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Constructing orders by means of inductive definitions

Guillaume Bonfante, François Lamarche*

Abstract

We present a class of algebraic theories that are enriched over a novel Symmetrical Monoidal Closed structure on the category of graphs, whose free models are posets that are equipped with an induction principle, which is easily formalized in type theory. We give examples.

The development of computer science has given a new impulse to the theory of inductive definitions. It was classically based on set theory à la Zermelo (for a survey see [Acz77]), but the needs of the theory of data types, and those of type theory, has compelled people to look towards universal algebra and category theory for inspiration and paradigms. In particular it has been known for a long time that the notion of free structure is closely related to that of induction principle, at the very least since Lawvere's categorical axiomatization of natural numbers [Law64]. But a lot of mathematical structures, be they algebraic or topological, admit a free model, and it is also known that those that can be said to define an induction principle look very much like free algebras for an absolutely free algebraic theory (a theory which is given by a choice of operations, but no equations between them), or free algebras for the very closely related concept of a free monad generated by an endofunctor on sets. Here we are not very specific about what we mean by sets, only that we assume a mathematical universe which is sufficiently rich for a number of standard constructions; it can be the naive set theory of current mathematical practice, or a topos, or a version of Martin-Löf theory. But in particular the framework of dependent type theory inaugurated by Martin-Löf has been very useful for providing different ways for getting well-formalized notions of "sets", and moreover the expressivity of dependent types has led to very rich and general notions of "absolutely free universal algebra" which allows the expression of elaborate inference rules and induction principles. There is already a sizeable amount of literature on this [Dyb97, PM93, PS89], for which the recent paper by Dybjer and Setzer [DS99] gives simple and general principles and a bibliography. The main thrust in that kind of generalized universal algebra is the obtention of more powerful notions of *signature*, which make use of the expressive power of dependent types, in order to define operations (or more appropriately, constructors) over them, whereas the other side of universal algebra, the presence of *equations* between these operations, has to be almost completely absent, because they are very dangerous with respect to induction rules. But this is here that the formalism of dependent types is useful: everything one can do in dependent types can be coded in, say, higher order logic, but at the price of the frequent use of the *equality predicate*, whereas it can be avoided when dependent types are used.

But in general the structures considered are in the world of sets, or should be thought of as sets. There is one important class of inductive definitions which could benefit from a universal algebraic treatment, and that is the ones that construct posets. Inductive posets are found in many places in logic and computer science. For example everybody knows that the ordinal ω is the free poset generated by a constant and an inflationary unary operator, successor. Also various kinds of ordinal notations, like tree ordinals [CT96, Sim93] can be generated by using induction. Of course order structures are very important in logic since they allow things to be compared for size. But generalized calculi for inductive ordered structures are still in their infancy, if not practically nonexistent[1].

The usual practice when constructing an order is to generate the underlying set as an inductive structure, and then define the order structure on it afterwards by using the available induction scheme. But this prevents the order structure itself from being used for induction, for example by having constructors that can only be applied to linearly ordered n-uples of elements.

*Loria, Callligramme project, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France,{`bonfante,lamarche`}`@loria.fr`.

[1]The final version of this paper will contain a discussion of the work by Pitts, Crole, et al. and why it cannot be said to be induction on posets.

Category theorists have known for years how to generalize universal algebra to many other universes than sets. There are actually two general approaches, namely that of monads, and that of algebraic theories, and while they are not identical in their methods and their results, there are large areas of application where they are interchangeable. One important generalization of the universe of sets where these techniques give rise to very useful and general techniques is that of enriched category theory over a complete symmetrical monoidal closed category (SMC). In this context, given a SMC as a base universe, an object of it can be considered either as the underlying set of an algebra, *and* as an arity for an operation. The idea that in category theory sets could be replace by any SMC dates back to Lawvere's seminal paper [Law73], and Kelly's general textbook is a good sourceon techniques needed to do this [Kel82]. In addition, the general theory of enriched universal algebra over an SMC has been developed by Kelly and Power [KP93]. There are also abstract notions of size (say when the SMC is locally presentable [GU71, KP93]), that allows one to consider the equivalent, say, of finitary operations.

