Characters and conjugacy classes of the symmetric group

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or *On some conjectures of J. Katriel*

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Résumé

Nous démontrons plusieurs conjectures dues à Jacob Katriel qui portent sur les classes de conjugaisons de $\mathfrak{S}_n$ vues comme opérateurs agissant par multiplication. La première conjecture exprime, pour une partition fixée ρ de la forme $r1^{n-r}$, les valeurs propres (ou caractères centraux) ω_ρ^λ en terme des contenus de λ. Tandis que Katriel a conjecturé une forme générique et un algorithme pour calculer les coefficients indéterminés, nous fournissons une formule explicite. La seconde conjecture (présentée au SFCA'98 à Toronto) donne une forme générale pour l'expression d'une classe de conjugaison en terme d'opérateurs élémentaires. Nous la prouvons en utilisant une description en termes d'opérateurs différentiels sur les polynômes symétriques. Finalement nous étendons partiellement nos résultats sur ω_ρ^λ à des partitions ρ quelconques.

Abstract

This article addresses several conjectures due to Jacob Katriel concerning conjugacy classes of $\mathfrak{S}_n$ viewed as operators acting by multiplication. The first conjecture expresses, for a fixed partition ρ of the form $r1^{n-r}$, the eigenvalues (or central characters) ω_ρ^λ in terms of contents of λ. While Katriel conjectured a generic form and an algorithm to compute missing coefficients, we provide an explicit expression. The second conjecture (presented at FPSAC'98 in Toronto) gives a general form for the expression of a conjugacy class in terms of *elementary* operators. We prove it using a convenient description by differential operators acting on symmetric polynomials. To conclude, we partially extend our results on ω_ρ^λ to arbitrary partitions ρ.

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1 Introduction

Although our aim is to prove Katriel's conjectures, we do not use his notations through this text: instead we keep closer to Macdonald's textbook [12] and provide translations when necessary.

1.1 Notations

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of weight n and length $\ell(\lambda) = k$ *i.e.* a finite non increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We write $\lambda \vdash n$ or $|\lambda| = n$ and $\lambda = 1^{\ell_1} 2^{\ell_2} \cdots n^{\ell_n}$ when ℓ_i parts of λ are equal to i ($i = 1 \ldots n$). (We shall consistently use greek letters for partitions and their parts and corresponding latin letters for the multiplicity notation.) We denote by C_λ the conjugacy class indexed by partition λ and by $\lambda(\sigma)$ the cycle-type of a permutation σ. Following [12], we let $z_\lambda = 1^{\ell_1} \ell_1! 2^{\ell_2} \ell_2! \cdots n^{\ell_n} \ell_n!$, so that $|C_\lambda| = n!/z_\lambda$.

Let $\mathbb{Q}[\mathfrak{S}_n]$ be the group algebra of the symmetric group over the rational numbers field $\mathbb{Q}$ and let $C[\mathfrak{S}_n]$ be the center of this group algebra. The formal sum of the permutations in a conjugacy class C_λ belongs to $C[\mathfrak{S}_n]$ and we abuse the notation and also denote it C_λ. The set $\{C_\lambda\}_{\lambda \vdash n}$ of these formal sums forms a linear basis for the center $C[\mathfrak{S}_n]$. Similarly the irreducible characters of the symmetric group $\mathfrak{S}_n$ are indexed by partitions of weight n and can be considered as elements of $C[\mathfrak{S}_n]$, in which the family $\{\chi^\lambda\}_{\lambda \vdash n}$ also forms a linear basis. For λ and μ two partitions of n we denote by χ^λ_μ the evaluation of the character χ^λ on any permutation of the class C_μ. In particular $\chi^\lambda_{1^n} = n!/h_\lambda$ where h_λ is the hook-length product of λ. The character table $[\chi^\lambda_\mu]$ gives natural formulae for changes of basis in $C[\mathfrak{S}_n]$: $\chi^\lambda = \sum_{\mu \vdash n} \chi^\lambda_\mu C_\mu$ and $C_\mu = \sum_{\lambda \vdash n} (\chi^\lambda_\mu / z_\lambda) \chi^\lambda$.

Partitions are usually represented by their Ferrers' diagrams. The content of a cell $x = (i, j) \in \lambda$ is $c(x) = i - j$. In order to state his conjectures, Katriel introduces the *content power-sum* σ_k defined for $k > 0$ by $\sigma_k(\lambda) = \sum_{x \in \lambda} c(x)^k$.

A partition is *reduced* if it contains no part 1. For $\lambda = 1^{\ell_1} 2^{\ell_2} \cdots k^{\ell_k}$, we denote $\bar{\lambda}$ the reduced partition $2^{\ell_2} \ldots k^{\ell_k}$. The reduced cycle-type of a permutation or of a conjugacy class is defined accordingly.

