Resultants of determinantal varieties

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Abstract

In this paper, a new kind of resultant, called the *determinantal resultant*, is introduced. This operator computes the projection of a determinantal variety under suitable hypothesis. As a direct generalization of the resultant of a very ample vector bundle [GKZ94], it corresponds to a necessary and sufficient condition so that a given morphism between two vector bundles on a projective variety X has rank lower or equal to a given integer in at least one point. First some conditions are given for the existence of such a resultant and it is showed how to compute explicitly its degree. Then a result of A. Lascoux [Las78] is used to obtain it as a determinant of a certain complex. Finally some more detailed results in the particular case where X is a projective space are exposed.

1 Introduction

Projection is one of the more used operation in effective algebraic geometry and more particularly in elimination theory. The Gröbner basis theory is a powerful tool to perform projections in all generality, however its computation complexity can be very high in practice. It is hence useful to develop some other tools being able to compute such projections, even if they apply only in particular cases. *Resultants* are such tools: they are used to eliminate a set of variables (often called parameters) of a given polynomial system, providing this elimination process leads to only one equation (in the parameters). The most known resultant is the Sylvester resultant of two homogeneous bivariate polynomials. Its generalization to n homogeneous polynomials in n variables has been stated by F.S. Macaulay in 1902 [Mac02]. Let $f_1, \ldots, f_n$ be polynomials in variables $x_1, \ldots, x_n$, their resultant is a polynomial in their coefficients which vanishes if and only if they have a common homogeneous root (in the algebraic closure of the ground field). An improvement of this resultant, taking account of the monomial supports of the input polynomials, has been exposed

in [KSZ92] and yields a tool even more efficient called the sparse resultant (see also [Stu93]).

In the book of I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky [GKZ94], a more general resultant, which encapsulates all the previous cited resultants, is presented : the resultant of a very ample vector bundle E of rank n on an irreducible projective variety X of dimension $n-1$ over an algebraically closed field $\mathbb{K}$. Denoting $V = H^0(X, E)$, this resultant is the divisor of the projective space $Y = \mathbb{P}(V)$ obtained as the projection on Y of the incidence variety

$$W = \{(x, f) \in X \times Y \ : \ f(x) = 0\},$$

by the canonical map $X \times Y \to Y$. For instance, the Macaulay resultant corresponds to the case $X = \mathbb{P}^{n-1}$ and $E = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^{n-1}}(d_i)$ with $d_i > 0$ for $i = 1, \ldots, n$ (in such a situation where the vector bundle E splits into n line bundles the resultant is called the *mixed* resultant in [GKZ94]). The aim of this paper is to extend this resultant of complete intersection varieties to the context of determinantal varieties. To be more precise, being given n homogeneous polynomials $f_1, \ldots, f_n$ in variables $x_1, \ldots, x_n$, the Macaulay resultant vanishes if and only if $f_1, \ldots, f_n$ have a common root on $\mathbb{P}^{n-1}$, that is the line matrix $(f_1, \ldots, f_n)$ is of rank zero in at least one point of $\mathbb{P}^{n-1}$. More generally the resultant of the vector bundle E can be interpreted as an operator which traduces a rank default of a morphism

$$E^* \xrightarrow{\ f\ } \mathcal{O}_X,$$

in at least one point of X. Consequently, we can ask for a "resultant" condition so that a morphism between two vector bundles on an irreducible projective variety X is of rank lower or equal to a given integer (which was for instance zero in the previous example). The first apparition of this problem in a concrete situation seems to be a talk of H. Lombardi. Notice that J.P. Jouanolou has also worked on such a problem, apparently in the particular case that we will call the principal case (unpublished).

This paper is organized as follows. In the section 2, we define what we will call a rth-determinantal resultant. Being given two vector bundles E and F of respective rank $m \geq n$ on an irreducible projective variety X and an integer $0 \leq r < n$, the rth-determinantal resultant of E and F on X is a divisor on the vector space of morphisms of E in F which gives a necessary and sufficient condition so that such a morphism has rank lower or equal to r in at least on point of X. We give suitable hypothesis on the dimension of X and on the vector bundles E and F such that this determinantal resultant exists. Then we show that, as for the resultant of a very ample vector bundle of [GKZ94], we can give the degree (and the multi-degree if E splits) of this determinantal resultant in terms of Chern classes of E and F. In section 3, we recall the resolution of a determinantal variety given by A. Lascoux in [Las78].

We use it in section 4 to show that, always as for the resultant of [GKZ94], the determinantal resultant can be computed as the determinant of a certain complex. Finally in section 5 we deal with the particular case where X is a projective space, it is possible to be more explicit in this situation.

This work is based on the sixth chapter of a thesis defended at the University of Nice [Bus01]. I would like to thank André Hirschowitz, learning mathematics with him was a real pleasure.

2   Construction of the determinantal resultant

Let X be an irreducible projective variety over an algebraically closed field $\mathbb{K}$. Let E and F be two vector bundles on X of respective rank m and n such that $m \geq n$. We denote by $H = \mathrm{Hom}(E, F)$ the finite dimensional vector space of morphisms from E to F. For all integer $k \geq 0$, and for all morphism $\varphi \in H$, we denote by $X_k(\varphi)$ the kth-determinantal variety of φ defined by

$$X_k(\varphi) = \{x \in X : \mathrm{rank}(\varphi(x)) \leq k\}.$$

It is well known that this variety has codimension at most $(m-k)(n-k)$. Let $Y = \mathbb{P}(H)$ be the projective space associated to H and let r be a fixed integer such that $m \geq n > r \geq 0$. We denote by ∇ the variety

$$\nabla = \{\varphi \in Y \ : \ \exists x \in X \ \mathrm{rank}(\varphi(x)) \leq r\} = \{\varphi \in Y \ : \ X_r(\varphi) \neq \emptyset\},$$

and call it the *resultant* variety. All along this paper, we will be interested in this variety in the particular case where it is a hypersurface of Y. Before stating the main theorem of this section which gives some conditions so that ∇ is a hypersurface of Y, we recall briefly the definition of the Hilbert scheme $\mathcal{H}_P(X)$ where P is a polynomial of the ring $\mathbb{K}[\nu]$ and refer to [EH00] for more details.

Consider the functor $h_{P,X}$ from the opposite category of schemes to the category of sets,

$$h_{P,X} : \begin{array}{rcl} (\mathfrak{S}chemes)^{opp} & \to & (\mathfrak{S}ets) \\ B & \mapsto & h_{P,X}(B), \end{array}$$

which associates to all scheme B the set $h_{P,X}(B)$ of subschemes $\mathcal{X} \subset X \times B$, flat over B, with fibers at each point of B admitting P as Hilbert polynomial. This functor is representable and the Hilbert scheme $\mathcal{H}_P(X)$ is defined as the scheme representing it. It parameterizes all the subschemes of X with the same Hilbert polynomial P. It comes with a universal subscheme $\mathcal{U} \subset X \times \mathcal{H}_P(X)$, flat over $\mathcal{H}_P(X)$ with Hilbert polynomial P, associated to the identity map. In this way, all subscheme $Y \subset X \times B$ flat over B with Hilbert polynomial

P is isomorphic to the fiber product $Y \simeq \mathcal{X} \times_{\mathcal{H}_P(X)} B \subset X \times B$ for a unique morphism $B \to \mathcal{H}_P(X)$. In what follows we will only consider the Hilbert scheme $\mathcal{H}_2(X)$ which parameterizes all the subschemes of X with Hilbert polynomial $P = 2$, that is all the zero-dimensional subschemes of degree 2 of X.

Theorem 1 *Let r be a positive integer such that $m \geq n > r \geq 0$, and suppose that X is an irreducible projective variety of dimension $(m - r)(n - r) - 1$. Suppose also that for all $z \in \mathcal{H}_2(X)$ the canonical restriction morphism*

$$H \longrightarrow H^0(z, \mathcal{H}om(E, F)_{|z})$$

is surjective, then ∇ is an irreducible hypersurface of Y.

PROOF. The proof of this theorem follows the same path than the proof of the similar result for more classical resultants (see [Jou91,GKZ94]). We first construct an associated incidence variety, compute its dimension, and show that it projects birationally on the resultant variety. The more technical point is to prove that this projection is birational.

Consider the incidence variety

$$W = \{(x, \varphi) \in X \times Y \ : \ \mathrm{rank}(\varphi(x)) \leq r\} \subset X \times Y,$$

and the two canonical projections

$$X \xleftarrow{p} W \xrightarrow{q} Y.$$

The resultant variety ∇ is obtained as the projection of W on Y, that is

$$q(W) = \{\varphi \in Y \ : \ \exists x \in X \ \mathrm{rank}(\varphi(x)) \leq r\} = \nabla.$$

Let W_x be the fiber of W at the point $x \in X$. Since, by hypothesis, H generates $\mathcal{H}om(E, F)$, the image of the evaluation morphism at x

$$W_x \xrightarrow{ev_x} \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$$

consists in all the morphisms of $\mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$ of rank lower or equal to r. These morphisms form an irreducible variety of $\mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$ of codimension $(m - r)(n - r)$. As the fibers of the evaluation morphism are isomorphic to morphisms from E to F which vanish at x, hence in particular irreducible, we deduce that W_x is irreducible. We deduce also that its codimension in H is the codimension of the morphisms of $\mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$ of rank lower or equal to r, that is $(m - r)(n - r)$. Denoting by $s + 1$ the dimension of H, Y has dimension s and we obtain that W is an irreducible projective variety of dimension

$$\dim(W) = (m - r)(n - r) - 1 + s - (m - r)(n - r) = s - 1.$$

Now if we can show that the morphism $W \to \nabla$ induced by q is birational, then we show that ∇ is an irreducible projective variety of codimension 1 in H, birational to W. Consequently, it remains to prove that the morphism $W \xrightarrow{q} \nabla$ is birational, that is its generic fiber is a smooth point (notice here that since X is a variety over a field its singular locus is of codimension at least one, see [Har77], II.8.16). To do this, we will show that the subvariety of Y consisting in all the morphism having rank lower or equal to r on a zero-dimensional subscheme of degree 2 of X (that is on two distinct points or on a double point) is of codimension greater or equal to 2.

