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On the irreducibility of multivariate subresultants

Sur l'irréductibilité des sous-résultants multivariés

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Abstract

Let P_1, \dots, P_n be generic homogeneous polynomials in n variables of degrees d_1, \dots, d_n respectively. We prove that if ν is an integer satisfying $\sum_{i=1}^n d_i - n + 1 - \min\{d_i\} < \nu$, then all multivariate subresultants associated to the family P_1, \dots, P_n in degree ν are irreducible. We show that the lower bound is sharp. As a byproduct, we get a formula for computing the residual resultant of $\binom{\rho - \nu + n - 1}{n - 1}$ smooth isolated points in \mathbb{P}^{n-1} .

Résumé

Soient P_1, \dots, P_n des polynômes homogènes génériques en n variables de degré respectif d_1, \dots, d_n . Nous montrons que si ν est un entier tel que $\sum_{i=1}^n d_i - n + 1 - \min\{d_i\} < \nu$, tous les sous-résultants multivariés de degré ν des polynômes P_1, \dots, P_n sont irréductibles. Nous montrons également que cette borne est atteinte dans des cas particuliers. Comme conséquence directe nous obtenons une nouvelle formule pour le calcul du résultant résiduel de $\binom{\rho - \nu + n - 1}{n - 1}$ points lisses isolés dans \mathbb{P}^{n-1} .

Classical subresultants of two univariate polynomials have been studied by Sylvester in the foundational work [13]. Multivariate subresultants, introduced in [2], provide a criterion for over-constrained polynomial systems to have Hilbert function of prescribed value, generalizing the classical case. To be more precise, let \mathbb{K} be a field. If P_1, \dots, P_s are homogeneous polynomials in $\mathbb{K}[X_1, \dots, X_n]$ with $d_i = \deg(P_i)$ and $s \leq n$, $H_{d_1, \dots, d_s}(\cdot)$ is the Hilbert function of a complete intersection given by s homogeneous polynomials in n variables of degrees d_1, \dots, d_s , and S is a set of $H_{d_1, \dots, d_s}(\nu)$ monomials of degree ν , the subresultant Δ_S^ν is a polynomial in the coefficients of the P_i 's of degree $H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(\nu - d_i)$ in the coefficients of P_i ($i = 1, \dots, s$) having the following universal property: $\Delta_S^\nu \neq 0$ if and only if $I_\nu + \mathbb{K}\langle S \rangle = \mathbb{K}[X_1, \dots, X_n]_\nu$, where I_ν is the degree ν part of the ideal generated by the P_i 's (see [2]).

Multivariate subresultants have been used in computational algebra for polynomial system solving ([10],[14]) as well as for providing explicit formulas for the representation of rational functions ([11,6,7,12]).

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The study of their properties is an active area of research ([3,4,6,7,8]). In particular, it is important to know which S verify $\Delta_S^\nu \neq 0$, and which of these Δ_S^ν are irreducible (see the final remarks and open questions in [2] and the conjectures in [7]). Partial results have been obtained in this direction. In [5] it is shown that, if $s = n$ and $\sum_{i=1}^n d_i - n - \min\{d_i\} < \nu$, then for every set S of monomials of degree ν and cardinal $H_{d_1, \dots, d_n}(\nu)$, the polynomial Δ_S^ν is not identically zero. Moreover, in [4], it is also proved that if $s = n$, $\nu = \sum_{i=1}^n d_i - n$, and $S = \{x_j^\nu\}$ for $j = 1, \dots, n$, then Δ_S^ν is an irreducible polynomial in the coefficients of the P_i' s. In [8, Lemma 4.2] the irreducibility of Δ_S^ν is shown for $s = n = 2$, $\max\{d_1, d_2\} \leq \nu$, and $S = \{X_2^\nu, X_1 X_2^{\nu-1}, \dots, X_1^{H_{d_1, d_2}(\nu)-1} X_2^{\nu-H_{d_1, d_2}(\nu)+1}\}$.

