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# On the irreducibility of multivariate subresultants

## Sur l'irréductibilité des sous-résultants multivariés

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### Abstract

Let  $P_1, \dots, P_n$  be generic homogeneous polynomials in  $n$  variables of degrees  $d_1, \dots, d_n$  respectively. We prove that if  $\nu$  is an integer satisfying  $\sum_{i=1}^n d_i - n + 1 - \min\{d_i\} < \nu$ , then all multivariate subresultants associated to the family  $P_1, \dots, P_n$  in degree  $\nu$  are irreducible. We show that the lower bound is sharp. As a byproduct, we get a formula for computing the residual resultant of  $\binom{\rho - \nu + n - 1}{n - 1}$  smooth isolated points in  $\mathbb{P}^{n-1}$ .

### Résumé

Soient  $P_1, \dots, P_n$  des polynômes homogènes génériques en  $n$  variables de degré respectif  $d_1, \dots, d_n$ . Nous montrons que si  $\nu$  est un entier tel que  $\sum_{i=1}^n d_i - n + 1 - \min\{d_i\} < \nu$ , tous les sous-résultants multivariés de degré  $\nu$  des polynômes  $P_1, \dots, P_n$  sont irréductibles. Nous montrons également que cette borne est atteinte dans des cas particuliers. Comme conséquence directe nous obtenons une nouvelle formule pour le calcul du résultant résiduel de  $\binom{\rho - \nu + n - 1}{n - 1}$  points lisses isolés dans  $\mathbb{P}^{n-1}$ .

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Classical subresultants of two univariate polynomials have been studied by Sylvester in the foundational work [13]. Multivariate subresultants, introduced in [2], provide a criterion for over-constrained polynomial systems to have Hilbert function of prescribed value, generalizing the classical case. To be more precise, let  $\mathbb{K}$  be a field. If  $P_1, \dots, P_s$  are homogeneous polynomials in  $\mathbb{K}[X_1, \dots, X_n]$  with  $d_i = \deg(P_i)$  and  $s \leq n$ ,  $H_{d_1, \dots, d_s}(\cdot)$  is the Hilbert function of a complete intersection given by  $s$  homogeneous polynomials in  $n$  variables of degrees  $d_1, \dots, d_s$ , and  $S$  is a set of  $H_{d_1, \dots, d_s}(\nu)$  monomials of degree  $\nu$ , the subresultant  $\Delta_S^\nu$  is a polynomial in the coefficients of the  $P_i$ 's of degree  $H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(\nu - d_i)$  in the coefficients of  $P_i$  ( $i = 1, \dots, s$ ) having the following universal property:  $\Delta_S^\nu \neq 0$  if and only if  $I_\nu + \mathbb{K}\langle S \rangle = \mathbb{K}[X_1, \dots, X_n]_\nu$ , where  $I_\nu$  is the degree  $\nu$  part of the ideal generated by the  $P_i$ 's (see [2]).

Multivariate subresultants have been used in computational algebra for polynomial system solving ([10],[14]) as well as for providing explicit formulas for the representation of rational functions ([11,6,7,12]).

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The study of their properties is an active area of research ([3,4,6,7,8]). In particular, it is important to know which  $S$  verify  $\Delta_S^\nu \neq 0$ , and which of these  $\Delta_S^\nu$  are irreducible (see the final remarks and open questions in [2] and the conjectures in [7]). Partial results have been obtained in this direction. In [5] it is shown that, if  $s = n$  and  $\sum_{i=1}^n d_i - n - \min\{d_i\} < \nu$ , then for every set  $S$  of monomials of degree  $\nu$  and cardinal  $H_{d_1, \dots, d_n}(\nu)$ , the polynomial  $\Delta_S^\nu$  is not identically zero. Moreover, in [4], it is also proved that if  $s = n$ ,  $\nu = \sum_{i=1}^n d_i - n$ , and  $S = \{x_j^\nu\}$  for  $j = 1, \dots, n$ , then  $\Delta_S^\nu$  is an irreducible polynomial in the coefficients of the  $P_i$ 's. In [8, Lemma 4.2] the irreducibility of  $\Delta_S^\nu$  is shown for  $s = n = 2$ ,  $\max\{d_1, d_2\} \leq \nu$ , and  $S = \{X_2^\nu, X_1 X_2^{\nu-1}, \dots, X_1^{H_{d_1, d_2}(\nu)-1} X_2^{\nu-H_{d_1, d_2}(\nu)+1}\}$ .