We denote by r (resp. s) the canonical projection of the universal subscheme $\mathcal{U} \subset X \times \mathcal{H}_2(X)$ of the Hilbert scheme $\mathcal{H}_2(X)$ on X (resp. $\mathcal{H}_2(X)$),

$$X \xleftarrow{r} \mathcal{U} \xrightarrow{s} \mathcal{H}_2(X).$$

By definition of $\mathcal{U}$ and $\mathcal{H}_2(X)$, the morphism s is flat and projective. As $r^*\mathcal{H}om(E, F)$ is a bundle over $\mathcal{U}$, it is flat over $\mathcal{U}$, and hence $r^*\mathcal{H}om(E, F)$ is flat over $\mathcal{H}_2(X)$. In this way all the hypothesis of the "changing basis" theorem are satisfied (see [Har77] theorem 12.11). For all $z \in \mathcal{H}_2(X)$ the fiber of the morphism s, denoted $\mathcal{U}_z$, is zero-dimensional in $\mathcal{U}$ and hence $H^1(\mathcal{U}_z, r^*(\mathcal{H}om(E, F))_z) = 0$; we deduce that $s_* r^*\mathcal{H}om(E, F)$ is a bundle over $\mathcal{H}_2(X)$ of rank $2mn$ and that its fiber at a point z of $\mathcal{H}_2(X)$ satisfies

$$s_* r^*\mathcal{H}om(E, F)_z \simeq H^0(\mathcal{U}_z, r^*\mathcal{H}om(E, F)_z) \simeq H^0(z, \mathcal{H}om(E, F)_{|z}).$$

We consider now the incidence variety

$$D = \{(z, \varphi) \in \mathcal{H}_2(X) \times Y \ : \ \mathrm{rank}(s_* r^*\varphi(z)) \leq r\} \subset \mathcal{H}_2(X) \times Y.$$

The image of the natural projection of D onto Y is exactly the set of $\varphi \in Y$ which vanish on a zero-dimensional subscheme of degree 2 of X. Let z be a point of the Hilbert scheme $\mathcal{H}_2(X)$. It is a zero-dimensional subscheme of degree 2 on X; it is hence associated to a dimension 2 $\mathbb{K}$-algebra, and hence of the form $\mathrm{Spec}(\mathbb{K} \oplus \mathbb{K})$ or $\mathrm{Spec}(\mathbb{K}[\epsilon]/\epsilon^2)$.

Let D_z be the fiber of the projection $D \to \mathcal{H}_2(X)$ above a point $z \in \mathcal{H}_2(X)$. If z is of type $\mathrm{Spec}(\mathbb{K} \oplus \mathbb{K})$, that is two $\mathbb{K}$-rational distinct points x and y of X, then $s_* r^*\mathcal{H}om(E, F)_z \simeq \mathcal{H}om(E, F)_x \oplus \mathcal{H}om(E, F)_y$. By hypothesis, the double evaluation at x and y

$$D_z \xrightarrow{ev_x \times ev_y} \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n)) \times \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$$

has image all pairs of morphisms (f, g) such that $f(x)$ and $g(y)$ are of rank lower or equal to r. The fibers of this morphism being isomorphic to morphisms vanishing at x and y, we deduce that the codimension of D_z in Y is the one of all pairs of morphisms (f, g) such that $f(x)$ and $g(y)$ are of rank lower or equal to r in $\mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n)) \times \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n))$, that is $2(m-r)(n-r)$. If z is now of type $\mathrm{Spec}(\mathbb{K}[\epsilon]/\epsilon^2)$, that is a $\mathbb{K}$-rational point x in X and a tangent

vector $t \in T_x X$ at x, then $s_* r^* \mathcal{H}om(E, F)_z \simeq (\mathbb{K}[\epsilon]/\epsilon^2)^{mn}$. A morphism $\varphi \in Y$ evaluated at $x + \epsilon t$ is identified to $\varphi(x + \epsilon t) = f(x) + g(t)\epsilon$. As for the case of two simple points, the evaluation morphism

$$D_z \to \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n)) \times \mathbb{P}(\mathrm{Hom}(\mathbb{K}^m, \mathbb{K}^n)) : \varphi \to (f, g)$$

has image all pairs of morphisms (f, g) such that $f(x)$ and $g(t)$ are of rank lower or equal to r and, by the same arguments, D_z is hence of codimension $2(m - r)(n - r)$ in Y.

Since the dimension of $\mathcal{H}_2(X)$ is twice the dimension of X, we deduce that

$$\dim(D) = 2(m - r)(n - r) - 2 + s - 2(m - r)(n - r) = s - 2,$$

and hence that the projection of D onto Y is of codimension greater or equal to 2.

Under the hypothesis of this theorem we define $\mathrm{Res}_{E,F,r}$, and call it the rth-*determinantal resultant of E and F*, to be the irreducible polynomial equation (defined up to a nonzero multiplicative constant of $\mathbb{K}$) of the hypersurface ∇. This determinantal resultant satisfies the property : for all $\varphi \in H = \mathrm{Hom}(E, F)$,

$$\mathrm{Res}_{E,F,r}(\varphi) = 0 \iff X_r(\varphi) \neq \emptyset.$$

Remark 2 In the particular case $F = \mathcal{O}_X$ the 0th-determinantal resultant is exactly the resultant associated to a vector bundle described in [GKZ94], chapter 3, section C. If we suppose moreover that E splits in $E = \oplus_{i=1}^m \mathcal{L}_i$ where each $\mathcal{L}_i$ is a line bundle such that its dual is very ample, the 0th-determinantal resultant corresponds to the mixed resultant ([GKZ94], chapter 3, section A) associated to $\mathcal{L}_i^*$, $i = 1, \ldots, m$. Continuing the specialization, if we suppose moreover that $X = \mathbb{P}^{m-1}$ and $\mathcal{L}_i \simeq \mathcal{O}_X(-d_i)$, with $d_i \geq 1$, for all $i = 1, \ldots, m$, then the 0th-determinantal resultant corresponds to the Macaulay resultant widely studied by J.P. Jouanolou (see [Jou91,Jou97]).

As we can see in the hypothesis of theorem 1, being given X, E and F, the rth-determinantal resultant $\mathrm{Res}_{E,F,r}$ not always exists. In fact such a determinantal resultant exists only if the integer r satisfies the inequality $m \geq n > r \geq 0$ and the equality $(m - r)(n - r) = \dim(X) + 1$. From this we can see directly that the $(n - 1)^{th}$-determinantal resultant, which we will often call the *principal determinantal resultant*, exists if $\dim(X) = m - n$. Also we can see that if $\dim(X) + 1$ is prime then the principal determinantal resultant is the only one possible (and exists if $\dim(X) = m - n$).

As $\mathrm{Res}_{E,F,r}$ represents, when it exists, an irreducible and reduced divisor of $Y = \mathbb{P}(H)$, it is quite natural to ask for its degree. We begin by fixing some notations.

If V is a vector bundle on a variety Z,

$$c_t(V) = 1 + c_1(V)t + c_2(V)t^2 + \ldots$$

denotes its Chern polynomial. We define the Chern polynomial of the virtual bundle $-V$ to be

$$c_t(-V) = \frac{1}{c_t(V)} = 1 - c_1(V)t + (c_1^2(V) - c_2(V))t^2 + \ldots.$$

For all formal series

$$s(t) = \sum_{k=-\infty}^{+\infty} c_k(s)t^k,$$

we set

$$\Delta_{p,q}(s) = \det \begin{pmatrix} c_p(s) & \cdots & c_{p+q-1}(s) \\ \vdots & & \vdots \\ c_{p-q+1}(s) & \cdots & c_p(s) \end{pmatrix}.$$

Proposition 3 *We suppose that all the assumptions of theorem 1 are satisfied. Let $f(t, \alpha)$ be the polynomial in two variables defined by*

$$f(t, \alpha) = \sum_{k=0}^{m} \left(c_k(E) - (m - k + 1)c_{k-1}(E)\alpha \right) t^k.$$

Then, the degree of the determinantal resultant $\mathrm{Res}_{E,F,r}$ is

$$(-1)^{(m-r)(n-r)} \int_X \Delta_{m-r,n-r}(f(t, \alpha)/c_t(F))_\alpha,$$

where the subscript α denotes the coefficient of α of the univariate polynomial $\Delta_{m-r,n-r}(f(t, \alpha)/c_t(F))$, and where $\int_X$ denotes the degree map of X.

PROOF. We begin again with the notations of the proof of theorem 1. The incidence variety

$$W = \{(x, \varphi) \in X \times Y \ : \ \mathrm{rank}(\varphi(x)) \le r\} \subset X \times Y,$$

where $Y = \mathbb{P}(\mathrm{Hom}(E, F))$, has the two canonical projections

$$X \xleftarrow{p} W \xrightarrow{q} Y.$$

We showed that the morphism q is birational on its image which is exactly the variety ∇ defined by the vanishing of the determinantal resultant $\mathrm{Res}_{E,F,r}$. We have also showed that $W \subset X \times Y$ is irreducible of codimension $(m-r)(n-r)$.

Now we denote also (abusing notations) by p and q both projections

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y,$$

and consider the vector bundle $p^*(\mathcal{H}om(E, F)) \otimes q^*(\mathcal{O}_Y(1))$ over $X \times Y$. The vector space of global sections of this vector bundle is naturally identified with $\mathrm{End}(H)$, the vector space of endomorphisms of $H = \mathrm{Hom}(E, F)$ (morphisms from H onto itself). Let

$$\sigma : p^*(E) \otimes q^*(\mathcal{O}_Y(-1)) \longrightarrow p^*(F)$$

be the section of the vector bundle $p^*(\mathcal{H}om(E, F)) \otimes q^*(\mathcal{O}_Y(1))$ which corresponds to the identity endomorphism. Its rth-determinantal variety, defined by

$$\mathcal{D}_r(\sigma) = \{(x, f) \in X \times Y : \mathrm{rank}(\sigma(x, f)) \leq r\},$$

is exactly the incidence variety W since $\sigma(x, f) = f(x)$ for all $(x, f) \in X \times Y$. It is of codimension $(m - r)(n - r)$, which is its waited codimension since the vector bundles $\mathcal{E} = p^*(E) \otimes q^*(\mathcal{O}_Y(-1))$ and $\mathcal{F} = p^*(F)$ on $X \times Y$ are respectively of rank m and n. If we denote by h the generic hyperplane of Y and $\alpha = q^*(h)$, the Thom-Porteous formula (see [ACGH85], chapter II) and the birationality of $q : W \to \nabla$ show that the degree of $\mathrm{Res}_{E,F,r}$ is given by the coefficient of h of

$$q_*(\Delta_{n-r,m-r}(c_t(\mathcal{F} - \mathcal{E}))) = q_*((-1)^{(m-r)(n-r)} \Delta_{m-r,n-r}(c_t(\mathcal{E} - \mathcal{F}))),$$

that is

$$(-1)^{(m-r)(n-r)} \int_Y q_* \left(\Delta_{m-r,n-r}(c_t(\mathcal{E})/c_t(\mathcal{F}))_\alpha \right),$$

where $\int_Y$ denotes the degree map over Y. Now the Chern polynomial of the bundle $\mathcal{E}$ equals

$$
\begin{aligned}
c_t(\mathcal{E}) &= \sum_{k=0}^{m} \left(\sum_{i=0}^{k} \mathbf{C}_{m-i}^{k-i} c_i(p^*(E))(-\alpha)^{k-i} \right) t^k \\
&= \sum_{k=0}^{m} \left(c_k(p^*(E)) - (m - k + 1)c_{k-1}(p^*(E))\alpha \right) t^k + O(\alpha^2).
\end{aligned}
$$

In this way we can see $\Delta_{m-r,n-r}(c_t(\mathcal{E} - \mathcal{F}))$ as a polynomial in the variable α. As p^* commutes with Chern classes, we obtain the desired result from the projection formula.

