On the closed image of a rational map and the implicitization problem

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Abstract

In this paper, we investigate some topics around the closed image S of a rational map λ given by some homogeneous elements $f_1, \ldots, f_n$ of the same degree in a graded algebra A. We first compute the degree of this closed image in case λ is generically finite and $f_1, \ldots, f_n$ define isolated base points in $\mathrm{Proj}(A)$. We then relate the definition ideal of S to the symmetric and the Rees algebras of the ideal $I = (f_1, \ldots, f_n) \subset A$, and prove some new acyclicity criteria for the associated approximation complexes. Finally, we use these results to obtain the implicit equation of S in case S is a hypersurface, $\mathrm{Proj}(A) = \mathbb{P}^{n-2}_k$ with k a field, and base points are either absent or local complete intersection isolated points.

1 Introduction

Let $f_1, \ldots, f_n$ be non-zero homogeneous polynomials in $k[X_1, \ldots, X_{n-1}]$ of the same degree d, where k is a field. They define a rational map

$$\begin{aligned} \mathbb{P}^{n-2}_k &\xrightarrow{\lambda} \mathbb{P}^{n-1}_k \\ (X_1 : \cdots : X_{n-1}) &\mapsto (f_1 : \cdots : f_n)(X_1 : \cdots : X_{n-1}), \end{aligned}$$

whose closed image (the algebraic closure of the image) is a hypersurface of $\mathbb{P}^{n-1}_k$, say H, providing λ is generically finite. The computation of the equation of H, up to a non-zero constant multiple, is known as the *implicitization problem*, and is the main purpose of this paper. We examine here this problem, under suitable assumptions, by studying the blow-up algebras associated to λ. More precisely, we use the acyclicity of the approximation complexes introduced in [24] to develop a new algorithm giving the implicit equation H, in case the ideal $(f_1, \dots, f_n)$ is zero-dimensional and locally a complete intersection. This algorithm, based on determinants of complexes (also called MacRae's invariants) computations, is the main contribution of this paper among other related results.

Implicitization in low dimensions, that is curve and surface implicitizations, have significant applications to the field of computer aided geometric design. For instance it helps drawing a curve or a surface nearby a singularity, computing autointersection of offsets and drafts, or computing the intersection with parameterized curves or surfaces. Consequently the implicitization problem has been widely studied and is always an active research area. It is basically an elimination problem, and thus can be solved by methods based on Gröbner basis computations, using for instance the Buchberger's algorithm, and this in all generality. However, in practice it appears that such methods involve heavy computations and high complexity, and are hence rarely used. For effective computations, algorithms from the old-fashioned approach of resultants, going back to the end of the nineteenth century, are preferred, even if they apply only under particular conditions. Moreover they usually give the implicit equation as a determinant of a matrix, which is a very useful representation with well-known adapted algorithms for numerical applications. We now survey these methods for curve and surface implicitizations in order to clarify what is known today, and also to shed light on the contribution of this paper.

1.1 Overview on curve implicitization

Curve implicitization is a quite easy problem which is well understood and solved. Let λ be the rational map

$$\begin{aligned} \mathbb{P}^1_k &\longrightarrow \mathbb{P}^2_k = \mathrm{Proj}(k[T_1, T_2, T_3]) \\ (X_1 : X_2) &\mapsto (f_1 : f_2 : f_3)(X_1, X_2), \end{aligned}$$

where $f_1, f_2, f_3 \in k[X_1, X_2]$ are homogeneous polynomials of same degree $d \geq 1$ such that $\gcd(f_1, f_2, f_3) = 1$, which implies that λ is regular (i.e. there is no base points). Notice that this last hypothesis is not restrictive because we can always obtain it by dividing each f_i, for $i = 1, 2, 3$, by $\gcd(f_1, f_2, f_3)$. In fact, as we will see, the difficulty of the implicitization problem comes from the presence of base points, and the case of curve is particularly simple since base points can be easily erased via a single gcd computation. There is roughly two types of methods to compute the degree d implicit equation of the closed image of λ, which is of the form $C(T_1, T_2, T_3)^{\deg(\lambda)}$, where C is an irreducible homogeneous polynomial in $k[T_1, T_2, T_3]$.

The first one is based on a resultant computation. We have the equality

$$\mathrm{Res}(f_1 - T_1 f_3, f_2 - T_2 f_3) = C(T_1, T_2, 1)^{\deg(\lambda)},$$

where Res denotes the classical resultant of two homogeneous polynomials in $\mathbb{P}^1_k$. We can thus obtain the implicit curve as the determinant of the well-known Sylvester's matrix of both polynomials $f_1 - T_1 f_3$ and $f_2 - T_2 f_3$ in variables X_1 and X_2. Another matricial formulation is known to compute such a resultant, the Bezout's matrix which is smaller than the Sylvester's matrix. If $P(X_1, X_2)$ and $Q(X_1, X_2)$ are two homogeneous polynomials in $k[X_1, X_2]$ of the same degree d, the Bezout's matrix $\mathbb{B}ez(P, Q)$ is the matrix $(c_{i,j})_{0 \le i,j \le d-1}$ where the $c_{i,j}$'s are the coefficients of the decomposition

$$\frac{P(S,1)Q(T,1) - P(T,1)Q(S,1)}{S - T} = \sum_{0 \le i,j \le n-1} c_{i,j} S^i T^j.$$

Since $\det(\mathbb{B}ez(P, Q)) = \mathrm{Res}(P, Q)$ (see e.g. [19]) we have

$$\det(\mathbb{B}ez(f_1 - T_1 f_3, f_2 - T_2 f_3)) = C(T_1, T_2, 1)^{\deg(\lambda)}.$$

A variation of this formula, due to Kravitsky (see [21], theorem 9.1.11), is

$$\det(T_1 \mathbb{B}ez(f_2, f_3) + T_2 \mathbb{B}ez(f_3, f_1) + T_3 \mathbb{B}ez(f_1, f_2)) = C(T_1, T_2, T_3)^{\deg(\lambda)}$$

(see the appendix A at the end of the paper for a proof in the general context of anisotropic resultant). Notice that these resultant-based formulations have a big advantage, they give a *generic* implicitization formula, that is computations can be made with generic coefficients of polynomials f_1, f_2, f_3, and then all possible implicit equations are obtained as a specialization of a generic formula.

More recently a new method has been developed for curve implicitization, called the "moving lines" method. It was introduced by T. Sederberg and F. Chen in [23]. This method is based on the study of the first syzygies of the homogeneous ideal $I = (f_1, f_2, f_3) \subset k[X_1, X_2]$ in appropriate degrees (recall that we always suppose that $\gcd(f_1, f_2, f_3) = 1$). A moving line of degree ν is a homogeneous polynomial

$$a_1(X_1, X_2)T_1 + a_2(X_1, X_2)T_2 + a_3(X_1, X_2)T_3,$$

where a_1, a_2, a_3 are homogeneous polynomials in $k[X_1, X_2]_\nu$, which is said to follow the implicit curve if it vanishes when substituting each T_i by f_i, for $i = 1, 2, 3$. In other words the triple (a_1, a_2, a_3) is a syzygy of degree ν of the ideal I. Choosing d moving lines $L_1, \ldots, L_d$ of degree $d - 1$ following the implicit curve, we can construct a square matrix M of size $d \times d$ corresponding to the $k[T_1, T_2, T_3]$-module map

$$
\begin{aligned}
\oplus_{i=1}^{d} k[T_1,T_2,T_3] &\longrightarrow (k[X_1,X_2]_{d-1}) \otimes_k k[T_1,T_2,T_3] \\
(p_1,p_2,\ldots,p_d) &\mapsto \sum_{i=1}^{d} p_i L_i.
\end{aligned}
$$

It can be shown that one can always choose $L_1,\ldots,L_d$ such that the determinant of M is non-zero, and in this case it equals what one wants, that is $C(T_1,T_2,T_3)^{\deg(\lambda)}$ (see [8]). This method hence yields an alternative to resultant computations, but however, it is not a generic method.

For curve implicitization the two methods we have recalled work well and give the desired result, but, as we are now going to see, it is no longer true for surface implicitization where the situation becomes more intricate.

1.2 Overview on surface implicitization

Consider the rational map λ

$$
\begin{aligned}
\mathbb{P}^2_k &\longrightarrow \mathbb{P}^3_k = \operatorname{Proj}(k[T_1,T_2,T_3,T_4]) \\
(X_1:X_2:X_3) &\mapsto (f_1:f_2:f_3:f_4)(X_1,X_2,X_3),
\end{aligned}
$$

where $f_1,f_2,f_3,f_4 \in k[X_1,X_2,X_3]$ are homogeneous polynomials of same degree $d \geq 1$, that we always suppose to be generically finite. Surface implicitization problem consists in computing the implicit equation of the closed image of λ, which is a hypersurface of $\mathbb{P}^3_k$ of the form $S(T_1,T_2,T_3,T_4)^{\deg(\lambda)}$, with S irreducible. It is much more difficult than curve implicitization problem and, as far as we know, is unsolved in all generality (always without using Gröbner basis). Here again we suppose that $\gcd(f_1,f_2,f_3,f_4) = 1$, which is not restrictive as we have seen in the preceding paragraph. However this hypothesis only erases base points in purely codimension 1 here, and hence it remains a finite set of isolated base points, possibly empty, that we denote by $\mathcal{B}$. We know then that the equation of the closed image of λ is a hypersurface of degree $(d^2 - \sum_{p\in\mathcal{B}} e_p)/\deg(\lambda)$, where e_p denotes the algebraic multiplicity of the point $p \in \mathcal{B}$ (see theorem 2.5, or [8], appendix). As for curve implicitization, two types of methods have been developed to solve the surface implicitization problem: methods based on resultant computations, and the method called "moving surfaces". We begin with the first, which is also the oldest one.

If there is no base point (i.e. $\mathcal{B} = \emptyset$), one easily shows that

$$
\operatorname{Res}(f_1 - T_1 f_4, f_2 - T_2 f_4, f_3 - T_3 f_4) = S(T_1,T_2,T_3,1)^{\deg(\lambda)},
$$

where Res denotes the classical resultant of three homogeneous polynomials in $\mathbb{P}^2$. This resultant can be computed with the well-known Macaulay's matrices, but this involves gcd computations since the determinant of each Macaulay's matrix gives only a multiple of this resultant. In order to be efficient, direct methods, avoiding these gcd computations, have been proposed. The general case of an implicit surface without base points is solved in [18], 5.3.17, using the anisotropic resultant; the implicit hypersurface of λ is then obtained as the determinant of a square matrix (notice that this result was rediscovered in [1], in a more geometrical setting, but only in some particular cases). As always with resultant-based methods, the formula is *generic*, i.e. computations can be done with indeterminate coefficients, giving the desired implicit equation as a specialization of a generic equation. If base points exist the preceding resultant vanishes identically, but we can use other constructions of resultants, taking into account base points, as the reduced resultant (see [18], 5.3.29) or the residual resultant (see [5]). In [6], the residual resultant is used to solve the surface implicitization problem if the ideal of base points is a local complete intersection satisfying some more technical conditions (see [6], theorem 3.2; for instance, if the ideal (f_1, f_2, f_3, f_4) is saturated the method failed). This method has the disadvantage of requiring the knowledge of the ideal of base points, but however it has the advantage of giving a generic formula for all parameterizations admitting the same ideal of base points.

We now briefly overview the second announced method, the "moving surfaces" method which has also been introduced in [23]. We first suppose that $\mathcal{B} = \emptyset$. The two main ingredients of this method are moving planes and moving quadrics. Like curves, a moving plane of degree ν following the surface is a polynomial in $k[X_1, X_2, X_3][T_1, T_2, T_3, T_4]$ of the form

$$a_1(X_1, X_2, X_3)T_1 + a_2(X_1, X_2, X_3)T_2 + a_3(X_1, X_2, X_3)T_3 + a_4(X_1, X_2, X_3)T_4,$$

where a_1, a_2, a_3 and a_4 are homogeneous polynomials in $k[X_1, X_2, X_3]_\nu$, such that it vanishes if we replace each T_i by f_i. In a similar way, a moving quadric of degree ν following the surface is a polynomial of the form

$$a_{1,1}(X_1, X_2, X_3)T_1^2 + a_{1,2}(X_1, X_2, X_3)T_1T_2 + \ldots + a_{4,4}(X_1, X_2, X_3)T_4^2,$$

where $a_{i,j}$'s, with $1 \leq i \leq j \leq 4$, are also homogeneous polynomials in $k[X_1, X_2, X_3]_\nu$, such that it vanishes if we replace each T_i by f_i. Choosing d moving planes $L_1, \ldots, L_n$ and $l = (d^2 - d)/2$ moving quadrics $Q_1, \ldots, Q_l$ of degree $d-1$ which follow the surface, we can construct a square matrix, say M, corresponding to the $k[\mathbf{T}]$-module map (where $k[\mathbf{T}]$ denotes $k[T_1, T_2, T_3, T_4]$)

$$\begin{aligned} \left(\oplus_{i=1}^{d} k[\mathbf{T}]\right) \bigoplus \left(\oplus_{j=1}^{l} k[\mathbf{T}]\right) &\to (k[X_1, X_2, X_3]_{d-1}) \otimes_k k[\mathbf{T}] \\ (p_1, \ldots, p_d, q_1, \ldots, q_l) &\to \sum_{i=1}^{d} p_i L_i + \sum_{j=1}^{l} q_i Q_i. \end{aligned}$$

It can be shown that it is always possible to choose $L_1, \ldots, L_n$ and $Q_1, \ldots, Q_l$ so that $\det(M)$ is non-zero and then equals $S^{\deg(\lambda)}$ (see [10]). In the recent paper [7], this method was improved: a similar matrix is constructed such that its determinant is the implicit surface, providing λ is birational and the base points ideal is locally a complete intersection satisfying also some other technical conditions (for instance the method rarely works if the ideal $(f_1, \ldots, f_4)$ is saturated).

1.3 Contents of the paper

As we have already mentioned, this article is centered on the implicitization problem. However we also develop some other results which are closely related. After this quite long introduction, the paper is organized as follows.

Section 2 regroups some results on the geometry of the closed image of a rational map λ from $\mathrm{Proj}(A)$, where A is a $\mathbb{Z}$-graded algebra, to $\mathrm{Proj}(k[T_1, \ldots, T_n])$, where k is only supposed to be a commutative ring, given by n homogeneous elements $f_1, \ldots, f_n$ in A of the same degree $d \geq 1$. We first give the ideal of definition of the scheme-theoretic image of λ, that we will also called hereafter the closed image of λ, and then prove that λ can be extended to a projective morphism by blowing-up. We end this section with a theorem on the degree of the closed image of λ, giving a closed formula to compute it, providing k is a field, A is a $\mathbb{N}$-graded finitely generated k-algebra without zero divisors, λ is generically finite (onto its image) and its base points are isolated.

In section 3, we link the definition ideal of the closed image of λ to the symmetric and Rees algebras associated to the ideal $I = (f_1, \ldots, f_n)$ of the $\mathbb{Z}$-graded algebra A. We also investigate what happens when the ideal I is of linear type, that is the Rees algebra and the symmetric algebra of I are isomorphic. We roughly prove that it is possible to "read" the definition ideal of the closed image of λ from these blow-up algebras.

The section 4 is devoted to the so-called approximation complexes introduced by A. Simis and W.V. Vasconcelos in [24]. In a first part we introduce these complexes, recalling their definitions and their basic, but very useful, properties. In a second part, having in mind to apply them to the implicitization problem, we prove two new acyclicity criterions for the first approximation complex associated to the ideal I, under suitable assumptions. The last part of this section deals with a problem introduced by D. Cox and solved by himself and H. Schenck in [9]: being given a homogeneous ideal $I = (a_1, a_2, a_3)$ of $\mathbb{P}^2$ of codimension two, the module of syzygies of I vanishing at the scheme locus $V(I)$ is generated by the Koszul sygyzies if and only if $V(I)$ is a local complete intersection in $\mathbb{P}^2$. We show that this problem appears naturally in the setting of the approximation complexes. This allows us to generalize this result to higher dimensions, and observe that the "natural" condition on I is not to be a local complete intersection, but to be syzygetic.

Finally, in section 5, we investigate the implicitization problem with the tools we have developed in the previous sections. We suppose here that k is a field, A is the polynomial ring $k[X_1, \dots, X_{n-1}]$, and λ is generically finite (so that the closed image of λ is a hypersurface of $\mathbb{P}_k^{n-1}$). Using the acyclicity criterions we proved in section 4, we show that an equation of the closed image of λ is obtained as the determinant of each degree ν, with $\nu \geq (n-2)(d-1)$, part of the first approximation complex associated to I, under the hypothesis that I is either such that $V(I) = \emptyset$ or a local complete intersection in $\mathrm{Proj}(A)$ of codimension $n-2$ (isolated base points). This new algorithm improves in particular the known techniques (not involving Gröbner basis) to solve curve and surface implicitizations. For curve implicitization, this algorithm is exactly the moving lines method in case $\gcd(f_1, f_2, f_3) = 1$, but its formalism shows clearly (in our point of view) why $d-1$ is the good degree of moving lines to consider. Moreover if we do not suppose $\gcd(f_1, f_2, f_3) = 1$, then our algorithm also applies. For surface implicitization, our algorithm works if the ideal $I = (f_1, \dots, f_4)$ defines local complete intersection base points of codimension 2. Comparing with the method of moving surfaces (and its improvement) and the resultant-based methods, it is more general since both preceding methods sometimes failed in this situation. We would like to mention that we have implemented our algorithm in the MAGMA language. This choice was guided by its needs: it just uses linear algebra, and more precisely linear systems solving routines.

This work is based on a course given by Jean-Pierre Jouanolou at the University Louis Pasteur of Strasbourg (France) during the academic year 2000-2001.

