HAL archives-ouvertes.fr

Hoàng-Reed conjecture holds for tournaments

Frédéric Havet, Stéphan Thomassé, Anders Yeo

To cite this version:

Frédéric Havet, Stéphan Thomassé, Anders Yeo. Hoàng-Reed conjecture holds for tournaments. [Research Report] RR-5976, INRIA. 2006, pp.7. <inria-00091366v2>

HAL Id: inria-00091366

https://hal.inria.fr/inria-00091366v2

Submitted on 15 Sep 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Hoàng-Reed conjecture holds for tournaments

Frédéric Havet — Stéphan Thomassé — Anders Yeo

N° 5976

Septembre 2006

Thème COM

Rapport de recherche

ISSN 0249-6399

Hoàng-Reed conjecture holds for tournaments

Frédéric Havet , Stéphan Thomassé , Anders Yeo*

Thème COM — Systèmes communicants
Projet Mascotte

Rapport de recherche n° ???? — Septembre 2006 — 7 pages

Abstract: Hoàng-Reed conjecture asserts that every digraph D has a collection $\mathcal{C}$ of circuits $C_1, \dots, C_{\delta^+}$, where δ^+ is the minimum outdegree of D, such that the circuits of $\mathcal{C}$ have a forest-like structure. Formally, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, for all $i = 2, \dots, \delta^+$. We verify this conjecture for the class of tournaments.

Key-words: tournament, forest-like structure, circuit

* Department of Computer Science, Royal Holloway University of London, Egham, Surrey TW20 OEX, UK, `anders@cs.rhul.ac.uk`

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 — Télécopie : +33 4 92 38 77 65

La conjecture de Hoàng-Reed est vraie pour les tournois

Résumé : La conjecture de Hoàng-Reed dit que chaque digraphe D admet une collection $\mathcal{C}$ de circuits $C_1, \ldots, C_{\delta^+}$, où δ^+ est le degré sortant minimum de D, telle que les circuits de $\mathcal{C}$ ont structure de forêt. Formellement, $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| \leq 1$, pour tout $i = 2, \ldots, \delta^+$. Nous vérifions cette conjecture pour la classe des tournois.

Mots-clés : tournoi, structure de forêt, circuit

1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph D on n vertices and with minimum outdegree n/k has a circuit of length at most k. Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A *circuit-tree* is either a singleton or consists of a set of circuits $C_1, \dots, C_k$ such that $|V(C_i) \cap (V(C_1) \cup \ldots \cup V(C_{i-1}))| = 1$ for all $i = 2, \dots, k$, where $V(C_j)$ is the set of vertices of C_j. A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique xy-directed path for every distinct vertices x and y. A vertex-disjoint union of circuit-trees is a *circuit-forest*. When all circuits have length three, we speak of a *triangle-tree*. For short, a k-circuit-forest is a circuit-forest consisting of k circuits.

Conjecture 1 (Hoàng and Reed [3]) *Every digraph has a δ^+-circuit-forest.*

This conjecture is not even known to be true for $\delta^+ = 3$. In the case $\delta^+ = 2$, C. Thomassen proved in [5] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament T, that is the 3-uniform hypergraph on vertex set V which edges are the 3-circuits of T.

Indeed, if a tournament T has a δ^+-circuit-forest, by the fact that every circuit contains a directed triangle, T also has a δ^+-triangle-forest. Observe that a δ^+-triangle-forest spans exactly $2\delta^+ + c$ vertices, where c is the number of connected components of the triangle-forest. When T is a regular tournament with outdegree δ^+, hence with $2\delta^+ + 1$ vertices, a δ^+-triangle-forest of T is necessarily a spanning δ^+-triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

Theorem 1 *Every tournament has a δ^+-triangle-tree.*

2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

Lemma 1 *Let $k \geq 1$ and let $a_1, a_2, \dots, a_k$ and $b_1, b_2, \dots, b_k$ be two sequences of positive reals. Let $A = \sum_{j=1}^k a_i$ and $B = \sum_{j=1}^k b_j$. If $\sum_{i=1}^k a_i b_i = \frac{AB}{2} + q$, where $q \geq 0$, then there is an i such that $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$.*