In this note we study the irreducibility problem in the case $s = n$. Let us introduce some notations in order to state our result. Let $\rho := \sum_{i=1}^n (d_i - 1)$. For $i = 1, \dots, n$ and $\alpha \in \mathbb{Z}_{\geq 0}^n$ such that $|\alpha| = d_i$, introduce a new variable $c_{i,\alpha}$. Let $\mathbb{A} := \mathbb{Z}[c_{i,\alpha}, i = 1, \dots, n, |\alpha| = d_i]$ and set

$$P_i(x_1, \dots, x_n) := \sum_{|\alpha|=d_i} c_{i,\alpha} x^\alpha. \quad (1)$$

Theorem *For every ν such that $\rho - \min\{d_i\} + 1 < \nu$ and every set S of monomials of degree ν and cardinality $H_{d_1, \dots, d_n}(\nu)$, the subresultant $\Delta_S^\nu(P_1, \dots, P_n)$ is irreducible in \mathbb{A} .*

Observe that, if $n = 2$, then $\rho - \min\{d_i\} + 1 = d_1 + d_2 - 2 - \min\{d_i\} + 1 = \max\{d_i\} - 1$, and this is equivalent to $\max\{d_i\} \leq \nu$, so our result contains those in [8].

Proof of the Theorem: For simplicity we assume hereafter that $d_1 \geq \dots \geq d_n \geq 1$. First observe that if $\nu > \rho$ then Δ_S^ν is simply a resultant, and is hence known to be irreducible. So, we can suppose w.l.o.g. that $d_n > 1$. We thus only have to consider integers ν such that

$$\rho \geq \nu > \rho - d_n + 1 = \sum_{i=1}^{n-1} (d_i - 1), \quad (2)$$

where we recall that $\rho = \sum_{i=1}^n (d_i - 1)$. We begin by computing the multi-degree of the subresultants Δ_S^ν ; we know (see [2]) that

$$\deg_{P_i}(\Delta_S^\nu) = H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(\nu - d_i).$$

But from the standard short exact sequence

$$0 \rightarrow \frac{R}{(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)}(-d_i) \xrightarrow{\times f_i} \frac{R}{(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n-1})} \rightarrow \frac{R}{(f_1, \dots, f_n)} \rightarrow 0,$$

where f_1, \dots, f_n are homogeneous polynomials of respective degree d_i in a graded polynomial ring R and f_1, \dots, f_n is a complete intersection in R , we deduce

$$H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(t - d_i) = H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(t) - H_{d_1, \dots, d_n}(t)$$

for all integer t . It follows that for all integer $\nu \geq \rho - d_n + 1$,

$$\deg_{P_i}(\Delta_S^\nu) = \frac{d_1 \dots d_n}{d_i} - H_{d_1, \dots, d_n}(\nu) = \frac{d_1 \dots d_n}{d_i} - \binom{\rho - \nu + n - 1}{n - 1}, \quad (3)$$

where that last equality comes from the facts that $H_{d_1, \dots, d_n}(\rho - t) = H_{d_1, \dots, d_n}(t)$ for all integer t , and $H_{d_1, \dots, d_n}(t) = \binom{t+n-1}{n-1}$ for all $0 \leq t < d_n$. We define $\mathbf{a} := \binom{\rho - \nu + n - 1}{n - 1}$. As \mathbf{a} does not depend on $i \in \{1, \dots, n\}$ and residual (or reduced) resultants of \mathbf{a} isolated points in \mathbb{P}^{n-1} have the same degree in the coefficients of P_i as the right hand side of (3), this suggest that we compare Δ_S^ν with residual resultants.

We will work with an ideal G defining \mathbf{a} points in \mathbb{P}^{n-1} which is generated in degree at most d_n and such that $G_{d_n-1} \neq 0$. Ideals defining \mathbf{a} points in sufficiently generic position are generated in degree exactly $\rho - \nu + 1$ (see [9, Proposition 4]). Since by (2) we have $d_n > \rho - \nu + 1$, we thus choose such an

ideal $G = (g_1, \dots, g_m)$, where $\deg(g_i) = \rho - \nu + 1$ for all $i = 1, \dots, m$, defining \mathbf{a} points in generic position (see [9] for the definition of “generic position”), and hence locally a complete intersection.

Now consider the following specialization of polynomials P_i 's

$$P_i \mapsto \bar{P}_i := \sum_{j=1}^m p_{ij}(x)g_j(x), \quad (4)$$

where $p_{ij}(x) = \sum_{|\alpha|=d_i-\rho+\nu-1} c_{ij}^{|\alpha|} x^\alpha$ is a generic polynomial of degree $d_i - \rho + \nu - 1$. There exists a resultant associated to the system $\bar{P}_1, \dots, \bar{P}_n$, called the *residual resultant*. We denote it by $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$. Let us recall its main properties (see [1] §3.1).

