n_S - 1) \geq i$ . As  $H$  is a forest the number of edges of  $H$  incident with some vertex of this subgraph is at most  $n_S - 1$ . So the hypothesis of Hall's theorem is always satisfied. This proves that  $H$  is a nice graph.  $\square$

As a corollary we have

**Corollary 4** There exists a connected map  $\sigma : G^* \rightarrow G$  of degree at most 3.

As a results from corollary 2 and 4 we have our main theorem:

**Theorem 6** For every 3-connected planar graph  $G$  we have  $\text{pw}(G) \leq 3 \text{pw}(G^*) + 2$

Furthermore our method proves the conjecture for planar graphs whose dual has an Hamiltonian path :



**Theorem 7** *If  $G$  has a Hamiltonian path, we have  $pw(G) \leq 2 pw(G^*) + 1$*

**Proof** The Hamiltonian path gives a nice subgraph of  $G$  of max-degree 2. □

**Corollary 5** *If  $G^*$  is 4-connected then we have  $pw(G) \leq 2 pw(G^*) + 1$*

**Proof** Thomassen proved in [?] that every 4-connected planar graph has a Hamiltonian cycle. Then by last theorem we have the result. □

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