Notation: Let R be a commutative ring. If I is an ideal of R, we will denote $I^{\sim}$ its associated sheaf of ideals on $\mathrm{Spec}(R)$. If the ring R is supposed to be $\mathbb{Z}$-graded, we will denote $\mathrm{Proj}(R)$ the quotient of $\mathrm{Spec}(R) \setminus V(\mathfrak{m})$ by the multiplicative group $\mathbb{G}_m$, where $\mathfrak{m}$ is the ideal of R generated by all the homogeneous elements of non-zero degree. If I is supposed to be an homogeneous ideal of R we will denote $I^{\#}$ its associated sheaf of ideals on $\mathrm{Proj}(R)$. If in addition the ring R is supposed to be $\mathbb{Z} \times \mathbb{Z}$-graded, we will denote $\mathrm{Biproj}(R)$ the quotient of $\mathrm{Spec}(R) \setminus V(\tau)$ by the group $\mathbb{G}_m \times \mathbb{G}_m$, where τ is the ideal of R generated by all the products of two bihomogeneous elements of bidegree $(0, a)$ and $(b, 0)$ such that $a \neq 0$ and $b \neq 0$. If I is supposed to be a bihomogeneous ideal of R, then we will denote $I^{\#\#}$ its associated sheaf of ideals on $\mathrm{Biproj}(R)$.
Finally, if R is a commutative ring and if J and I are two ideals of R, we define the ideal of inertia forms (or Trägheitsformen) of I with respect to J

to be

$$\mathrm{TF}_J(I) := \cup_{n\in\mathbb{N}}(I :_R J^n) = \{a \in R \text{ such that } \exists n \in \mathbb{N}\ \forall \xi \in J^n \ :\ a\xi \in I\}.$$

2 The geometry of the closed image of a rational map

Let k be a commutative ring, A a $\mathbb{Z}$-graded k-algebra, and denote by $\tau : k \to A_0$ the canonical morphism of rings. Suppose given two integers $n \geq 1$ and $d \geq 1$, and also an element $f_i \in A_d$ for each $i \in \{1, 2, \ldots, n\}$. The k-algebra morphism

$$\begin{aligned} h : k[T_1, \ldots, T_n] &\longrightarrow A \\ T_i \qquad &\mapsto f_i \end{aligned}$$

yields a k-scheme morphism

$$\mu : \cup_{i=1}^n D(f_i) \longrightarrow \cup_{i=1}^n D(T_i) = (\mathbb{A}_k^n)^*,$$

where $\mathbb{A}_k^n$ denotes the affine space of dimension n over k, and $(\mathbb{A}_k^n)^*$ this affine space without the origin. If we quotient the morphism μ by the action of the multiplicative group $\mathbb{G}_m$, we obtain another k-scheme morphism,

$$\lambda \ :\ \cup_{i=1}^n D_+(f_i) \longrightarrow \mathbb{P}_k^{n-1} = \mathrm{Proj}(k[T_1, \ldots, T_n]), \tag{1}$$

and the following commutative diagram:

$$\begin{array}{ccc} D(\mathbf{f}) := \cup_{i=1}^n D(f_i) & \xrightarrow{\mu} & (\mathbb{A}_k^n)^* \\ {\scriptstyle \mathrm{Proj}}\downarrow & & {\scriptstyle \mathrm{Proj}}\downarrow \\ D_+(\mathbf{f}) := \cup_{i=1}^n D_+(f_i) & \xrightarrow{\lambda} & \mathbb{P}_k^{n-1}. \end{array}$$

The main aim of this paper is to provide a "formula" for the closure of the set-theoretic images of the maps λ and μ in case they are hypersurfaces. It turns out that, under suitable assumption, the closure of a set-theoretic image of a scheme morphism have a natural scheme structure called the *scheme-theoretic image*. More precisely, if $X \xrightarrow{\rho} Y$ is a morphism of schemes, then its scheme-theoretic image exists if, for instance, $\rho_*(\mathcal{O}_X)$ is quasi-coherent. It is the smallest closed subscheme of Y containing the set-theoretic image of ρ. Its associated ideal sheaf is given by $\ker(\rho^\# : \mathcal{O}_Y \to \rho_*\mathcal{O}_X)$ and it is supported on the closure of its set-theoretic image (see [13] I.6.10.5 for more details). The first part of this section is devoted to the description of the scheme-theoretic image of the map λ (and μ): we give its sheaf of ideals on $\mathbb{P}_k^{n-1}$, and a projective morphism which extends it. In a second part we prove a formula to compute the degree of the scheme-theoretic image of λ in case elements $f_1, \ldots, f_n$ define a finite number of points in A (possibly zero). This formula generalizes the one commonly uses when dealing with the implicitization problem to obtain the degree of a parameterized surface of $\mathbb{P}^3$ (see e.g. [8], appendix).

2.1 The ideal of the closed image of λ

We begin here by the description of the ideal sheaf of the scheme-theoretic image, that we will also often called the closed image, of λ (and μ).

Theorem 2.1 *The scheme-theoretic images of the respective morphisms μ and λ are given by*

$$V(\ker(h)^{\sim})_{|(\mathbb{A}^n_k)^*} = V\left(\mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h))^{\sim}\right)_{|(\mathbb{A}^n_k)^*},$$

and

$$V(\ker(h)^{\#}) = V\left(\mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h))^{\#}\right).$$

Proof. First notice that for all $i \in \{1, 2, \ldots, n\}$, the restriction $\mu_{|D(f_i)}$ is the spectrum of the morphism of rings $k[T_1, \ldots, T_n]_{T_i} \to A_{f_i}$, obtained from h by localization. Since $\mu^{-1}(D(T_i)) = D(f_i)$ for all $i \in \{1, 2, \ldots, n\}$, it follows that μ is an affine morphism. Now denote by j the canonical inclusion $(\mathbb{A}^n_k)^* \hookrightarrow \mathbb{A}^n_k$. The sheaf $\mathcal{O}_{D(\mathbf{f})}$ is quasi-coherent and the morphism $j \circ \mu$ is quasi-compact and separated (since it is affine), so we deduce that $(j \circ \mu)_*(\mathcal{O}_{D(\mathbf{f})})$ is quasi-coherent (see [14], prop. II.5.8), and we obtain

$$(j \circ \mu)_*(\mathcal{O}_{D(\mathbf{f})}) = \ker(\prod_{i=1}^{n} A_{f_i} \to \prod_{1 \le i,j \le n,\, i \ne j} A_{f_i f_j})^{\sim},$$

and also

$$\mu_*(\mathcal{O}_{D(\mathbf{f})}) = \ker(\prod_{i=1}^{n} A_{f_i} \to \prod_{1 \le i,j \le n,\, i \ne j} A_{f_i f_j})^{\sim}{}_{|(\mathbb{A}^n_k)^*}.$$

Notice that $\ker(\prod A_{f_i} \to \prod A_{f_i f_j})$ is a $k[T_1, \ldots, T_n]$-module via the map h. From here the kernel of the canonical morphism $\mathcal{O}_{(\mathbb{A}^n_k)^*} \to \mu_* \mathcal{O}_{D(\mathbf{f})}$ is exactly

$$\begin{aligned}
&\ker(\ k[T_1, \ldots, T_n] \xrightarrow{h} \ker(\prod_{i=1}^{n} A_{f_i} \to \prod_{1 \le i,j \le n} A_{f_i f_j})\ )^{\sim}{}_{|(\mathbb{A}^n_k)^*} \\
&= \ker(k[T_1, \ldots, T_n] \xrightarrow{h} \frac{A}{H^0_{(f_1,\ldots,f_n)}(A)})^{\sim}{}_{|(\mathbb{A}^n_k)^*} \\
&= (\mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h)))^{\sim}{}_{|(\mathbb{A}^n_k)^*} \\
&= \ker(h)^{\sim}{}_{|(\mathbb{A}^n_k)^*},
\end{aligned}$$

and this proves the first statement of the proposition. If we quotient by the action of the multiplicative group $\mathbb{G}_m$ we obtain

$$\ker(\mathcal{O}_{\mathbb{P}^{n-1}_k} \to \lambda_*(\mathcal{O}_{D_+(\mathbf{f})})) = (\mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h)))^\# = \ker(h)^\#$$

which is the second statement. $\square$

Remark 2.2 *The ideal* $\mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h))$ *is often called the saturated ideal of* $\ker(h)$ *in* $k[T_1,\ldots,T_n]$. *Also, in the case where* $H^0_{(f_1,\ldots,f_n)}(A) = 0$ *we have* $\ker(h) = \mathrm{TF}_{(T_1,\ldots,T_n)}(\ker(h))$, *i.e.* $\ker(h)$ *is saturated, and hence, for all* $\nu \in \mathbb{Z}$, $\ker(h)_\nu \simeq H^0(\mathbb{P}^{n-1}_k, \ker(h)^\#(\nu))$.

Now we show that the map λ can be extended to a projective morphism whose image is the closed image of λ. This result is quite classical (see for instance [14], II.7.17.3), but we prefer to detail it, giving a particular attention to graduations.

First of all we set the indeterminates $T_1,\ldots,T_n$ to degree 1, the usual choice to construct $\mathbb{P}^{n-1}_k = \mathrm{Proj}(k[T_1,\ldots,T_n])$. It follows that the ring

$$A[T_1,\ldots,T_n] = A \otimes_k k[T_1,\ldots,T_n]$$

is naturally bi-graded by the tensor product graduation associated to the graduations of A and $k[T_1,\ldots,T_n]$. Let Z be a new indeterminate and set its degree to $1-d$. In this way the $\mathbb{Z}\times\mathbb{Z}$-graded A-algebra morphism

$$\begin{aligned} A[T_1,\ldots,T_n] &\longrightarrow A[Z,Z^{-1}] \\ T_i &\mapsto f_i Z \end{aligned} \tag{2}$$

is homogeneous, and hence gives a natural $\mathbb{Z}\times\mathbb{Z}$-graduation to its image: the Rees algebra $\mathrm{Rees}_A(I)$, where $I = (f_1,\ldots,f_n) \subset A$. For this graduation, we denote by Bl_I the blow-up

$$Bl_I = \mathrm{Biproj}(\mathrm{Rees}_A(I)) \subset \mathrm{Biproj}(A[T_1,\ldots,T_n]) = \mathrm{Proj}(A) \times_k \mathbb{P}^{n-1}_k,$$

and respectively by u and v the two canonical projections

$$Bl_I \xrightarrow{u} \mathrm{Proj}(A) \quad \text{and} \quad Bl_I \xrightarrow{v} \mathbb{P}^{n-1}_k.$$

Since $I_p = A_p$ for all $p \in D_+(\mathbf{f})$, the restriction of u to $\Omega = u^{-1}(D_+(\mathbf{f}))$ is an isomorphism from Ω to $D_+(\mathbf{f})$. The following proposition shows that the morphism of k-schemes v extends the morphism λ by identifying Ω with $D_+(\mathbf{f})$ via $u_{|\Omega}$.

Proposition 2.3 *With the above notations, the following diagram is commutative:*

$$\begin{tikzcd} \Omega \arrow[d, "u_{|\Omega}"', "\wr"] \arrow[dr, "v_{|\Omega}"] & \\ D_+(\mathbf{f}) \arrow[r, "\lambda"] & \mathbb{P}_k^{n-1} \end{tikzcd}$$

Proof. Setting the weights of the indeterminates $T_1, \ldots, T_n$ to d and Z to 0, we obtain two graded k-algebra morphisms

$$\begin{array}{rcl} incl \circ h : k[T_1, \ldots, T_n] & \xrightarrow{h} & A \hookrightarrow \mathrm{Rees}_A(I) \subset A[Z, Z^{-1}] \\ T_i & \mapsto & f_i \end{array}$$

and

$$\begin{array}{rcl} \rho : k[T_1, \ldots, T_n] & \longrightarrow & \mathrm{Rees}_A(I) \subset A[Z, Z^{-1}] \\ T_i & \mapsto & f_i Z. \end{array}$$

Let $i \in \{1, 2, \ldots, n\}$ be fixed. Over $D_+(T_i) \subset \mathbb{P}_k^{n-1}$, the map $\lambda \circ u_{|\Omega}$ is affine, associated to the k-algebra morphism

$$\begin{array}{rcl} (incl \circ h)_i : k[T_1, \ldots, T_n]_{(T_i)} & \longrightarrow & (\mathrm{Rees}_A(I))_{(f_i)} \hookrightarrow A_{(f_i)}[Z, Z^{-1}] \\ T_j/T_i & \mapsto & f_j/f_i, \end{array}$$

and the map $v_{|\Omega}$ is also affine, associated to the k-algebra morphism

$$\begin{array}{rcl} \rho_i : k[T_1, \ldots, T_n]_{(T_i)} & \longrightarrow & \mathrm{Rees}_A(I)_{(f_i Z)} \subset A_{(f_i)}[Z, Z^{-1}] \\ T_j/T_i & \mapsto & f_j Z/f_i Z. \end{array}$$

This clearly shows that $\rho_i(T_j/T_i) = f_j/f_i = (incl \circ h)_i$ in $A_{(f_i)}[Z, Z^{-1}]$, which implies the proposition. □

2.2 The degree of the closed image of λ

In what follows we give an explicit formula to compute the degree of the closed image of λ, that is the image of v, providing that $\mathrm{Proj}(A/I)$ is a zero-dimensional scheme (possibly empty), where I denotes the ideal $(f_1, \ldots, f_n)$.

For this we first fix some notation about multiplicities (we refer to [3] for a complete treatment on the subject), and recall some classical results.

From now we suppose that k is a field and we denote by C the $\mathbb{N}$-graded polynomial ring $k[X_1, \ldots, X_r]$ with $\deg(X_i) = 1$ for all $i = 1, \ldots, r$. For all $\mathbb{Z}$-graded finite C-module M one defines the *Hilbert series* of M:

$$\mathrm{H}_M(T) = \sum_{\nu \in \mathbb{Z}} \dim_k(M_\nu) T^\nu.$$

If δ denotes the Krull dimension of M, there exists a unique polynomial $L_M(T)$ such that $L_M(1) \neq 0$ and

$$\mathrm{H}_M(T) = \sum_{\nu \in \mathbb{Z}} \dim_k(M_\nu) T^\nu = \frac{L_M(T)}{(1-T)^\delta}.$$

The number $L_M(1)$ is an invariant of the module M called the *multiplicity* of M; we will denote it by $\mathrm{mult}_k(M) := L_M(1)$. Another way to obtain this invariant is the *Hilbert polynomial* of M, denoted $\mathrm{P}_M(X)$. It is a polynomial of degree $\delta - 1$ such that $\mathrm{P}_M(\nu) = \dim_k(M_\nu)$ for all sufficiently large $\nu \in \mathbb{N}$. The Hilbert polynomial is of the form

$$\mathrm{P}_M(X) = \frac{a_{\delta-1}}{(\delta-1)!} X^{\delta-1} + \ldots + a_0,$$

and we have the equality $\mathrm{mult}_k(M) = L_M(1) = a_{\delta-1}$. One can also define it as the Euler characteristic of $M^\sharp$ on $\mathbb{P}_k^{r-1}$, that is for all $\nu \in \mathbb{Z}$ we have:

$$\mathrm{P}_M(\nu) = \chi(\mathbb{P}_k^{r-1}, M^\sharp(\nu)) = \sum_{i \geq 0} (-1)^i \dim_k H^i(\mathbb{P}_k^{r-1}, M^\sharp(\nu)). \tag{3}$$

A well-known formula relates the Hilbert series and the Hilbert polynomial: for all $\nu \in \mathbb{Z}$

$$H_M(T)_{|T^\nu} - P_M(\nu) = \sum_{i \geq 0} (-1)^i \dim_k H^i_{\mathfrak{m}}(M)_\nu, \tag{4}$$

where $H_M(T)_{|T^\nu}$ is the coefficient of T^ν in the Hilbert series $H_M(T)$, that is $\dim_k(M_\nu)$, and $\mathfrak{m}$ is the irrelevant ideal of $k[X_1, \ldots, X_r]$, that is $\mathfrak{m} = (X_1, \ldots, X_r)$. Recall also that for all $i > 0$

$$H^i(\mathbb{P}_k^{n-1}, M^\sharp(\nu)) \simeq H^{i+1}_{\mathfrak{m}}(M)_\nu,$$

and that, for all $\nu \in \mathbb{Z}$, we have an exact sequence:

$$0 \to H^0_{\mathfrak{m}}(M)_\nu \to M_\nu \to H^0(\mathbb{P}_k^{n-1}, M^\sharp(\nu)) \to H^1_{\mathfrak{m}}(M)_\nu \to 0. \tag{5}$$

Such a definition of multiplicity for M is called a *geometric* multiplicity and is also often called the degree of M because of its geometric meaning. Indeed let J

be a graded ideal of C and consider the quotient ring $R = C/J$. If δ denotes the dimension of R then the subscheme $\mathrm{Proj}(R)$ of $\mathbb{P}_k^{n-1}$ is of dimension $\delta - 1$. The degree of $\mathrm{Proj}(R)$ over $\mathbb{P}_k^{n-1}$ is defined to be the number of points obtained by cutting $\mathrm{Proj}(R)$ by $\delta - 1$ generic linear forms. To be more precise, if $l_1, \dots, l_{\delta-1}$ are generic linear forms of $\mathbb{P}_k^{n-1}$, then the scheme $S = \mathrm{Proj}(R/(l_1, \dots, l_{\delta-1}))$ is finite and we set

$$\deg_{\mathbb{P}_k^{r-1}}(\mathrm{Proj}(R)) := \dim_k \Gamma(S, \mathcal{O}_S) = \dim_k \left(\frac{R_\nu}{(l_1, \dots, l_{\delta-1})_\nu} \right),$$

for all sufficiently large ν. This geometric degree is in fact exactly what we called the multiplicity of R, i.e. we have

$$\deg_{\mathbb{P}_k^{r-1}}(\mathrm{Proj}(R)) = \mathrm{mult}_k(R) = L_M(1) = a_{d-1}.$$

This equality follows immediately from the exact sequence

$$0 \to R(-1) \xrightarrow{\times l_1} R \to R/(l_1) \to 0$$

which gives $\mathrm{H}_{R/(l_1)}(T) = (1-T)\mathrm{H}_R(T) = L_M(T)/(1-T)^{\delta-1}$, and an easy recursion.

At this point we can compute the multiplicity of a Veronese module which will be useful later.

Lemma 2.4 *Let M be a $\mathbb{Z}$-graded finite C-module. We denote by $M^{(d)}$, for all integers d, the Veronese module $\bigoplus_{\nu \in \mathbb{Z}} M_{d\nu}$ which is a $k[X_1^d, \dots, X_n^d]$-module. Setting $\delta = \dim_k(M)$ we have:*

$$\mathrm{mult}_k(M^{(d)}) = d^{\delta-1} \mathrm{mult}_k(M).$$

Proof. Let d be a fixed integer. It is known that for all integers ν the sum $\sum_{\xi^d=1} \xi^\nu$, where the sum is taken over all the d-unit roots, equals d if ν divides d and zero otherwise. Using this result one deduces

$$\mathrm{H}_{M^{(d)}}(T^d) = \frac{L_{M^{(d)}}(T^d)}{(1-T^d)^\delta} = \frac{1}{d} \sum_{\xi^d=1} \mathrm{H}_M(\xi T) = \frac{1}{d} \sum_{\xi^d=1} \frac{L_M(\xi T)}{(1-\xi T)^\delta}.$$

We obtain that $L_{M^{(d)}}(T^d)$ equals

$$\frac{1}{d} \sum_{\xi^d=1} \frac{(1-T^d)^\delta}{(1-\xi T)^\delta} L_M(\xi T) = \frac{1}{d} \sum_{\xi^d=1} \frac{(1-T)^\delta (1+T+\ldots+T^{d-1})^\delta}{(1-\xi T)^\delta} L_M(\xi T).$$

As $\mathrm{mult}_k(M^{(d)}) = L_{M^{(d)}}(1)$ it remains only to compare the preceding equality when $T \mapsto 1$, and one obtains

$$\mathrm{mult}_k(M^{(d)}) = \frac{d^\delta}{d} L_M(1) = d^{\delta-1} \mathrm{mult}_k(M),$$

the desired result. $\square$

In order to state the main theorem of this section we need to introduce another notion of multiplicity which is called the *algebraic* multiplicity. Let $(R, \mathfrak{m})$ be a local noetherian ring and $M \neq 0$ a finite R-module. Let $I \subset \mathfrak{m}$ be an ideal of R such that there exists an integer t satisfying $\mathfrak{m}^t M \subset IM$ (any such ideal is called a definition ideal of M), the numerical function length$(M/I^\nu M)$ is a polynomial function for sufficiently large values of $\nu \in \mathbb{N}$. This polynomial, denoted $\mathrm{S}_M^I(X)$, is called the *Hilbert-Samuel* polynomial of M with respect to I. It is of degree $\delta = \dim_k(M)$ and of the form:

$$\mathrm{S}_M^I(X) = \frac{e(I, M)}{\delta!} X^\delta + \textit{terms of lower powers in } X.$$

The algebraic multiplicity of I in M is the number $e(I, M)$ appearing in this polynomial. With such a definition of algebraic multiplicity one can define the algebraic multiplicity of a zero-dimensional scheme as follows: let J be a graded ideal of a $\mathbb{N}$-graded ring R, then if $T = \mathrm{Proj}(R/J)$ is a finite subscheme of $\mathrm{Proj}(R)$, its algebraic multiplicity is

$$e(T, \mathrm{Proj}(R)) = e(J^\sharp, R^\sharp) = \sum_{t \in T} e(J_t^\sharp, \mathcal{O}_{\mathrm{Proj}(R),t}) = \sum_{t \in T} e(J_t, R_t).$$

We are now ready to state the main result of this section.