- $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$ is a homogeneous and *irreducible* polynomial in the ring of all the coefficients $\mathbb{Q}[c_{ij}^{|\alpha|}]$,
- For any given specialization of the coefficients $c_{ij}^{|\alpha|}$'s sending \bar{P}_i to Q_i , we have

$$\text{Res}_G(Q_1, \dots, Q_n) = 0 \text{ if and only if } (Q_1, \dots, Q_n)^{\text{sat}} \subsetneq G = G^{\text{sat}},$$

- $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$ is multi-homogeneous: it is homogeneous in the coefficients of each polynomials \bar{P}_i , $i = 1, \dots, n$, and we have

$$\deg_{\bar{P}_i}(\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)) = \frac{d_1 \cdots d_n}{d_i} - \mathbf{a}.$$

We are now going to compare this residual resultant with the specialized subresultant $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$, which is non-zero as proved in [4]. We claim that we have the following implications:

$$\Delta_S^\nu(Q_1, \dots, Q_n) \neq 0 \Rightarrow H_{(Q)}(\nu) = \mathbf{a} \Rightarrow H_{(Q)}(t) = \mathbf{a} \text{ for all } t \geq \nu \Rightarrow \text{Res}_G(Q_1, \dots, Q_n) \neq 0, \quad (5)$$

where $H_{(Q)}(\cdot)$ denotes the Hilbert function associated to the ideal (Q_1, \dots, Q_n) . Only the second implication needs to be proved, the others follow directly from the algebraic properties of resultants and subresultants. We know that $H_G(t) = \mathbf{a}$ for all $t \geq \rho - \nu + 1$ (see [9]), and since we have supposed (2), it is a straightforward computation to show that $\nu \geq \rho - \nu + 1$. It follows that, by hypothesis, the ideals G and (Q) coincide in degree ν and have Hilbert function \mathbf{a} in this degree. As they are both generated in degree at most ν this implies that they coincide in all higher degrees, and therefore they both have Hilbert function equal to \mathbf{a} in these degrees, because G is the defining ideal of a set of points.

Due to (5) and the irreducibility of the residual resultant, we deduce that $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$ divides $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$. But both polynomials have the same degree, so they must be equal up to a rational number (giving a new formula for computing this residual resultant using [3]). Since this residual resultant is irreducible, and since Δ_S^ν and $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$ have the same multi-degree, this shows that Δ_S^ν is irreducible in $\mathbb{Q}[\text{coeff}(P_i)]$.

It remains to prove that Δ_S^ν is irreducible in $\mathbb{Z}[\text{coeff}(P_i)]$. As it is irreducible in $\mathbb{Q}[\text{coeff}(P_i)]$, we only have to show that Δ_S^ν has content ± 1 . Suppose that this is not the case, and let $p \in \mathbb{Z}$ be a prime dividing the content of Δ_S^ν . Let k be the algebraic closure of \mathbb{Z}_p . This implies that $\Delta_S^\nu = 0$ in $K := k(\text{coeff}(P_i))$, and hence S is linearly dependent in $K[x_1, \dots, x_n]/\langle P_1, \dots, P_n \rangle$, contradicting the main result of [4].

Reducibility in lower degrees: We now exhibit some sets S of degree $\nu = \rho - \min\{d_i\} + 1$ such that Δ_S^ν factorizes. This shows that the lower bound in our theorem is sharp.

- **$\mathbf{n} = 2, \mathbf{d}_1 > \mathbf{d}_2$:** In this case, $\nu = d_1 - 1 \geq d_2$, and $H_{d_1, d_2}(\nu) = d_2$. Thus Δ_S^ν can be here computed with Sylvester type matrices [13]. However, setting $f_2 = c_0 x_1^{d_2} + c_1 x_1^{d_2-1} x_2 + \dots + c_{d_2} x_2^{d_2}$, the universal property of the subresultant Δ_S^ν shows immediately that it is a power of c_0 , and we have already seen that its degree is $d_1 - d_2 + 1$; it follows that $\Delta_S^\nu = c_0^{d_1 - d_2 + 1}$, so it can not be irreducible.

- $\mathbf{n} > 2$, $\mathbf{d}_1 - 1 > \mathbf{d}_2 = \mathbf{d}_3 = \dots = \mathbf{d}_n = 1$: Again in this case, $\nu = d_1 - 1$ and $H_{d_1, d_2}(\nu) = 1$. Choose $S = \{x_1^\nu\}$ and, if $f_i = c_{1i}x_1 + \dots + c_{ni}x_n$, $i = 2, \dots, n$, we set $\delta := \det(c_{ij})_{2 \leq i, j \leq n}$. Applying Lemma 4.4 in [6] to this situation, we get that $\Delta_S^\nu = \delta^\nu$. So, Δ_S^ν is not irreducible.

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