Remark 4 If we focus on the principal case, the preceding formula shows that the degree of a principal determinantal resultant is given by

$$(-1)^{m-n+1} c_{m-n+1} \left(f(t, \alpha)/c_t(F) \right)_\alpha.$$

If we suppose moreover that $F = \mathcal{O}_X^{\oplus n}$ then this formula specializes in the simple expression $(-1)^{m-n} n c_{m-n}(E)$, that is $n c_{m-n}(E^*)$. Finally if we suppose moreover that $n = 1$ (i.e. $F = \mathcal{O}_X$ and $r = 0$) we obtain $c_{m-1}(E^*)$ which is well the degree of the resultant associated to a very ample vector bundle given in [GKZ94], chapter 3, theorem 3.10).

Classical resultants of polynomial systems are known to be multi-homogeneous, depending of the geometry of the system. This property can be stated with the notion of mixed resultant (see [GKZ94], chapter 3). Let X be an irreducible projective variety of dimension $m - 1$, let $\mathcal{L}_1, \ldots, \mathcal{L}_m$ be m very ample line bundles on X and denote by $V_i = H^0(X, \mathcal{L}_i)$. Then the mixed resultant of $\mathcal{L}_1, \ldots, \mathcal{L}_m$ is a polynomial over $V = \oplus_{i=1}^m V_i$ which is multi-homogeneous in each V_i of degree $\int_X \prod_{j \neq i} c_1(\mathcal{L}_j)$ (recall that the mixed resultant can be seen as the principal determinantal resultant of $E = \oplus_{i=1}^m \mathcal{L}_i^*$ and $F = \mathcal{O}_X$). In the context of the determinantal resultant this multi-homogeneity appears as follows.

Proposition 5 *We suppose that all the assumptions of theorem 1 are satisfied and that $E = \oplus_{i=1}^m E_i$ where each E_i is a line bundle on X. Let $f(t, \alpha_1, \ldots, \alpha_m)$ be the polynomial in $m + 1$ variables defined by*

$$f(t, \alpha_1, \ldots, \alpha_m) = \prod_{i=1}^m (1 + (c_1(E_i) - \alpha_i)t).$$

The rth-determinantal resultant is multi-homogeneous in each vector space $H_i = \mathrm{Hom}(E_i, F)$, $i = 1, \ldots, m$, and its degree with respect to H_i is

$$(-1)^{(m-r)(n-r)} \int_X \Delta_{m-r,n-r}(f(t, \alpha_1, \ldots, \alpha_n)/c_t(F))_{\alpha_i},$$

where the subscript α_i denotes the coefficient of α_i of the multivariate polynomial $\Delta_{m-r,n-r}(f(t, \alpha_1, \ldots, \alpha_n)/c_t(F))$ (in variables $\alpha_1, \ldots, \alpha_m$).

PROOF. This proof is just a refinement of the proof of proposition 3. If the vector bundle E splits into $E = \oplus_{i=1}^m E_i$ then clearly $H = \mathrm{Hom}(E, F) = \oplus_{i=1}^m \mathrm{Hom}(E_i, F) = \oplus_{i=1}^m H_i$. Each morphism $\varphi \in H$ can be decomposed as $\varphi = \oplus_{i=1}^m \varphi_i$ where each $\varphi_i \in H_i$. Now multiplying each morphism φ_i by its own nonzero constant λ_i do not change the rank, that is

$$\forall x \in X \quad \mathrm{rank}(\varphi(x)) = \mathrm{rank}(\oplus_{i=1}^m \lambda_i \varphi_i(x)).$$

It follows that $\mathrm{Res}_{E,F,r}$ is multi-homogeneous with respect to each H_i. Consequently we can see our determinantal resultant in $Y_1 \times \ldots \times Y_m$, where Y_i denotes the projective space $\mathbb{P}(H_i)$, instead of $Y = \mathbb{P}(H)$.

Taking again the proof of proposition 3 the incidence variety W is obtained

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as the zero locus of the canonical section

$$\sigma : \mathcal{E} = \bigoplus_{i=1}^{m} (p^*(E_i) \otimes q^*(\mathcal{O}_{Y_i}(-1))) \to \mathcal{F} = p^*(F).$$

Denote by h_i the generic hyperplane of Y_i for all $i = 1, \ldots, m$. The Chern polynomial of the bundle $\mathcal{E}$ can be written $c_t(\mathcal{E}) = \prod_{i=1}^{m}(1 + (c_1(E_i) - \alpha_i)t)$, where $\alpha_i = q^*(h_i)$, and we obtain the desired result as in proposition 3.

3 Resolution of a determinantal variety

In this section we will focus on the following problem: being given two vector bundles E and F on a scheme X and a sufficiently generic morphism $\varphi : E \to F$, describe the locus Y of points of X where φ has rank lower or equal to a given integer r; more precisely, define an exact complex $L(\varphi, r)^\bullet$ of vector bundles over X which gives a resolution of the trivial vector bundle of Y, $\mathcal{O}_Y$. This problem has been solved by Alain Lascoux in its paper titled *"Syzygies des variétés déterminantales"*, [Las78]. Its result is based on the fact that Y is birational to a subscheme Z of a grassmannian G on X, that is there exists a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & G \\ \downarrow & & \pi \downarrow \\ Y & \longrightarrow & X, \end{array}$$

such that the restriction of the map π to Z is birational on Y, and such that $\mathcal{O}_Z$ admits a known resolution $K^\bullet$ on G (this resolution is the Koszul complex of a section of a vector bundle over G). The author then studies the spectral sequence associated to the projection of the complex $K^\bullet$ on X, and constructs in this way a resolution of $\mathcal{O}_Y$ on X. Notice that we have to suppose here that the algebraically closed field $\mathbb{K}$ is of characteristic zero, hypothesis needed in [Las78]. However we will see at the end of this section that this hypothesis is not needed in the particular case of interest $r = \min(\operatorname{rank}(E), \operatorname{rank}(F)) - 1$ (the principal case).

In what follows we gather (from the original paper of Alain Lascoux [Las78]) the principal ingredients in order to state the nice results obtained by Alain Lascoux we will use in the next section to compute the determinantal resultant. Before giving the resolution $L(\varphi, r)^\bullet$ itself we recall first some definitions on partitions and Schur functors.

3.1   Partitions and Schur functors

3.1.1   Partitions

Let r be an integer. A *partition* is a r-uples $I = (i_1, \ldots, i_r)$ in $\mathbb{N}^r$ such that $0 \leq i_1 \leq \ldots \leq i_r$; its *length* is the number of its nonzero elements, and its *weight*, denoted $|I|$, is $i_1 + \ldots + i_r$. We will identify naturally two partitions which differ only by adding zeros on the left, but however it will be useful to consider the partition $(0, \ldots, 0) \in \mathbb{N}^r$ which will be denoted O_r.

A partition is usually represented by an interval of $\mathbb{N} \times \mathbb{N}$, its *Ferrer diagram*; the first line contains i_r boxes, the second i_{r-1} boxes, and so on... This yields the following picture :
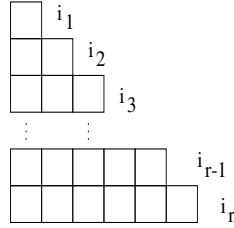
A partition J is said to be *contained* in I, denoted $J \subseteq I$, if it is true for their Ferrer diagrams.

On the set of all partitions we define an involution map which sends each partition I to the partition $I^\star$, called the *dual* partition of I, such that its Ferrer diagram is the transposed Ferrer diagram of I. For instance $(1, 2, 4)^\star = (1, 1, 2, 3)$.

If I and J are two partitions $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_q)$ such that $i_r \leq j_1$, the *concatenation* $IJ = conc(I, J) = (i_1, \ldots, i_r, j_1, \ldots, j_q)$ is a partition. It is possible to extend this operator as follows. Let r and q be two integers, $I \in \mathbb{Z}^r$, $J \in \mathbb{Z}^q$, and consider the set $\{i_1, i_2 + 1, \ldots, i_r + r - 1, j_1 + r, \ldots, j_q + r + q - 1\}$. If all these numbers are positive and distinct, this set can be write in a unique way as $\{h_1, \ldots, h_{r+q} + r + q - 1\}$, with $H = (h_1, \ldots, h_{r+q})$ a partition. We say that H is the *concatenation* of I and J, with *ampleness* $n(I, J)$, the minimal number of transpositions to rectify the first set onto the second. Otherwise we set $conc(I, J) = \emptyset$ and $n(I, J) = \infty$.

3.1.2   Tabloïds - Young diagrams

Let E be a set and I be a partition. A *tabloïd* of diagram I with values in E is a filling of the diagram of I with elements of E.

Let τ be a tabloïd and μ be a permutation of the boxes of the diagram of I. We denote by τ^μ the tabloïd induced by this permutation and we define $Tabl_I E$ as the free $\mathbb{Z}$-module with basis all tabloïds of diagram I.

Suppose now the set E has a total ordering. A *Young diagram* of a diagram I is a tabloïd of diagram I such that each column read from the low to the top is a strictly growing sequence, and each line, from the left to the right, is a growing sequence.

3.1.3 Schur functors

Let E be a vector space, r be an integer and I be an element of $\mathbb{N}^r$. We denote by $S^I E$ the tensor product $S^{i_1} E \otimes \ldots \otimes S^{i_r} E$, where $S^\bullet E$ is the symmetric algebra of E. In the same way $\wedge^I E = \wedge^{i_1} E \otimes \ldots \otimes \wedge^{i_r} E$, where $\wedge^\bullet E$ is the exterior algebra of E.