1.2 Multiplication by a conjugacy class

Let λ be a partition of n. We are interested in the element ω^λ of $C[\mathfrak{S}_n]$ which is defined ([12, p126]) by

$$\forall \rho \vdash n, \qquad \omega^\lambda(\rho) = \omega^\lambda_\rho = \frac{h_\lambda}{z_\rho} \chi^\lambda_\rho.$$

These elements are called *central characters* or *eigenvalues* by J. Katriel[1]. Because the family $\{\chi^\lambda / h_\lambda\}_{\lambda \vdash n}$ forms a basis of orthogonal idempotents in $C[\mathfrak{S}_n]$, we have

$$\forall \rho, \lambda \vdash n, \qquad C_\rho \cdot \chi^\lambda = \sum_{\mu \vdash n} \frac{\chi^\mu_\rho}{z_\mu} (\chi^\mu \cdot \chi^\lambda) = \omega^\lambda_\rho \chi^\lambda,$$

explaining why the evaluation ω^λ_ρ may be called the eigenvalue of the conjugacy class C_ρ associated to the eigenvector χ^λ. Here we consider C_ρ as an operator acting on $C[\mathfrak{S}_n]$ by multiplication. The multiplicative structure of $C[\mathfrak{S}_n]$ has been largely studied in terms of *connexion coefficients* [2, and ref. therein], also called

[1] For a partition $\rho \vdash n$ and an irreducible representation Γ, indexed by the partition $\gamma \vdash n$, our ω^γ_ρ is denoted λ^Γ_ρ in [8].

structure constants [3, 4, and ref. therein]. These coefficients are defined for all triples of partitions (λ, μ, ν) of n by

$$C_\lambda \cdot C_\mu = \sum_\nu \mathfrak{a}^\nu_{\lambda,\mu} C_\nu.$$

The first set of conjectures that we consider are [8, Conj.1–2] (see also [5]). In these articles, Katriel suggests that for a fixed integer r, the eigenvalues $\omega^\lambda_{r1^{n-r}}$ are given by evaluations of a polynomial of $\mathbb{Q}[n][\sigma_1, \ldots, \sigma_{r-1}]$ on the contents power-sums $\sigma_k(\lambda)$. In [6] an algorithm is given, which is used in [7] to produce numerical expressions supporting these conjectures up to $r = 18$. In Section 2 we give an explicit expression (Theorem 1) for the polynomials considered by Katriel and prove some of their conjectured properties.

In a second set of conjectures, which we consider in Section 3, Katriel looks for expressions of the conjugacy classes as sums of more *elementary* operators. He requires that these expressions depend only on the reduced cycle-type. These conjectures where presented in Toronto at FPSAC'98 [11] and derive from previous weaker conjectures [9, 10, and ref. therein]. In order to state more easily Katriel's formulae we use a representation of the action of conjugacy classes on $C[\mathfrak{S}_n]$ by an action of some differential operators on the space of symmetric functions. Once correctly stated in this form (Theorem 3), these conjectures are relatively easy to prove. Our approach is reminiscent of that of I.P. Goulden and D.M. Jackson (see [2] and reference therein).

Finally in Section 4, we consider a last set of conjectures [8, Conj.3–6], which are the natural extensions of the first ones, from partitions of the form $r1^{n-r}$ to arbitrary partitions ρ. As opposed to the case $r1^{n-r}$, we have not found for all ρ an explicit expression of ω_ρ for arbitrary ρ in terms of the σ_i. However we prove a weak version of [8, Conj.4] on the general form of ω_ρ, which implies [8, Conj.5].

Numerous examples of decompositions of ω_ρ or C_ρ for small ρ can be found *e.g.* in [8]. Thus we have not included many examples. Instead we provide `Maple` procedures based on our theorems at `http://www.loria.fr/~schaeffe`.

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2 Central characters for partitions $r1^{n-r}$

Throughout this section let r denote a fixed integer. We define the content weight of a partition ν to be $w(\nu) = |\nu| + 2\ell(\nu)$. Given a family of indeterminate $\sigma = \{\sigma_i\}_{i\geq 1}$, we use the notation $\sigma_\lambda = \prod_i \sigma_{\lambda_i}$ for any partition λ. Given a polynomial $P(\sigma)$ in $\mathbb{Q}[\sigma]$, by $P(\sigma(\lambda))$ we mean $P(\sigma)$ in which σ_i is substituted for the ith content power-sum $\sigma_i(\lambda)$.

Let Ω_r be the following polynomial of $\mathbb{Q}[n][\sigma_1, \ldots, \sigma_{r-1}]$:

$$\Omega_r(n, \sigma) = \sum_{\substack{k,\nu \\ k+w(\nu)=r+1}} \frac{(-1)^{|\nu|}\sigma_\nu}{z_\nu} \sum_{i+j=k} q^r_i(n)\phi^r_{\nu,j}(n),$$

where $q^r_i(n)$ and $\phi^r_{\nu,i}(n)$ are polynomials from $\mathbb{Q}[n]$ with respective degrees $i-1$ and i that are explicitly given in the proof. They are respectively the coefficients of Y^i in $Q^r(Y, n)$ (Formula (5)) and of $Y^{2\ell(\nu)+i}$ in $\phi^r_\nu(Y, n)$ (Formula (8)).

Theorem 1 (Part of [8, Conj.1]) *The eigenvalue $\omega_{\bar{r}}$ belongs to $\mathbb{Q}[n][\sigma]$. More precisely, for $n \geq r$, and for all partitions λ of n,*

$$\omega^{\lambda}_{r1^{n-r}} = \Omega_r(n, \sigma(\lambda)).$$

Moreover, for $n < r$, and all partitions λ of n, $\Omega_r(n, \sigma(\lambda)) = 0$.

Corollary 1 ([8, Conj.2]) *The coefficient of σ_{r-1} in Ω_r is 1.*

Conjecture 1 (Remaining from [8, Conj.1]) *In the definition of Ω_r, the inner sum, which is a polynomial from $\mathbb{Q}[n]$ of degree at most k, is in fact*

- *null if k is odd,*
- *of degree $k/2$ if k is even.*

Although we have an explicit form for the inner sum, we have been unable to prove these cancellations.