Theorem 2.5 *Suppose that k is a field and A a $\mathbb{N}$-graded k-algebra of the form $k[X_1, \ldots, X_r]/J$, where J is a prime homogeneous ideal and each X_i is of degree one. Denote by δ the dimension of A and let $I = (f_1, \ldots, f_n)$ be an ideal of A such that each f_i is of degree $d \geq 1$. Then, if $T = \mathrm{Proj}(A/I)$ is finite over k, the number $d^{\delta-1}\deg_{\mathbb{P}_k^{r-1}}(\mathrm{Proj}(A)) - e(T, \mathrm{Proj}(A))$ equals*

$$\begin{cases} \deg(\lambda).\deg_{\mathbb{P}_k^{n-1}}(S) \text{ if } \lambda \text{ is generically finite} \\ 0 \text{ if } \lambda \text{ is not generically finite,} \end{cases}$$

where S denotes the closed image of λ (i.e. the image of v).

Proof. We denote by C the polynomial ring $k[X_1, \ldots, X_r]$ and consider it as a $\mathbb{N}$-graded ring by setting $\deg(X_i) = 1$ for all $i = 1, \ldots, r$. We denote also by $\mathfrak{m}$ its irrelevant ideal $\mathfrak{m} = (X_1, \ldots, X_r)$, and for simplicity we set $X = \mathrm{Proj}(A)$. The k-algebra A is a $\mathbb{N}$-graded C-module and combining (3) and (4) we obtain for all $\nu \in \mathbb{N}$

$$\dim_k(A_{d\nu}) = \chi(X, \mathcal{O}_X(d\nu)) + \sum_{i \geq 0} (-1)^i \dim_k H_{\mathfrak{m}}^i(A)_{d\nu}. \tag{6}$$

Similarly, for all $\nu \in \mathbb{N}$ the ideal I^ν of A is also a $\mathbb{N}$-graded C-module. We hence deduce that for all $\nu \in \mathbb{N}$

$$\dim_k((I^\nu)_{d\nu}) = \chi(X, (I^\nu)^\sharp(d\nu)) + \sum_{i\geq 0}(-1)^i \dim_k H^i_{\mathfrak{m}}(I^\nu)_{d\nu}. \tag{7}$$

Now the exact sequence of sheaves on X

$$0 \to (I^\nu)^\#(d\nu) \to \mathcal{O}_X(d\nu) \to \frac{\mathcal{O}_X}{(I^\nu)^\#}(d\nu) \to 0,$$

shows that, always for all $\nu \in \mathbb{N}$,

$$\chi(X, \mathcal{O}_X(d\nu)) - \chi(X, (I^\nu)^\sharp(d\nu)) = \chi(X, \frac{\mathcal{O}_X}{(I^\nu)^\#}(d\nu)).$$

Since $T = \mathrm{Proj}(A/I)$ is supposed to be finite, for all integers ν we have

$$\chi(X, \frac{\mathcal{O}_X}{(I^\nu)^\sharp}(d\nu)) = \chi(X, \mathcal{O}_X/(I^\nu)^\sharp) = \dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp),$$

where the last equality comes from (3). Subtracting (7) to (6), it follows that for all integers ν

$$\begin{aligned}\dim_k(A^{(d)}_\nu) - \dim_k((I^\nu)_{d\nu}) = {} & \dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp) + \sum_{i\geq 0}(-1)^i \dim_k H^i_{\mathfrak{m}}(A)_{d\nu} \\ & - \sum_{i\geq 0}(-1)^i \dim_k H^i_{\mathfrak{m}}(I^\nu)_{d\nu},\end{aligned}$$

and for all $\nu \in \mathbb{N}$ sufficiently large (recall that $H^i_{\mathfrak{m}}(A)_\nu = 0$ for $i \geq 0$ and $\nu \gg 0$)

$$\dim_k(A^{(d)}_\nu) - \dim_k((I^\nu)_{d\nu}) = \dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp) - \sum_{i\geq 0}(-1)^i \dim_k H^i_{\mathfrak{m}}(I^\nu)_{d\nu}. \tag{8}$$

At this point, notice that the algebraic multiplicity $e(T, X)$ appears naturally in this formula. Indeed, for all $\nu \gg 0$,

$$\dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp) = \sum_{t\in T} \mathrm{length}(A_t/I^\nu_t) = \frac{e(T, X)}{\delta!}\nu^\delta + R(\nu),$$

where R is a polynomial in $k[\nu]$ of degree stricly less than δ.

Now we relate (8) with the Rees algebra of I. Recall that we have a map $h : k[T_1, \ldots, T_n] \to A$ which sends each T_i to the polynomial $f_i \in A_d$. The polynomial ring $k[T_1, \ldots, T_n]$ is naturally $\mathbb{N}$-graded by setting $\deg(T_i) = 1$ for $i = 1, \ldots, n$. We have the following exact sequence of $\mathbb{N}$-graded $k[T_1, \ldots, T_n]$-modules

$$0 \to \ker(h) \to k[T_1, \ldots, T_n] \xrightarrow{h} \mathrm{Im}(h) \to 0.$$

From the definition of the map h it comes $\mathrm{Im}(h)_\nu = I^\nu \cap A_{d\nu} = (I^\nu)_{d\nu}$, and hence $\dim_k((I^\nu)_{d\nu}) = \dim_k(\mathrm{Im}(h)_\nu)$, for all $\nu \in \mathbb{N}$. The Rees algebra of I, that we denote hereafter by B, is the image of the following A-algebra morphism

$$\begin{aligned} A[T_1,\ldots,T_n] &\xrightarrow{\beta} B := \mathrm{Rees}_I(A) \subset A[Z] \\ T_i &\mapsto f_i Z. \end{aligned}$$

As A is $\mathbb{N}$-graded, B is naturally $\mathbb{Z}\times\mathbb{Z}$-graded if β is a homogeneous map, that is if $\deg(T_i) = d + \deg(Z)$, for all $i = 1,\ldots,n$. We choose $\deg(T_i) = 0$ for $i = 1,\ldots,n$, and hence $\deg(Z) = -d$ (notice that B has its own grading as a A-algebra, so this choice do not interfere with the grading of C). In this way $B_{p,q} = I^q_{p+dq}$ for all $(p,q) \in \mathbb{Z}\times\mathbb{Z}$, and hence $B_{p,\bullet} = \bigoplus_{q\in\mathbb{Z}} I^q(dq)_p$. It follows that $B = \bigoplus_{q\geq 0} I^q(dq)$ as a $\mathbb{N}$-graded A-module. Hereafter we will denote by B_p the graded part of B as a A-module, that is $B_p := B_{p,\bullet}$. It is then easy to check that $B_0 = \bigoplus_{q\geq 0}(I^q)\cap A_{dq} = \mathrm{Im}(h)$ and $\bigoplus_{q\geq 0} H^i_{\mathfrak{m}}(I^q)_{dq} = H^i_{\mathfrak{m}}(B)_0$, both as $\mathbb{N}$-graded $k[T_1,\ldots,T_n]$-modules. Finally (8) can be rewritten as

$$\dim_k(A^{(d)}_\nu) - \dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp) = \dim_k(B_{0,\nu}) - \sum_{i\geq 0}(-1)^i \dim_k(H^i_{\mathfrak{m}}(B)_{0,\nu}), \quad (9)$$

always for all integers ν sufficiently large.

We are now ready to prove the second assertion of this theorem. Denoting $\mathfrak{p} = (\ker(h)) \subset k[T_1,\ldots,T_n]$, if we suppose that the map λ is not generically finite, then $\dim(k[T_1,\ldots,T_n]/\mathfrak{p}) < \delta = \dim(A)$. Since B_0 and $H^i_{\mathfrak{m}}(B)_0$, with $i \geq 0$, are $k[T_1,\ldots,T_n]/\mathfrak{p}$-modules, we deduce that B_0 and $H^i_{\mathfrak{m}}(B)_0$, for all $i \geq 0$, have Hilbert polynomials of degree strictly less than $\dim(A)$. But the left side of equality (9) is a polynomial of degree δ with leading coefficient

$$\frac{d^{\delta-1}\mathrm{mult}_k(A) - e(T, \mathrm{Proj}(A))}{\delta!},$$

and thus we obtain the second statement of this theorem (notice that we did not use here the hypothesis J is prime).

We now concentrate on the first point of this theorem. We denote $\mathrm{Proj}_{\mathrm{A}}(B)$ the Proj of B as a $\mathbb{N}$-graded A-module, so that (5) gives (in a more general setting, see [11] theorem A4.1), for all $\nu \in \mathbb{N}$, the exact sequence of $\mathbb{N}$-graded $k[T_1,\ldots,T_n]$-modules:

$$0 \to H^0_{\mathfrak{m}}(B)_0 \to B_0 \to \Gamma(\mathrm{Proj}_{\mathrm{A}}(B), \mathcal{O}_{\mathrm{Proj}_{\mathrm{A}}(B)}) \to H^1_{\mathfrak{m}}(B)_0 \to 0. \quad (10)$$

Now remark that λ is generically finite if and only if the morphism $\overline{v}$: $\mathrm{Proj}_A(B) \to \mathrm{Spec}(k[T_1,\ldots,T_n]/\mathfrak{p})$, associated to the canonical morphism $k[T_1,\ldots,T_n]/\mathfrak{p} \to B$ sending each T_i to $f_i Z$, is also generically finite. Thus there exists an element $L \in k[T_1, \dots, T_n]/\mathfrak{p}$ such that $\overline{v}_L$ is finite. We obtain that $H^i_{\mathfrak{m}}(B_L) = 0$ for all $i \geq 2$, since $\mathrm{Proj}_A(B)_{|D(L)}$ is affine and hence $H^i(\mathrm{Proj}_A(B)_{|D(L)}, \mathcal{O}_{\mathrm{Proj}_A(B)_{|D(L)}}) = 0$ for all $i \geq 1$ (see [14], III.3), and it follows that, for all $i \geq 2$, $H^i_{\mathfrak{m}}(B)_0$ has Hilbert polynomial of degree less than or equal to $\delta - 1$. This and (10) show that (9) reduces to the equality

$$\dim_k(A^{(d)}_\nu) - \dim_k H^0(X, \mathcal{O}_X/(I^\nu)^\sharp) = \dim_k \Gamma(\mathrm{Proj_A}(B), \mathcal{O}_{\mathrm{Proj_A}(B)})_\nu, \quad (11)$$

which is true for all $\nu \gg 0$.

Consider now the generic point s of the image $S = V(\mathfrak{p}^\#) \subset \mathbb{P}^{n-1}_k$ of v, and denote by Y the fibred product

$$Y := Bl_{\mathbb{P}^{n-1}_k} \times_{\mathbb{P}^{n-1}_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^{n-1}_k, s}) = Bl_I \times_S \mathrm{Spec}(\mathcal{O}_{S,s}).$$

We claim that, if the map λ is generically finite, there exists an isomorphism of $\left(\frac{k[T_1,\dots,T_n]}{\mathfrak{p}}\right)_{\mathfrak{p}}$-modules:

$$\Gamma(\mathrm{Proj}_A(B), \mathcal{O}_{\mathrm{Proj}_A(B)})_{\mathfrak{p}} \simeq \left(\frac{k[T_1, \dots, T_n]}{\mathfrak{p}}\right)_{\mathfrak{p}} \otimes_{\mathcal{O}_{S,s}} \Gamma(Y, \mathcal{O}_Y). \quad (12)$$

Suppose for the moment that this claim is true. From proposition 2.3 λ is generically finite if and only if v is. Supposing now that J is prime, this is equivalent to say that the scheme Y is the spectrum of a field which is a finite extension of the field $\mathcal{O}_{S,s}$; the degree of λ is then the degree of the extension $\Gamma(Y, \mathcal{O}_Y)$ of $\mathcal{O}_{S,s}$. By (12) we deduce that there exists a morphism of $\mathbb{N}$-graded $k[T_1, \dots, T_n]/\mathfrak{p}$-modules of finite type

$$\gamma : \frac{k[T_1, \dots, T_n]}{\mathfrak{p}} \otimes_{\mathcal{O}_{S,s}} \Gamma(Y, \mathcal{O}_Y) \to \Gamma(\mathrm{Proj}_A(B), \mathcal{O}_{\mathrm{Proj}_A(B)}),$$

which becomes an isomorphism by localisation at $\mathfrak{p}$. The kernel E and the cokernel F of γ are hence graded $k[T_1, \dots, T_n]/\mathfrak{p}$-modules of finite type annihilated by an homogeneous element of non-zero degree. It follows that $\dim(E)$ and $\dim(F)$ are strictly lower than $\dim(A)$, which equals $\dim(S)$ by hypothesis. From here we may deduce that the Hilbert polynomial of the $k[T_1, \dots, T_n]$-module $\Gamma(\mathrm{Proj}_A(B), \mathcal{O}_{\mathrm{Proj}_A(B)})$ is of degree δ with $\deg(\lambda)\deg(S)/\delta!$ as leading coefficient, since the source of γ consists in $\deg(\lambda)$ copies of $k[T_1, \dots, T_n]/\mathfrak{p}$ which has Hilbert polynomial of degree δ with leading coefficient $\deg(S)/\delta!$. From (11) the first statement of the theorem follows. To complete the proof it thus remains to prove the claim (12). Hereafter, for convenience, we will write, as usually, $k[\mathbf{T}]$ for the polynomial ring $k[T_1, \dots, T_n]$.

We denote by $\hat{v} : \mathrm{Proj}_A(B) \to \mathbb{A}^n_k = \mathrm{Spec}(k[\mathbf{T}])$ the canonical projection, whose closed image is $V(\mathfrak{p}^\sim)$, which is $\{1\} \times \mathbb{G}_m$-equivariant. Recall that, by definition,

$$Bl_I := \mathrm{Biproj}(B) = (\mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*})/\{1\} \times \mathbb{G}_m,$$

and that the morphism $v : Bl_I \to \mathbb{P}^{n-1}_k$ is obtained from the morphism $\hat{v}_{|(\mathbb{A}^n_k)^*}$: $\mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*} \to (\mathbb{A}^n_k)^*$ by passing to the quotient. In fact, as the variables T_i are all of degree 1, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*} \arrow[r, "\hat{v}_{|(\mathbb{A}^n_k)^*}"] \arrow[d, "f"'] & (\mathbb{A}^n_k)^* \arrow[d, "\pi"'] \\
Bl_I = \mathrm{Biproj}(B) \arrow[r, "v"] & \mathbb{P}^{n-1}_k,
\end{tikzcd}$$

where the canonical projections f and π are (trivial) $\mathbb{G}_m$-torsors (also called principal $\mathbb{G}_m$-bundles, see [22]). The ring $B_\mathfrak{p}$ is naturally $\mathbb{Z} \times \mathbb{Z}$-graded and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}_A(B_\mathfrak{p}) \arrow[r] \arrow[d, "\hat{v}_\mathfrak{p}"'] & \mathrm{Proj}_A(B) \arrow[d, "\hat{v}"'] \\
\mathrm{Spec}(k[\mathbf{T}]_\mathfrak{p}) \arrow[r, "\theta_\mathfrak{p}"] & \mathrm{Spec}(k[\mathbf{T}])
\end{tikzcd}$$

which shows, since $\theta_\mathfrak{p}$ is flat, that

$$\Gamma(\mathrm{Proj}_A(B_\mathfrak{p}), \mathcal{O}_{\mathrm{Proj}_A(B_\mathfrak{p})}) \simeq \Gamma(\mathrm{Proj}_A(B), \mathcal{O}_{\mathrm{Proj}_A(B)})_\mathfrak{p}. \tag{13}$$

The morphism $\mathrm{Proj}_A(B_\mathfrak{p}) \to \mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*}$ is clearly $\{1\} \times \mathbb{G}_m$-equivariant, and hence, passing to the quotient, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Proj}_A(B_\mathfrak{p}) \arrow[r] \arrow[d, "f_\mathfrak{p}"'] & \mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*} \arrow[d, "f"'] \\
\mathrm{Biproj}(B_\mathfrak{p}) \arrow[r] & \mathrm{Biproj}(B),
\end{tikzcd}$$

where both vertical arrows are $\mathbb{G}_m$-torsors. We deduce the isomorphism

$$\mathcal{O}_{\mathrm{Proj}_A(B_\mathfrak{p})} \simeq f^*_\mathfrak{p}({B_{\mathfrak{p}(0,0)}}^{\#\#}). \tag{14}$$