It is possible to represent the elements of $S^i E$ by line tabloïds, and the elements of $\wedge^i E$ by column tabloïds (such a tabloïd is not unique in general). Let I be a partition and $J = I^\star$ its dual, we have canonical morphisms

$$\psi_S : Tabl_I E \to S^I E$$

$$\psi_\wedge : Tabl_I E \to \wedge^J E$$

which are obtained by reading the lines or the columns of a tabloïd of diagram I.

Let ρ_I' be the endomorphism of $Tabl_I E$

$$
\begin{aligned}
Tabl_I E &\to & Tabl_I E \\
\tau &\mapsto & \sum_\mu (-1)^\mu \tau^\mu
\end{aligned}
$$

where the sum is running over all permutations μ preserving the columns of the diagram, with $(-1)^\mu = \mathrm{signature}(\mu)$. Then there exists a unique morphism which makes the following diagram commutative :

$$
\begin{array}{ccc}
Tabl_I E & \xrightarrow{\rho_I'} & Tabl_I E \\
\psi_\wedge \downarrow & & \psi_S \downarrow \\
\wedge^J E & \xrightarrow{\rho_I} & S^I E.
\end{array}
$$

Choosing a total ordering on a basis of E, one shows that the image of the map ρ_I is a vector space with basis the images of Young diagrams by the composed map $\psi_S \circ \rho_I'$.

This construction for a vector space E is easily extended to a vector bundle E on a scheme X. We define the *Schur functor of index I* (from the category of vector bundles on X to itself), denoted indifferently S_I or $\wedge_J$, as the image functor of $\wedge^J$ in S^I by ρ_I (we set $S_\emptyset = 0$).

Notice that if the length of I is strictly greater than the rank of E then $S_I E = 0$, since $\wedge^{length(l)} E = 0$.

3.2   The resolution

Let E and F be two vector bundles of respective rank m and n, $m \geq n$, on a scheme X and φ a morphism between them $\varphi : E \to F$. Let also r be a positive integer strictly lower to n and Y be the subscheme of X defined by

$$Y = \{x \in X : \mathrm{rank}(\varphi(x)) \leq r\}.$$

Its trivial bundle $\mathcal{O}_Y$ is defined as $\mathrm{coker}(\varphi')$ where

$$\varphi' : \wedge^{r+1} E \otimes \wedge^{r+1} F^* \to \mathcal{O}_X$$

is associated to the morphism $\wedge^{r+1}\varphi$.

Consider now the relative grassmannian $G = G_r(F) \xrightarrow{\pi} X$ of quotients of F of rank $q = n - r$, and

$$0 \to S \to \pi^*(F) \to Q \to 0$$

its tautological exact sequence. Let Z be the subscheme of $G_r(F)$ defined as the zero locus of the composed morphism $\pi^*E \to \pi^*F \to Q$, i.e. $\mathcal{O}_Z$ is defined as the cokernel of the induced map $\psi : \pi^*E \otimes Q^* \to \mathcal{O}_G$. We suppose that Z is *locally a complete intersection*, that is the Koszul complex associated to ψ,

$$0 \to K^{-m(n-r)} \to \ldots \to K^p = \wedge^{-p}(\pi^*E \otimes Q^*) \to \ldots \to K^0 = \mathcal{O}_G,$$

is exact (this hypothesis implies that the restriction of π to $Z \to Y$ is birational, see [Las78]). We can then construct a complex $L(\varphi, r)^\bullet$ from the non-degenerated spectral sequence

$$\mathcal{E}^{s,t} = R^t\pi_*(\wedge^{-s}(\pi^*E \otimes Q^*)),$$

by setting $L^p = \bigoplus_{s+t=p} \mathcal{E}^{s,t}$.

Denoting $n(I) = n(I, O_r)$ and $I' = conc(I, O_r)$ the Cauchy formula and the Bott theorem (see [Las78]) show that

$$L^p = \bigoplus_{s+t=p} \mathcal{E}^{s,t} = \bigoplus_{-|I|+n(I)=p} \wedge_I E \otimes S_{I'} F^*,$$

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where $S_{I'}F^* = R^{n(I)}\pi_*(S_I Q^*)$ (which implies that I must be of length lower or equal to q so that $S_{I'}F^*$ is non-zero). The following lemma gives an explicit description of I' and $n(I)$:

Lemma 6 ([Las78], 5.10) *Let $I = (i_1, \ldots, i_q)$ and let $p(I)$ be the dimension of the greatest square contained in the diagram of I, then :*

- $p = p(I)$ *is such that $i_{q-p} + q - p - 1 \leq n - 1$ and $i_{q-p+1} + q - p \geq n$,*
- *If $i_{q-p+1} < p + r$ then $I' = \emptyset$,*
- *If $i_{q-p+1} \geq p + r$ then $n(I) = pr$ and*

$$I' = (i_1, \ldots, i_{q-p}, \underbrace{p, \ldots, p}_{r}, i_{q-p+1} - r, \ldots, i_q - r).$$

Now we have to link the terms L^p to obtain the complex $L(\varphi, r)^\bullet$. The lemma 5.13 of [Las78] tells us that if (I, H) is a pair of partitions such that $H \subset I$, $H' \neq \emptyset$, $I' \neq \emptyset$ and $|I| - n(I) = |H| - n(H) - 1$, then I and H have the same respective parts, excepted one, and the same is true for $I'^\star$ and $H'^\star$. More precisely we have

$$I = \bullet - \bullet i \bullet - \bullet, \ I'^\star = \circ - \circ i' \circ - \circ,$$

$$H = \bullet - \bullet h \bullet - \bullet, \ H'^\star = \circ - \circ h' \circ - \circ,$$

where $\bullet - \bullet$ (resp. $\circ - \circ$) is the set where I and H are the same (resp. $I'^\star$ and $H'^\star$). For such a pair (I, H) we can hence define the morphism

$$\psi_{I,H} : \wedge_I E \otimes S_{I'} F^* \to \wedge_H E \otimes S_{H'} F^*$$

as the composition of the canonical injective and surjective maps (coming from the Pieri formula, see [Las78] (1.5.3)) and the contraction morphism

$$d_s : \wedge^i E \otimes \wedge^{i'} F \to \wedge^{i-h} E \otimes \wedge^h E \otimes \wedge^{i'-h'} F^* \otimes \wedge^{h'} F^* \to \wedge^h E \otimes \wedge^{h'} F,$$

(with $s = i - h = i' - h'$) :

$$\wedge_I E \otimes \wedge_{I'} F \longrightarrow \wedge_{\bullet - \bullet} E \otimes \wedge^i E \otimes \wedge^{i'} F^* \otimes \wedge_{\circ - \circ} F^*$$

$$\xrightarrow{d_s} \wedge_{\bullet - \bullet} E \otimes \wedge^h E \otimes \wedge^{h'} F^* \otimes \wedge_{\circ - \circ} F^* \longrightarrow \wedge_H E \otimes \wedge_{H'} F.$$

For all pair (I, H) such that H is not contained in I we set $\psi_{I,H} = 0$. In this way we define the differential maps of the complex $L(\varphi, r)^\bullet$ by :

$$L^p = \bigoplus_{-|I|+n(I)=p} \wedge_I E \otimes S_{I'} F^* \xrightarrow{\psi_p = \oplus \psi_{I,H}} L^{p+1} = \bigoplus_{-|H|+n(H)=p+1} \wedge_H E \otimes S_{H'} F^*.$$

The complex $L(\varphi, r)^\bullet$ that we obtain is such that $L^0 = \mathcal{O}_X$, and such that the term on the far left corresponds to the partition $I = (\underbrace{m, \ldots, m}_{q})$ for which

$n(I) = qr$, that is L^{qr-mq}. We can now state the following theorem :

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Theorem 7 [Las78] *Suppose that the subscheme Z of $G_r(F)$ is locally a complete intersection, then the complex $L(\varphi, r)^\bullet$ is a minimal resolution of $\mathcal{O}_Y$.*

Before ending this section we would like to say a little more on the morphism $L^{-1} \to \mathcal{O}_X$ and on the principal case.

• The morphism $L^{-1} \to \mathcal{O}_X$.
Lemma 6 describes all pairs of partitions (I, I') such that $|I| - n(I) = 1$. In fact if we want I' to be zero we need that $i_{p-q+1} \geq p + r$ and hence we obtain $|I| \geq p(p + r)$. As in this case $n(I) = pr$, the unique partition I such that $|I| - n(I) = 1$ and $I' \neq 0$ is the line partition $I = (r + 1)$. Its dual partition I' is then the column partition $I' = (\underbrace{1, \ldots, 1}_{r+1})$, and hence we deduce that $L^{-1} = \wedge^{r+1} E \otimes \wedge^{r+1} F^*$. The morphism $L^{-1} \to L^0 = \mathcal{O}_X$ is then reduced, by the description of the complex, to the contraction morphism $\wedge^{r+1} E \otimes \wedge^{r+1} F^* \to \mathcal{O}_X$ associated to the morphism $\wedge^{r+1} \varphi$. We can hence check that the cokernel of $L^{-1} \to \mathcal{O}_X$ is well $\mathcal{O}_Y$ as desired.

• The principal case.
We focus here on the case where $r = n - 1$, i.e. $q = 1$. Let s be a fixed integer such that $1 \leq s \leq m - n + 1$. We want to explicit all pairs of partitions (I, I') such that $I \neq \emptyset$, $I' \neq \emptyset$ and $|I| - n(I) = s$. As $q = 1$, I is always a line partition (i_1) and hence $p(I) = 1$. Moreover, if we want I' to be zero, we must have $i_1 \geq n$ and then $n(I) = n - 1$. We deduce that $I = (n + s - 1)$ and $I' = (\underbrace{1, \ldots, 1}_{r-1}, s)$. We obtain in this way

$$L^{-s} = \wedge^{n+s-1} E \otimes S^{s-1} F^* \otimes \wedge^n F^*.$$

We can then easily see that the complex we obtain in this case is the well known Eagon-Northcott complex. Consequently in this case we do not have to suppose that the ground field $\mathbb{K}$ is of characteristic zero since the Eagon-Northcott complex gives a resolution of $\mathcal{O}_Y$ even if $\mathbb{K}$ is not of characteristic zero (see for instance [BV80]).