The category of posets and monotone functions looks at first like the right kind of category for this kind of universal algebra, being cartesian closed (and thus SMC) and locally presentable. So an arity for an operation/constructor can be any finite poset like the ordinal n mentioned above. One can also have infinitary arities, like ω, and construct the free poset over a given set of the operations with given arities, with additional equation-like conditions that force not identification of elements (pretty dangerous for induction, as we have said), but enrich the order structure between the elements (an example being decreeing inflationarity of a unary operation).

But the results are disappointing in practice. There are many reasons for this, but they all revolve around the fact that the structure of poset does not merge well at all with the internal logic of inductive definitions. In other words, induction schemes are not easy to express, when we think in terms of posets. Because of transitivity, pairs of elements are related too easily in the free structure, independently of the control of induction, so to speak. Also, dependent types and posets are not on friendly terms at all. This may be because, if the category of posets is cartesian closed, unlike sets it is not *locally cartesian closed*, or more generally *relatively locally cartesian closed* [Pit89], which is the kind of structure that allows the direct use of dependent type theory.

In this paper we propose a solution to this problem, by using universal algebra not on posets, but on the category of binary relations and that of multigraphs, which are to binary relations what categories are to posets. Multigraphs (oriented graphs with possibly many edges between pairs of vertices, often simply called *graphs*) can be thought of as the constructive analog of binary relations, where an arrow betwen two vertices is a proof that the vertices are related. This permits a direct translation of this paper into type theory, to be described in further work.

There already has been some work on universal algebra over graphs, notably by Burroni [Bur81], mostly for showing that some classes of categories with structure, like categories with finite limits, or elementary toposes, are monadic over graphs. Another area where universal algebra over graphs has shown itself to be important is higher-order category theory, and Burroni's higher-order generalisation of the theory of automata and languages [Bur93]. But as far as we know this work could always be formulated in the language of ordinary (i.e., set-enriched) monads.

The main technical innovation of our approach is the use of a novel SMC structure on multigraphs, which restricts to ordinary oriented graphs (binary relations), and seems not to have been noticed by computer scientists so far[2]. This structure is what permits binary relations/graphs to be considered not only as arities, which is already possible in the theory of ordinary monads, due to Linton's classic result [Lin69], but in addition allowing us to define operations/constructors with these arities that act as *monotone* operations and not just functions on the sets of vertices. The aforementioned categories of graphs and multigraphs do have a standard SMC structure (they are cartesian closed, and binary relations form a full subcategory of multigraphs), but whether X, Y are graphs or multigraphs, the function spaces $X \Rightarrow Y$ does not have as "underlying set" (i.e. set of vertices) the set $\mathrm{Hom}(X, Y)$, which makes a kind of graph-enriched universal algebra possible but prevents the ordinary naive use of graphs as arities. In order to get this good correspondence between underlying sets of function spaces and sets of morphisms, one has to resort to the cartesian closed category of *reflexive* graphs or multigraphs. These categories (and the enrichment associated

[2] A query on the "categories" list server a few weeks ago brought no information whatsoever about its possible previous appearance in the literature.

to their cartesian closed structures) have also been studied, due in particular to their geometrical interest [Law89, Bro92]. but reflexivity is a hindrance for us, just like transitivity (see above) because it complicates the inductive construction by automatically adding the relation $a \leq a$ for every new object a we construct. Then these proofs/ordered pairs interact with the construction process, complicating things appreciably if not beyond hope. By far the simplest way we have found to get induction principles for ordered sets is to use the SMC structure on multigraphs we have mentioned.

So in this paper we define a class of universal algebraic structures over that SMC structure such that their free algebras have a natural structure of "constructive posets", in which are the ordinary way of defining posets inside depedent type theory; in other words there is a two-place predicate which represents order, but it is not interpreted as a relation, but as the graph structure. In addition, we give results that say that the constructive posets thus constructed by induction are "the posets we think they should be", by showing that their poset collapse are the free posets satisfying the same induction principles.