Proof. If $k = 1$, then the lemma follows immediately as $q = \frac{AB}{2}$ and $A + B \geq \frac{A+B}{2} + \sqrt{AB}$. So assume that $k > 1$. Without loss of generality, we may assume that $(a_1, b_1) \geq (a_2, b_2) \geq \ldots \geq (a_k, b_k)$ in the lexicographical order. Let r be the minimum value such that $b_r \geq b_i$ for all $i = 1, 2, \ldots, r$. Note that $a_1 \geq |A|/2$, since otherwise $\sum_{i=1}^{k} a_i b_i < \sum_{i=1}^{k} A b_i/2 = AB/2$. Analogously $b_r \geq |B|/2$. Define a' and b' so that $a_1 = A/2 + a'$ and $b_r = B/2 + b'$.

If $r \neq 1$, then the following holds:

$$\begin{array}{rcl} \sum_{i=1}^{k} a_i b_i & \leq & a_1 b_1 + \sum_{i=2}^{k} a_i b_r \\ & \leq & a_1(B - b_r) + (A - a_1) b_r \\ & = & (\frac{A}{2} + a')(\frac{B}{2} - b') + (\frac{A}{2} - a')(\frac{B}{2} + b') \\ & = & \frac{AB}{2} - 2a'b' \\ & \leq & \frac{AB}{2} \end{array}$$

As $q \geq 0$, this implies we have equality everywhere above, which means that $b_1 = B - b_r$. As $B = b_1 + b_r$, we must have $k = 2$. As there was equality everywhere above we have $b' = 0$ or $a' = 0$ which implies that $a_1 = a_2 = A/2$ or $b_1 = b_2 = B/2$. In both cases we would have $r = 1$, a contradiction.

Suppose now that $r = 1$. Then

$$\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = (\frac{A}{2} + a')(\frac{B}{2} + b') + (\frac{A}{2} - a')(\frac{B}{2} - b')$$

This implies that $q \leq 2a'b'$. The minimum value of $a' + b'$ is obtained when $a' = b' = \sqrt{q/2}$. Therefore the minimum value of $a_1 + b_1$ is $A/2 + B/2 + 2\sqrt{q/2}$. This completes the proof of the lemma. ■

Corollary 1 *Let G be a bipartite graph with partite sets A and B. If $|E(G)| = \frac{|A||B|}{2} + q$, where $q \geq 0$, then there is a connected component in G of size at least $|V(G)|/2 + \sqrt{2q}$.*

Proof. Let $Q_1, Q_2, \ldots, Q_k$ be the connected components of G. Let $a_i = |A \cap Q_i|$ and $b_i = |B \cap Q_i|$ for all $i = 1, 2, \ldots, k$. We note that $\sum_{i=1}^{k} a_i b_i \geq \frac{|A||B|}{2} + q$. By Lemma 1, we have $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$ for some i. This completes the proof. ■

Lemma 2 *Let T be a triangle-tree in a digraph D, and let $X \subseteq V(T)$ and $Y \subseteq V(T)$ be such that $|X| + |Y| \geq |V(T)| + 2$. Then there exists a triangle C in T such that the three disjoint triangle-trees in $T - E(C)$ can be named T_1, T_2, T_3 such that Y intersects both T_1 and T_2 and X intersects both T_2 and T_3.*

Proof. We show this by induction. As $|X| + |Y| \geq |V(T)| + 2$, we note that T contains at least one triangle. If T only contains one triangle then the lemma holds as either X or Y equals $V(T)$, and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that T contains at least two triangles. Let $T = T_1 \cup C$, where C is a triangle and T_1 is a triangle-tree. If $|X \cap T_1| + |Y \cap T_1| \geq |V(T_1)| + 2$, then we are done by induction. So assume that this is not the case. As $|V(T_1)| = |V(T)| - 2$ this implies that $|X \setminus V(T_1)| + |Y \setminus V(T_1)| \geq 3$.