4 Computation of the determinantal resultant

In section 2 we have defined the rth-determinantal resultant which gives a necessary and sufficient condition so that a given morphism between two vector bundles on an irreducible projective variety is of rank lower or equal to the integer r in at least one point. We will now show that it is possible to give an explicit representation of the determinantal resultant $\mathrm{Res}_{E,F,r}$ (when it exists), more precisely we will show that this resultant is the determinant of a certain

complex. The key point here is to obtain a resolution of the incidence variety, resolution that we will get from the preceding section, and to project it on the space of parameters, that is on $Y = \mathbb{P}(\mathrm{Hom}(E, F))$.

X is always an irreducible and projective variety of dimension $(m-r)(n-r)-1$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, where m, n and r are three positive integers such that $m \geq n > r \geq 0$. Let E and F be two vector bundles on X of respective rank m and n. We suppose that for all $z \in \mathcal{H}_2(X)$ the restriction morphism $H \longrightarrow H^0(z, \mathcal{H}om(E, F)_{|z})$ is surjective. Then, by theorem 1 the rth-determinantal resultant $\mathrm{Res}_{E,F,r}$ exists; for all $\varphi \in Y = \mathbb{P}(H)$, it satisfies

$$\mathrm{Res}_{X,E,F}(\varphi) = 0 \Longleftrightarrow X_r(\varphi) \neq \emptyset,$$

and define a divisor ∇ on Y.

We denote by p and q both projections

$$X \xleftarrow{p} X \times Y \xrightarrow{q} Y,$$

and we consider the incidence variety

$$W = \{(x, \varphi) \in X \times Y \ : \ \mathrm{rank}(\varphi(x)) \leq r\} \subset X \times Y.$$

In the proof of proposition 3, we have seen that the canonical section

$$\sigma : p^*(E) \otimes q^*(\mathcal{O}_Y(-1)) \longrightarrow p^*(F)$$

of the vector bundle $p^*(\mathcal{H}om(E, F)) \otimes q^*(\mathcal{O}_Y(1))$ on $X \times Y$ is such that its rth-determinantal variety defined by

$$\mathcal{D}_r(\sigma) = \{(x, \varphi) \in X \times Y : \mathrm{rank}(\sigma(x, \varphi)) \leq r\},$$

is exactly the incidence variety W, which is of codimension $(m - r)(n - r)$. This section can be seen as the "universal section" since the restriction of σ to a fiber $X \times \{\varphi\}$ is just the section $\varphi \in Y$. To simplify the notations we set $E' = p^*(E) \otimes q^*(\mathcal{O}_Y(-1))$ and $F' = p^*(F)$, which are two vector bundles over $X \times Y$ of respective rank m and n.

We consider now the relative grassmannian

$$\pi : G_r(F') \to X \times Y,$$

of quotients of rank $n - r$, which is isomorphic to $G_r(F) \times Y$. Denoting

$$0 \to S \to \pi^*(F') \to Q \to 0$$

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its tautological exact sequence, we define the closed subscheme Z of $G_r(F')$ as the zero locus of the composed morphism :

$$\sigma^\sharp \ : \ \pi^*(E') \xrightarrow{\pi^*(\sigma)} \pi^*(F') \to Q.$$

The morphism $\sigma^\sharp$ is hence a section of the vector bundle $\pi^*(E') \otimes Q^*$ of rank $m(n-r)$. We obtain the following commutative diagram

$$\begin{array}{ccc}
Z & \longrightarrow & G_r(F) \times Y \\
\pi_{|Z} \downarrow & & \pi \downarrow \\
\mathcal{D}_r(\sigma) & \longrightarrow & X \times Y,
\end{array}$$

where the two horizontal arrows are the canonical injections. The morphism $\pi_{|Z}$, restriction of π to Z on $\mathcal{D}_r(\sigma)$, is birational and hence Z is of codimension $m(n-r)$, that is of codimension the rank of the vector bundle $\pi^*(E') \otimes Q^*$. If we show that Z is locally a complete intersection, i.e. that the Koszul complex associated to $\sigma^\sharp$,

$$0 \to K^{-m(n-r)} \to \ldots \to K^p = \wedge^{-p}(\pi^*E \otimes Q^*) \to \ldots \to K^0 = \mathcal{O}_{G_r(F')},$$

is exact, then the theorem 7 gives us a resolution of $\mathcal{O}_W$ by vector bundles over $X \times Y$, which is the complex $L(\sigma, r)^\bullet$ (see section 3.2).

Lemma 8 *Under the hypothesis of theorem 1, Z is locally a complete intersection.*

PROOF. As the variety Y is smooth, we deduce that the projection τ : $G_r(F) \times Y \to G_r(F)$ is a smooth morphism. The restriction $\tau_{|Z} : Z \to G_r(F)$ is hence a flat morphism. Its geometric fiber at the point (x, V_x), where V_x is a vector subspace of dimension r of F_x, is identified to the set of morphisms $f \in Y$ such that $\mathrm{Im}(f(x)) \subset V_x$; this fiber is hence smooth. It follows that the morphism $\tau_{|Z} : Z \to G_r(F)$ is smooth (see [Har77], theorem III,10.2). Moreover the restriction of $\sigma^\sharp$ at each of its geometric fibers is transversal to the zero section; as the codimension of Z equals the rank of $\pi^*(E') \otimes Q^*$ we deduce that the Koszul complex associated to $\sigma^\sharp$ is exact (see [GKZ94], proposition II,1.4), and hence that Z is locally a complete intersection.

The complex $L(\sigma, r)^\bullet$ is hence a resolution of $\mathcal{O}_W$ over $X \times Y$. It is of the form

$$\ldots \to \bigoplus_{-|I|+n(I)=p} \wedge_I E' \otimes S_{I'} F'^* \to \ldots \to \mathcal{O}_{X \times Y}.$$

Let $\mathcal{M}$ be a line bundle over X. We denote by $\mathcal{L}^\bullet$ the complex $L(\sigma, r)^\bullet \otimes p^*(\mathcal{M})$; it is a resolution of $\mathcal{O}_W \otimes p^*(\mathcal{M})$. The complex $\mathcal{L}^\bullet$ of $\mathcal{O}_{X \times Y}$-modules induces

the complex $q_*(\mathcal{L}^\bullet)$ of $\mathcal{O}_Y$-modules. Its term on the far right is

$$q_*(\mathcal{L}^0) = q_*(\mathcal{O}_{X \times Y} \otimes p^*(\mathcal{M})) = \mathcal{O}_Y \otimes H^0(X, \mathcal{M}),$$

and for all $p = -1, \ldots, -(m-r)(n-r)$, we have

$$q_*(\mathcal{L}^p) = q_* \left(\bigoplus_{-|I|+n(I)=p} \wedge_I \left(p^*(E) \otimes q^*(\mathcal{O}(-1))\right) \otimes S_{I'}(p^*(F)) \otimes p^*(\mathcal{M}) \right)$$

$$= \bigoplus_{-|I|+n(I)=p} q_* \left(\wedge_I(p^*(E)) \otimes q^*(\mathcal{O}(-1))^{\otimes|I|} \otimes S_{I'}(p^*(F)) \otimes p^*(\mathcal{M}) \right)$$

$$= \bigoplus_{-|I|+n(I)=p} q_* \left(p^*(\wedge_I(E) \otimes S_{I'}(F) \otimes \mathcal{M}) \otimes q^*(\mathcal{O}_Y(-1))^{\otimes|I|} \right)$$

$$= \bigoplus_{-|I|+n(I)=p} H^0(X, \wedge_I(E) \otimes S_{I'}(F) \otimes \mathcal{M}) \otimes \mathcal{O}_Y(-|I|).$$

Remark 9 The Schur functors definition we gave show immediately that if I is a partition, E a vector bundle and L a line bundle, then $\wedge_I(E \otimes L) \simeq L^{\otimes|I|} \otimes \wedge_I(E)$.

We come to the following definition :

Definition 10 *We will say that the line bundle $\mathcal{M}$ stabilizes the complex $q_*(\mathcal{L}^\bullet)$ if for all $p = 0, \ldots, -(m-r)(n-r)$, and all partition I such that $-|I|+n(I) = p$, all the cohomology groups $H^i(X, \wedge_I(E) \otimes S_{I'}(F) \otimes \mathcal{M})$ vanish for all $i > 0$.*

We will now show that if $\mathcal{M}$ stabilizes the complex $q_*(\mathcal{L}^\bullet)$ then this complex is acyclic. To do this consider an injective resolution $I^{\bullet\bullet}$ of the complex $\mathcal{L}^\bullet$. We can choose this resolution

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 \to I^{-(m-r)(n-r),s} \to \ldots \to I^{0,s} \to & & J^s & & \to 0 \\
\uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow \\
0 \to I^{-(m-r)(n-r),0} \to \ldots \to I^{0,0} \to & & J^0 & & \to 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 \to \mathcal{L}^{-(m-r)(n-r)} \to \ldots \to \mathcal{L}^0 \to \mathcal{O}_W \otimes p^*(\mathcal{M}) \to 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0
\end{array}
$$

such that :

- the complex $J^\bullet$ is an injective resolution of $\mathcal{O}_W \otimes p^*(\mathcal{M})$,
- each column is an exact complex,
- each line is also an exact complex (since $\mathcal{L}^\bullet$ is a resolution of $\mathcal{O}_W \otimes p^*(\mathcal{M})$).

There exists two spectral sequences which converge to the hyper-direct image of the complex $\mathcal{L}^\bullet$, $\mathbb{R}^i q_*(\mathcal{L}^\bullet)$, which is by definition the ith cohomology sheaf of the total complex associated to the double complex $q_*(I^{\bullet\bullet})$. These two spectral sequences correspond respectively to column and line filtrations of this double complex $q_*(I^{\bullet\bullet})$.

The first spectral sequence is of the form

$$
E_1^{p,q} = R^q q_*(\mathcal{L}^p) \Rightarrow \mathbb{R}^i q_*(\mathcal{L}^\bullet).
$$

The cohomology of the columns of the double complex $q_*(I^{\bullet\bullet})$ vanishes for $q > 0$ since we have supposed that $\mathcal{M}$ stabilizes the complex $q_*(\mathcal{L}^\bullet)$, which implies that $R^i q_*(\mathcal{L}^j) = 0$ for $i > 0$ (using proposition 4.2.2 of [LP97]). This spectral sequence hence degenerates at the second step and we obtain

$$
E_2^{p,q} = 0 \text{ if } q \neq 0, \quad E_2^{p,0} = H^p(q_*(\mathcal{L}^\bullet)),
$$

where $E_2^{p,0}$ vanishes for $p > 0$. It follows $\mathbb{R}^i q_*(\mathcal{L}^\bullet) = 0$ if $i > 0$, and $\mathbb{R}^i q_*(\mathcal{L}^\bullet) = H^i(q_*(\mathcal{L}^\bullet))$ if $i \leq 0$.