Now we have $\mathrm{Proj}_A(B_\mathfrak{p}) = \mathrm{Spec}(k[\mathbf{T}]_\mathfrak{p}) \times_{(\mathbb{A}^n_k)^*} \mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*}$ and $\mathrm{Spec}(k[\mathbf{T}]_\mathfrak{p}) = (\mathbb{A}^n_k)^* \times_{\mathbb{P}^{n-1}_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^{n-1}_k, s})$. It follows that

$$\mathrm{Proj}_A(B_\mathfrak{p}) = \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^{n-1}_k, s}) \times_{\mathbb{P}^{n-1}_k} \mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*},$$

and the diagram (where both squares are commutative)

$$\begin{tikzcd}
\mathrm{Proj}_A(B_{\mathfrak{p}}) \arrow[r] \arrow[d, "f_{\mathfrak{p}}"'] & \mathrm{Proj}_A(B)_{|(\mathbb{A}^n_k)^*} \arrow[d, "f"'] \\
\mathrm{Biproj}(B_{\mathfrak{p}}) \arrow[r] \arrow[d] & \mathrm{Biproj}(B) \arrow[d, "v"'] \\
\mathrm{Spec}(\mathcal{O}_{\mathbb{P}^{n-1}_k,s}) \arrow[r] & \mathbb{P}^{n-1}_k
\end{tikzcd}$$

shows that $\mathrm{Biproj}(B_{\mathfrak{p}}) = Bl_I \times_{\mathbb{P}^{n-1}_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^{n-1}_k,s}) = Bl_I \times_S \mathrm{Spec}(\mathcal{O}_{S,s}) = Y$. From this we can also deduce $\mathrm{Proj}_A(B_{\mathfrak{p}}) = Bl_I \times_S \mathrm{Spec}((k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}})$ where the projection of $\mathrm{Spec}((k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}})$ on S is given by the composed morphism $\mathrm{Spec}((k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}}) \xrightarrow{\alpha} \mathrm{Spec}(\mathcal{O}_{S,s}) \to S$. Moreover the localized morphism $f_{\mathfrak{p}}$ is then $f_{\mathfrak{p}} : Bl_I \times_S \mathrm{Spec}((k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}}) \xrightarrow{\mathrm{Id}\times\alpha} Bl_I \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$. By hypothesis the scheme $Y = Bl_I \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is finite on $\mathrm{Spec}(\mathcal{O}_{S,s})$, it is hence the spectrum of a semi-local ring and it follows (by [2], proposition 5, II.5)

$$B_{\mathfrak{p}(0,0)}{}^{\#\#} \simeq \mathcal{O}_{\mathrm{Biproj}(B_{\mathfrak{p}})} \simeq \mathcal{O}_Y.$$

By (14) we deduce $\mathcal{O}_{\mathrm{Proj}_A(B_{\mathfrak{p}})} \simeq f_{\mathfrak{p}}^*(\mathcal{O}_{\mathrm{Biproj}}(B_{\mathfrak{p}}))$. As $B_{\mathfrak{p}}$ contains an invertible homogeneous element of non-zero degree with respect to the grading of $k[\mathbf{T}]$, $f_{\mathfrak{p}}$ is an affine morphism and we obtain the following isomorphisms of $(k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}}$-modules

$$\begin{aligned}\Gamma(\mathrm{Proj}_A(B_{\mathfrak{p}}), \mathcal{O}_{\mathrm{Proj}_A(B_{\mathfrak{p}})}) &\simeq \Gamma(\mathrm{Biproj}(B_{\mathfrak{p}}), f_{\mathfrak{p}*}(\mathcal{O}_{\mathrm{Proj}_A(B_{\mathfrak{p}})})) \\ &\simeq \Gamma(Y, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S,s}} (k[\mathbf{T}]/\mathfrak{p})_{\mathfrak{p}}.\end{aligned}$$

Comparing with (13), this proves the claim (12). □

Applying this theorem with $A = k[X_1, \dots, X_r]$ (or equivalently $J = (0)$), we can compute the degree of the closed image of a generically finite rational map

$$\begin{aligned}\mathbb{P}^{r-1}_k &\xrightarrow{\lambda} \mathbb{P}^{n-1}_k \\ (X_1 : \cdots : X_r) &\mapsto (f_1 : \cdots : f_n)(X_1 : \cdots : X_r),\end{aligned}$$

if $\dim(A/(f_1, \dots, f_n)) \leq 1$, and moreover decide if this map is generically finite by computing $d^{r-1} - e(T, \mathrm{Proj}(A))$. For instance if we suppose in addition $r = 3$ and $n = 4$ we recover the well-known formula to compute the degree of a surface of $\mathbb{P}^3_k$ parameterized by 4 homogeneous polynomials of $\mathbb{P}^2_k$ of the same degree and without common factor (see e.g. [8], appendix). This theorem also yields a more general formula to compute the degree of the closed image of a generically finite rational map from an irreducible subvariety $X = \mathrm{Proj}(A)$ of $\mathbb{P}^{r-1}_k = \mathrm{Proj}(k[X_1, \ldots, X_r])$ to $\mathbb{P}^{n-1}_k$ with a finite number of base points (possibly zero). For instance the closed image of a regular map λ from a curve $\mathcal{C}$ of $\mathbb{P}^2_k$ to $\mathbb{P}^2_k$ given by homogeneous polynomials of same degree $d \geq 1$ (this implies that λ is generically finite by the second point of the theorem) is $d.\deg(\mathcal{C})/deg(\lambda)$.

3 Blow-up algebras associated to a rational map

Like at the beginning of the preceding section, we suppose that k is a commutative ring and A is a $\mathbb{Z}$-graded k-algebra. We denote also by $\tau : k \to A_0$ the canonical morphism of rings and consider the k-algebra morphism

$$\begin{aligned} h : k[T_1, \ldots, T_n] &\longrightarrow A \\ T_i &\mapsto f_i, \end{aligned}$$

where each f_i is supposed to be of degree $d \geq 1$. We will focus on two blow-up algebras associated to the ideal $I = (f_1, \ldots, f_n)$ of A, the Rees algebra $\mathrm{Rees}_A(I)$ and the symmetric algebra $\mathrm{Sym}_A(I)$, and show their close relation with the ideal $\ker(h)$ of $k[T_1, \ldots, T_n]$ we would like to study. We have already seen that the Rees algebra of I appears naturally in our situation, and we begin to deal with it. Hereafter, for simplicity, we will often denote the sequence $T_1, \ldots, T_n$ by the bold letter $\mathbf{T}$ and for instance write $k[\mathbf{T}]$ (resp. $A[\mathbf{T}]$) instead of $k[T_1, \ldots, T_n]$ (resp. $A[T_1, \ldots, T_n]$).

Introducing a new indeterminate Z, the Rees algebra of I can be described as the image of the A-algebra morphism

$$\begin{aligned} \beta : A[T_1, \ldots, T_n] &\longrightarrow A[Z, Z^{-1}] \\ T_i &\mapsto f_i Z^{-1}. \end{aligned}$$

(Note that, compared with the definition (2) of β, we are switching to Z^{-1} in order to not overload notations in the sequel.) The kernel of β has the following simple description that

$$\ker(\beta) = (T_1 - f_1 Z^{-1}, \ldots, T_n - f_n Z^{-1}) \cap A[\mathbf{T}].$$

This gives an easy way to compute explicitly $\ker(\beta)$ since we only need to eliminate the variable Z^{-1} from the ideal $(T_1 - f_1 Z^{-1}, \ldots, T_n - f_n Z^{-1})$. This description also shows how $\ker(\beta)$ and what is called "moving hypersurfaces" in different works dealing with the implicitization problem (see 1.1 and 1.2) are related. A moving hypersurface following the parameterization $f_1, \ldots, f_n$ is a polynomial $F \in A[\mathbf{T}]$ homogeneous in the $T_i's$ and which satisfies $F(f_1, \ldots, f_n) = 0$. The latter condition implies that $F \in (T_1 - f_1, \ldots, T_n - f_n)$, so moving hypersurfaces following the parameterization are exactly the homogeneous elements of $(T_1 - f_1, \ldots, T_n - f_n) \cap A[\mathbf{T}]$. But it is easy to check that these homogeneous elements generate $(T_1 - f_1 Z^{-1}, \ldots, T_n - f_n Z^{-1}) \cap A[\mathbf{T}] = \ker(\beta)$ by adding the new variable Z^{-1}. Thus it follows that $\ker(\beta)$ is generated by the moving hypersurfaces following the parameterization.

We now turn to a second description of $\ker(\beta)$ in terms of inertia forms, which will enable us to relate it with the kernel of the map h.

Lemma 3.1 $\ker(\beta) = \mathrm{TF}_{(Z)}((f_1 - T_1 Z, \ldots, f_n - T_n Z)) \cap A[T_1, \ldots, T_n]$ *as ideals of the ring* $A[T_1, \ldots, T_n]$.

Proof. Let us consider the A-algebra morphism

$$A[T_1, \ldots, T_n, Z] \xrightarrow{\phi} A[Z, Z^{-1}] : T_i \mapsto f_i Z^{-1},$$

and the quotient algebra $D = A[\mathbf{T}, Z]/(f_1 - T_1 Z, \ldots, f_n - T_n Z)$. The kernel of the localization morphism $D \to D_Z$ is $H^0_{(Z)}(D)$. The A-algebra isomorphism $D_Z \to A[Z, Z^{-1}]$ obtained by sending each T_i to f_i/Z makes the following diagram commutative:

$$\begin{tikzcd} A[\mathbf{T}, Z] \arrow[r, "\phi"] \arrow[d] & A[Z, Z^{-1}] \\ D \arrow[r] & D_Z \arrow[u, "\wr"'] \end{tikzcd}$$

We hence deduce that $\ker(\phi) = \mathrm{TF}_{(Z)}(f_1 - T_1 Z, \ldots, f_n - T_n Z)$. Since $\ker(\beta) = \ker(\phi) \cap A[\mathbf{T}]$, the lemma follows immediately. $\square$

Remark 3.2 *We saw that* $H^0_{(Z)}(D) = \ker(D \to D_Z)$ *and is hence prime if* A *is assumed to be an integral domain. It follows then easily that the ideal* $\mathrm{TF}_{(Z)}(f_1 - T_1 Z, \ldots, f_n - T_n Z)$ *is also prime since it is isomorphic to the kernel of the canonical map* $A[\mathbf{T}, Z] \to D_Z$.

Recall that h is the map $k[T_1, \ldots, T_n] \to A$ which sends each T_i to f_i. If we denote by $\tau[\mathbf{T}]$ the canonical polynomial extension

$$\begin{aligned}\tau[\mathbf{T}] : k[T_1,\dots,T_n] &\longrightarrow A[T_1,\dots,T_n]\\ T_i &\mapsto T_i,\end{aligned}$$

which is a $k[\mathbf{T}]$-algebra morphism, we can easily check that

$$\begin{aligned}\ker(h) &= \tau[\mathbf{T}]^{-1}((T_1 - f_1,\dots,T_n - f_n))\\ &= \{P \in k[T_1,\dots,T_n] : P(f_1,\dots,f_n) = 0\}.\end{aligned} \tag{15}$$

If we denote also by Θ the composed morphism $i\circ\tau[\mathbf{T}]$, where i is the canonical inclusion $A[T_1,\dots,T_n] \hookrightarrow A[T_1,\dots,T_n,Z]$, we have the following description of $\ker(h)$:

Lemma 3.3 $\ker(h) = \Theta^{-1}(\mathrm{TF}_{(Z)}(f_1 - T_1Z,\dots,f_n - T_nZ))$.

Proof. First let $P \in k[\mathbf{T}]$ such that $\Theta(P) \in \mathrm{TF}_{(Z)}(f_1 - T_1Z,\dots,f_n - T_nZ))$, that is there exists $N \in \mathbb{N}^*$ such that $Z^N\tau[\mathbf{T}](P) \in (f_1 - T_1Z,\dots,f_n - T_nZ)$ in $A[\mathbf{T}, Z]$. Specializing Z to 1 we obtain $P \in \ker(h)$ by (15).

Now let $P \in \ker(h) \subset k[\mathbf{T}]$. As the ideal $\ker(h)$ is homogeneous in $k[\mathbf{T}]$, we can suppose that P is homogeneous of degree m. In this case we can write

$$\begin{aligned}Z^mP(\mathbf{T}) &= P(ZT_1,\dots,ZT_n)\\ &= P(ZT_1,\dots,ZT_n) - P(f_1,\dots,f_n)\\ &\in (f_1 - T_1Z,\dots,f_n - T_nZ).\end{aligned}$$

We hence deduce that $P \in \mathrm{TF}_{(Z)}((f_1 - T_1Z,\dots,f_n - T_nZ))$. $\square$

Remark 3.4 *If $k \subset A_0$ then we have immediately*

$$\ker(h) = \mathrm{TF}_{(Z)}(f_1 - T_1Z,\dots,f_n - T_nZ) \cap k[\mathbf{T}].$$

Moreover if we suppose that $k = A_0$, $\deg(T_i) = 0$ and $\deg(Z) = d \geq 1$, then we obtain $k[\mathbf{T}] = (A[\mathbf{T}, Z])_0$ and so

$$\ker(h) = (\mathrm{TF}_{(Z)}(f_1 - T_1Z,\dots,f_n - T_nZ))_0.$$

We are now ready to relate $\ker(h)$ and $\ker(\beta)$.

Proposition 3.5 *Suppose that $k \subset A_0$, then*

$$\ker(h) = \ker(\beta) \cap k[\mathbf{T}] = \mathrm{TF}_{(Z)}(f_1 - T_1Z,\dots,f_n - T_nZ)) \cap k[\mathbf{T}].$$

Moreover, if J is an ideal of A such that $H^0_J(A) = 0$, then we have $\ker(\beta) =$

$\mathrm{TF}_J(\ker(\beta))$ *and hence*

$$\ker(h) = \ker(\beta) \cap k[\mathbf{T}] = \mathrm{TF}_J(\ker(\beta)) \cap k[\mathbf{T}].$$

Proof. By lemma 3.1 and remark 3.4 the first claim is proved. To prove the second part of this lemma we consider the A-algebra

$$D = A[\mathbf{T}, Z]/(f_1 - T_1Z, \ldots, f_n - T_nZ).$$

This yields an isomorphism of A-algebras, that we have already used in lemma 3.1, $D_Z \xrightarrow{\sim} A[Z, Z^{-1}]$ obtained by sending each T_i to f_i/Z. Now suppose that J is an ideal of A such that $H^0_J(A) = 0$, then $H^0_J(D_Z) = H^0_J(A)[Z, Z^{-1}] = 0$. This implies that $H^0_J(D) \subset H^0_{(Z)}(D)$, from we deduce

$$\mathrm{TF}_J((f_1 - T_1Z, \ldots, f_n - T_nZ)) \subset \mathrm{TF}_{(Z)}((f_1 - T_1Z, \ldots, f_n - T_nZ)).$$

Now it comes easily

$$\begin{aligned}\mathrm{TF}_{(Z)}((f_1 - T_1Z, \ldots, f_n - T_nZ)) &= \mathrm{TF}_{(Z)}\mathrm{TF}_J((f_1 - T_1Z, \ldots, f_n - T_nZ)) \\ &= \mathrm{TF}_J\mathrm{TF}_{(Z)}((f_1 - T_1Z, \ldots, f_n - T_nZ)),\end{aligned}$$

and intersecting with $A[\mathbf{T}]$ we obtain $\ker(\beta) = \mathrm{TF}_J(\ker(\beta))$. □

This result shows the close relation between $\ker(h)$ and $\ker(\beta)$. In fact, as we have already mentionned, a lot of information about the closed image of λ is contained in the kernel of β (which can be obtained by Gröbner basis computations for instance). However $\ker(\beta)$ is difficult to study in general, and we prefer to work with an "approximation" of it which involves the study of the symmetric algebra of I that we now describe.

Classically, we have the well known canonical surjective morphism of A-algebras

$$\begin{aligned}\alpha : A[T_1, \ldots, T_n] &\longrightarrow \mathrm{Sym}_A(I) \\ T_i &\mapsto f_i,\end{aligned}$$

whose kernel is described by

$$\ker(\alpha) = \{T_1g_1 + \ldots + T_ng_n : g_i \in A[\mathbf{T}], \sum_{i=1}^{n} f_ig_i = 0\}.$$

Note here that $\ker(\alpha)$ is generated by the moving hyperplanes (i.e. moving hypersurfaces of degree 1) following the parameterization $f_1, \ldots, f_n$. The symmetric algebra of I appears naturally by its link with the Rees algebra of I

(see for instance [25]). We have the following commutative diagram

$$\begin{tikzcd}
\ker(\beta) \arrow[dr] & & \ker(\alpha) \arrow[ll, hook'] \arrow[dl] \\
& A[T_1,\dots,T_n] \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathrm{Sym}_A(I) \arrow[rr, "\sigma"] & & \mathrm{Rees}_A(I)
\end{tikzcd}$$

where σ denotes the canonical map from $\mathrm{Sym}_A(I)$ to $\mathrm{Rees}_A(I)$. In fact the quotient $\ker(\beta)/\ker(\alpha)$ has been widely studied as it gives a measure of the difficulty in examining the Rees algebra of I. We recall that the ideal I is said to be of *linear type* if the canonical map σ is an isomorphism. The following proposition can be summarizes by "the ideal $\ker(\alpha)$ is as an approximation of the ideal $\ker(\beta)$".

Proposition 3.6 *Let J be an ideal of A such that the ideal I is of linear type outside $V(J)$ then*

$$\mathrm{TF}_J(\ker(\alpha)) = \mathrm{TF}_J(\ker(\beta)).$$

If moreover $H^0_J(A) = 0$ then

$$\ker(\beta) = \mathrm{TF}_J(\ker(\alpha)).$$

Proof. The first assertion comes by definition: if I is of linear type outside $V(J)$ then the $A[\mathbf{T}]$-module $\ker(\beta)/\ker(\alpha)$ is supported in $V(J)$, that is

$$J.A[\mathbf{T}] \subset \sqrt{J.A[\mathbf{T}]} \subset \sqrt{\mathrm{ann}_{A[\mathbf{T}]}(\ker(\beta)/\ker(\alpha))},$$

which implies $\mathrm{TF}_J(\ker(\alpha)) = \mathrm{TF}_J(\ker(\beta))$. The second statement is a consequence of the first one and proposition 3.5. $\square$

Remark 3.7 *We will see later, as a particular case, that if the first homology group of the Koszul complex associated to the sequence $f_1, \dots, f_n$ is zero outside $V(J)$, then I is of linear type outside $V(J)$.*

In view of propositions 3.6 and 3.5, we end this section with the following corollary which summarizes the relations between $\ker(h)$, $\ker(\beta)$ and $\ker(\alpha)$:

Corollary 3.8 *Suppose that $k \subset A_0$. If J is an ideal of A such that $H^0_J(A) = 0$*

and I is of linear type outside $V(J)$, then

$$\ker(h) = \ker(\beta) \cap k[\mathbf{T}] = \mathrm{TF}_J(\ker(\beta)) \cap k[\mathbf{T}] = \mathrm{TF}_J(\ker(\alpha)) \cap k[\mathbf{T}].$$

4 Approximation complexes

In the first part of this section we give the definition and some basic properties of the *approximation complexes*. These complexes was introduced in [24] and systematically developed in [15] and [16]. At their most typical, they are projective resolutions of the symmetric algebras of ideals and allow an in-depth study of the canonical morphism $\sigma : \mathrm{Sym}_A(I) \to \mathrm{Rees}_A(I)$. In what follows we only develop (sometimes without proof) those properties that directly affect the applications we are interested in. For a complete treatment on the subject we refer the reader to the previously cited articles. In the second part of this section we prove two new acyclicity lemmas about the approximation complexes that we will use later to deal with the implicitization problem. Finally, we end this section with a problem introduced by David Cox in the study of the method of "moving surfaces", and called *Koszul syzygies* (see [9]).