Without loss of generality assume that $|X \setminus V(T_1)| \geq 2$ and $|Y \setminus V(T_1)| \geq 1$. Let T_2 be the singleton-tree consisting of a vertex in $Y \setminus V(T_1)$ and let T_3 be the singleton-tree $X \setminus (V(T_1) \cup V(T_2))$. Note that $T - E(C)$ consists of the triangle-trees T_1, T_2 and T_3. By definition, X intersects both T_2 and T_3 and Y intersects T_2. If Y also intersects T_1, we have our conclusion. If not, since $|X| + |Y| \geq |V(T)| + 2$, we have $Y = T_2 \cup T_3$ and $X = V(T)$, and free to rename T_1, T_2, T_3, we have our conclusion. ■

3 Proof of Theorem 1.

We will need the following result:

Theorem 2 (Guo and Volkmann [2]) *Let D be a strong p-partite tournament. For every partite set V_i in D there exists a vertex $x \in V_i$ which belongs to a k-cycle for all $3 \leq k \leq p$.*

Now, we assume that D is a strong tournament as otherwise we just consider the terminal strong component. Let T be a maximum size triangle-tree in D, and assume for the sake of contradiction that $|V(T)| < 2\delta^+(D)+1$. Let D^{MT} be the multipartite tournament obtained from D by deleting all the arcs with both endpoints in $V(T)$. Let $V_1, V_2, \ldots, V_l$ be the partite sets in D^{MT} such that $V_1 = V(T)$ and $|V_i| = 1$ for all $i > 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be a partition of $V(D^{MT})$ such that each Q_i induces an independent set or a strong component and there are no arcs from Q_j to Q_i for all $j > i$ (such a partition exists as we may take a topological ordering of its strong components and merge neighbouring components as long as there are no arcs between them). Furthermore assume that if Q_i is an independent set then it is maximum possible. This implies that there is an arc from Q_i to Q_{i+1} for all $i = 1, 2, \ldots, k-1$.

If there is a Q_i with $Q_i \cap V_1 \neq \emptyset$ and $Q_i \not\subseteq V_1$ then we obtain the following contradiction. Note that at least three partite sets intersect Q_i as $D^{MT}\langle Q_i \rangle$ would not be strong if there were only two partite sets since $|V_i| = 1$ for all $i > 1$. By Theorem 2, there is a 3-circuit in $D^{MT}\langle Q_i \rangle$ containing exactly one vertex from V_1. This contradicts the maximality of T. So every set Q_i is either a subset of V_1 or is disjoint from V_1.

Note that $Q_1 \cup Q_k \subset V_1$, as otherwise D would not be strong. Let $D' = D\langle V_1 \rangle$. If there is a vertex $x \in Q_k$ with $d^+_{D'}(x) \leq \frac{|V_1|-1}{2}$, then $d^+_D(x) \leq \frac{|V_1|-1}{2}$, which implies that $|V(T)| \geq 2\delta^+(D) + 1$, a contradiction. So $d^+_{D'}(x) \geq \frac{|V_1|+1}{2}$ for all $x \in Q_k$, as $|V_1|$ is odd.

Let G_1 denote the bipartite graph with partite sets Q_k and $V_1 - Q_k$, and with $E(G_1) = \{uv \mid u \in Q_k,\ v \in V_1 - Q_k,\ uv \in A(D)\}$. Note that the following now holds by the above.

$$|Q_k|\frac{|V_1|+1}{2} \leq \sum_{u \in Q_k} d^+_{D'}(u) = \binom{|Q_k|}{2} + |E(G_1)| \tag{1}$$

This implies that $|E(G_1)| \geq \frac{|Q_k|(|V_1|-|Q_k|)}{2} + |Q_k|$, which by Lemma 1 implies that there is a connected component in G_1 of size at least $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$. As the size of the maximum component in G_1 is an integer it is at least $|V_1|/2 + 3/2$. Two cases can now occur:

- If $|Q_{k-1}| > 1$ or $Q_{k-2} \not\subseteq V_1$ (or both). If $|Q_{k-1}| > 1$ then let $Z = \{z_1, z_2\}$ be any two distinct vertices in Q_{k-1} otherwise let Z be any two distinct vertices in $Q_{k-1} \cup Q_{k-2}$. By the definition of the Q_i's we note that $Z \cap V_1 = \emptyset$ and there are all arcs from $(V_1 - Q_k)$ to Z and from Z to Q_k. We let $X = Y$ be the vertices of a connected component in G_1 of size at least $(|V_1| + 3)/2$ and use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T_1, T_2 and T_3, of $T - E(C)$ all intersect X (as $X = Y$). As X are the vertices of a connected component in G_1 there are edges, u_1v_1 and u_2v_2, from G_1 such that the following holds. The edge u_1v_1 connects T_3 and T_j, where $j \in \{1, 2\}$ and u_2v_2 connects T_{3-j} and $T_j \cup T_3$. Without loss of generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C) \cup v_1z_1u_1v_1 \cup v_2z_2u_2v_2$ is a triangle-tree in D with more triangles than T, a contradiction.

- If $|Q_{k-1}| = 1$ and $Q_{k-2} \subseteq V_1$. Note that $k > 3$, as otherwise $|V(D) \setminus V(T)| = 1$ and we would be done. This implies that $k > 4$ as $Q_1 \subseteq V_1$, which implies that $Q_2 \not\subseteq V_1$. Now let $Q_{k-1} = \{z_1\}$ and let $z_2 \in Q_{k-3}$ be arbitrary. Let G_2 denote the bipartite graph with partite sets $A = Q_k \cup Q_{k-2}$ and $B = V_1 - A$, and with $E(G_2) = \{uv \mid u \in A,\ v \in B,\ uv \in A(D)\}$. Recall that $d^+_{D'}(x) \geq \frac{|V_1|+1}{2}$ for all $x \in Q_k$. Analogously we get that $d^+_{D'}(y) \geq \frac{|V_1|+1}{2} - 1$ for all $y \in Q_{k-2}$ (as $|Q_{k-1}| = 1$). This implies the following.

$$|A|\frac{|V_1| + 1}{2} - |Q_{k-2}| \leq \sum_{u \in A} d^+_{D'}(u) = \binom{|A|}{2} + |E(G_2)| \tag{2}$$

This implies that $|E(G_2)| \geq \frac{|A|(|V_1|-|A|)}{2} + |A| - |Q_{k-2}|$, which by Corollary 1 implies that there is a connected component in G_2 of size at least $|V_1|/2 + \sqrt{2|Q_k|}$, as $|A| - |Q_{k-2}| = |Q_k|$. Note that $|Q_k| > 1$, as otherwise the vertex in Q_{k-1} only has out-degree one, and we would be done. Therefore there is a connected component in G_2 of size at least $|V_1|/2 + 2$.

Let X be the vertices of a connected component in G_1 of size at least $|V_1|/2 + 3/2$ and let Y be the vertices in a connected component of G_2 of size at least $|V_1|/2 + 2$. Now use Lemma 2 to find a triangle C in T, such that the three disjoint triangle-trees, T_1, T_2 and T_3, of $T - E(C)$ have the following property. The set Y intersects T_1 and T_2 and the set X intersects T_2 and T_3. Due to the definition of X and Y there exists edges, $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$, such that the following holds. The edge u_1v_1 connects T_3 and T_j, where $j \in \{1, 2\}$ and u_2v_2 connects T_{3-j} and $T_j \cup T_3$. Without loss of generality assume that $u_1, u_2 \in Q_k$ and $v_1, v_2 \in V_1 - Q_k$. Now $T - E(C) \cup v_1z_1u_1v_1 \cup v_2z_2u_2v_2$ is a triangle-tree in D with more triangles than T, a contradiction. This completes the proof.

■

References

[1] L. Caccetta and R. Häggkvist, On minimal digraphs with given girth, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, *Congress. Numer.*, **XXI** (1978), 181–187.

[2] Y. Guo and L. Volkmann, Cycles in multipartite tournaments. *Journal of Combinatorial Theory, Series B*, **62** (1994), 363-366.

[3] C.T. Hoàng and B. Reed, A note on short cycles in digraphs, *Discrete Math.*, **66** (1987), 103-107.

[4] B.D. Sullivan, A summary of results and problems related to the Caccetta-Häggkvist conjecture, preprint.

[5] C. Thomassen, The 2-linkage problem for acyclic digraphs, *Discrete Math.*, **55** (1985), 73-87.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399