The second spectral sequence corresponds to the filtration of the double com-

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plex $q_*(I^{\bullet\bullet})$ by lines. Each complexes

$$0 \to I^{-(m-r)(n-r),i} \to \ldots \to I^{0,i} \to J^i \to 0$$

is an exact complex of injective sheaves. We deduce that this complex splits (if we have an injective morphism $0 \to I \xrightarrow{s} K$, with I an injective object, then the identity $Id : I \to I$ can be lifted to a morphism $p : K \to I$ such that $p \circ s = Id$). Applying the functor q_* to this complex gives hence an exact split complex. Our spectral sequence, at the first step, is hence such that

$$E'^{0,q}_1 = q_*(J^q), \text{ if } E'^{p,q}_1 = 0 \text{ if } p \neq 0.$$

It degenerates at the second step, we have

$$E'^{p,q}_2 = 0 \text{ if } p \neq 0, \text{ and } E'^{0,q}_2 = R^q q_*(\mathcal{O}_W \otimes p^*(\mathcal{M})).$$

We deduce $\mathbb{R}^i q_*(\mathcal{L}^\bullet) = 0$ if $i < 0$, and $\mathbb{R}^i q_*(\mathcal{L}^\bullet) = R^i q_*(\mathcal{O}_W \otimes p^*(\mathcal{M}))$ if $i \geq 0$.

The comparison of our two spectral sequences show that $H^i(q_*(\mathcal{L}^\bullet)) = 0$ if $i < 0$, and that

$$H^0(q_*(\mathcal{L}^\bullet)) = q_*(\mathcal{O}_W \otimes p^*(\mathcal{M})) = H^0(X, \mathcal{M}) \otimes q_*(\mathcal{O}_W).$$

The complex $q_*(\mathcal{L}^\bullet)$ is hence acyclic and its cohomology in degree 0 is $H^0(X, \mathcal{M}) \otimes q_*(\mathcal{O}_W)$. We consider now its associated graded complex which is a complex of $S^\bullet(H^*)$-modules, that we denote $\mathcal{C}(\mathcal{M})^\bullet$, and which is such that

$$\mathcal{C}(\mathcal{M})^0 = H^0(X, \mathcal{M}) \otimes S^\bullet(H^*), \text{ and,}$$

$$C(\mathcal{M})^p = \bigoplus_{-|I|+n(I)=p} H^0(X, \wedge_I(E) \otimes S_{I'}(F) \otimes \mathcal{M}) \otimes S^\bullet(H^*).$$

We denote also by $\mathbf{C}(H)$ the fraction field of $S^\bullet(H^*)$. We have the following theorem :

Theorem 11 *Suppose the hypothesis of theorem 1 are satisfied. If $\mathcal{M}$ stabilizes the complex $q_*(\mathcal{L}^\bullet)$ then*

$$\det(\mathcal{C}(\mathcal{M})^\bullet \otimes \mathbf{C}(H)) = \mathrm{Res}_{E,F,r}.$$

PROOF. The morphism $q : W \to \nabla$ being birational, the complex $\mathcal{C}(\mathcal{M})^\bullet$ is generically exact, that is the complex of vector spaces (of finite dimensions) $\mathcal{C}(\mathcal{M})^\bullet \otimes \mathbf{C}(H)$ is exact. By the theorem 30 of [GKZ94] we deduce that

$$\det(\mathcal{C}(\mathcal{M})^\bullet \otimes \mathbf{C}(H)) = \sum_D \left(\sum_i (-1)^i \mathrm{mult}_{(Z_D)}(H^i(\mathcal{C}(\mathcal{M})^\bullet) \right).D,$$

where the sum is running over all the irreducible polynomials D of $S^\bullet(H^*)$, and where $\text{mult}_{(Z_D)}$ denotes the multiplicity along the irreducible hypersurface associated to the polynomial D. The results of [Ser55] show that all the modules $H^i(\mathcal{C}(\mathcal{M})^\bullet)$ for $i \neq 0$ are supported on the irrelevant ideal and hence that

$$\det(\mathcal{C}(\mathcal{M})^\bullet \otimes \mathbf{C}(H)) = \sum_D \text{mult}_{(Z_D)} H^0(q_*(\mathcal{L}^\bullet)).D.$$

Using again that the morphism q is birational from W to the resultant divisor $q(W) = \nabla$, we deduce the theorem.

Remark 12 The theorem 11 generalizes the construction given for the resultant of a very ample vector bundle in [GKZ94]. Indeed the complex that we obtain here degenerates in the Koszul complex of the universal section of the vector bundle $\text{Hom}(E, \mathcal{O}_X)$, by setting $F = \mathcal{O}_X$ and $r = 0$.

This result shows that the computation of the determinantal resultant can be done by computing the determinant of a certain complex (which can be done by using the method of Cayley, see [GKZ94], appendix B). As a consequence the determinantal resultant is also obtained as the gcd of all the determinants of the maximal minors of the surjective map

$$\mathcal{C}(\mathcal{M})^{-1} \otimes \mathbf{C}(H) \to \mathcal{C}(\mathcal{M})^0 \otimes \mathbf{C}(H),$$

coming from the complex $\mathcal{C}(\mathcal{M})^\bullet \otimes \mathbf{C}(H)$. In this way, being given $\varphi \in H$, the problem of testing if $X_r(\varphi)$ is empty or not is traduced in a problem of linear algebra since it corresponds to test if a matrix is surjective or not (which is basically done by rank computations). In the following section we will detail these results in the case where X is a projective space.

5 Determinantal resultant on projective spaces

In this section we focus on the particular case where $X = \mathbb{P}^{(m-r)(n-r)-1}$, where m, n and r are three positive integers such that $m \geq n > r \geq 0$, and where the vector bundles E and F are given by

$$E = \bigoplus_{i=1}^m \mathcal{O}_X(-d_i), \ \ F = \bigoplus_{j=1}^n \mathcal{O}_X(-k_j),$$

the integers d_i and k_j (not necessary positives) satisfying $d_i > k_j$ for all i, j. The vector space $H = \text{Hom}(E, F)$ is identified with the vector space of matrices of size $n \times m$ with entry i, j a homogeneous polynomial on X of degree

$d_j - k_i$, i.e. the matrices

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \ldots & h_{1,m} \\ h_{2,1} & h_{2,2} & \ldots & h_{2,m} \\ \vdots & \vdots & & \vdots \\ h_{n,1} & h_{n,2} & \ldots & h_{n,m} \end{pmatrix},$$

where $h_{i,j} \in H^0(X, \mathcal{O}_X(d_j - k_i))$. By theorem 1 the rth-determinantal resultant $\mathrm{Res}_{E,F,r}$ exists and vanishes for all matrix with rank lower or equal to r in at least on point of X. It is a generalization of the classical Macaulay resultant (corresponding to the case $n = 1$, $r = 0$ and $k_1 = 0$).

Remark 13 Notice that for all $l \in \mathbb{Z}$ both determinantal resultants $\mathrm{Res}_{E,F,r}$ and $\mathrm{Res}_{E \otimes \mathcal{O}(l), F \otimes \mathcal{O}(l), r}$ equal. Consequently we can always suppose that one of the integers d_i or one of the integers k_j is zero.

In what follows we will first look at the degree of the determinantal resultant in this case and then explicit its computation. We will end with the two particular and historical examples of Sylvester and Dixon resultants in the determinantal context, and with the application of the computation of the Chow form of a rational normal scroll. Notice that other applications of determinantal resultants exist, for instance the detection of two curves in $\mathbb{P}^3$, one of them being arithmetically Cohen Macaulay (and hence being determinantal by the Hilbert-Burch theorem).

5.1 The degree

Observing proposition 5 it appears that the degree of the rth-determinantal resultant (when it exists) is given by a closed formula which only depends on the integers d_i and k_j. Indeed the Chern polynomials of E and F are given by the formulas :

$$c_t(E) = \prod_{i=1}^{m}(1 - d_i t) \quad, \quad c_t(F) = \prod_{i=1}^{n}(1 - k_i t).$$

To be more precise, the degree of the rth-determinantal resultant in the coefficients of the column number i (that is in the coefficients of the polynomials $h_{1,i}, h_{2,i}, \ldots, h_{n,i}$) is the coefficient of α_i of the multivariate polynomial (in variables $\alpha_1, \ldots, \alpha_m$)

$$(-1)^{(m-r)(n-r)} \Delta_{m-r,n-r} \left(\frac{\prod_{i=1}^{m}(1 - (d_i + \alpha_i)t)}{\prod_{i=1}^{n}(1 - k_i t)} \right). \tag{1}$$

For instance the degree of the principal determinantal resultant is obtained as the coefficient of α_i of the multivariate polynomial (in variables $\alpha_1, \ldots, \alpha_m$) computed as the coefficient of the monomial t^{m-n+1} in the univariate polynomial (in the variable t)

$$(-1)^{(m-r)(n-r)} \frac{\prod_{i=1}^m (1-(d_i+\alpha_i)t)}{\prod_{i=1}^n (1-k_i t)}.$$

Let us see what we obtain if we take the simple example of $X = \mathbb{P}^1$. The arithmetic conditions on m, n and r implies that $r = n-1$ (principal case) and $m - n = 1$. Hence the simplest determinantal resultant is obtained with $E = \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2)$ and $F = \mathcal{O}$. It is the well-known Sylvester resultant of two polynomials. We see easily that the coefficient of t^2 in (1) is $(d_1 + \alpha_1)(d_2 + \alpha_2)$ and hence recover that the degree of the Sylvester resultant is d_1 in the coefficients of the second column and d_2 in the coefficients of the first one.

We can also look at the principal determinantal resultant corresponding to $E = \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \mathcal{O}(-d_3)$ and $F = \mathcal{O}(-k) \oplus \mathcal{O}$. Denoting by N_i the degree of this resultant in the coefficients of the column number i, for $i = 1, 2, 3$, and applying (1) we obtain $N_1 = d_2 + d_3 - k$, $N_2 = d_1 + d_3 - k$ and $N_3 = d_1 + d_2 - k$.