4.1 Definition and basic properties

Let A be a commutative ring and J be an ideal of A generated by r elements $a_1, \ldots, a_r$ (which we will often denote by the bold letter $\mathbf{a}$). Both applications

$$u : A[T_1, \ldots, T_r]^r \xrightarrow{(a_1,\ldots,a_r)} A[T_1, \ldots, T_r] : (b_1, \ldots, b_r) \mapsto \sum_{i=1}^{r} b_i a_i,$$

$$v : A[T_1, \ldots, T_r]^r \xrightarrow{(T_1,\ldots,T_r)} A[T_1, \ldots, T_r] : (b_1, \ldots, b_r) \mapsto \sum_{i=1}^{r} b_i T_i,$$

give two Koszul complexes $K(\mathbf{a}; A[\mathbf{T}])$ and $K(\mathbf{T}; A[\mathbf{T}])$ with respective differentials $d_{\mathbf{a}}$ and $d_{\mathbf{T}}$. One can check easily that these differentials satisfy the property $d_{\mathbf{a}} \circ d_{\mathbf{T}} + d_{\mathbf{T}} \circ d_{\mathbf{a}} = 0$, and hence there exists three complexes, the so-called approximation complexes, which we denote

$$\begin{aligned}
&\mathcal{Z}_\bullet = (\ker d_{\mathbf{a}}, d_{\mathbf{T}})\\
&\mathcal{B}_\bullet = (\mathrm{Im}\, d_{\mathbf{a}}, d_{\mathbf{T}})\\
&\mathcal{M}_\bullet = (H_\bullet(K(\mathbf{a}; A[\mathbf{T}])), d_{\mathbf{T}}).
\end{aligned}$$

The $\mathcal{Z}$-complex ends with the sequence $\ker(u) \xrightarrow{v} A[T_1, \ldots, T_r] \to 0$. Since

$v(\ker(u)) = \{\sum_{i=1}^{r} b_i T_i \ s.t. \ \sum_{i=1}^{r} b_i a_i = 0\}$, we deduce that

$$H_0(\mathcal{Z}) = \frac{A[T_1, \ldots, T_r]}{v(\ker(u))} \simeq \mathrm{Sym}_A(J).$$

A similar argument show that $H_0(\mathcal{M}) \simeq \mathrm{Sym}_{A/J}(J/J^2)$. More generally one can check that $v(\ker(u))$ annihilates the homology $A[\mathbf{T}]$-modules of $\mathcal{Z}, \mathcal{B}$ and $\mathcal{M}$ which are hence modules over $\mathrm{Sym}_A(J)$. These homology modules have also a nice property, which is probably one of the most important of the approximation complexes:

Proposition 4.1 *The homology modules of $\mathcal{Z}, \mathcal{B}$ and $\mathcal{M}$ do not depend on the generating set chosen for the ideal J.*

Proof. See proposition 3.2.6 and corollary 3.2.7 of [25]. □

Some other nice properties of the approximation complexes are due to the exactness of the standard Koszul complex $\mathcal{L} = K(\mathbf{T}; A[\mathbf{T}])$. By definition we have the exact sequence of complexes

$$0 \to \mathcal{Z} \hookrightarrow \mathcal{L} \xrightarrow{d_{\mathbf{a}}} \mathcal{B}[-1] \to 0, \tag{16}$$

where $\mathcal{B}[-1]$ denotes the translate of $\mathcal{B}$ such that $\mathcal{B}[-1]_n = \mathcal{B}_{n-1}$. Moreover, setting $\deg(T_i) = 1$ for all i, the complexes $\mathcal{L}, \mathcal{Z}$ and $\mathcal{B}$ (and also $\mathcal{M}$) are graded and give for all $t \in \mathbb{N}$ an exact sequence

$$0 \to (\mathcal{Z})_t \hookrightarrow (\mathcal{L})_t \xrightarrow{d_{\mathbf{a}}} (\mathcal{B}[-1])_{t-1} \to 0,$$

where $(\mathcal{L})_t, (\mathcal{Z})_t$ and $(\mathcal{B})_t$ are the part of degree t of the complexes $\mathcal{L}, \mathcal{Z}$ and $\mathcal{B}$. For instance $(\mathcal{L})_t$ is the complex of A-modules

$$0 \to A[\mathbf{T}]_{t-r} \xrightarrow{d_{\mathbf{T}}} \ldots \xrightarrow{d_{\mathbf{T}}} A[\mathbf{T}]_{t-1}^{r} \xrightarrow{d_{\mathbf{T}}} A[\mathbf{T}]_t \to 0.$$

Now from the exactness of $\mathcal{L}$ in positive degrees and the long exact sequence associated to (16) we deduce, for all $i \geq 1$, an isomorphism of graded modules $H_i(\mathcal{B}) \xrightarrow{\sim} H_i(\mathcal{Z})(1)$, where $H_i(\mathcal{Z})(1)$ denotes the graded module $H_i(\mathcal{Z})$ with degree shifted by 1 (i.e. $H_i(\mathcal{Z})(1)_t = H_i(\mathcal{Z})_{t+1}$). For the case $i = 0$ the long exact sequence ends with

$$H_1(\mathcal{L}) = 0 \to H_0(\mathcal{B}) \to H_0(\mathcal{Z}) \xrightarrow{\pi} H_0(\mathcal{L}) \to 0,$$

hence $H_0(\mathcal{B}) \simeq \ker(\pi)$. Since $H_0(\mathcal{L})_i = 0$ if $i > 0$ and is A if $i = 0$, we deduce

Proposition 4.2 *If for all $i \geq 0$ we define the graded modules $\tilde{H}_i(\mathcal{Z})$ by $\tilde{H}_i(\mathcal{Z})_t = H_i(\mathcal{Z})_t$ for all $i \geq 1$ and all t, $\tilde{H}_0(\mathcal{Z})_t = H_0(\mathcal{Z})_t$ if $t \neq 0$ and*

$\tilde{H}_0(\mathcal{Z})_0 = 0$, *then we have an isomorphism of graded modules*

$$H_i(\mathcal{B}) \xrightarrow{\sim} \tilde{H}_i(\mathcal{Z})(1) \text{ for all } i \geq 0.$$

We have another natural exact sequence of complexes involving the complex $\mathcal{M}$:

$$0 \to \mathcal{B} \to \mathcal{Z} \to \mathcal{M} \to 0. \tag{17}$$

Proposition 4.2 and the long exact sequence associated to (17) give the following graded long exact sequence:

$$\begin{aligned} \ldots \to \tilde{H}_i(\mathcal{Z})(1) \to H_i(\mathcal{Z}) \to H_i(\mathcal{M}) \to \tilde{H}_{i-1}(\mathcal{Z})(1) \to \ldots \\ \ldots \to \tilde{H}_0(\mathcal{Z})(1) \to H_0(\mathcal{Z}) \to H_0(\mathcal{M}) \to 0. \end{aligned} \tag{18}$$

This sequence yields two results on these approximation complexes. The first one is contained in the following proposition.

Proposition 4.3 *Suppose that A is a noetherian ring and $i \geq 1$. If $H_i(\mathcal{M}) = 0$ then $H_i(\mathcal{Z}) = 0$. In particular, if $\mathcal{M}$ is acyclic then $\mathcal{Z}$ is also acyclic.*

Proof. From the long exact sequence (18) we have, for all $i \geq 1$, the exact sequence

$$\tilde{H}_i(\mathcal{Z})(1) \to H_i(\mathcal{Z}) \to H_i(\mathcal{M}).$$

By hypothesis $H_i(\mathcal{M}) = 0$, and since $i \geq 1$, we deduce a surjective morphism of $A[\mathbf{T}]$-modules $H_i(\mathcal{Z})(1) \to H_i(\mathcal{Z})$. Now A is noetherian, so $A[\mathbf{T}]$ is also and we deduce that $H_i(\mathcal{Z})$ is a $A[\mathbf{T}]$-module of finite type. This implies that our surjective morphism is in fact bijective. Taking into account the degrees we obtain isomorphisms $H_i(\mathcal{Z})_{t+1} \xrightarrow{\sim} H_i(\mathcal{Z})_t$ for all t. As $H_i(\mathcal{Z})_{-1} = 0$ we deduce by iteration that $H_i(\mathcal{Z})_t = 0$ for all t. □

Remark 4.4 *Notice that, by definition of $\mathcal{M}$, if $H_i(K(\mathbf{a}; A[\mathbf{T}])) = 0$ then $H_i(\mathcal{M}) = 0$.*

The second result given by (18) is a condition so that the canonical morphism $\sigma : \mathrm{Sym}_A(J) \to \mathrm{Rees}_A(J)$ becomes an isomorphism: in this case we say that J is of *linear type*. By definition we have

$$\tilde{H}_0(\mathcal{Z})(1) = \oplus_{t\geq 1} H_0(\mathcal{Z})_t = \oplus_{t\geq 1}\mathrm{Sym}^t_A(J) = \mathrm{Sym}^+_A(J),$$

and hence, examining the end of (18), we obtain the exact sequence

$$H_1(\mathcal{M}) \to \mathrm{Sym}^+_A(J) \xrightarrow{\mu} \mathrm{Sym}_A(J) \to \mathrm{Sym}_{A/J}(J/J^2) \to 0.$$

The map μ is called the *downgrading* map (see [25], chap. 3), it is induced by the morphism

$$T_A^n(J) \to T_A^{n-1}(J) : x_1 \otimes \ldots \otimes x_n \mapsto x_1x_2 \otimes x_3 \otimes \ldots \otimes x_n,$$

where $T_A^n(J)$ denotes the degree n part of the tensor algebra associated to J over A. Denoting $\mathrm{gr}_J(A) := \oplus_{i\geq 0} J^i/J^{i+1}$ and $\mathrm{Rees}_A^+(J) := \oplus_{i\geq 1} J^i$, we obtain the commutative diagram

$$\begin{tikzcd}
H_1(\mathcal{M}) \arrow[r] & \mathrm{Sym}_A^+(J) \arrow[r, "\mu"] \arrow[d] & \mathrm{Sym}_A(J) \arrow[r, "\pi"] \arrow[d, "\sigma"] & \mathrm{Sym}_{A/J}(J/J^2) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathrm{Rees}_A^+(J) \arrow[r] & \mathrm{Rees}_A(J) \arrow[r] & \mathrm{gr}_J(A) \arrow[r] & 0,
\end{tikzcd}$$

which gives the following proposition.

Proposition 4.5 *If* $H_1(\mathcal{M}) = 0$ *then* $\mathrm{Sym}_A(J) \simeq \mathrm{Rees}_A(J)$*, that is* J *is of linear type.*

Proof. For all $i \geq 0$ we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sym}_A^{i+1}(J) \arrow[r, "\mu"] \arrow[d, "\sigma_{i+1}"] & \mathrm{Sym}_A^i(J) \arrow[d, "\sigma_i"] \\
0 \arrow[r] & J^{i+1} \arrow[r] & J^i.
\end{tikzcd}$$

But σ_0 is an isomorphism, which implies that σ_1 is injective. As σ_1 is already surjective it is an isomorphism. By iteration we show that σ_t is an isomorphism for all $t \in \mathbb{N}$, and hence that $\mathrm{Sym}_A(J) \simeq \mathrm{Rees}_A(J)$. □

This result gives a criterion for an ideal to be of *linear type*, that is σ is an isomorphism. If the ring A is noetherian one can show that σ is an isomorphism if and only if γ is (see [25], theorem 2.2.1). There also exists other criterions for an ideal to be of linear type based on some properties of a set of generators of J (see [25], paragraph 3.3) but we will not go further in this direction here.

4.2 Acyclicity criteria

We will here prove two acyclicity results that we will apply later to the implicitization problem. We first recall the acyclicity criterion of Peskine-Szpiro:

Lemma 4.6 (Acyclicity lemma) *Suppose that R is a commutative noetherian ring, I a proper ideal, and*

$$C_\bullet : 0 \to M_n \to M_{n-1} \to \ldots \to M_0$$

a complex of finitely generated R-modules with homology $H_k = H_k(C_\bullet)$ such that each non-zero module H_k, for all $k = 1, \ldots, n$, satisfies $\operatorname{depth}_I(H_k) = 0$. *Then* $\operatorname{depth}_I(M_k) \geq k$ *for all $k = 1, \ldots, n$ implies $C_\bullet$ is exact.*

Proof. We refer to [4], lemma 3. □

Hereafter we assume that A is a noetherian commutative $\mathbb{N}$-graded ring, and we denote by $\mathfrak{m}$ its ideal generated by elements of strictly positive degree, that is $\mathfrak{m} = A_+ = \oplus_{\nu>0} A_\nu$. The following proposition is our first acyclicity result.

Proposition 4.7 *Let $I = (a_1, \ldots, a_n)$ be an ideal of A such that both ideals I and $\mathfrak{m}$ have the same radical, and suppose that $\sigma =$ $\operatorname{depth}_{\mathfrak{m}}(A) \geq 1$. Then the homology modules $H_i(\mathcal{Z})$ vanish for $i \geq$ $\max(1, n-\sigma)$, where $\mathcal{Z}$ denotes the $\mathcal{Z}$-complex associated to the ideal I.*
In particular if $n \geq 2$ and $\sigma =$ $\operatorname{depth}_{\mathfrak{m}}(A) \geq n-1$ then the complex $\mathcal{Z}$ is acyclic.

Proof. This proof is based on chasing depths and the acyclicity lemma 4.6. Let $K_\bullet(\mathbf{a}; A)$ be the Koszul complex associated to the sequence $a_1, \ldots, a_n$ in A. We denote by $H_i(\mathbf{a}; A)$ its homology modules and by Z_i (resp. B_i) the i-cycles (resp. the i-boundaries) of $K_\bullet(\mathbf{a}; A)$. The $\mathcal{Z}$-complex is of the form

$$0 \to \mathcal{Z}_{n-1} \to \mathcal{Z}_{n-2} \to \ldots \to \mathcal{Z}_1 \to \mathcal{Z}_0 \to 0.$$

Since I and $\mathfrak{m}$ have the same radical the homology modules $H_i(\mathbf{a}; A)$ are supported on $V(\mathfrak{m})$ for I annihilates them. Hence, by proposition 4.3 and remark 4.4, homology modules $H_i(\mathcal{Z})$ are supported on $V(\mathfrak{m})$ for all $i \geq 1$, and consequently we have

$$\operatorname{depth}_{\mathfrak{m}}(H_i(\mathcal{Z})) = 0 \ \text{ for all } H_i(\mathcal{Z}) \neq 0, \ i \geq 1. \tag{19}$$

Always since I and $\mathfrak{m}$ have the same radical we have $\sigma = \operatorname{depth}_{\mathfrak{m}}(A) = \operatorname{depth}_I(A)$ and hence deduce by lemma 4.6 that $H_i(\mathbf{a}; A) = 0$ for $i > n - \sigma$. It follows that the following truncated Koszul complex is exact:

$$0 \to K_n \to K_{n-1} \to \ldots \to K_{n-\sigma+1} \to K_{n-\sigma},$$

and, since $\sigma \geq 1$, we have exact sequences

$$
\begin{array}{l}
0 \to K_n \to B_{n-1} \to 0, \\
0 \to B_{n-1} \to K_{n-1} \to B_{n-2} \to 0, \\
\qquad \vdots \qquad\qquad\qquad \vdots \\
0 \to B_{n-\sigma+1} \to K_{n-\sigma+1} \to B_{n-\sigma} \to 0.
\end{array}
$$

We can now use standard properties of depth (see [11] corollary 18.6). As $\mathrm{depth}_{\mathfrak{m}}(K_i) \geq \sigma$ for all i, it follows by iterations that $\mathrm{depth}_{\mathfrak{m}}(B_i) \geq \sigma - (n - i) + 1$ for i from $n - \sigma$ to $n - 1$. Since $\sigma \geq 1$ we have $Z_n = 0$ and $Z_i = B_i$ for $n - 1 \geq i > n - \sigma$. Moreover $Z_{n-\sigma} \subset K_{n-\sigma}$ and $\mathrm{depth}_{\mathfrak{m}}(K_{n-\sigma}) \geq 1$ hence $\mathrm{depth}_{\mathfrak{m}}(Z_{n-\sigma}) \geq 1$. We finally obtain, using the fact that $\mathcal{Z}_i = Z_i[\mathbf{T}]$ for all $i = 0, \ldots, n$,

$$
\mathcal{Z}_n = 0 \ \text{ and } \ \mathrm{depth}_{\mathfrak{m}}(\mathcal{Z}_i) \geq i - (n - \sigma - 1) \ \text{ for } \ n > i \geq n - \sigma. \tag{20}
$$

By lemma 4.6, (19) and (20) show that the complex

$$
0 \to \mathcal{Z}_{n-1} \to \ldots \to \mathcal{Z}_{n-\sigma+1} \to \mathcal{Z}_{n-\sigma-1}
$$

is exact, i.e. $H_i(\mathcal{Z}) = 0$ for $i \geq \max(1, n - \sigma)$. □

The preceding result is valid only if both ideals I and $\mathfrak{m}$ have the same radical. Consequently, when we will apply this result to the implicitization problem, it will be useful only in the absence of base points. In order to deal with certain cases where there exists base points we prove a second acyclicity lemma.

For any ideal J of a ring R we denote by $\mu(J)$ the minimal number of generators of J. We recall the standard definition of a projective local complete intersection ideal.

