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## *Hoàng-Reed conjecture holds for tournaments*

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## Hoàng-Reed conjecture holds for tournaments

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**Abstract:** Hoàng-Reed conjecture asserts that every digraph  $D$  has a collection  $\mathcal{C}$  of circuits  $C_1, \dots, C_{\delta^+}$ , where  $\delta^+$  is the minimum outdegree of  $D$ , such that the circuits of  $\mathcal{C}$  have a forest-like structure. Formally,  $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| \leq 1$ , for all  $i = 2, \dots, \delta^+$ . We verify this conjecture for the class of tournaments.

**Key-words:** tournament, forest-like structure, circuit

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## La conjecture de Hoàng-Reed est vraie pour les tournois

**Résumé :** La conjecture de Hoàng-Reed dit que chaque digraphe  $D$  admet une collection  $\mathcal{C}$  de circuits  $C_1, \dots, C_{\delta^+}$ , où  $\delta^+$  est le degré sortant minimum de  $D$ , telle que les circuits de  $\mathcal{C}$  ont structure de forêt. Formellement,  $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| \leq 1$ , pour tout  $i = 2, \dots, \delta^+$ . Nous vérifions cette conjecture pour la classe des tournois.

**Mots-clés :** tournoi, structure de forêt, circuit

## 1 Introduction.

One of the most celebrated problems concerning digraphs is the Caccetta-Häggkvist conjecture (see [1]) asserting that every digraph  $D$  on  $n$  vertices and with minimum outdegree  $n/k$  has a circuit of length at most  $k$ . Little is known about this problem, and, more generally, questions concerning digraphs and involving the minimum outdegree tend to be intractable. As a consequence, many open problems flourished in this area, see [4] for a survey. The Hoàng-Reed conjecture [3] is one of these.

A *circuit-tree* is either a singleton or consists of a set of circuits  $C_1, \dots, C_k$  such that  $|V(C_i) \cap (V(C_1) \cup \dots \cup V(C_{i-1}))| = 1$  for all  $i = 2, \dots, k$ , where  $V(C_j)$  is the set of vertices of  $C_j$ . A less explicit, yet concise, definition is simply that a circuit-tree is a digraph in which there exists a unique  $xy$ -directed path for every distinct vertices  $x$  and  $y$ . A vertex-disjoint union of circuit-trees is a *circuit-forest*. When all circuits have length three, we speak of a *triangle-tree*. For short, a  $k$ -circuit-forest is a circuit-forest consisting of  $k$  circuits.

**Conjecture 1** (Hoàng and Reed [3]) *Every digraph has a  $\delta^+$ -circuit-forest.*

This conjecture is not even known to be true for  $\delta^+ = 3$ . In the case  $\delta^+ = 2$ , C. Thomassen proved in [5] that every digraph with minimum outdegree two has two circuits intersecting on a vertex (i.e. contains a circuit-tree with two circuits). The motivation of the Hoàng-Reed conjecture is that it would imply the Caccetta-Häggkvist conjecture, as the reader can easily check. Our goal in this paper is to show Conjecture 1 for the class of tournaments, i.e. orientations of complete graphs. Since this class is notoriously much simpler than general digraphs, our result is by no means a first step toward a better understanding of the problem. However, it gives a little bit of insight in the triangle-structure of a tournament  $T$ , that is the 3-uniform hypergraph on vertex set  $V$  which edges are the 3-circuits of  $T$ .

Indeed, if a tournament  $T$  has a  $\delta^+$ -circuit-forest, by the fact that every circuit contains a directed triangle,  $T$  also has a  $\delta^+$ -triangle-forest. Observe that a  $\delta^+$ -triangle-forest spans exactly  $2\delta^+ + c$  vertices, where  $c$  is the number of connected components of the triangle-forest. When  $T$  is a regular tournament with outdegree  $\delta^+$ , hence with  $2\delta^+ + 1$  vertices, a  $\delta^+$ -triangle-forest of  $T$  is necessarily a spanning  $\delta^+$ -triangle-tree. The main result of this paper establish the existence of such a tree for every tournament.

**Theorem 1** *Every tournament has a  $\delta^+$ -triangle-tree.*

## 2 Components in bipartite graphs.

We first need two lemmas in order to get lower bounds on the largest component of a bipartite graph in terms of the number of edges.