Finally we finish with an example where we know by advance the multidegrees. This example corresponds to the case $E = \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2)$, $F = \mathcal{O}(-k) \oplus \mathcal{O}$ with $r = 0$ (it is not a principal case). We are hence over $X = \mathbb{P}^3$. Applying our results we can find $N_1 = d_2(d_2 - k)(2d_1 - k)$ and $N_2 = d_1(d_1 - k)(2d_2 - k)$. In fact this determinantal condition corresponds to the classical resultant of the polynomials $h_{1,1}, h_{1,2}, h_{2,1}, h_{2,2}$ of respective degree $d_1 - k, d_2 - k, d_1$ and d_2 since we have $r = 0$. For such a resultant we know that the degree in the coefficient of one of these polynomial is the product of the degrees of the others and we can in this way check our formulas for N_1 and N_2.

5.2 The computation

By theorem 11 determinantal resultants on projective spaces can be computed as the determinant of a certain complex defined in the preceding section. This complex involves a line bundle $\mathcal{M}$ that we are going to precise here. In fact we have to choose this line bundle such that

$$H^i(X, \wedge_I(E) \otimes S_{I'}(F^*) \otimes \mathcal{M}) = 0,$$

for all $i > 0$, and for all partition I such that $-|I| + n(I) = p$ (see definition 10). The integers $d_j - k_i$ being all positives, we deduce that we have only to have that $\mathcal{M}$ stabilizes the term on the far left of the complex $\mathcal{L}^\bullet$ to stabilize

all the complex (this is a direct consequence of the cohomology of line bundles on projective spaces, see [Har77], theorem III, 5.1). In this way, we have to determine all the integers d such that

$$H^i(X, \wedge_I(E) \otimes S_{I'}(F^*) \otimes \mathcal{O}(d)) = 0,$$

for all $i > 0$, and where I is the partition $I = (\underbrace{m, \ldots, m}_{q})$ (recall that $q = n-r$).

To do this we have to state another property of the Schur functors, the additivity property : if A and B are two vector bundles on a scheme Z, and I a partition, then

$$S_I(A \oplus B) \simeq \bigoplus_J S_{I/J}(A) \oplus S_J(B),$$

where the sum is limited to partitions J contained in I (see [Las78]). If now L is a line bundle on Z, we deduce that for all pair of partitions (I, H),

$$S_{I/H}(A \oplus L) \simeq \bigoplus_J L^{\otimes |J/H|} \otimes S_{I/J}(A),$$

the sum being on all partitions J such that

$$H \subset J \subset I, \text{ if } h_1 \leq j_1 \leq h_2 \leq j_2 \ldots .$$

We hence deduce, for $I = (\underbrace{m, \ldots, m}_{q})$, that

$$\wedge_I(E) \simeq \mathcal{O}(-q(\sum_{i=1}^{m} d_i)) \bigoplus_s \mathcal{O}(-r_s),$$

where the integers r_s satisfy $r_s \leq q(\sum_{i=1}^{m} d_i)$.
The partition I' associated to the partition I is defined by

$$I' = (\underbrace{n-r}_{r}, \underbrace{m-r}_{q}).$$

Supposing $k_1 \geq k_2 \geq \ldots \geq k_n$, it follows

$$S_{I'}(F^*) \simeq \mathcal{O}((n-r)(k_1 + \ldots + k_r) + (m-r)(k_{r+1} + \ldots + k_n)) \bigoplus_s \mathcal{O}(t_s),$$

where the integers t_s satisfy $t_s \leq (n-r)(k_1+\ldots+k_r)+(m-r)(k_{r+1}+\ldots+k_n)$.
The cohomology of line bundles on projective spaces shows that the integer d must be such that :

$$d - q(\sum_{i=1}^{m} d_i) + (n-r)(\sum_{i=1}^{n} k_i) + (m-n)(k_{r+1} + \ldots k_n) \geq -(m-r)(n-r) + 1,$$

that is

$$d \geq q(\sum_{i=1}^{m} d_i) - (n - r)(\sum_{i=1}^{n} k_i) - (m - n)(k_{r+1} + \ldots + k_n) - (m - r)(n - r) + 1.$$

We denote by $\nu_{\mathbf{d},\mathbf{k}}$ the integer on the right of this last equality, that is

$$\nu_{\mathbf{d},\mathbf{k}} = (n - r)(\sum_{i=1}^{m} d_i - \sum_{i=1}^{n} k_i) - (m - n)(k_{r+1} + \ldots + k_n) - (m - r)(n - r) + 1.$$

Proposition 14 *Suppose that $k_1 \geq k_2 \geq \ldots \geq k_n$, then for all integer $d \geq \nu_{\mathbf{d},\mathbf{k}}$, we have*

$$\det(\mathcal{C}(\mathcal{O}_X(d))^{\bullet} \otimes \mathbf{C}(H)) = \mathrm{Res}_{E,F,r}.$$

Remark 15 The determinantal resultant is invariant if we twist both vector bundles E and F by the same line bundle $\mathcal{O}(l)$, with $l \in \mathbb{Z}$. We can check by an easy computation that the integer $\nu_{\mathbf{d},\mathbf{k}}$ we just defined is invariant by this transformation, i.e.

$$\nu_{\mathbf{d},\mathbf{k}}(E, F) = \nu_{\mathbf{d},\mathbf{k}}(E \otimes \mathcal{O}(l), F \otimes \mathcal{O}(l)).$$

Notice also that the formula of the integer $\nu_{\mathbf{d},\mathbf{k}}$ for the determinantal resultant is

$$\nu_{\mathbf{d},\mathbf{k}} = (d_1 + \ldots + d_m - m) - (k_1 + \ldots + k_n - n).$$

Here again, for the case $n = 1$ we recover that $\nu_{\mathbf{d},\mathbf{k}}$ is the known critical degree for the Macaulay resultant, that is $d_1 + \ldots + d_m - m + 1$ (see [Jou97]), since we can suppose that $k_1 = k_n = 0$ without changing the determinantal resultant.

We can now explicit how to compute the determinantal resultant. Denoting $R = \mathbb{K}[x_0, \ldots, x_{(m-r)(n-r)-1}]$, the first map σ_d (the one of the far right) of the complex $\mathcal{C}(\mathcal{O}_X(d))^{\bullet} \otimes \mathbf{C}(H)$ is the map

$$\bigoplus_{i_1 < \ldots < i_{r+1}, j_1 < \ldots < j_{r+1}} R_{[d - \sum_{t=1}^{r+1} d_{i_t} + \sum_{t=1}^{r+1} k_{i_t}]} e_{i_1, \ldots, i_{r+1}, j_1, \ldots, j_{r+1}} \xrightarrow{\sigma_d} R_{[d]}$$

which associates to each $e_{i_1, \ldots, i_{r+1}, j_1, \ldots, j_{r+1}}$ the polynomial $\Delta_{i_1, \ldots, i_{r+1}, j_1, \ldots, j_{r+1}}$ denoting the determinant of the minor

$$\begin{pmatrix} h_{j_1, i_1} & h_{j_1, i_2} & \ldots & h_{j_1, i_{r+1}} \\ h_{j_2, i_1} & h_{j_2, i_2} & \ldots & h_{j_2, i_{r+1}} \\ \vdots & \vdots & & \vdots \\ h_{j_{r+1}, i_1} & h_{j_{r+1}, i_2} & \ldots & h_{j_{r+1}, i_{r+1}} \end{pmatrix},$$

$R_{[t]}$ being the vector space of homogeneous polynomials of fixed degree t. By properties of complex determinants (see [GKZ94], appendix A), we deduce the following algorithm :

Proposition 16 *Choose an integer $d \geq \nu_{\mathbf{d},\mathbf{k}}$. All nonzero maximal minor (of size $\sharp R_{[d]}$) of the map σ_d is a multiple of the rth-determinantal resultant $\operatorname{Res}_{E,F,r}$. Moreover the greatest common divisor of all the determinants of these maximal minors is exactly $\operatorname{Res}_{E,F,r}$.*

This proposition gives us an algorithm to compute explicitly the determinantal resultant, completely similar to the one giving the expression of the Macaulay resultant. Notice that it is also possible to give the equivalent (in a less explicit form) of the so-called Macaulay matrices (of the Macaulay resultant) for the principal determinantal resultant. These Macaulay matrices are in fact some particular maximal minors of the map σ_d of the Macaulay resultant, i.e $n = 1$ and $r = 0$. They are obtained by specializing the input polynomials in $x_0^{d_0}, x_1^{d_1}, \ldots, x_m^{d_m}$ (see [Jou97]). For the principal determinantal resultant ($r = n-1$) it is also possible to give a specialization of polynomials $(h_{i,j})$ so that we can obtain maximal minors explicitly. The specialization is the following one :

$$\begin{pmatrix} x_0^{d_1-k_1} \ x_1^{d_2-k_1} & \cdots & x_{m-n}^{d_{m-n+1}-k_1} & 0 & \cdots \ \cdots & 0 \\ 0 & x_0^{d_2-k_2} \ x_1^{d_3-k_2} & \cdots & x_{m-n}^{d_{m-n+2}-k_2} & 0 & \cdots & 0 \\ \vdots & \vdots \quad \vdots & \vdots & \vdots & \vdots \ \vdots & \vdots \\ 0 & \cdots & 0 \quad x_0^{d_n-k_n} & x_1^{d_{n+1}-k_n} & \cdots \ \cdots & x_{m-n}^{d_m-k_n} \end{pmatrix}.$$

It is easy to check that the rank of this matrix is n at all points of $X = \mathbb{P}^{m-n}$ and hence that its maximal minors can be left to the matrix σ_d.

However, from a computational point of view we have to notice that developed determinants are not efficient. In practice it is preferable to work with the matrix of σ_d itself. For instance to see if some particular values of the parameters annihilate the determinantal resultant we just have to put them in the matrix and check its rank.

5.3 Two particular examples

We mention here how the classical Sylvester resultants and Dixon resultants, whose names mainly refer to a way of computing some particular resultants, are generalized to the determinantal case.