Definition 4.8 *Let I be an ideal of A. I is said to be a local complete intersection in* $\mathrm{Proj}(A)$ *if and only if for all $\mathfrak{p} \in \mathrm{Spec}(A) \setminus V(\mathfrak{m})$ we have $\mu(I_{\mathfrak{p}}) = \mathrm{depth}_{I_{\mathfrak{p}}}(A_{\mathfrak{p}})$.*

Proposition 4.9 *Let $I = (a_1, \ldots, a_n)$ be an ideal of A such that I is a local complete intersection in* $\mathrm{Proj}(A)$ *and $n \geq 2$. Suppose $\mathrm{depth}_{\mathfrak{m}}(A) \geq n - 1$ and $\mathrm{depth}_I(A) = n - 2$, then the $\mathcal{Z}$-complex associated to I is acyclic.*

Proof. Notice first that, since $\mathrm{depth}_I(A) = n - 2$, the minimal number of generators of I is at least $n - 2$. If $\mu(I) = n - 2$ then by proposition 4.1 we obtain immediately that $\mathcal{M}$ is acyclic and so $\mathcal{Z}$, since I is a complete intersection ideal. Consequently we suppose that $\mu(I) \geq n - 1$. The proof is now very similar to the one of proposition 4.7 and is also based on chasing depths and the acyclicity lemma 4.6. We keep the notation of proposition 4.7, that is $K_\bullet(\mathbf{a}; A)$ denotes the Koszul complex associated to the sequence $a_1, \ldots, a_n$ in A, for all $i \geq 0$ $H_i(\mathbf{a}; A)$ denote its homology modules and Z_i

(resp. B_i) its i-cycles (resp. its i-boundaries). The $\mathcal{Z}$-complex is of the form

$$0 \to \mathcal{Z}_{n-1} \to \mathcal{Z}_{n-2} \to \ldots \to \mathcal{Z}_1 \to \mathcal{Z}_0 \to 0.$$

Since I is a local complete intersection in $\mathrm{Proj}(A)$, proposition 4.1 shows that the homology modules $H_i(\mathcal{M})$ for all $i \geq 1$ are supported on $V(\mathfrak{m})$, where $\mathcal{M}$ is the approximation complex associated to the homology of the ideal I. Now by proposition 4.3 it follows that the homology modules $H_i(\mathcal{Z})$ are also supported on $V(\mathfrak{m})$ for all $i \geq 1$, and consequently we have

$$\mathrm{depth}_{\mathfrak{m}}(H_i(\mathcal{Z})) = 0 \quad \text{for all } H_i(\mathcal{Z}) \neq 0,\ i \geq 1. \tag{21}$$

Since we have supposed $\mathrm{depth}_I(A) = n-2$, the homology modules $H_i(\mathbf{a}; A)$ vanish for $i > 2$. The following truncated Koszul complex is hence exact:

$$0 \to K_n \to K_{n-1} \to \ldots \to K_3 \to K_2,$$

and we have exact sequences

$$\begin{array}{l} 0 \to K_n \to B_{n-1} \to 0, \\ 0 \to B_{n-1} \to K_{n-1} \to B_{n-2} \to 0, \\ \quad\ \vdots \qquad\qquad\qquad \vdots \\ 0 \to B_3 \to K_3 \to B_2 \to 0. \end{array}$$

As $\mathrm{depth}_{\mathfrak{m}}(K_i) \geq n-1$ for all $i = 1, \ldots, n$, it follows by iterations that $\mathrm{depth}_{\mathfrak{m}}(B_i) \geq i$ for i from 2 to $n-1$. But we have $Z_n = 0$ and $Z_i = B_i$ for $n-1 \geq i > 2$, so $\mathrm{depth}_{\mathfrak{m}}(Z_i) \geq i$ for $n-1 \geq i \geq 3$. We have also $Z_1 \subset K_1$ and $\mathrm{depth}(K_1) \geq 1$, so we deduce $\mathrm{depth}_{\mathfrak{m}}(Z_1) \geq 1$.
Now since $\mathcal{Z}_i = Z_i[\mathbf{T}]$ we obtain

$$\mathcal{Z}_n = 0 \quad \text{and} \quad \mathrm{depth}_{\mathfrak{m}}(\mathcal{Z}_i) \geq i \quad \text{for} \quad i \geq 3 \quad \text{and } i = 1. \tag{22}$$

If we prove that $\mathrm{depth}(\mathcal{Z}_2) \geq 2$, (21) and (22) imply that the $\mathcal{Z}$-complex associated to I is exact by lemma 4.6. To show this last inequality we prove that in both cases $\mu(I) = n-1$ and $\mu(I) = n$ we have $H_2(\mathcal{M}) = 0$. For this we need the two lemmas stated just after this proposition. If $\mu(I) = n-1$, lemma 4.11 implies immediately $H_2(\mathcal{M}) = 0$. If $\mu(I) = n$ lemma 4.11 implies that $H_2(\mathcal{M}) = H^0_{\mathfrak{m}}(H_2(\mathcal{M}))$, and by lemma 4.10 we obtain that $H^0_{\mathfrak{m}}(H_2(\mathcal{M})) = 0$, so $H_2(\mathcal{M}) = 0$. It follows that in both cases $\mathcal{B}_2 = \mathcal{Z}_2$, and since we have proved $\mathrm{depth}_{\mathfrak{m}}(B_2) \geq 2$, we deduce $\mathrm{depth}_{\mathfrak{m}}(\mathcal{B}_2) \geq 2$, and finally $\mathrm{depth}_{\mathfrak{m}}(\mathcal{Z}_2) \geq 2$. $\square$

Lemma 4.10 *Let $I = (a_1, \ldots, a_n)$ be an ideal of A such that* $\mathrm{depth}_{\mathfrak{m}}(A) >$ $\mathrm{depth}_I(A)$*, then* $H^0_{\mathfrak{m}}(H_{n-\mathrm{depth}_I(A)}(\mathbf{a}; A)) = 0$.

Proof. This lemma is just a standard use of the classical spectral sequences associated to the double complex

$$
\begin{array}{ccccccccc}
0 \to & \mathcal{C}^0_{\mathfrak{m}}(\wedge^n A^n) & \xrightarrow{d_{\mathbf{a}}} & \ldots & \xrightarrow{d_{\mathbf{a}}} & \mathcal{C}^0_{\mathfrak{m}}(\wedge^1 A^n) & \xrightarrow{d_{\mathbf{a}}} & \mathcal{C}^0_{\mathfrak{m}}(A) & \to 0 \\
 & \downarrow & & & & \downarrow & & \downarrow & \\
0 \to & \mathcal{C}^1_{\mathfrak{m}}(\wedge^n A^n) & \to & \ldots & \to & \mathcal{C}^1_{\mathfrak{m}}(\wedge^1 A^n) & \to & \mathcal{C}^1_{\mathfrak{m}}(A) & \to 0 \\
 & \downarrow & & & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & & & \downarrow & & \downarrow & \\
0 \to & \mathcal{C}^r_{\mathfrak{m}}(\wedge^n A^n) & \to & \ldots & \to & \mathcal{C}^r_{\mathfrak{m}}(\wedge^1 A^n) & \to & \mathcal{C}^r_{\mathfrak{m}}(A) & \to 0.
\end{array}
$$

Its first row is the Koszul complex associated to the sequence $a_1, \ldots, a_n$, and its columns are the classical Čech complexes (recall that $C^0_{\mathfrak{m}}(A) = A$, and if $x_1, \ldots, x_r$ is a system of parameters of A, then for all $t \geq 1$ we have $\mathcal{C}^t_{\mathfrak{m}}(A) = \bigoplus_{1 \leq i_1 < \cdots < i_t \leq r} A_{x_{i_1} x_{i_2} \cdots x_{i_t}}$). We know that $H_i(\mathbf{a}; A) = 0$ for $i > n - \mathrm{depth}_I(A)$, and also that $H^i_{\mathfrak{m}}(A) = 0$ if $i < \mathrm{depth}_{\mathfrak{m}}(A)$ which implies $H^i_{\mathfrak{m}}(A) = 0$ for all $i \leq \mathrm{depth}_I(A)$. Examining the two filtrations by rows and by columns we deduce that $H^0_{\mathfrak{m}}(H_{n-\mathrm{depth}_I(A)}(\mathbf{a}; A)) = 0$. $\square$

Lemma 4.11 *Let $I = (a_1, \ldots, a_n)$ be an ideal of A and let $\mathcal{M}$ be the $\mathcal{M}$-complex associated to I. We have the two following properties:*

a) For all $i > \zeta := \mu(I) - \mathrm{depth}_I(A)$ we have $H_i(\mathcal{M}) = 0$.
b) If we suppose that for all $\mathfrak{p} \in Spec(A) \setminus V(\mathfrak{m})$ we have $\zeta > \zeta_{\mathfrak{p}} := \mu(I_{\mathfrak{p}}) - \mathrm{depth}_{I_{\mathfrak{p}}}(A_{\mathfrak{p}})$, then $H_\zeta(\mathcal{M}) = H^0_{\mathfrak{m}}(H_\zeta(\mathcal{M}))$.

Proof. This lemma is a direct consequence of the proposition 4.1. Indeed, for the first statement, we can construct the complex $\mathcal{M}$ from the Koszul complex of a sequence of elements $a'_1, \ldots, a'_{\mu(I)}$ which generate I. This Koszul complex has all its homology groups $H_p(\mathbf{a}'; A) = 0$ for $p > \mu(I) - \mathrm{depth}_I(A)$, and we deduce, by definition of $\mathcal{M}$, that $H_p(\mathcal{M}) = 0$ for such p. Now for all $\mathfrak{p} \in \mathrm{Spec}(A)$ and all $i \geq 0$ we have $H_i(\mathcal{M}(I))_{\mathfrak{p}} \simeq H_i(\mathcal{M}(I)_{\mathfrak{p}}) \simeq H_i(\mathcal{M}(I_{\mathfrak{p}}))$, where the last isomorphism is true by proposition 4.1. From the first statement of this lemma we deduce that $H_\zeta(\mathcal{M}(I_{\mathfrak{p}})) = 0$ for all $\mathfrak{p} \notin V(\mathfrak{m})$, since $\zeta > \zeta_{\mathfrak{p}}$. It follows that $H_\zeta(\mathcal{M})_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \notin V(\mathfrak{m})$, which implies $H_\zeta(\mathcal{M}) = H^0_{\mathfrak{m}}(H_\zeta(\mathcal{M}))$. $\square$

We will now investigate a problem introduced by David Cox, always in relation with the implicitization problem, and solved by himself and Hal Schenck in [9], theorem 1.7, in $\mathbb{P}^2$. Being given a homogeneous ideal $I = (a_1, a_2, a_3)$ of $\mathbb{P}^2$ of codimension two, the authors showed that the module of syzygies of I vanishing at the scheme locus $V(I)$ is generated by the Koszul sygyzies if and only if $V(I)$ is a local complete intersection in $\mathbb{P}^2$. In other words, if Z_1 (resp. B_1) denotes the first syzygies (resp. the first boundaries) of the Koszul complex associated to the sequence a_1, a_2, a_3, they showed that $Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n = B_1$ if and only if I is a local complete intersection in $\mathbb{P}^2$, where $\mathfrak{m}$ is the irrelevant ideal of the ring $A = k[X_1, X_2, X_3]$ defining $\mathbb{P}^2$. This result is interesting in at least two points: the first one is that it gives an algorithmic way to test if the ideal I is a local complete intersection of $\mathbb{P}^2$ or not, and the second is that it is the key point of a new formula for the implicitization problem (see e.g. [7]).

In what follows we show that this problem is closely related to syzygetic ideals and generalize the preceding result. Let A be a $\mathbb{N}$-graded commutative noetherian ring, $\mathfrak{m} = \oplus_{\nu>0} A_\nu$ be its irrelevant ideal, and let I be another ideal of A generated by n elements, say $a_1, \ldots, a_n$. We denote by Z_1 (resp. B_1) the first syzygies (resp. the first boundaries) of the Koszul complex over A associated to sequence $a_1, \ldots, a_n$. Our aim is to formulate a necessary and sufficient condition so that the inclusion

$$B_1 \subset Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n$$

becomes an equality (notice that we always have $B_1 \subset Z_1 \cap I.A^n$ and also $Z_1 \cap I.A^n \subset Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n$).

We have an exact sequence of A-modules:

$$0 \to B_1 \hookrightarrow I.A^n \to \mathrm{Sym}^2_A(I) \to 0,$$

where the last map sends $(x_1, \ldots, x_n) \in I.A^n$ to the equivalence class of $(x_1 \otimes a_1 + \ldots + x_n \otimes a_n)$ in $\mathrm{Sym}^2_A(I)$. We have the other exact sequence

$$0 \to Z_1 \cap I.A^n \to I.A^n \to I^2 \to 0,$$

where the last map sends $(x_1, \ldots, x_n) \in I.A^n$ to $x_1 a_1 + \ldots + x_n a_n \in I^2$. Putting these two exact sequences together we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_1 & \longrightarrow & I.A^n & \longrightarrow & \mathrm{Sym}_A^2(I) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow & & \\ 0 & \longrightarrow & Z_1 \cap I.A^n & \longrightarrow & I.A^n & \longrightarrow & I^2 & \longrightarrow & 0, \end{array}$$

where the vertical maps are the canonical ones. From the snake lemma we deduce immediately the isomorphism

$$\frac{Z_1 \cap I.A^n}{B_1} \simeq \ker(\mathrm{Sym}_A^2(I) \to I^2).$$

Following [24], we denote $\ker(\mathrm{Sym}_A^2(I) \to I^2)$ by $\delta(I)$ (even if it is not the real definition of this invariant of I but a property) and will say that I is *syzygetic* if $\delta(I) = 0$. We just proved:

Lemma 4.12 *$B_1 = Z_1 \cap I.A^n$ if and only if I is syzygetic.*

We state now the main result of this paragraph.

Theorem 4.13 *If $\mathrm{depth}_{\mathfrak{m}}(A) > \mathrm{depth}_I(A) = n - 1$ then the following statements are equivalent:*

a) $B_1 = Z_1 \cap I.A^n$
b) $B_1 = Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n$.
c) I is syzygetic.

Proof. We have already prove that a) is equivalent to c), and as

$$B_1 \subset Z_1 \cap I.A^n \subset Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n,$$

it remains only to show that a) implies b).

Let $(x_1, \dots, x_n) \in Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n$. By definition there exists an integer ν such that

$$\mathfrak{m}^\nu(x_1, \dots, x_n) \subset Z_1 \cap I.A^n = B_1.$$

This last implies that for all $\xi \in \mathfrak{m}$ the equivalence class of $\xi^\nu.(x_1, \dots, x_n)$ in $H_1(\mathbf{a}, A)$ is 0. But by lemma 4.10 we have $H^0_{\mathfrak{m}}(H_1(\mathbf{a}, A)) = 0$, and we deduce that the equivalence class of $(x_1, \dots, x_n)$ in $H_1(\mathbf{a}, A)$ is 0, that is $(x_1, \dots, x_n) \in B_1$. □

We now give a more explicit condition on the ideal I to obtain the desired property $B_1 = Z_1 \cap \mathrm{TF}_{\mathfrak{m}}(I).A^n$. For this we consider the approximation complex $\mathcal{M}$

attached to the ideal I. From now we suppose that $A = k[X_1, \ldots, X_n]$ with k a noetherian commutative ring, $I = (a_1, \ldots, a_n)$, and we take $\mathfrak{m} = (X_1, \ldots, X_n)$ (in this case we have $\operatorname{depth}_{\mathfrak{m}}(A) = n$).

Proposition 4.14 *Suppose k is a Cohen-Macaulay ring. If I is a local complete intersection in $\mathbb{P}_k^{n-1}$ and $\operatorname{codim}(I) = n-1$, then $B_1 = Z_1 \cap \operatorname{TF}_{\mathfrak{m}}(I).A^n$.*

Proof. Since $\operatorname{depth}_I(A) = n-1$, we have only to prove that I is syzygetic by theorem 4.13. Now lemma 4.11 shows that $H_1(\mathcal{M}) = 0$ in both cases $\mu(I) = n-1$ and $\mu(I) = n$: if $\mu(I) = n-1$ we have directly $H_1(\mathcal{M}) = 0$ by a) and, if $\mu(I) = n$, b) implies $H_1(\mathcal{M}) = H^0_{\mathfrak{m}}(H_1(\mathcal{M}))$ and lemma 4.10 shows that this last is zero. By proposition 4.5 it follows that I is of linear type, which implies that I is syzygetic. $\square$

In the case $n = 3$ the preceding proposition becomes an equivalence, due to a particular property of $\delta(I)$ which is stated in theorem 2.2 of [24]. The following proposition generalizes the theorem 1.7 of [9], where k is a field, to the case k is a noetherian regular ring.

Proposition 4.15 *Suppose k is a regular ring, $n = 3$ and $\operatorname{codim}(I) = 2$. Then I is a local complete intersection in $\mathbb{P}_k^2$ if and only if $B_1 = Z_1 \cap \operatorname{TF}_{\mathfrak{m}}(I).A^3$.*

Proof. The only point we have to prove is that, under our hypothesis, if I is syzygetic then I is a local complete intersection in $\mathbb{P}_k^2$. For this we will use theorem 2.2 of [24]: let R be a noetherian local ring of depth two and let P an ideal of codimension two and projective dimension one, then P is generated by a regular sequence if and only if $\delta(P) = 0$.

Clearly $I_{\mathfrak{p}} \simeq \operatorname{TF}_{\mathfrak{m}}(I)_{\mathfrak{p}}$ for all $\mathfrak{p} \notin V(\mathfrak{m})$. As the invariant $\delta(I)$ of I satisfies $\delta(I)_{\mathfrak{p}} \simeq \delta(I_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$, if we prove that $TF_{\mathfrak{m}}(I)$ has projective dimension 1 then theorem 2.2 of [24] implies that $\operatorname{TF}_{\mathfrak{m}}(I)$ is a local complete intersection in $\mathbb{P}_k^2$, which is equivalent to say that I is.

From the definition of $\operatorname{TF}_{\mathfrak{m}}(I)$, it follows that $\mathfrak{m}$ is not associated to $\operatorname{TF}_{\mathfrak{m}}(I)$ (it is easy to see that $\operatorname{TF}_{\mathfrak{m}}(I) : \mathfrak{m} = \operatorname{TF}_{\mathfrak{m}}(I)$) and hence

$$\operatorname{depth}_{\mathfrak{m}}(A/\operatorname{TF}_{\mathfrak{m}}(I)) > 0.$$

But we know that $\operatorname{depth}_{\mathfrak{m}}(A/\operatorname{TF}_{\mathfrak{m}}(I)) \leq \dim(A/\operatorname{TF}_{\mathfrak{m}}(I))$, and we know also that $\dim(A/\operatorname{TF}_{\mathfrak{m}}(I)) = \dim(A) - \operatorname{codim}(\operatorname{TF}_{\mathfrak{m}}(I)) = 1$. We deduce

$$\dim(A/\operatorname{TF}_{\mathfrak{m}}(I)) - \operatorname{depth}_{\mathfrak{m}}(A/\operatorname{TF}_{\mathfrak{m}}(I)) = 0.$$

Now, since k is assumed to be regular, the Auslander-Buchsbaum formula gives

$$\operatorname{pd}(A/\operatorname{TF}_{\mathfrak{m}}(I)) = \operatorname{depth}_{\mathfrak{m}}(A) - \operatorname{depth}_{\mathfrak{m}}(A/\operatorname{TF}_{\mathfrak{m}}(I)).$$

As $\mathrm{depth}_{\mathfrak{m}}(A) = \dim(A) = \dim(A/\mathrm{TF}_{\mathfrak{m}}(I)) + \mathrm{codim}(\mathrm{TF}_{\mathfrak{m}}(I))$ we deduce $\mathrm{pd}(A/\mathrm{TF}_{\mathfrak{m}}(I)) = \mathrm{codim}(\mathrm{TF}_{\mathfrak{m}}(I)) = 2$, whence $\mathrm{pd}(\mathrm{TF}_{\mathfrak{m}}(I)) = 1$. □

5 The implicitization problem via the approximation complexes

As we have seen, the kernel of the map $h : k[\mathbf{T}] \to A$ which sends T_i to $f_i \in A_d$ for $i = 1, \dots, n$, defines the closed image of its associated map λ (1). In this section we will investigate the so-called implicitization problem. Hereafter we suppose that k is a field and that A is the polynomial ring $k[X_1, \dots, X_{n-1}]$, with $n \geq 3$, whose irrelevant ideal is denoted $\mathfrak{m} = (X_1, \dots, X_{n-1}) \subset A$. If the morphism λ is generically finite then the closed image of λ is a hypersurface of $\mathbb{P}^{n-1}_k$, and the implicitization problem consists in computing explicitly its equation (up to a nonzero multiple in k). In what follows we will apply techniques using the approximation complexes we have introduced in section 4. We will denote by I the ideal of A generated by the polynomials $f_1, \dots, f_n$ which are supposed to be of the same degree $d \geq 1$, considering each X_i of degree 1 in A. We will also denote by $\mathcal{Z}$ and $\mathcal{M}$ the two approximation complexes associated to I. The basic idea is to show that the $\mathcal{Z}$-complex is acyclic under certain conditions, giving thus a resolution of the symmetric algebra $\mathrm{Sym}_A(I)$, and then obtain the implicit equation as the determinant of some of its homogeneous components. Before going further into different particular cases we give the main ingredient of this section.