Proof. Let us borrow the following result from [12, p118]. Let $\rho = r1^{n-r}$, λ be a partition of n, μ be its shifted partition $\mu_i = \lambda_i + n - i$ for $1 \leq i \leq n$ and $\phi(X) = \prod_{i=1}^{n}(X - \mu_i)$. Then

$$\omega^{\lambda}_{r1^{n-r}} = \frac{h_\lambda}{z_{r1^{n-r}}}\chi^{\lambda}_{\rho}$$

is equal to

$$\frac{-1}{r^2}\sum_{i=1}^{n}\mu_i(\mu_i - 1)\dots(\mu_i - r + 1)\frac{\phi(\mu_i - r)}{\phi'(\mu_i)}, \tag{1}$$

which is also the coefficient of X^{-1} in the expansion of

$$\frac{-1}{r^2}X(X-1)\dots(X-r+1)\frac{\phi(X-r)}{\phi(X)}$$

in descending powers of X. Changing the sign of X, the latter formula can be explicitly written as

$$\frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(X+n+i)\prod_{i=1}^{n}\frac{X+r+\mu_i}{X+r+n-i}\prod_{i=1}^{n}\frac{X+n-i}{X+\mu_i}. \tag{2}$$

Recall now that the content polynomial $c_\lambda(X)$ of the partition λ is the polynomial in the indeterminate X defined ([12, p15]) by

$$c_\lambda(X) = \prod_{x\in\lambda}(X + c(x)).$$

From [12, p15], for $\xi_i = \lambda_i + m - i$, $1 \leq i \leq m$, we have

$$\frac{c_\lambda(X+m)}{c_\lambda(X+m-1)} = \prod_{i=1}^{m}\frac{X+\xi_i}{X+m-i}.$$

On one hand, if we take $m = n$, we get $\xi_i = \mu_i$ for $1 \leq i \leq n$. On the other hand, if $m = n + r$, then $\xi_i = r + \mu_i$ for $1 \leq i \leq n$ and $\xi_{n+i} = r - i$ for $0 \leq i < r$. Therefore Formula (2) can be rewritten into

$$\frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(X+n+i)\frac{c_\lambda(X+n+r)}{c_\lambda(X+n+r-1)}\frac{c_\lambda(X+n-1)}{c_\lambda(X+n)} \tag{3}$$

In view of setting $X = 1/Y$ in this formula, let us introduce the reciprocal content polynomials $\tilde{c}_\lambda(Y) = \prod_{x\in\lambda}(1+c(x)Y)$. We obtain $\omega^\lambda_{r1^{n-r}}$ as the coefficient of Y^{r+1} in the Taylor expansion of

$$Q^r(Y,n)\cdot\mathcal{C}^r_\lambda(Y,n) \tag{4}$$

where

$$Q^r(Y,n) = \frac{(-1)^r}{r^2}\prod_{i=0}^{r-1}(1+(n+i)Y)\left(\frac{(1+(n+r)Y)(1+(n-1)Y)}{(1+(n+r-1)Y)(1+nY)}\right)^n \tag{5}$$

and

$$\mathcal{C}^r_\lambda(Y,n) = \frac{\tilde{c}_\lambda\left(\frac{Y}{1+(n+r)Y}\right)}{\tilde{c}_\lambda\left(\frac{Y}{1+(n+r-1)Y}\right)}\frac{\tilde{c}_\lambda\left(\frac{Y}{1+(n-1)Y}\right)}{\tilde{c}_\lambda\left(\frac{Y}{1+nY}\right)}. \tag{6}$$

In terms of λ-rings the following development of $\mathcal{C}^r_\lambda(Y,n)$ in power-sums is immediate:

$$\mathcal{C}^r_\lambda(Y,n) = \sum_\nu \frac{(-1)^{|\nu|}}{z_\nu}\sigma_\nu\phi^r_\nu(Y,n)Y^{|\nu|}, \tag{7}$$

where $\phi^r_\nu(Y,n) = \prod_i \phi^r_{\nu_i}(Y,n)$ and

$$\begin{aligned}\phi^r_j(Y,n) \quad &= \quad (1+nY)^{-j} + (1+(n+r-1)Y)^{-j} \\ &\qquad -(1+(n+r)Y)^{-j} - (1+(n-1)Y)^{-j}.\end{aligned} \tag{8}$$

Let us compute the coefficient of Y^i in $\phi^r_j(Y,n)$: from (8),

$$\phi^r_j(Y,n) \quad = \quad \sum_{i\geq 0}\binom{j+i-1}{i}\left(n^i + (n+r-1)^i - (n+r)^i - (n-1)^i\right)(-Y)^i.$$

We see that terms of degree 0 and 1 in Y cancels, and for Y^2 we obtain $-2r\binom{j+1}{2}$. Therefore $\phi^r_\nu(Y,n)$ has a lowest degree term of degree $2\ell(\nu)$. Let $\phi^r_{\nu,i}(n)$ be the coefficient of $Y^{2\ell(\nu)+i}$ in $\phi^r_\nu(Y,n)$. The polynomial $\phi^r_{j,i}$ has degree i in n, so that $\phi^r_{\nu,i}(n)$ has degree i.

Let us turn now to the coefficient $q^r_i(n)$ of Y^i in $Q^r(Y,n)$. Expansions using the binomial theorem show that $q^r_i(n)$ is a polynomial of degree $i-1$ without constant term.