In this note we study the irreducibility problem in the case  $s = n$ . Let us introduce some notations in order to state our result. Let  $\rho := \sum_{i=1}^n (d_i - 1)$ . For  $i = 1, \dots, n$  and  $\alpha \in \mathbb{Z}_{\geq 0}^n$  such that  $|\alpha| = d_i$ , introduce a new variable  $c_{i, \alpha}$ . Let  $\mathbb{A} := \mathbb{Z}[c_{i, \alpha}, i = 1, \dots, n, |\alpha| = d_i]$  and set

$$P_i(x_1, \dots, x_n) := \sum_{|\alpha|=d_i} c_{i, \alpha} x^\alpha. \quad (1)$$

**Theorem** *For every  $\nu$  such that  $\rho - \min\{d_i\} + 1 < \nu$  and every set  $S$  of monomials of degree  $\nu$  and cardinality  $H_{d_1, \dots, d_n}(\nu)$ , the subresultant  $\Delta_S^\nu(P_1, \dots, P_n)$  is irreducible in  $\mathbb{A}$ .*

Observe that, if  $n = 2$ , then  $\rho - \min\{d_i\} + 1 = d_1 + d_2 - 2 - \min\{d_i\} + 1 = \max\{d_i\} - 1$ , and this is equivalent to  $\max\{d_i\} \leq \nu$ , so our result contains those in [8].

**Proof of the Theorem:** For simplicity we assume hereafter that  $d_1 \geq \dots \geq d_n \geq 1$ . First observe that if  $\nu > \rho$  then  $\Delta_S^\nu$  is simply a resultant, and is hence known to be irreducible. So, we can suppose w.l.o.g. that  $d_n > 1$ . We thus only have to consider integers  $\nu$  such that

$$\rho \geq \nu > \rho - d_n + 1 = \sum_{i=1}^{n-1} (d_i - 1), \quad (2)$$

where we recall that  $\rho = \sum_{i=1}^n (d_i - 1)$ . We begin by computing the multi-degree of the subresultants  $\Delta_S^\nu$ ; we know (see [2]) that

$$\deg_{P_i}(\Delta_S^\nu) = H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(\nu - d_i).$$

But from the standard short exact sequence

$$0 \rightarrow \frac{R}{(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_n)}(-d_i) \xrightarrow{\times f_i} \frac{R}{(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{n-1})} \rightarrow \frac{R}{(f_1, \dots, f_n)} \rightarrow 0,$$

where  $f_1, \dots, f_n$  are homogeneous polynomials of respective degree  $d_i$  in a graded polynomial ring  $R$  and  $f_1, \dots, f_n$  is a complete intersection in  $R$ , we deduce

$$H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(t - d_i) = H_{d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n}(t) - H_{d_1, \dots, d_n}(t)$$

for all integer  $t$ . It follows that for all integer  $\nu \geq \rho - d_n + 1$ ,

$$\deg_{P_i}(\Delta_S^\nu) = \frac{d_1 \dots d_n}{d_i} - H_{d_1, \dots, d_n}(\nu) = \frac{d_1 \dots d_n}{d_i} - \binom{\rho - \nu + n - 1}{n - 1}, \quad (3)$$

where that last equality comes from the facts that  $H_{d_1, \dots, d_n}(\rho - t) = H_{d_1, \dots, d_n}(t)$  for all integer  $t$ , and  $H_{d_1, \dots, d_n}(t) = \binom{t+n-1}{n-1}$  for all  $0 \leq t < d_n$ . We define  $\mathbf{a} := \binom{\rho - \nu + n - 1}{n - 1}$ . As  $\mathbf{a}$  does not depend on  $i \in \{1, \dots, n\}$  and residual (or reduced) resultants of  $\mathbf{a}$  isolated points in  $\mathbb{P}^{n-1}$  have the same degree in the coefficients of  $P_i$  as the right hand side of (3), this suggest that we compare  $\Delta_S^\nu$  with residual resultants.

We will work with an ideal  $G$  defining  $\mathbf{a}$  points in  $\mathbb{P}^{n-1}$  which is generated in degree at most  $d_n$  and such that  $G_{d_n-1} \neq 0$ . Ideals defining  $\mathbf{a}$  points in sufficiently generic position are generated in degree exactly  $\rho - \nu + 1$  (see [9, Proposition 4]). Since by (2) we have  $d_n > \rho - \nu + 1$ , we thus choose such an

ideal  $G = (g_1, \dots, g_m)$ , where  $\deg(g_i) = \rho - \nu + 1$  for all  $i = 1, \dots, m$ , defining  $\mathbf{a}$  points in generic position (see [9] for the definition of “generic position”), and hence locally a complete intersection.

Now consider the following specialization of polynomials  $P_i$ 's

$$P_i \mapsto \bar{P}_i := \sum_{j=1}^m p_{ij}(x)g_j(x), \quad (4)$$

where  $p_{ij}(x) = \sum_{|\alpha|=d_i-\rho+\nu-1} c_{ij}^{|\alpha|} x^\alpha$  is a generic polynomial of degree  $d_i - \rho + \nu - 1$ . There exists a resultant associated to the system  $\bar{P}_1, \dots, \bar{P}_n$ , called the *residual resultant*. We denote it by  $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$ . Let us recall its main properties (see [1] §3.1).