Determinantal Sylvester resultants: Let n be a positive integer, we consider matrices with polynomial entries in $\mathbb{P}^1$ of size $n \times (n+1)$. The case $n = 1$, that is a 1×2 matrix, corresponds to the classical Sylvester resultant. If we add one line and one column we obtain a new Sylvester resultant corresponding to the vanishing of all the 2×2 minors of a 2×3 matrix, and

so on... We obtain the following result: let $(d_1, \ldots, d_{n+1})$ and $(k_1, \ldots, k_n)$ be two sequences of integers such that $d_i - k_j > 0$ for all i, j. For any morphism $\phi : \oplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(-d_i) \to \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(-k_i)$ and for any integer

$$m \geq \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - \min_{j=1,\ldots,n} k_j - 1,$$

the determinantal resultant of ϕ is the determinant of the finite-dimensional vector spaces complex (Hilbert-Burch complex)

$$\bigoplus_{i=1}^{n} S\left(m + k_i - \sum_{j=1}^{n+1} d_j + \sum_{j=1}^{n} k_j\right) \xrightarrow{\phi^*} \bigoplus_{i=1}^{n+1} S\left(m + d_i - \sum_{j=1}^{n+1} d_j + \sum_{j=1}^{n} k_j\right) \xrightarrow{\wedge^n \phi} S(m),$$

where $S(d)$ denotes the vector space of polynomials of degree d in $\mathbb{P}^1$. Morevover it is an irreducible polynomial in $\mathbb{Z}[S(d_i - k_j); \forall i, j]$ which is, for all $i = 1, \ldots, n+1$, homogeneous in $\oplus_{j=1}^{n} S(d_i - k_j)$ of degree $N_i := \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - d_i$. Its total degree is $n \sum_{j=1}^{n+1} d_j - (n+1) \sum_{j=1}^{n} k_j$.

Determinantal Dixon resultants: Following the classical Dixon resultant, we consider here $n \times (n + 2)$ matrices of polynomials on $\mathbb{P}^1 \times \mathbb{P}^1$ of the same bidegree (d_1, d_2). After some straightforward computations (using the Künneth formula) we obtain the following result: For any morphism $\phi : \oplus_{i=1}^{n+2} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-d_1, -d_2) \to \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$, the determinantal resultant is the determinant of both square matrices

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(d_1 - 1, 2d_2 - 1) \xrightarrow{\wedge^n \phi} S((n+1)d_1 - 1, (n+2)d_2 - 1),$$

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(2d_1 - 1, d_2 - 1) \xrightarrow{\wedge^n \phi} S((n+2)d_1 - 1, (n+1)d_2 - 1),$$

of size $(n+2)(n+1)d_1 d_2$, where $S(d, k)$ denotes the vector space of polynomials of bidegree (d, k) in $\mathbb{P}^1 \times \mathbb{P}^1$.

Moreover it is homogeneous in the coefficients of each column of ϕ of degree $(n+1)nd_1 d_2$; its total degree is $(n+2)(n+1)nd_1 d_2$.

5.4 Chow forms of rational normal scrolls

We now illustrate the determinantal resultant by showing how it computes the Chow form of a rational normal scroll. Notice that such formulas are exposed in [ES01], examples 2.4 and 2.5.

A rational normal scroll can be basically defined by its equations. Let $d_1, \ldots, d_r$ be r positive integers such that at least one of them is strictly positive. Denoting $N + 1 = \sum_{i=1}^{r}(d_i + 1)$, we take the homogeneous coordinates on $\mathbb{P}^N$ to be

$$X_{1,0}, X_{1,1}, \ldots, X_{1,d_1}, X_{2,0}, \ldots, X_{2,d_2}, \ldots, X_{r,0}, \ldots, X_{r,d_r}.$$

Define a $2 \times (\sum_{i=1}^{r} d_i)$ matrix $M(d_1, \ldots, d_r)$ of linear forms on $\mathbb{P}^N$ by

$$\begin{pmatrix} X_{1,0} & \ldots & X_{1,d_1-1} & X_{2,0} & \ldots & X_{2,d_2-1} & \ldots & X_{r,0} & \ldots & X_{r,d_r-1} \\ X_{1,1} & \ldots & X_{1,d_1} & X_{2,1} & \ldots & X_{2,d_2} & \ldots & X_{r,1} & \ldots & X_{r,d_r} \end{pmatrix}.$$

The rational normal scroll $S(d_1, \ldots, d_r) \subset \mathbb{P}^N$ is the variety defined by the ideal of 2×2 minors of $M(d_1, \ldots, d_r)$. It is an irreducible variety of dimension r and degree $\sum_{i=1}^{r} d_i$, as it is proved in [EH87] (notice that $\deg(S) = 1 + \text{codim}(S)$, a rational normal scroll is a variety of minimal degree). In this paper another (equivalent) definition is given. Let F be the vector bundle $F = \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i)$. By hypothesis we made on the integers d_i, $i = 1, \ldots, r$, the vector bundle F is generated by $N + 1$ global sections. The projectivized vector bundle $\mathbb{P}(F)$ is a smooth variety of dimension r mapping to $\mathbb{P}^1$ with fibers $\mathbb{P}^{r-1}$. Its tautological line bundle $\mathcal{O}_{\mathbb{P}(F)}(1)$ is generated by its global sections and defines a tautological map $\mathbb{P}(F) \to \mathbb{P}^N$. This map is birational and $S(d_1, \ldots, d_r)$ is its image.

Recall now that the Chow divisor of a k-dimensional variety $X \subset \mathbb{P}^n$ is the hypersurface in the Grassmannian G of planes of codimension $k + 1$ in $\mathbb{P}^n$, consisting of those planes meeting X. The Chow form of X, denoted $\mathcal{C}(X)$, is its defining equation, defined up to the multiplication by a non-zero constant. If X is of degree e then $\mathcal{C}(X)$ is also of degree e in G. From the definition of the rational normal scroll $S(d_1, \ldots, d_r)$ we deduce that its Chow form $\mathcal{C}(S(d_1, \ldots, d_r)))$ can be computed as a principal determinantal resultant (for which we do not have to suppose that the ground field is of characteristic zero). Indeed, a $(r + 1)$-dimensional space of sections of F, say $\alpha : \mathcal{O}^{r+1} \to F$, corresponds to a plane of codimension $r + 1$ in $\mathbb{P}^N$ which meets X if and only if there exists a point p of $\mathbb{P}^1$ such that $\text{rank}(\alpha(p)) \leq r$. We have the following result :

Proposition 17 *The Chow form of the rational normal scroll $S(d_1, \ldots, d_r)$ is given by the principal determinantal resultant $\text{Res}_{E,F,r}$, where $E = \mathcal{O}_{\mathbb{P}^1}^{r+1}$ and $F = \oplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_i)$.*

Let us illustrate this proposition with the example of the rational normal scroll $S(2, 1)$ (see also [ES01], example 2.5). The "generic" map $\alpha : \mathcal{O}^3 \to$

$\mathcal{O}(2) \oplus \mathcal{O}(1)$ is given by the matrix

$$
\begin{pmatrix}
a_0x^2 + a_1xy + a_2y^2 & b_0x^2 + b_1xy + b_2y^2 & c_0x^2 + c_1xy + c_2y^2 \\
a_3x + a_4y & b_3x + b_4y & c_3x + c_4y
\end{pmatrix},
$$

where $(x : y)$ denotes the homogeneous coordinates of $\mathbb{P}^1$. The matrix associated to its principal determinantal resultant $(r = 1)$ is given by the 5×6 matrix :

$$
\begin{pmatrix}
a_3b_0 - a_0b_3 & 0 & a_3c_0 - a_0c_3 \\
a_4b_0 + a_3b_1 - a_1b_3 - a_0b_4 & a_3b_0 - a_0b_3 & a_4c_0 + a_3c_1 - a_1c_3 - a_0c_4 \\
a_4b_1 + a_3b_2 - a_2b_3 - a_1b_4 & a_4b_0 + a_3b_1 - a_1b_3 - a_0b_4 & a_4c_1 + a_3c_2 - a_2c_3 - a_1c_4 \\
a_4b_2 - a_2b_4 & a_4b_1 + a_3b_2 - a_2b_3 - a_1b_4 & a_4c_2 - a_2c_4 \\
0 & a_4b_2 - a_2b_4 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & b_3c_0 - b_0c_3 & 0 \\
a_3c_0 - a_0c_3 & b_4c_0 + b_3c_1 - b_1c_3 - b_0c_4 & b_3c_0 - b_0c_3 \\
a_4c_0 + a_3c_1 - a_1c_3 - a_0c_4 & b_4c_1 + b_3c_2 - b_2c_3 - b_1c_4 & b_4c_0 + b_3c_1 - b_1c_3 - b_0c_4 \\
a_4c_1 + a_3c_2 - a_2c_3 - a_1c_4 & b_4c_2 - b_2c_4 & b_4c_1 + b_3c_2 - b_2c_3 - b_1c_4 \\
a_4c_2 - a_2c_4 & 0 & b_4c_2 - b_2c_4
\end{pmatrix}.
$$

We can compute the gcd of the maximal minors (size 5×5) of this matrix and we obtain the Chow form of $S(2,1)$ in the Stiefel coordinates (see [GKZ94], chapter 3.1 for different coordinates of Grasmannians). It is of degree 3 in each set of variables $(a_i)_{i=0\ldots4}$, $(b_i)_{i=0\ldots4}$ and $(c_i)_{i=0\ldots4}$:

$$
-a_4^3b_2^2b_3c_0^3 + a_3a_4^2b_2^2b_4c_0^3 + 2a_2a_4^2b_2b_3b_4c_0^3 - 2a_2a_3a_4b_2b_4c_0^3 - a_2^2a_4b_3b_4^2c_0^3 + \ldots \;.
$$

The Plücker coordinates are obtained from the Stiefel coordinates as the 3×3 minors of the matrix

$$
\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 \\
b_0 & b_1 & b_2 & b_3 & b_4 \\
c_0 & c_1 & c_2 & c_3 & c_4
\end{pmatrix},
$$

and we can check that the Chow form of $S(2,1)$ is well of total degree 3 in these coordinates. From a computational point of view it is not efficient to compute such an extended formula for $\mathcal{C}(S(2,1))$. It is better to work with a matricial representation of this Chow form, here with the 5×6 matrix we have computed, and use it. For instance, to test if a point of the grassmannian (in the Stiefel coordinates) is on $\mathcal{C}(S(2,1))$ is done by replacing the a_i's, b_i's and c_i's by their value and computing the rank of the matrix : if the rank is 5 the point is not on the Chow form, otherwise it is.

Looking to the general case of proposition 17, the Chow form of the rational normal scroll $S(d_1, \ldots, d_r)$ is obtained in this way in the Stiefel coordinates of the Grassmannian of planes of codimension $r+1$ in $\mathbb{P}^N$. By (1) we see that the principal determinant we compute is of degree $\sum_{i=1}^r d_i$ with respect to each column of the "generic" map $\mathcal{O}_{\mathbb{P}^1}^{r+1} \to \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(d_i)$, and as a polynomial of total degree $\sum_{i=1}^r d_i$ (the degree of $S(d_1, \ldots, d_r)$) in the Plücker coordinates.

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