Finally, from (4) and (7) we have $\omega^\lambda_{r1^{n-r}}$ as the coefficient of Y^{r+1} in

$$\sum_\nu \frac{(-1)^{|\nu|}\sigma_\nu(\lambda)}{z_\nu}Y^{|\nu|}\phi^r_\nu(Y,n)Q^r(Y,n).$$

This gives the expected result

$$\omega^\lambda_{r1^{n-r}} \quad = \quad \sum_{\substack{k,\nu\\ k+w(\nu)=r+1}} \frac{(-1)^{|\nu|}\sigma_\nu(\lambda)}{z_\nu}\sum_{i+j=k} q^r_i(n)\phi^r_{\nu,j}(n).$$

For the second part of the theorem, observe that, starting from Formula (1), all manipulations are valid even when $n < r$. But in Formula (1), the nullity for $n < r$ is easily proven: in the summand, either $m_i < r$, and the falling power does the job, or there exists j such that $m_i = m_j + r$.

For the corollary, notice that the constant term of $Q^r(Y,n)$ is $(-1)^r/r^2$ by (5) while that of ϕ_{r-1} has been said to be $-2r\binom{r}{2}$. With $z_{(r-1)} = r-1$ the coefficient 1 is found. $\square$

3 Elementary operators

3.1 Symmetric functions

Let $\mathbf{x}=\{x_1, x_2, \dots\}$ be a set of indeterminates and let $\Lambda = \Lambda_{\mathbb{Q}}[\mathbf{x}]$ be the ring of symmetric functions on $\{x_1, x_2, \dots\}$ over the field of rational numbers $\mathbb{Q}$. The ring Λ is the direct sum of its homogeneous components of degree n: $\Lambda = \oplus_{n\geq 0}\Lambda^n$. Following [12], we denote by $p_\lambda = p_\lambda(\mathbf{x})$ the power sum symmetric functions. The family $\{p_\lambda\}_\lambda$ forms a linear basis of Λ. The usual scalar product $<,>$ on Λ satisfies

$$\forall n \quad \forall \lambda, \mu \qquad < p_\lambda, p_\mu >= z_\lambda \delta_{\lambda,\mu}.$$

We need also the following differential operators known in the litterature as Hammond's operators (see [13] or [12]):

Definition 1 *For each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, let ${p_\lambda}^\perp$ be the adjoint operator to the multiplication by $p_\lambda(\mathbf{x})$ with respect to the usual scalar product $<,>$ on Λ:*

$$\forall f, g \in \Lambda \qquad < p_\lambda f, g >=< f, {p_\lambda}^\perp g > .$$

For any partition λ, let $\pi_\lambda = \lambda_1\lambda_2 \cdots \lambda_k$ be the product of its parts λ_i. Then the operator ${p_\lambda}^\perp$ is conveniently described as a differential operator on the basis $\{p_\lambda(\mathbf{x})\}_\lambda$:

$${p_\lambda}^\perp = \pi_\lambda \frac{\partial^k}{\partial p_{\lambda_1} \partial p_{\lambda_2} \dots \partial p_{\lambda_k}}.$$

The use of such operators in relation with connexion coefficients is not new and can be found for instance in [2]. We are interested in representing the multiplication by a conjugacy class as an action of an operator on a basis of symmetric functions: more precisely we look for operators G_α, satisfying

$$\forall \beta, \gamma, \qquad G_\alpha \cdot q_\beta|_{q_\gamma} \;=\; C_\alpha \cdot C_\beta|_{C_\gamma}.$$

There is a trivial way to do this:

Definition 2 *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n. For any permutation $\rho \in C_\lambda$, define the operator $G_\lambda : \Lambda^n \to \Lambda^n$ by*

$$G_\lambda \;=\; \frac{1}{z_\lambda} \sum_{\sigma \in \mathfrak{S}_n} p_{\lambda(\rho\sigma)} {p_{\lambda(\sigma)}}^\perp \;=\; \frac{1}{z_\lambda} \sum_{\nu,\mu \vdash n} \frac{\mathfrak{a}_{\lambda,\mu}^\nu}{z_\mu} p_\nu {p_\mu}^\perp. \tag{9}$$

From this definition and the orthogonality relation $p_\mu^\perp p_\lambda = z_\lambda \delta_{\lambda,\mu}$ for partitions of the same weight, it is immediate that for any partitions λ,μ of n

$$G_\lambda q_\mu \;=\; \sum_{\gamma \vdash n} \mathfrak{a}_{\lambda,\mu}^\nu q_\nu,$$

where $\{q_\lambda = \frac{p_\lambda}{z_\lambda}\}_\lambda$ is an orthonormal basis of Λ.

The operators G_λ are not interesting because their definition uses the structure constants $\mathfrak{a}_{\lambda,\mu}^\nu$ which they are meant to produce; however they provide an easy introduction to what we mean by representing the multiplication by the action of an operator on symmetric functions.

Our aim is to define a more interesting family H of such operators, satisfying for all partitions α

$$\forall \beta, \gamma, \qquad H_{\bar{\alpha}} \cdot q_\beta|_{q_\gamma} \;=\; C_\alpha \cdot C_\beta|_{C_\gamma}.$$

Observe here that we require that, in some loose sense, the H are defined in terms of $\bar{\alpha}$ and not of α.