The symmetric algebra of I is naturally bi-graded by the exact sequence:

$$0 \to \ker(\alpha) \to A[T_1, \dots, T_n] \xrightarrow{\alpha} \mathrm{Sym}_A(I) \to 0,$$

where $\ker(\alpha) = (\sum b_i T_i \mid b_i \in A \ \sum b_i f_i = 0) \subset A[T_1, \dots, T_n]$. We denote by $\mathrm{Sym}_A(I)_\nu$ the graded part of $\mathrm{Sym}_A(I)$ corresponding to the grading of A, that is, to be more precise, $\mathrm{Sym}_A(I)_\nu = \oplus_{l \geq 0} A_\nu \mathrm{Sym}^l_A(I)$.

Proposition 5.1 *Suppose that I is of linear type outside $V(\mathfrak{m})$, and let η be an integer such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all $\nu \geq \eta$ then*

$$\mathrm{ann}_{k[\mathbf{T}]}(\mathrm{Sym}_A(I)_\nu) = \ker(h) \quad \text{for all } \nu \geq \eta.$$

Proof. By definition, for $\nu \in \mathbb{N}$,

$$\mathrm{ann}_{k[\mathbf{T}]}(\mathrm{Sym}_A(I)_\nu) = \{f \in k[\mathbf{T}] \, : \, f.\mathrm{Sym}_A(I)_\nu = 0\},$$

and from the description of $\mathrm{Sym}_A(I)$ we deduce

$$\mathrm{ann}_{k[\mathbf{T}]}(\mathrm{Sym}_A(I)_\nu) = \{f \in k[\mathbf{T}] \, : \, f.A_\nu[\mathbf{T}] \subset \ker(\alpha)\}. \tag{23}$$

In the same way, $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ is equivalent to

$$\{f \in A_\nu[\mathbf{T}] \,:\, \exists n\, \mathfrak{m}^n f \subset \ker(\alpha)\} = \ker(\alpha)_\nu. \tag{24}$$

From (23) and (24) we easily check that

$$\begin{aligned} \mathrm{ann}_{k[\mathbf{T}]}(\mathrm{Sym}_A(I)_\nu) &= \{f \in k[\mathbf{T}] \,:\, f.A_\eta[\mathbf{T}] \subset \ker(\alpha)\} \\ &= \{f \in k[\mathbf{T}] \,:\, f.A_{\geq \eta}[\mathbf{T}] \subset \ker(\alpha)\}, \end{aligned} \tag{25}$$

for all $\nu \geq \eta$.

Now since I is of linear type outside $V(\mathfrak{m})$ and $H^0_{\mathfrak{m}}(A) = 0$, corollary 3.8 implies that $\ker(h)$ is exactly $\mathrm{TF}_{\mathfrak{m}}(\ker(\alpha)) \cap k[\mathbf{T}]$ (observe here that $\mathrm{TF}_{\mathfrak{m}} = \mathrm{TF}_{\mathfrak{m}[\mathbf{T}]}$), and we have

$$\begin{aligned} \ker(h) &= \{f \in k[\mathbf{T}] \,:\, \exists n\, \mathfrak{m}^n.f \subset \ker(\alpha)\} \\ &= \{f \in k[\mathbf{T}] \,:\, \exists n\, f.A_{\geq n}[\mathbf{T}] \subset \ker(\alpha)\}. \end{aligned} \tag{26}$$

Now it is easy to check that (25) and (26) are the same. $\square$

In what follows we will always suppose that the map λ is generically finite, this implies that the ideal $\ker(h)$ of $k[\mathbf{T}]$ is principal. Indeed, it is clear that $\ker(h)$ is a prime ideal of codimension one since λ is supposed to be generically finite. As $k[\mathbf{T}]$ is a factorial domain, $\ker(h)$ is principal by theorem 1, VII 3.1 [2].

5.1 *Implicitization of a hypersurface without base points*

Here we suppose that both ideals I and $\mathfrak{m}$ of A have the same radical. This condition is equivalent to say that the variety $V(I)$ is empty in $\mathrm{Proj}(A)$, that is the polynomials $f_1, \ldots f_n$ have no common roots in $\mathrm{Proj}(A)$. Under this condition proposition 4.7 shows that the approximation complex $\mathcal{Z}$ is acyclic, since $\mathrm{depth}_{\mathfrak{m}}(A) = n-1$, and hence gives the exact complex

$$0 \to \mathcal{Z}_{n-1} \xrightarrow{d_{\mathbf{T}}} \ldots \xrightarrow{d_{\mathbf{T}}} \mathcal{Z}_1 \xrightarrow{d_{\mathbf{T}}} A[\mathbf{T}] \to \mathrm{Sym}_A(I) \to 0.$$

Recall that $\mathcal{Z}_i$ and $\mathrm{Sym}_A(I)$ are bigraded. We denote respectively by $(\mathcal{Z}_i)_\nu$ and $\mathrm{Sym}_A(I)_\nu$ the graded part of $\mathcal{Z}_i$ and $\mathrm{Sym}_A(I)$ corresponding to the graduation of A. We have the following theorem:

Theorem 5.2 *Suppose that both ideals I and $\mathfrak{m}$ have the same radical. Let η be an integer such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all $\nu \geq \eta$, and denote by H the* *reduced equation (defined up to a nonzero multiple in k) of the closed image of the map λ. Then the determinant of the degree ν part of the $\mathcal{Z}$-complex associated to I, which is a complex of $k[\mathbf{T}]$-modules of the form*

$$0 \to (\mathcal{Z}_{n-1})_\nu \to \ldots \to (\mathcal{Z}_1)_\nu \to A_\nu[\mathbf{T}],$$

is exactly $H^{\deg(\lambda)}$, of degree d^{n-2}.

Proof. First notice that since I and $\mathfrak{m}$ have the same radical, I is of linear type outside $V(\mathfrak{m})$. Indeed for all prime $\mathfrak{p} \notin V(\mathfrak{m})$ we have $I_\mathfrak{p} = A_\mathfrak{p}$ and hence, by proposition 4.1, the approximation complex $\mathcal{M}_\mathfrak{p}$ is acyclic. This implies, by proposition 4.5, that I is of linear type outside $V(\mathfrak{m})$.

Now let ν be a fixed integer greater or equal to η. By proposition 5.1 we know that

$$\mathrm{ann}_{k[\mathbf{T}]}(H_0(\mathcal{Z}_\bullet)_\nu) = \ker(h)$$

which is a principal ideal generated by H, that we denote hereafter by $\mathfrak{p}$. Moreover by proposition 4.7 the $\mathcal{Z}$-complex associated to I is acyclic and so, by standard properties of determinants of complexes (see e.g. [20]), we deduce that

$$\det((\mathcal{Z}_\bullet)_\nu) = \mathrm{div}(H_0(\mathcal{Z}_\bullet)_\nu) = \mathrm{length}((\mathrm{Sym}_A(I)_\nu)_\mathfrak{p}).[H].$$

It remains to prove that $\mathrm{length}((\mathrm{Sym}_A(I)_\nu)_\mathfrak{p}) = \deg(\lambda)$, but this last equality follows from the proof of theorem 2.5. Using the notation of the theorem 2.5, as I is assumed to be of linear type outside $V(\mathfrak{m})$,

$$\begin{aligned}\mathrm{length}((\mathrm{Sym}_A(I)_\nu)_\mathfrak{p}) &= \dim_{(k[\mathbf{T}]/\mathfrak{p})_\mathfrak{p}} \Gamma(\mathrm{Proj}_A(\mathrm{Sym}_A(I)_\mathfrak{p}), \mathcal{O}_{\mathrm{Proj}_A(\mathrm{Sym}_A(I)_\mathfrak{p})}(\nu)) \\ &= \dim_{(k[\mathbf{T}]/\mathfrak{p})_\mathfrak{p}} \Gamma(\mathrm{Proj}_A(B_\mathfrak{p}), \mathcal{O}_{\mathrm{Proj}_A(B_\mathfrak{p})}(\nu)) \\ &= \dim_{(k[\mathbf{T}]/\mathfrak{p})_\mathfrak{p}} \Gamma(\mathrm{Proj}_A(B_\mathfrak{p}), \mathcal{O}_{\mathrm{Proj}_A(B_\mathfrak{p})}) = \deg(\lambda),\end{aligned}$$

where the last equality follows from (12) (and the five following lines). Finally theorem 2.5 shows also that $H^{\deg(\lambda)}$ is of degree d^{n-2}. □

Remark 5.3 *Notice that theorem 2.5 implies that λ is always generically finite if $d \geq 1$ and λ is a regular map (i.e. there is no base points).*

We deduce immediately this corollary which is standard when dealing with determinants of complexes (see [12], appendix A).

Corollary 5.4 *Under the hypothesis of theorem 5.2, $H^{\deg(\lambda)}$ is obtained as the gcd of the maximal minors of the surjective $k[\mathbf{T}]$-module morphism*

$$(\mathcal{Z}_1)_\nu \xrightarrow{d_\mathbf{T}} A_\nu[\mathbf{T}],$$

for all $\nu \geq \eta$.

It is also possible to give an explicit bound for the integer η, bound which is here the best possible (we will see that it is obtained for particular examples).

Proposition 5.5 *($n \geq 3$) Suppose that both ideals I and $\mathfrak{m}$ have the same radical, then*

$$H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0 \quad \forall \nu \geq (n-2)(d-1).$$

Proof. From the standard spectral sequences associated to the bicomplex

$$\begin{array}{ccccccccc}
0 \rightarrow & \mathcal{C}^0_{\mathfrak{m}}(\mathcal{Z}_n) & \xrightarrow{d_{\mathbf{T}}} \ldots \xrightarrow{d_{\mathbf{T}}} & \mathcal{C}^0_{\mathfrak{m}}(\mathcal{Z}_1) & \xrightarrow{d_{\mathbf{T}}} & \mathcal{C}^0_{\mathfrak{m}}(\mathcal{Z}_0) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{C}^1_{\mathfrak{m}}(\mathcal{Z}_n) & \rightarrow \ldots \rightarrow & \mathcal{C}^1_{\mathfrak{m}}(\mathcal{Z}_1) & \rightarrow & \mathcal{C}^1_{\mathfrak{m}}(\mathcal{Z}_0) & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& \vdots & \vdots & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & \mathcal{C}^{n-1}_{\mathfrak{m}}(\mathcal{Z}_n) & \rightarrow \ldots \rightarrow & \mathcal{C}^{n-1}_{\mathfrak{m}}(\mathcal{Z}_1) & \rightarrow & \mathcal{C}^{n-1}_{\mathfrak{m}}(\mathcal{Z}_0) & \rightarrow 0,
\end{array}$$

where the first row is the $\mathcal{Z}$-complex associated to I, and columns are the usual Čech complexes (as in the proof of lemma 4.10), it is easy to see that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all ν such that $H^i_{\mathfrak{m}}(\mathcal{Z}_i)_\nu = 0$ for all $i \geq 1$. We consider the Koszul complex $K_\bullet(\mathbf{f}; A)$ associated to the sequence $f_1, \ldots, f_n$ over A and denote by Z_i (resp. B_i) its i-cycles (resp. its i-boundaries). As $\mathrm{depth}_{\mathfrak{m}}(A) = \mathrm{depth}_I(A) = n-1$, we know that $B_i = Z_i$ for all $i \geq 2$. We deduce first that Z_n equals 0 (and hence $\mathcal{Z}_n = 0$), and also that $B_{n-1} \simeq A(-d)$. It follows $H^i_{\mathfrak{m}}(B_{n-1}) = 0$ for $i \neq n-1$ and $H^{n-1}_{\mathfrak{m}}(B_{n-1})_\nu = H^{n-1}_{\mathfrak{m}}(Z_{n-1})_\nu = 0$ for all $\nu \geq d-n+2$ (recall that $H^{n-1}(A)_\nu = 0$ for all $\nu > -(n-1)$). Now by iterations from the exact sequences

$$0 \rightarrow B_{i+1}(-d) \rightarrow K_{i+1}(-d) \rightarrow B_i \rightarrow 0,$$

for $i \geq 2$, we deduce that

$$H^i_{\mathfrak{m}}(B_i)_\nu = H^i_{\mathfrak{m}}(Z_i)_\nu = 0, \quad \forall\, i \geq 2 \text{ and } \forall\, \nu \geq (n-2)d - n + 2.$$

Finally the exact sequence $0 \rightarrow Z_1(-d) \rightarrow A(-d)^n \rightarrow I \rightarrow 0$ shows that $H^1_{\mathfrak{m}}(Z_1)_\nu = 0$ for all $\nu \geq d-n+2$, and the proposition is proved. $\square$

Before examining the case where base points exist, we look at the two particular situations of curves and surfaces, and link our result to the so-called method of *moving surfaces* (see 1.1 and 1.2).

5.1.1 Implicitization of curves in $\mathbb{P}^2_k$ without base points

We consider here the particular case $n = 3$. We have $\mathfrak{m} = (X_1, X_2) \subset A = k[X_1, X_2]$. We suppose that the three homogeneous polynomials f_1, f_2, f_3 of the same degree $d \geq 1$ have no base points, that is they have no common factors (in fact base points in this case are easily "erased" by dividing each f_i by the gcd of f_1, f_2, f_3; however we will see in 5.2.1 that this computation is not necessary). The f_i's define hence a regular map

$$\mathbb{P}^1_k \xrightarrow{\lambda} \mathbb{P}^2_k : (X_1 : X_2) \mapsto (f_1 : f_2 : f_3)(X_1, X_2),$$

whose image is a curve of reduced implicit equation C. All the hypothesis made in the preceding paragraph are valid and it follows that the determinant of each complex $(\mathcal{Z}_\bullet)_\nu$, for all $\nu \geq d-1$, is exactly $C^{\deg(\lambda)}$. These complexes are of the form

$$0 \to (\mathcal{Z}_2)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_1)_\nu \xrightarrow{d_{\mathbf{T}}} A_\nu[\mathbf{T}].$$

But $\mathcal{Z}_2 \simeq A(-d)[\mathbf{T}]$ since $\operatorname{depth}_I(A) \geq 2$, and hence $(\mathcal{Z}_2)_\nu = 0$ for all $\nu \leq d-1$. We deduce that the determinant of the complex $(\mathcal{Z}_\bullet)_{d-1}$, which is $C^{\deg(\lambda)}$, is in fact the determinant of the matrix $(\mathcal{Z}_1)_{d-1} \xrightarrow{d_{\mathbf{T}}} A_{d-1}[\mathbf{T}]$ of $k[\mathbf{T}]$-modules, i.e. it is obtained as the determinant of the first syzygies of f_1, f_2, f_3 in degree $d-1$. This result is exactly the method of *moving lines* which we have recalled in 1.1.

Consider the simple example $f_1 = X_1^2$, $f_2 = X_1X_2$ and $f_3 = X_2^2$. Applying the method, the matrix of syzygies of degree $d-1$ is:

$$\begin{pmatrix} -T_2 & T_3 \\ T_1 & T_2 \end{pmatrix},$$

and the implicit equation is hence $T_2^2 - T_1T_3$. The method fails if we try $\nu = d-2$.

5.1.2 Implicitization of surfaces in $\mathbb{P}^3_k$ without base points

We consider here the particular case $n = 4$. We suppose that the four polynomials f_1, f_2, f_3, f_4 of the same degree d have no common roots in $\operatorname{Proj}(A)$, where A is here the polynomial ring $A = k[X_1, X_2, X_3]$. These polynomials define a regular map

$$\mathbb{P}^2_k \xrightarrow{\lambda} \mathbb{P}^3_k : (X_1 : X_2 : X_3) \mapsto (f_1 : f_2 : f_3 : f_4)(X_1, X_2, X_3),$$

whose image is a surface of reduced implicit equation S. Applying the preceding results we obtain that the determinant of each complex $(\mathcal{Z}_\bullet)_\nu$, for all

$\nu \geq 2(d-1)$, is exactly $S^{\deg(\lambda)}$. These complexes are of the form

$$0 \to (\mathcal{Z}_3)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_2)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_1)_\nu \xrightarrow{d_{\mathbf{T}}} A_\nu[\mathbf{T}].$$

As in the case of curves we have also $\mathcal{Z}_3 \simeq A(-d)[\mathbf{T}]$, but here $2(d-1) \geq d$ since $d \geq 1$. Consequently we obtain here the equation $S^{\deg(\lambda)}$ as the product of two determinants divided by another one (see [12], appendix A, to compute the determinant of a complex). We illustrate it with the two following examples.

First consider the example given by $f_1 = X_1^2$, $f_2 = X_2^2$, $f_3 = X_3^2$ and $f_4 = X_1^2 + X_2^2 + X_3^2$. Clearly the implicit equation is $T_1 + T_2 + T_3 - T_4 = 0$. Applying our method in degree $\nu = 2(2-1) = 2$ we compute the three matrices of the complex which are respectively (from the right to the left) of size 6×9, 9×4 and 4×1. We obtain $(T_1 + T_2 + T_3 - T_4)^4$ as the product of two determinants of size 6×6 and 1×1 divided by another one of size 3×3. If we try the method in degree $\nu = 1$ the first matrix (on the right) is square of determinant $(T_1 + T_2 + T_3 - T_4)^3$ and the others are zero, but 3 is different from 4 which is the degree of the parameterization map.

As another example we consider $f_1 = X_1^2 X_2$, $f_2 = X_2^2 X_3$, $f_3 = X_1 X_3^2$ and $f_4 = X_1^3 + X_2^3 + X_3^3$. Applying our method in degree $4 = 2(3-1)$ we find the irreducible implicit equation of degree 9 as the product of two determinants of size 15×15 and 3×3 divided by another one of size 9×9. The method fails in degree strictly less than 4.