**Lemma 1** *Let  $k \geq 1$  and let  $a_1, a_2, \dots, a_k$  and  $b_1, b_2, \dots, b_k$  be two sequences of positive reals. Let  $A = \sum_{j=1}^k a_j$  and  $B = \sum_{j=1}^k b_j$ . If  $\sum_{i=1}^k a_i b_i = \frac{AB}{2} + q$ , where  $q \geq 0$ , then there is an  $i$  such that  $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$ .*

**Proof.** If  $k = 1$ , then the lemma follows immediately as  $q = \frac{AB}{2}$  and  $A + B \geq \frac{A+B}{2} + \sqrt{AB}$ . So assume that  $k > 1$ . Without loss of generality, we may assume that  $(a_1, b_1) \geq (a_2, b_2) \geq \dots \geq (a_k, b_k)$  in the lexicographical order. Let  $r$  be the minimum value such that  $b_r \geq b_i$  for all  $i = 1, 2, \dots, r$ . Note that  $a_1 \geq |A|/2$ , since otherwise  $\sum_{i=1}^k a_i b_i < \sum_{i=1}^k A b_i / 2 = AB/2$ . Analogously  $b_r \geq |B|/2$ . Define  $a'$  and  $b'$  so that  $a_1 = A/2 + a'$  and  $b_r = B/2 + b'$ .

If  $r \neq 1$ , then the following holds:

$$\begin{aligned} \sum_{i=1}^k a_i b_i &\leq a_1 b_1 + \sum_{i=2}^k a_i b_r \\ &\leq a_1 (B - b_r) + (A - a_1) b_r \\ &= \left(\frac{A}{2} + a'\right) \left(\frac{B}{2} - b'\right) + \left(\frac{A}{2} - a'\right) \left(\frac{B}{2} + b'\right) \\ &= \frac{AB}{2} - 2a'b' \\ &\leq \frac{AB}{2} \end{aligned}$$

As  $q \geq 0$ , this implies we have equality everywhere above, which means that  $b_1 = B - b_r$ . As  $B = b_1 + b_r$ , we must have  $k = 2$ . As there was equality everywhere above we have  $b' = 0$  or  $a' = 0$  which implies that  $a_1 = a_2 = A/2$  or  $b_1 = b_2 = B/2$ . In both cases we would have  $r = 1$ , a contradiction.

Suppose now that  $r = 1$ . Then

$$\frac{AB}{2} + q \leq a_1 b_1 + (A - a_1)(B - b_1) = \left(\frac{A}{2} + a'\right) \left(\frac{B}{2} + b'\right) + \left(\frac{A}{2} - a'\right) \left(\frac{B}{2} - b'\right)$$

This implies that  $q \leq 2a'b'$ . The minimum value of  $a' + b'$  is obtained when  $a' = b' = \sqrt{q}/2$ . Therefore the minimum value of  $a_1 + b_1$  is  $A/2 + B/2 + 2\sqrt{q}/2$ . This completes the proof of the lemma.  $\blacksquare$

**Corollary 1** *Let  $G$  be a bipartite graph with partite sets  $A$  and  $B$ . If  $|E(G)| = \frac{|A||B|}{2} + q$ , where  $q \geq 0$ , then there is a connected component in  $G$  of size at least  $|V(G)|/2 + \sqrt{2q}$ .*

**Proof.** Let  $Q_1, Q_2, \dots, Q_k$  be the connected components of  $G$ . Let  $a_i = |A \cap Q_i|$  and  $b_i = |B \cap Q_i|$  for all  $i = 1, 2, \dots, k$ . We note that  $\sum_{i=1}^k a_i b_i \geq \frac{|A||B|}{2} + q$ . By Lemma 1, we have  $a_i + b_i \geq \frac{A+B}{2} + \sqrt{2q}$  for some  $i$ . This completes the proof.  $\blacksquare$

**Lemma 2** *Let  $T$  be a triangle-tree in a digraph  $D$ , and let  $X \subseteq V(T)$  and  $Y \subseteq V(T)$  be such that  $|X| + |Y| \geq |V(T)| + 2$ . Then there exists a triangle  $C$  in  $T$  such that the three disjoint triangle-trees in  $T - E(C)$  can be named  $T_1, T_2, T_3$  such that  $Y$  intersects both  $T_1$  and  $T_2$  and  $X$  intersects both  $T_2$  and  $T_3$ .*

**Proof.** We show this by induction. As  $|X| + |Y| \geq |V(T)| + 2$ , we note that  $T$  contains at least one triangle. If  $T$  only contains one triangle then the lemma holds as either  $X$  or  $Y$  equals  $V(T)$ , and the other has at least two vertices. Assume now that the lemma holds for all smaller triangle-trees and that  $T$  contains at least two triangles. Let  $T = T_1 \cup C$ , where  $C$  is a triangle and  $T_1$  is a triangle-tree. If  $|X \cap T_1| + |Y \cap T_1| \geq |V(T_1)| + 2$ , then we are

done by induction. So assume that this is not the case. As  $|V(T_1)| = |V(T)| - 2$  this implies that  $|X \setminus V(T_1)| + |Y \setminus V(T_1)| \geq 3$ .