3.2 Restricted permutations

In order to define our operators we need some elementary results on restricted permutations. For a subset S of $[n] = \{1, \dots, n\}$, and a permutation $\sigma \in \mathfrak{S}_n$, let $\sigma_{|S}$ be the permutation of the elements of S such that, for all $i \in S$, $\sigma_{|S}(i) = \sigma^k(i)$ where k is the least positive integer such that $\sigma^k(i)$ is in S.

The idea rests on the following remark: let ρ and σ be permutations in $\mathfrak{S}_n$, and consider the product $\tau = \rho\sigma$. Then, if $[n] \setminus S$ contains only fixed points of ρ, we have $\tau_{|S} = \rho_{|S}\sigma_{|S}$, and conversely τ can be obtained from $\tau_{|S}$ by inserting after each $i \in S$ the block that separates i and $\sigma_{|S}(i)$ in the decomposition of σ into disjoint cycles.

Example. If $\rho|_3 = (1\ 2\ 3)$ and $\sigma = (1\ aaa\ 2\ bbb)\ (3\ ccc)$, then $\sigma_{|3} = (1\ 2)\ (3)$, $\tau_{|3} = (1\ 3)\ (2)$ and $\tau = (1\ aaa\ 3\ ccc)\ (2\ bbb)$.

Given a permutation σ_0 of $\mathfrak{S}_p$, and $n \geq p$, the permutation σ_0 can be extended naturally to a permutation of $\mathfrak{S}_n$ by adding fixed points. Therefore any $\sigma_0 \in \mathfrak{S}_p$ acts by left multiplication on $\mathfrak{S}_n$. We need one last definition: given a partition $\alpha = (\alpha_1, \dots, \alpha_\ell)$, the canonical permutation of type α is the permutation of cycle type α whose k-th cycle is $(\alpha_1 + \dots + \alpha_{k-1} + 1, \dots, \alpha_1 + \dots + \alpha_k)$.

3.3 The operator $H_{\bar{\alpha}}$ and Katriel's notations

We give two equivalent definitions of H. The first one uses usual notations.

Definition 3 *Let $\bar{\alpha}$ be a reduced partition of weight p. Let ρ_0 be the canonical permutation of cycle type $\bar{\alpha}$. Then the operator $H_{\bar{\alpha}} : \Lambda \to \Lambda$ is defined by:*

$$H_{\bar{\alpha}} \quad = \quad \frac{1}{z_{\bar{\alpha}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \quad \sum_{i_1, \dots, i_p \geqslant 1} p_{\gamma'} \cdot p_{\beta'}^{\perp} \tag{10}$$

where β' is the cycle type of any permutation τ obtained from σ_0 by inserting $i_j - 1$ elements after each $j \in \{1, \dots, p\}$, and γ' is the cycle type $\rho_0\tau$.

The fact that the cycle type γ' depends only on the integers $i_1, \dots, i_p$, and not on the elements that we choose to insert in σ_0 is a consequence of the previous discussion on restricted permutations.

This operator is closely related to Katriel's bracket operators (which are not completely rigorously defined). A simple variation on his notation is:

$$\langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle \quad \text{stands for} \quad \sum_{i_1, i_2, i_3 \geqslant 1} p_{[i_1+i_2, i_3]} p_{[i_1, i_2+i_3]}^{\perp},$$

where the brackets $[\,,]$ denote multisets of integers (*i.e.* partitions). A further simplification of this notation (even closer to Katriel's) is to replace each variable by its index and write sums as cycles:

$$\langle\!\langle (1,2)(3) \mid (1)(2,3) \rangle\!\rangle \quad \text{stands for} \quad \langle\!\langle i_1 + i_2; i_3 \mid i_1; i_2 + i_3 \rangle\!\rangle$$

Let us rewrite Definition 3 with this notation

Definition 4 *Let $\bar{\alpha}$ be a reduced partition of weight p. Let ρ_0 be the canonical permutation of cycle type $\bar{\alpha}$. Then*

$$H_{\bar{\alpha}} \quad = \quad \frac{1}{z_{\bar{\alpha}}} \sum_{\sigma_0 \in \mathfrak{S}_p} \langle\!\langle \rho_0\sigma_0 \mid \sigma_0 \rangle\!\rangle. \tag{11}$$

Finally, Katriel conjectured a symmetry in the coefficients, which allows to introduce a last notation:

$$\langle P \mid Q\rangle \quad \text{stands for} \quad \langle\!\langle P \mid Q\rangle\!\rangle + \langle\!\langle Q \mid P\rangle\!\rangle$$

Examples. We keep with the intermediate notation, which we find more descriptive, even though we used Formula 11 to get quickly all terms.