Remark 5.6 *As we have seen, our method gives in general the implicit equation as the product of two determinants divided by another one. However, we point out that the method of moving surfaces of T. Sederberg gives here (i.e. in the case without base points) the implicit equation as a determinant of a square matrix, as we have recalled in 1.2.*

5.2 Implicitization of a hypersurface with isolated local complete intersection base points

In this paragraph we suppose that the ideal $I = (f_1, \ldots, f_n)$ of the polynomial ring $A = k[X_1, \ldots, X_{n-1}]$ is a local complete intersection in $\mathrm{Proj}(A)$ of codimension $n-2$ (see definition 4.8), that is the ideal I defines points in $\mathbb{P}_k^{n-2}$ which are locally generated by a regular sequence. Under this condition we have $\mathrm{depth}_I(A) = n-2 < \mathrm{depth}_{\mathfrak{m}}(A) = n-1$, so proposition 4.9 shows that the $\mathcal{Z}$ approximation complex associated to I is acyclic. As for the case without base points we obtain the exact complex

$$0 \to \mathcal{Z}_{n-1} \xrightarrow{d_{\mathbf{T}}} \ldots \xrightarrow{d_{\mathbf{T}}} \mathcal{Z}_1 \xrightarrow{d_{\mathbf{T}}} A[\mathbf{T}] \to \mathrm{Sym}_A(I) \to 0.$$

Recalling that $\mathrm{Sym}_A(I)_\nu$ denotes the graded part of $\mathrm{Sym}_A(I)$ corresponding to the graduation of A, we have the following theorem:

Theorem 5.7 *Suppose that the ideal $I = (f_1, \ldots, f_n)$ is a local complete intersection in $\mathrm{Proj}(A)$ of codimension $n-2$ such that the map λ is generically finite. Let η be an integer such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all $\nu \geq \eta$, and denote by H the reduced equation (defined up to a nonzero multiple in k) of the closed image of the map λ. Then the determinant of the degree ν part of the $\mathcal{Z}$-complex associated to I, which is a complex of $k[\mathbf{T}]$-modules of the form*

$$0 \to (\mathcal{Z}_{n-1})_\nu \to \ldots \to (\mathcal{Z}_1)_\nu \to A_\nu[\mathbf{T}],$$

is exactly $H^{\deg(\lambda)}$, of degree $d^{n-2} - \dim_k \Gamma(T, \mathcal{O}_T)$ where $T = \mathrm{Proj}(A/I)$.

Proof. First notice that since I is a local complete intersection in $\mathrm{Proj}(A)$ we deduce that $H_i(\mathcal{M})$ are supported on $V(\mathfrak{m})$ for all $i \geq 1$, and hence that I is of linear type outside $V(\mathfrak{m})$. Let ν be a fixed integer greater or equal to η. By proposition 5.1 we know that

$$\mathrm{ann}_{k[\mathbf{T}]}(H_0(\mathcal{Z}_\bullet)_\nu) = \ker(h)$$

which is a principal ideal generated by H. Moreover by proposition 4.9 the $\mathcal{Z}$-complex associated to I is acyclic and hence, by the same argument as the one given in the proof of theorem 5.2, we deduce that

$$\det((\mathcal{Z}_\bullet)_\nu) = H^{\deg(\lambda)}.$$

In the same way, theorem 2.5 shows that the polynomial $H^{\deg(\lambda)}$ is of degree $d^{n-2} - e(T, \mathrm{Proj}(A))$, and $e(T, \mathrm{Proj}(A)) = \dim_k \Gamma(T, \mathcal{O}_T)$ since T is locally a complete intersection. $\square$

Remark 5.8 *The hypothesis saying that λ is generically finite can be tested using theorem 2.5. In the particular case $n = 4$ the hypothesis saying that I is a local complete intersection of codimension 2 can be tested using proposition 4.15.*

We deduce the standard corollary:

Corollary 5.9 *Under the hypothesis of theorem 5.7, $H^{\deg(\lambda)}$ is obtained as the gcd of the maximal minors of the surjective $k[\mathbf{T}]$-module morphism*

$$(\mathcal{Z}_1)_\nu \xrightarrow{d_{\mathbf{T}}} A_\nu[\mathbf{T}],$$

for all $\nu \geq \eta$.

We now show that the bound given for the integer η in proposition 5.5 is also available here.

Proposition 5.10 *($n \geq 3$) Suppose that the ideal $I = (f_1, \ldots, f_n)$ is a local complete intersection in* $\mathrm{Proj}(A)$ *of codimension $n-2$, then*

$$H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0 \quad \forall \nu \geq (n-2)(d-1).$$

Proof. This proof is quite similar to the proof of proposition 5.5. The same first argument shows that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$ for all ν such that $H^i_{\mathfrak{m}}(\mathcal{Z}_i)_\nu = 0$ for all $i \geq 1$. We consider the Koszul complex $K_\bullet(\mathbf{f}; A)$ associated to the sequence $f_1, \ldots, f_n$ over A, and denote by Z_i (resp. B_i) its i-cycles (resp. its i-boundaries). As $\mathrm{depth}_{\mathfrak{m}}(A) = n-1$ and $\mathrm{depth}_I(A) = n-2$, we know that $B_i = Z_i$ for all $i > 2$. We deduce first that Z_n equals 0 (and hence $\mathcal{Z}_n = 0$), and second that $B_{n-1} \simeq A(-d)$. It follows $H^i_{\mathfrak{m}}(B_{n-1}) = 0$ for $i \neq n-1$, and $H^{n-1}_{\mathfrak{m}}(B_{n-1})_\nu = H^{n-1}_{\mathfrak{m}}(Z_{n-1})_\nu = 0$ for all $\nu \geq d-n+2$ (recall that $H^{n-1}_{\mathfrak{m}}(A)_\nu = 0$ for all $\nu > -(n-1)$). Now by iterations from the exact sequences

$$0 \to B_{i+1}(-d) \to K_{i+1}(-d) \to B_i \to 0,$$

for $i \geq 2$, we deduce that

$$H^i_{\mathfrak{m}}(B_i)_\nu = H^i_{\mathfrak{m}}(Z_i)_\nu = 0 \ \forall\ i \geq 3 \text{ and } \forall\ \nu \geq (n-3)d - n + 2,$$

and that $H^2_{\mathfrak{m}}(B_2)_\nu = 0$ for all $\nu \geq (n-2)d - n + 2$. Also the exact sequence $0 \to Z_1(-d) \to A(-d)^n \to I \to 0$ shows that $H^1_{\mathfrak{m}}(Z_1)_\nu = 0$ for all $\nu \geq d - n + 2$. Finally $H^2_{\mathfrak{m}}(H_2(\mathbf{a}; A)) = 0$ since $H_2(\mathbf{a}; A)$ is supported on $V(I)$ which of dimension 1, and the exact sequence $0 \to B_2 \to Z_2 \to H_2 \to 0$ shows that $H^2_{\mathfrak{m}}(Z_2)_\nu = 0$ for all ν such that $H^2_{\mathfrak{m}}(B_2)_\nu = 0$, that is for all $\nu \geq (n-2)d - n + 2$. □

We now focus on the particular cases of interest of curves and surfaces.

5.2.1 *Implicitization of curves in $\mathbb{P}^2_k$ with base points*

We consider here the particular case $n = 3$. We have $\mathfrak{m} = (X_1, X_2) \subset A = k[X_1, X_2]$. We suppose that the three homogeneous polynomials f_1, f_2, f_3 of the same degree $d \geq 1$ have base points, that is they have a common factor. All the hypothesis made in the preceding paragraph are valid and it follows that the determinant of each complex $(\mathcal{Z}_\bullet)_\nu$, for all $\nu \geq d-1$, is exactly $C^{\deg(\lambda)}$, where C denotes the reduced implicit curve. These complexes are of the form

$$0 \to (\mathcal{Z}_2)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_1)_\nu \xrightarrow{d_{\mathbf{T}}} A_\nu[\mathbf{T}].$$

Comparing to 5.1.1, we do not have $\mathcal{Z}_2 \simeq A(-d)[\mathbf{T}]$ because here $\mathrm{depth}_I(A) = 1$, whereas $\mathrm{depth}_I(A) = 2$ in 5.1.1. The determinant of the complex $(\mathcal{Z}_\bullet)_{d-1}$, which is $C^{\deg(\lambda)}$, is hence generally obtained as the quotient of two determinants. We can illustrate this with the simple example of 5.1.1 where we multiply each equation by X_1: $f_1 = X_1^3$, $f_2 = X_1^2 X_2$ and $f_3 = X_1 X_2^2$. The first matrix on the right is given by:

$$\begin{pmatrix} -T_2 & -T_3 & -T_3 & 0 \\ T_1 & 0 & T_2 & -T_3 \\ 0 & T_1 & 0 & T_2 \end{pmatrix},$$

and the second one is given by:

$$\begin{pmatrix} -T_3 \\ T_2 \\ 0 \\ -T_1 \end{pmatrix}.$$

It follows that the implicit equation is obtained as the quotient

$$\frac{\begin{vmatrix} -T_2 & -T_3 & -T_3 \\ T_1 & 0 & T_2 \\ 0 & T_1 & 0 \end{vmatrix}}{|-T_1|} = -T_2^2 + T_1 T_3.$$

5.2.2 *Implicitization of surfaces in $\mathbb{P}^3_k$ with lci base points*

We suppose here that $n = 4$. As pointed out in remark 5.8, it is here possible to know the degree of the implicit equation we are looking for and to test if the base points are locally complete intersection or not. Under the hypothesis of theorem 5.7 we obtain here again that the implicit equation is the determinant of the complex $(\mathcal{Z}_\bullet)_\nu$, for all $\nu \geq 2(d-1)$. As in the case where there is no base points, these complexes are of the form

$$0 \to (\mathcal{Z}_3)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_2)_\nu \xrightarrow{d_{\mathbf{T}}} (\mathcal{Z}_1)_\nu \xrightarrow{d_{\mathbf{T}}} A_\nu[\mathbf{T}].$$

with $\mathcal{Z}_3 \simeq A(-d)[\mathbf{T}]$. Consequently we deduce the implicit equation as the product of two determinants divided by another one.

Notice that recently it has been proved in [7] that the method of moving surfaces works with local complete intersection base points which satisfy additional conditions (see 1.2). The algorithm we propose here avoids these technical hypothesis. To illustrate it we end this section with the following example taken from [7], for which the (improved) moving surfaces method failed. We take $f_1 = X_1X_3^2$, $f_2 = X_2^2(X_1 + X_3)$, $f_3 = X_1X_2(X_1 + X_3)$ and $f_4 = X_2X_3(X_1+X_3)$. The ideal $I = (f_1, f_2, f_3, f_4)$, which is not saturated, is a local complete intersection in $\mathbb{P}^2$ of codimension 2 defining 6 points (counted with multiplicity). Applying our method in degree $2(3-1) = 4$, we obtain the implicit equation as the quotient $\frac{\Delta_0\Delta_2}{\Delta_1}$, where Δ_0 is the determinant of the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_1 & 0 & 0 & 0 & 0 & 0 & -T_3 & T_2 & -T_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & T_1 & 0 & 0 & 0 & 0 & 0 & -T_3 & 0 & T_2 & -T_4 & 0 & 0 & 0 & 0 \\
0 & -T_4 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & -T_4 & 0 & 0 & 0 \\
T_4 & 0 & -T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 \\
0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & -T_3 & 0 & 0 & T_2 & 0 & 0 \\
0 & 0 & 0 & -T_4 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & T_4 & 0 & 0 & -T_4 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_4 & 0 \\
-T_4 & 0 & T_4 & 0 & 0 & -T_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_3 & 0 & 0 \\
0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -T_1 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_1 & 0 & -T_1 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2
\end{pmatrix},$$

Δ_1 is the determinant of

$$\begin{pmatrix}
0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_3 & 0 & 0 & 0 \\
0 & T_4 & 0 & -T_4 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_3 & 0 \\
0 & -T_1 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_1 & 0 & -T_1 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_4 & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & 0 & T_2 & -T_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -T_4 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 & 0 \\
0 & T_4 & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 & 0 \\
-T_4 & 0 & 0 & 0 & 0 & 0 & -T_4 & T_1 & 0 & 0 & 0 & 0 & -T_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & T_4 & 0 & -T_4 & 0 & 0 & 0 & 0 & T_2 & 0 & 0 \\
0 & -T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 & 0 \\
T_1 & 0 & 0 & 0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -T_1 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & -T_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2
\end{pmatrix},$$

and Δ_3 is the determinant of

$$\begin{pmatrix}
0 & -T_4 & T_1 \\
0 & T_1 & 0 \\
-T_1 & 0 & T_1
\end{pmatrix}.$$

We deduce that the desired equation is $T_1T_2T_3 + T_1T_2T_4 - T_3T_4^2$. Point out that, trying our method empirically, we find the latter equation by applying theorem 5.7 with $\nu = 3$, and even $\nu = 2$. For the case $\nu = 3$ the equation is obtained as a similar quotient $\frac{\Delta_0\Delta_2}{\Delta_1}$, where here Δ_0 is of size 10×10, Δ_1 of size 8×8, and Δ_2 of size 1×1. For the case $\nu = 2$, the equation is obtained as a quotient $\frac{\Delta_0}{\Delta_1}$ (the third map of the $\mathcal{Z}$-complex in this degree degenerates to zero), where Δ_0 is of size 6×6 and Δ_1 of size 3×3.

A Kravitsky's formula and anisotropic resultant

Let k be a commutative ring, X and Y be two indeterminates, and $P, Q, R \in k[X,Y]_d$ be three homogeneous polynomials of same given degree $d \geq 1$. Introducing three new indeterminates u, v, w, we may consider the following $d \times d$ matrix with coefficients in $k[u, v, w]$:

$$\Omega := u\mathbb{B}ez(Q,R) + v\mathbb{B}ez(R,P) + w\mathbb{B}ez(P,Q).$$

Let Z be another new indeterminate. Setting the weight of X, Y, Z respectively to $1, 1, d$, the polynomials $P - uZ, Q - vZ, R - wZ$ are isobaric of weight d; we can hence consider the anisotropic resultant (see [17] and [18])

$$\mathcal{R} := {}^a\mathrm{Res}(P - uZ, Q - vZ, R - wZ) \in k[u, v, w].$$

Proposition *With the preceding notations, we have the equality*

$$\mathcal{R} = (-1)^{\frac{d(d+1)}{2}} \det(\Omega)$$

in $k[u, v, w]$.

Proof. By specialization, it is sufficient to prove this proposition in case P, Q, R have indeterminate coefficients and $k = \mathbb{Z}[\mathrm{coeff}(P,Q,R)]$, that we suppose hereafter. The generic polynomials $P - uZ, Q - vZ, R - wZ$ are isobar of weight d, and d is a multiple of the lcm of the weights of X, Y, Z. In this way $\mathcal{R}$ is *irreducible*, and normalized by the condition

$${}^a\mathrm{Res}(X^d, Y^d, Z) = 1.$$

We denote $A := \mathbb{Z}[\mathrm{coeff}(P,Q,R)][u,v,w]$ and $B := A[X,Y,Z]/(P - uZ, Q - vZ, R - wZ)$. We know that the ideal $H^0_{\mathfrak{m}}(B)$, where $\mathfrak{m} = (X, Y, Z)$, is prime, and that, by definition, $\mathcal{R}$ is a generator of the ideal $H^0_{\mathfrak{m}}(B)_0$ of A. For all $L \in k[X,Y]$ we denote $\tilde{L} := L(X,1) \in k[X]$, and it follows that the ring

$$\tilde{B} := A[X,Z]/(\tilde{P} - uZ, \tilde{Q} - vZ, \tilde{R} - wZ)$$

has no zero divisors, and that

$$H^0_{\mathfrak{m}}(B)_0 = A \cap (\tilde{P} - uZ, \tilde{Q} - vZ, \tilde{R} - wZ) \text{ in } A[X,Z].$$

Finally, we know that

$$\mathrm{Res}(P - uZ^d, Q - vZ^d, R - wZ^d) = \mathcal{R}^d,$$

where Res denotes the classical resultant of three homogeneous polynomials, which shows that $\mathcal{R}$ is homogeneous of degree $3d^2/d = 3d$ in the coefficients of the polynomials $P - uZ, Q - vZ, R - wZ$.

Now introduce new indeterminates $T_1, \ldots, T_d$, and denote by

$$\Delta(T_1, \ldots, T_d) := (T^{d-j})_{1 \le i,j \le d}$$

the Vandermonde's matrix. We have clearly

$$\Delta(T_1, \ldots, T_d) \circ \Omega \circ \begin{pmatrix} X^{d-1} \\ X^{d-2} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}, \tag{A.1}$$

with, for all $1 \le i \le d$,

$$\begin{aligned}(T_i - X)a_i = {} & u(\tilde{Q}(T_i)\tilde{R}(X) - \tilde{R}(T_i)\tilde{Q}(X)) \\ & + v(\tilde{R}(T_i)\tilde{P}(X) - \tilde{P}(T_i)\tilde{R}(X)) \\ & + w(\tilde{P}(T_i)\tilde{Q}(X) - \tilde{Q}(T_i)\tilde{P}(X)),\end{aligned}$$

which can be rewritten

$$\begin{aligned}(T_i - X)a_i = {} & u(\tilde{Q}(T_i)(\tilde{R}(X) - wZ) - \tilde{R}(T_i)(\tilde{Q}(X) - vZ)) \\ & + v(\tilde{R}(T_i)(\tilde{P}(X) - uZ) - \tilde{P}(T_i)(\tilde{R}(X) - wZ)) \\ & + w(\tilde{P}(T_i)(\tilde{Q}(X) - vZ) - \tilde{Q}(T_i)(\tilde{P}(X) - uZ)).\end{aligned}$$

This shows that $(T_i - X)a_i$ is in the ideal of $A[T_1, T_2, \ldots, T_d, X, Z]$ generated by the polynomials $\tilde{P} - uZ, \tilde{Q} - vZ, \tilde{R} - wZ$. Composing (A.1) on the left by the adjoint matrix of $\Delta \circ \Omega$, we deduce that, for all $0 \le l \le d - 1$,

$$\Big(\prod_{1 \le i \le d} (T_i - X) \prod_{1 \le i < j \le d} (T_i - T_j)\Big)\det(\Omega)X^l \in (\tilde{P} - uZ, \tilde{Q} - vZ, \tilde{R} - wZ).$$

As $(\prod_{1 \le i \le d}(T_i - X) \prod_{1 \le i < j \le d}(T_i - T_j))$ is not a zero divisor in $B[T_1, \ldots, T_d]$ (by the Dedekind-Mertens lemma), it follows in particular (case $l = 0$) that

$$\det(\Omega) \in (\tilde{P} - uZ, \tilde{Q} - vZ, \tilde{R} - wZ),$$

and thus $\mathcal{R}$ divides $\det(\Omega)$ in the ring A. The entries of the matrix Ω are homogeneous of degree 3 with respect to u, v, w and $\mathrm{coeff}(P, Q, R)$, thus $\det(\Omega)$ is homogeneous of degree $3d$ with respect to those coefficients. Consequently, there exists $c \in \mathbb{Z}$ such that $\det(\Omega) = c\mathcal{R}$. We can determine c with the specialization $P \mapsto X^d$, $Q \mapsto Y^d$, $R \mapsto 0$, $u \mapsto 0$, $v \mapsto 0$ and $w \mapsto w$. On one hand the matrix Ω specializes in $w\mathbb{B}ez(X^d, Y^d)$, and hence $\det(\Omega)$ specializes in

$$(-1)^{\frac{d(d-1)}{2}} w^d \mathrm{Res}(X^d, Y^d) = (-1)^{\frac{d(d-1)}{2}} w^d,$$

and on the other hand $\mathcal{R}$ specializes in

$$^a\mathrm{Res}(X^d, Y^d, -wZ) = (-1)^d w^d . {}^a\mathrm{Res}(X^d, Y^d, Z) = (-1)^d w^d,$$

following the properties of the anisotropic resultant. The comparison of these two last equalities gives the proposition. $\square$

Acknowledgement

The authors are grateful to the referee for useful comments and remarks.

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