- $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$  is a homogeneous and *irreducible* polynomial in the ring of all the coefficients  $\mathbb{Q}[c_{ij}^{|\alpha|}]$ ,
- For any given specialization of the coefficients  $c_{ij}^{|\alpha|}$ 's sending  $\bar{P}_i$  to  $Q_i$ , we have

$$\text{Res}_G(Q_1, \dots, Q_n) = 0 \text{ if and only if } (Q_1, \dots, Q_n)^{\text{sat}} \subsetneq G = G^{\text{sat}},$$

- $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$  is multi-homogeneous: it is homogeneous in the coefficients of each polynomials  $\bar{P}_i$ ,  $i = 1, \dots, n$ , and we have

$$\deg_{\bar{P}_i}(\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)) = \frac{d_1 \cdots d_n}{d_i} - \mathbf{a}.$$

We are now going to compare this residual resultant with the specialized subresultant  $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$ , which is non-zero as proved in [4]. We claim that we have the following implications:

$$\Delta_S^\nu(Q_1, \dots, Q_n) \neq 0 \Rightarrow H_{(Q)}(\nu) = \mathbf{a} \Rightarrow H_{(Q)}(t) = \mathbf{a} \text{ for all } t \geq \nu \Rightarrow \text{Res}_G(Q_1, \dots, Q_n) \neq 0, \quad (5)$$

where  $H_{(Q)}(\cdot)$  denotes the Hilbert function associated to the ideal  $(Q_1, \dots, Q_n)$ . Only the second implication needs to be proved, the others follow directly from the algebraic properties of resultants and subresultants. We know that  $H_G(t) = \mathbf{a}$  for all  $t \geq \rho - \nu + 1$  (see [9]), and since we have supposed (2), it is a straightforward computation to show that  $\nu \geq \rho - \nu + 1$ . It follows that, by hypothesis, the ideals  $G$  and  $(Q)$  coincide in degree  $\nu$  and have Hilbert function  $\mathbf{a}$  in this degree. As they are both generated in degree at most  $\nu$  this implies that they coincide in all higher degrees, and therefore they both have Hilbert function equal to  $\mathbf{a}$  in these degrees, because  $G$  is the defining ideal of a set of points.

Due to (5) and the irreducibility of the residual resultant, we deduce that  $\text{Res}_G(\bar{P}_1, \dots, \bar{P}_n)$  divides  $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$ . But both polynomials have the same degree, so they must be equal up to a rational number (giving a new formula for computing this residual resultant using [3]). Since this residual resultant is irreducible, and since  $\Delta_S^\nu$  and  $\Delta_S^\nu(\bar{P}_1, \dots, \bar{P}_n)$  have the same multi-degree, this shows that  $\Delta_S^\nu$  is irreducible in  $\mathbb{Q}[\text{coeff}(P_i)]$ .

It remains to prove that  $\Delta_S^\nu$  is irreducible in  $\mathbb{Z}[\text{coeff}(P_i)]$ . As it is irreducible in  $\mathbb{Q}[\text{coeff}(P_i)]$ , we only have to show that  $\Delta_S^\nu$  has content  $\pm 1$ . Suppose that this is not the case, and let  $p \in \mathbb{Z}$  be a prime dividing the content of  $\Delta_S^\nu$ . Let  $k$  be the algebraic closure of  $\mathbb{Z}_p$ . This implies that  $\Delta_S^\nu = 0$  in  $K := k(\text{coeff}(P_i))$ , and hence  $S$  is linearly dependent in  $K[x_1, \dots, x_n]/\langle P_1, \dots, P_n \rangle$ , contradicting the main result of [4].

**Reducibility in lower degrees:** We now exhibit some sets  $S$  of degree  $\nu = \rho - \min\{d_i\} + 1$  such that  $\Delta_S^\nu$  factorizes. This shows that the lower bound in our theorem is sharp.

- **$\mathbf{n} = 2$ ,  $\mathbf{d}_1 > \mathbf{d}_2$ :** In this case,  $\nu = d_1 - 1 \geq d_2$ , and  $H_{d_1, d_2}(\nu) = d_2$ . Thus  $\Delta_S^\nu$  can be here computed with Sylvester type matrices [13]. However, setting  $f_2 = c_0 x_1^{d_2} + c_1 x_1^{d_2-1} x_2 + \dots + c_{d_2} x_2^{d_2}$ , the universal property of the subresultant  $\Delta_S^\nu$  shows immediately that it is a power of  $c_0$ , and we have already seen that its degree is  $d_1 - d_2 + 1$ ; it follows that  $\Delta_S^\nu = c_0^{d_1 - d_2 + 1}$ , so it can not be irreducible.

- $\mathbf{n} > 2$ ,  $\mathbf{d}_1 - 1 > \mathbf{d}_2 = \mathbf{d}_3 = \dots = \mathbf{d}_n = 1$ : Again in this case,  $\nu = d_1 - 1$  and  $H_{d_1, d_2}(\nu) = 1$ . Choose  $S = \{x_1^\nu\}$  and, if  $f_i = c_{1i}x_1 + \dots + c_{ni}x_n$ ,  $i = 2, \dots, n$ , we set  $\delta := \det(c_{ij})_{2 \leq i, j \leq n}$ . Applying Lemma 4.4 in [6] to this situation, we get that  $\Delta_S^\nu = \delta^\nu$ . So,  $\Delta_S^\nu$  is not irreducible.

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