$$\begin{aligned}
H_2 &= \frac{1}{2}\langle\!\langle i_1; i_2 \mid i_1+i_2\rangle\!\rangle + \frac{1}{2}\langle\!\langle i_1+i_2 \mid i_1; i_2\rangle\!\rangle \;=\; \frac{1}{2}\langle i_1+i_2 \mid i_1; i_2\rangle \\
&= \frac{1}{2}\left(\sum_{i_1,i_2\geq 1} p_{i_1,i_2}p^{\perp}_{i_1+i_2} + \sum_{i_1,i_2\geq 1} p_{i_1+i_2}p^{\perp}_{i_1,i_2}\right) \\
&= \frac{1}{2}p_2p_{11}^{\perp} + \frac{1}{2}p_{11}p_2^{\perp} + p_3p_{21}^{\perp} + p_{21}p_3^{\perp} + p_4p_{31}^{\perp} \\
&\qquad + p_{31}p_4^{\perp} + \frac{1}{2}p_4p_{22}^{\perp} + \frac{1}{2}p_{22}p_4^{\perp} + \cdots \\
H_3 &= \frac{1}{3}\langle\!\langle i_1+i_2+i_3 \,|\, i_1,i_2,i_3\rangle\!\rangle + \frac{1}{3}\langle\!\langle i_1,i_2,i_3 \,|\, i_1+i_2+i_3\rangle\!\rangle + \frac{1}{3}\langle\!\langle i_1+i_2+i_3 \,|\, i_1+i_2+i_3\rangle\!\rangle \\
&\quad + \frac{1}{3}\langle\!\langle i_1+i_2,i_3 \,|\, i_1,i_2+i_3\rangle\!\rangle + \frac{1}{3}\langle\!\langle i_1+i_3,i_2 \,|\, i_2,i_1+i_3\rangle\!\rangle + \frac{1}{3}\langle\!\langle i_2+i_3,i_1 \,|\, i_3,i_1+i_2\rangle\!\rangle \\
&= \langle\!\langle i_1+i_2,i_3 \,|\, i_1,i_2+i_3\rangle\!\rangle + \frac{1}{3}\langle i_1+i_2+i_3 \,|\, i_1,i_2,i_3\rangle + \frac{1}{3}\langle\!\langle i_1+i_2+i_3 \,|\, i_1+i_2+i_3\rangle\!\rangle \\
&= \frac{1}{3}\sum_{\substack{(i_1,i_2,i_3)\geqslant 1\\ n=i_1+i_2+i_3}} \left(p_np^{\perp}_{(i_1,i_2,i_3)} + p_{(i_1,i_2,i_3)}p_n^{\perp} + p_np_n^{\perp} + 3p_{(i_1+i_3,i_2)}p^{\perp}_{(i_1+i_2,i_3)}\right) \\
H_{2^2} &= \frac{1}{8}\langle i_1;i_2;i_3;i_4 \,|\, i_1+i_2;i_3+i_4\rangle + \frac{1}{4}\langle\!\langle i_1+i_2;i_3;i_4 \,|\, i_1;i_2;i_3+i_4\rangle\!\rangle \\
&\quad + \frac{1}{4}\langle\!\langle i_1+i_2;i_3+i_4 \,|\, i_1+i_3;i_2+i_4\rangle\!\rangle + \langle\!\langle i_1+i_2+i_3;i_4 \,|\, i_1;i_2+i_3+i_4\rangle\!\rangle \\
&\quad + \frac{1}{2}\langle i_1+i_2+i_3+i_4 \,|\, i_1+i_2;i_3;i_4\rangle + \frac{1}{4}\langle\!\langle i_1+i_2+i_3+i_4 \,|\, i_1+i_2+i_3+i_4\rangle\!\rangle
\end{aligned}$$

Katriel's global conjecture of [11] is that C_α "=" $H_{\bar{\alpha}}$. More formally, we shall prove the following theorem.

Theorem 2 (Global Conjecture) *Let α, β and γ be partitions of n, then*

$$C_\alpha \cdot C_\beta|_{C_\gamma} = H_{\bar{\alpha}} \cdot q_\beta|_{q_\gamma}.$$

Observe that applying a permutation of indices in an elementary bracket operator do not change it. Collecting equivalent terms to form a sum over "distinct contributions" (as we did in the examples), we immediately obtain from (11) a proof of Katriel's *central conjecture* on the resulting coefficients.

Proof. Let p be the weight of $\bar{\alpha}$, ρ_0 the associated canonical permutation, and ρ its natural extension in $\mathfrak{S}_n$. On one hand,

$$\begin{aligned}
C_\alpha \cdot C_\beta|_{C_\gamma} &= \frac{|C_\alpha|}{|C_\gamma|}\, C_\beta \cdot C_\gamma|_{C_\alpha} \;=\; \frac{z_\gamma}{z_\alpha}\,\mathrm{Card}\,\{(\sigma,\tau)\in C_\beta\times C_\gamma \mid \rho\sigma=\tau\} \\
&= \frac{z_\gamma}{z_\alpha}\sum_{\sigma_0\in\mathfrak{S}_p} \mathrm{Card}\,\left\{(\sigma,\tau)\in C_\beta\times C_\gamma \mid \sigma_{|p}=\sigma_0,\ \tau_{|p}=\rho_0\sigma_{|p}\right\}
\end{aligned}$$

Our discussion on restricted permutations implies that, for all $\sigma_0 \in \mathfrak{S}_p$,

$$\begin{aligned}\mathrm{Card}\left\{(\sigma,\tau) \in C_\beta \times C_\gamma \mid \sigma_{|p} = \sigma_0,\ \tau_{|p} = \rho_0\sigma_{|p}\right\} &= \\ \sum_{(i_0,\dots,i_p)\in\mathcal{C}(\beta,\gamma)} \binom{n-p}{i_0} \cdot (n-p-i_0)! \cdot \frac{i_0!}{z_{\beta-\beta'}}\end{aligned}$$

where the sum runs over the compositions $(i_0,\dots,i_p)$ of n such that inserting i_j elements of $\{p+1,\dots,n\}$ after each j of $\{1,\dots,p\}$ in $\sigma_{|p}$ and $\tau_{|p}$ leads to permutations of respective cycle types β' and γ' with the following properties: $\forall i, b'_i \leqslant b_i,\ c'_i \leqslant c_i$ and $\beta-\beta' = \gamma-\gamma'$, where $\beta = 1^{b_1}\dots n^{b_n}$ and so on, and $\beta-\beta'$ denotes the partition $1^{b_1-b'_1}\dots n^{b_n-b'_n}$.