Without loss of generality assume that  $|X \setminus V(T_1)| \geq 2$  and  $|Y \setminus V(T_1)| \geq 1$ . Let  $T_2$  be the singleton-tree consisting of a vertex in  $Y \setminus V(T_1)$  and let  $T_3$  be the singleton-tree  $X \setminus (V(T_1) \cup V(T_2))$ . Note that  $T - E(C)$  consists of the triangle-trees  $T_1, T_2$  and  $T_3$ . By definition,  $X$  intersects both  $T_2$  and  $T_3$  and  $Y$  intersects  $T_2$ . If  $Y$  also intersects  $T_1$ , we have our conclusion. If not, since  $|X| + |Y| \geq |V(T)| + 2$ , we have  $Y = T_2 \cup T_3$  and  $X = V(T)$ , and free to rename  $T_1, T_2, T_3$ , we have our conclusion. ■

### 3 Proof of Theorem 1.

We will need the following result:

**Theorem 2** (Guo and Volkmann [2]) *Let  $D$  be a strong  $p$ -partite tournament. For every partite set  $V_i$  in  $D$  there exists a vertex  $x \in V_i$  which belongs to a  $k$ -cycle for all  $3 \leq k \leq p$ .*

Now, we assume that  $D$  is a strong tournament as otherwise we just consider the terminal strong component. Let  $T$  be a maximum size triangle-tree in  $D$ , and assume for the sake of contradiction that  $|V(T)| < 2\delta^+(D) + 1$ . Let  $D^{MT}$  be the multipartite tournament obtained from  $D$  by deleting all the arcs with both endpoints in  $V(T)$ . Let  $V_1, V_2, \dots, V_l$  be the partite sets in  $D^{MT}$  such that  $V_1 = V(T)$  and  $|V_i| = 1$  for all  $i > 1$ .

Let  $Q_1, Q_2, \dots, Q_k$  be a partition of  $V(D^{MT})$  such that each  $Q_i$  induces an independent set or a strong component and there are no arcs from  $Q_j$  to  $Q_i$  for all  $j > i$  (such a partition exists as we may take a topological ordering of its strong components and merge neighbouring components as long as there are no arcs between them). Furthermore assume that if  $Q_i$  is an independent set then it is maximum possible. This implies that there is an arc from  $Q_i$  to  $Q_{i+1}$  for all  $i = 1, 2, \dots, k - 1$ .

If there is a  $Q_i$  with  $Q_i \cap V_1 \neq \emptyset$  and  $Q_i \not\subseteq V_1$  then we obtain the following contradiction. Note that at least three partite sets intersect  $Q_i$  as  $D^{MT} \langle Q_i \rangle$  would not be strong if there were only two partite sets since  $|V_i| = 1$  for all  $i > 1$ . By Theorem 2, there is a 3-circuit in  $D^{MT} \langle Q_i \rangle$  containing exactly one vertex from  $V_1$ . This contradicts the maximality of  $T$ . So every set  $Q_i$  is either a subset of  $V_1$  or is disjoint from  $V_1$ .

Note that  $Q_1 \cup Q_k \subset V_1$ , as otherwise  $D$  would not be strong. Let  $D' = D \langle V_1 \rangle$ . If there is a vertex  $x \in Q_k$  with  $d_{D'}^+(x) \leq \frac{|V_1| - 1}{2}$ , then  $d_D^+(x) \leq \frac{|V_1| - 1}{2}$ , which implies that  $|V(T)| \geq 2\delta^+(D) + 1$ , a contradiction. So  $d_{D'}^+(x) \geq \frac{|V_1| + 1}{2}$  for all  $x \in Q_k$ , as  $|V_1|$  is odd.

Let  $G_1$  denote the bipartite graph with partite sets  $Q_k$  and  $V_1 - Q_k$ , and with  $E(G_1) = \{uv \mid u \in Q_k, v \in V_1 - Q_k, uv \in A(D)\}$ . Note that the following now holds by the above.

$$|Q_k| \frac{|V_1| + 1}{2} \leq \sum_{u \in Q_k} d_{D'}^+(u) = \binom{|Q_k|}{2} + |E(G_1)| \quad (1)$$

This implies that  $|E(G_1)| \geq \frac{|Q_k|(|V_1| - |Q_k|)}{2} + |Q_k|$ , which by Lemma 1 implies that there is a connected component in  $G_1$  of size at least  $|V_1|/2 + \sqrt{2|Q_k|} \geq |V_1|/2 + \sqrt{2}$ . As the size



of the maximum component in  $G_1$  is an integer it is at least  $|V_1|/2 + 3/2$ . Two cases can now occur:

- If  $|Q_{k-1}| > 1$  or  $Q_{k-2} \not\subseteq V_1$  (or both). If  $|Q_{k-1}| > 1$  then let  $Z = \{z_1, z_2\}$  be any two distinct vertices in  $Q_{k-1}$  otherwise let  $Z$  be any two distinct vertices in  $Q_{k-1} \cup Q_{k-2}$ . By the definition of the  $Q_i$ 's we note that  $Z \cap V_1 = \emptyset$  and there are all arcs from  $(V_1 - Q_k)$  to  $Z$  and from  $Z$  to  $Q_k$ . We let  $X = Y$  be the vertices of a connected component in  $G_1$  of size at least  $(|V_1| + 3)/2$  and use Lemma 2 to find a triangle  $C$  in  $T$ , such that the three disjoint triangle-trees,  $T_1, T_2$  and  $T_3$ , of  $T - E(C)$  all intersect  $X$  (as  $X = Y$ ). As  $X$  are the vertices of a connected component in  $G_1$  there are edges,  $u_1v_1$  and  $u_2v_2$ , from  $G_1$  such that the following holds. The edge  $u_1v_1$  connects  $T_3$  and  $T_j$ , where  $j \in \{1, 2\}$  and  $u_2v_2$  connects  $T_{3-j}$  and  $T_j \cup T_3$ . Without loss of generality assume that  $u_1, u_2 \in Q_k$  and  $v_1, v_2 \in V_1 - Q_k$ . Now  $T - E(C) \cup v_1z_1u_1v_1 \cup v_2z_2u_2v_2$  is a triangle-tree in  $D$  with more triangles than  $T$ , a contradiction.
- If  $|Q_{k-1}| = 1$  and  $Q_{k-2} \subseteq V_1$ . Note that  $k > 3$ , as otherwise  $|V(D) \setminus V(T)| = 1$  and we would be done. This implies that  $k > 4$  as  $Q_1 \subseteq V_1$ , which implies that  $Q_2 \not\subseteq V_1$ . Now let  $Q_{k-1} = \{z_1\}$  and let  $z_2 \in Q_{k-3}$  be arbitrary. Let  $G_2$  denote the bipartite graph with partite sets  $A = Q_k \cup Q_{k-2}$  and  $B = V_1 - A$ , and with  $E(G_2) = \{uv \mid u \in A, v \in B, uv \in A(D)\}$ . Recall that  $d_{D'}^+(x) \geq \frac{|V_1|+1}{2}$  for all  $x \in Q_k$ . Analogously we get that  $d_{D'}^+(y) \geq \frac{|V_1|+1}{2} - 1$  for all  $y \in Q_{k-2}$  (as  $|Q_{k-1}| = 1$ ). This implies the following.

$$|A| \frac{|V_1|+1}{2} - |Q_{k-2}| \leq \sum_{u \in A} d_{D'}^+(u) = \binom{|A|}{2} + |E(G_2)| \quad (2)$$

This implies that  $|E(G_2)| \geq \frac{|A|(|V_1|-|A|)}{2} + |A| - |Q_{k-2}|$ , which by Corollary 1 implies that there is a connected component in  $G_2$  of size at least  $|V_1|/2 + \sqrt{2|Q_k|}$ , as  $|A| - |Q_{k-2}| = |Q_k|$ . Note that  $|Q_k| > 1$ , as otherwise the vertex in  $Q_{k-1}$  only has out-degree one, and we would be done. Therefore there is a connected component in  $G_2$  of size at least  $|V_1|/2 + 2$ .

Let  $X$  be the vertices of a connected component in  $G_1$  of size at least  $|V_1|/2 + 3/2$  and let  $Y$  be the vertices in a connected component of  $G_2$  of size at least  $|V_1|/2 + 2$ . Now use Lemma 2 to find a triangle  $C$  in  $T$ , such that the three disjoint triangle-trees,  $T_1, T_2$  and  $T_3$ , of  $T - E(C)$  have the following property. The set  $Y$  intersects  $T_1$  and  $T_2$  and the set  $X$  intersects  $T_2$  and  $T_3$ . Due to the definition of  $X$  and  $Y$  there exists edges,  $u_1v_1 \in E(G_1)$  and  $u_2v_2 \in E(G_2)$ , such that the following holds. The edge  $u_1v_1$  connects  $T_3$  and  $T_j$ , where  $j \in \{1, 2\}$  and  $u_2v_2$  connects  $T_{3-j}$  and  $T_j \cup T_3$ . Without loss of generality assume that  $u_1, u_2 \in Q_k$  and  $v_1, v_2 \in V_1 - Q_k$ . Now  $T - E(C) \cup v_1z_1u_1v_1 \cup v_2z_2u_2v_2$  is a triangle-tree in  $D$  with more triangles than  $T$ , a contradiction. This completes the proof. ■

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