This simplifies to:

$$C_\alpha \cdot C_\beta|_{C_\gamma} = \frac{z_\gamma}{z_{\bar\alpha}} \sum_{\bar\sigma\in\mathfrak{S}_p} \sum_{(i_0,\dots,i_p)\in\mathcal{C}(\beta,\gamma)} \frac{1}{z_{\beta-\beta'}}.$$

On the other hand,

$$H_{\bar\alpha} \cdot q_\beta = \frac{1}{z_{\bar\alpha}} \sum_{\sigma_0\in\mathfrak{S}_p} \sum_{i_1,\dots,i_p\geqslant 0} p_{\gamma'} \frac{\pi_{\beta'}}{z_\beta} \frac{\partial p_\beta}{\partial p_{\beta'}}.$$

Since $\dfrac{\partial p_\beta}{\partial p_{\beta'}} = 0$ unless $\beta \geqslant \beta'$, in which case

$$\frac{\partial p_\beta}{\partial p_{\beta'}} = \frac{b_1!\dots b_n!}{(b_1-b'_1)!\dots(b_n-b'_n)!}\, p_{\beta-\beta'},$$

we obtain:

$$H_{\bar\alpha} \cdot q_\beta = \frac{1}{z_{\bar\alpha}} \sum_{\bar\sigma\in\mathfrak{S}_p} \sum_{(i_0,\dots i_p)\in\mathcal{C}(\beta)} \frac{p_{\gamma'+\beta-\beta'}}{z_{\beta-\beta'}},$$

where the sums runs over the compositions of n such that $\forall i, b'_i \leqslant b_i$. So finally:

$$H_{\bar\alpha} \cdot q_\beta|_{q_\gamma} = \frac{z_\gamma}{z_{\bar\alpha}} \sum_{\sigma\in\mathfrak{S}_p} \sum_{(i_0,\dots i_p)\in\mathcal{C}(\beta,\gamma)} \frac{1}{z_{\beta-\beta'}}.$$

□

An immediate consequence of Theorem 2 is that the operators $H_{\overline{\lambda}}$ are self adjoint and therefore their expansion $H_{\overline{\lambda}} = \sum_{\nu,\mu} a^\nu_{\lambda,\mu} p_\nu p_\mu^\perp$, are symmetric in ν and μ, a fact that can also be proven directly form their definition. This is also part of Katriel's conjecture.

3.4 Families of connexion coefficients

For a reduced partition $\bar\alpha$, let $K_{\bar\alpha}(n)$ be the sum in $C[\mathfrak{S}_n]$ of all permutations with reduced cycle-type $\bar\alpha$ if $n \geq |\bar\alpha|$, and 0 otherwise.

Let $\bar\alpha$, $\bar\beta$ be reduced partitions and define the coefficients $\mathfrak{a}^{\gamma}_{\bar\alpha,\bar\beta}(n)$ by

$$K_{\bar\alpha}(n) \cdot K_{\bar\beta}(n) = \sum_{\bar\gamma} \mathfrak{a}^{\bar\gamma}_{\bar\alpha,\bar\beta}(n) K_{\bar\gamma}(n). \tag{12}$$

In [1] it is proved that the $\mathfrak{a}^{\bar\gamma}_{\bar\alpha,\bar\beta}(n)$ are polynomials in n. This follows for instance from Theorem 2: let k (resp. h) be the largest part of $\bar\beta$ (resp. $\bar\gamma$), then the elementary operator $H_{\bar\alpha}$ is applied to $q_{\bar\beta 1^{n-|\bar\beta|}}$, the terms with non-zero contributions are of the form

$$p_\mu p_\lambda^\perp q_{\bar\beta 1^{n-|\bar\beta|}}\Big|_{q_{\bar\gamma 1^{n-|\bar\gamma|}}}$$

where λ and μ are partitions of length $|\bar\alpha|$, $\lambda = 1^{\ell_1}\dots k^{\ell_k}$ having parts of size at most k and $\mu = 1^{m_1}\dots h^{m_h}$ at most h. There are finitely many such partitions and the contribution of this term is a polynomial of degree m_1 in n.

Theorem 2 is a generalisation of this result in the sense that it proves that other families of coefficients are polynomial. For instance, for any reduced partition $\bar\alpha$ with even weight the coefficient

$$\mathfrak{b}_{\bar\alpha}(n) = K_{\bar\alpha}(2n)\cdot C_{2^n}\Big|_{C_{2^n}}$$

is a polynomial in n. The expression of H_{2^2} presented before gives

$$\mathfrak{b}_{2^2}(n) = \frac{1}{4}p_{2^2}p_{2^2}^\perp q_{2^n}\Big|_{q_{2^n}} = n(n-1).$$

4 Central characters for general partitions

In view of Section 3.4 we have obtained in Theorem 1 the eigenvalues of K_r, as a polynomial of $\mathbb{Q}[n][\sigma]$. We seek now similar results for the eigenvalues of $K_{\bar\alpha}$ for any reduced partition $\bar\alpha$.

Following the suggestion of [8] this can be done as follows. Let $\bar\alpha = (2^{a_2},\dots,k^{a_k})$ be a reduced partition. We prove first by recurrence on $|\bar\alpha|$ that $K_{\bar\alpha}(n)$ can be expressed as a linear combination of products of the $K_r(n)$:

$$K_{\bar\alpha}(n) = \frac{K_{\alpha_1}(n)\cdots K_{\alpha_\ell}(n)}{\prod_i a_i!} + \sum_{\bar\beta} b^{\bar\beta}(n)\prod_i K_{\beta_i}(n), \tag{13}$$

where the sum is over reduced partitions $\bar\beta$ such that $|\bar\beta| < |\bar\alpha|$. Indeed for $|\bar\alpha| = 2$, $\bar\alpha$ being reduced is the partition 2 and we are done. Otherwise linearize $K_{\alpha_1}(n)\cdots K_{\alpha_\ell}(n)$ to form relations like (12). According to Theorem 2, we obtain

$$K_{\alpha_1}(n)\cdots K_{\alpha_\ell}(n) = \textstyle\prod_i a_i!\cdot K_{\bar\alpha}(n) + \displaystyle\sum_{\bar\beta} a^{\bar\beta}(n)K_{\bar\beta}(n), \tag{14}$$

where the sum ranges over reduced partitions $\bar\beta$ such that $|\bar\beta| < |\bar\alpha|$. Indeed, the coefficient of $K_{\bar\alpha}(n)$ is given by successive applications of the elementary operators $\langle\langle\alpha_i|1;\dots;1\rangle\rangle$ while other operators result in a lower weight (because some fix point are not removed). Similarly the partitions $\bar\beta$ also satisfy $|\bar\beta| + \ell(\bar\beta) \le |\bar\alpha| + \ell(\bar\alpha)$ because, in the application of an elementary operator, each time a cycle broken (thus increasing the length by one) it prevent the incorporation of a fix-point (thus reducing the final weight by one). By induction hypothesis the $K_{\bar\beta}(n)$ can be linearized and we obtain (13).

Now the $\omega_{\bar\alpha}$ are eigenvalues of $K_{\bar\alpha}$, *i.e.* for $n > |\bar\alpha|$, and $\lambda \vdash n$,

$$K_{\bar\alpha}(n)\cdot\chi^\lambda = \omega^\lambda_{\bar\alpha 1^{n-|\bar\alpha|}}\chi^\lambda.$$

Using Formula (13) we obtain, with $n_a = n - |\bar\alpha|$,

$$\omega^\lambda_{\bar\alpha 1^{n_a}} = \frac{\omega^\lambda_{\alpha_1 1^{n_{a_1}}}\cdots\omega^\lambda_{\alpha_\ell 1^{n_{a_\ell}}}}{\prod_i a_i!} + \sum_{\bar\beta} b^{\bar\beta}(n)\prod_i \omega^\lambda_{\bar\beta_i 1^{n_\beta}},$$

so that we are led to define the polynomial $\Omega_{\bar\alpha}$ from $\mathbb{Q}[n][\sigma]$ by

$$\Omega_{\bar\alpha} = \frac{\Omega_{\alpha_1}\cdots\Omega_{\alpha_\ell}}{\prod_i a_i!} + \sum_{\bar\beta} b^{\bar\beta}(n)\prod_i \Omega_{\beta_i}, \tag{15}$$

where the sum ranges over reduced partitions $\bar\beta$ with $|\bar\beta| < |\bar\alpha|$. In the each Ω_{α_i} of the product, the monomials σ_ν satisfy $w(\nu) \le \alpha_i + 1$. Therefore the monomials σ_ν in this product satisfy $w(\nu) \le |\bar\alpha| + \ell(\bar\alpha)$. By the same induction as before, it is also the case of the monomials in the $\Omega_{\bar\beta}$. Finally, applying Theorem 1, we obtain the following theorem.

Theorem 3 (Part of [8, Conj.3]) *Let $\bar\alpha$ be a reduced partition. For $n \ge |\bar\alpha|$, and for all partitions λ of n,*

$$\omega^\lambda_{\bar\alpha 1^{n-|\bar\alpha|}} = \Omega_{\bar\alpha}(n, \sigma(\lambda)),$$

where $\Omega_{\bar\alpha}$ is a polynomial from $\mathbb{Q}[n][\sigma]$ involving only monomials σ_ν such that $w(\nu) \le |\bar\alpha| + \ell(\bar\alpha)$.

Corollary 2 ([8, Conj.4]) *Let $\bar\alpha = (2^{a_2}, \ldots, k^{a_k})$ be a reduced partition with k parts and $\alpha' = (1^{a_2}, \ldots, (k-1)^{a_k})$. Then the coefficient of $\sigma_{\alpha'}$ in $\Omega_{\bar\alpha}$ is*

$$\prod_{i\ge 2} \frac{1}{a_i!}.$$

Conjecture 2 (Remaining from [8, Conj.3]) *The coefficient of a monomial σ_ν of $\Omega_{\bar\alpha}$, which is a polynomial from $\mathbb{Q}[n]$, is in fact*

- *null if $k = |\bar\alpha| + \ell(\bar\alpha) - w(\nu)$ is even,*
- *of degree at most $k/2$ otherwise.*

This conjecture is implied by Conjecture 1 of the present article, using (15).

Conjecture 3 (From [8, Conj.5]) *The polynomial $\Omega_{\bar\alpha}$ vanishes on too small partitions,* i.e. *for $n < |\bar\alpha|$ and $\lambda \vdash n$, $\Omega_{\bar\alpha}(n, \sigma(\lambda)) = 0$.*

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