Our treatment is elementary and needs only the most basic knowledge of category theory. One reason things are easier here than in general enriched universal algebra is that our SMC universe has a very simple set of generators, namely the point and one-vertex graph. Our mathematical base universe is naive sets, but we make everything explicit enough for easy translation into type theory.

1 The category of multigraphs

Definition 1.1. A *multigraph* G is a 4-tuple $G = (|G|, R_G, dom_G, codom_G)$, where $|G|$ and R_G are sets, dom_G and $codom_G$ two functions $R_G \overset{dom_G}{\underset{codom_G}{\rightrightarrows}} |G|$. Usually, $|G|$ is called the set of *nodes*, or equivalently of *vertices*, R_G is the set of *arrows* or *edges*. This is what Mac Lane [ML91] calls a graph, and we will often follow this practice.

By $p : x - y \in G$, we mean that p is an edge from x to y in the multigraph G. More precisely, $p \in R_G$ and $dom_G(p) = x$ and $codom_G(p) = y$. We will forget the subscript G for domains and codomains when it is not necessary.

A *homomorphism* $G \to G'$ of multigraphs is a pair of functions $(|f|, R_f)$, $|f| : |G| \to |G'|$, $R_f : R_G \to R_{G'}$ such that the diagram commute.

$$\begin{tikzcd}
|G| \arrow[d, "|f|"'] & R_G \arrow[l, "dom_G"'] \arrow[r, "codom_G"] \arrow[d, "R_f"'] & |G| \arrow[d, "|f|"] \\
|G'| & R_{G'} \arrow[l, "dom_{G'}"'] \arrow[r, "codom_{G'}"] & |G'|
\end{tikzcd}$$

We will simplify the notation by taking the symbol f for both $|f|$ and R_f, in functor style.

$\mathcal{M}$ is the category whose objects are the multigraphs and arrows the homomorphisms of multigraphs. They compose under the usual (and expected) composition, and identities are given by the identities on the underlying sets.

Definition 1.2. A graph is said to be *skeletal* when there is at most one edge between two vertices. Naturally these are also very often called directed, or oriented *graphs*, but we will always add the term skeletal to avoid all confusion in this paper. They form a full subcategory $\mathcal{S}$ such that the inclusion has a left adjoint $(\breve{-})$: given $G \in \mathcal{M}$, $\breve{G}$ is obtained by identifying all the edges of G between a pair of vertices to a single one.

We recall that a pre-order is a skeletal graph which is reflexive and transitive, and that an order is a pre-order for which antisymmetry holds. Pre-orders and orders form full subcategories of $\mathcal{M}$ and $\mathcal{S}$.

Example 1.1. In particular, we consider four particular (skeletal) graphs:

$$\begin{tikzcd}
\mathbf{0}: & & \mathbf{I}: & \bullet & \mathbf{2}: & 0 \arrow[r] & 1 \\
& & \mathbf{1}: & 0 \arrow[loop above, "1_0"] & & &
\end{tikzcd}$$

As each vertex k in a graph G may be identified with the unique morphism of graph $\mathbf{I} \xrightarrow{k} G$ which maps $\bullet \mapsto k$, we use the same notation for both. Note also that $\mathbf{1}$ is the terminal object in $\mathcal{M}$.

1.1 A symmetric monoidal closure structure in multigraphs

It is well-known that the category of (multi)graphs is cartesian closed, being a topos of presheaves, but the problem for us with that structure is that the set of vertices of the arrow object $X \Rightarrow Y$ is not the set of all morphisms $X \to Y$.

Definition 1.3. Given two multigraphs X and Y, we define $X \otimes Y$ to be:

- $|X \otimes Y| = |X| \times |Y|$,
- $\begin{cases} (u,v) : (x,y) - (x',y') & \text{if } u : x - x' \text{ and } v : y - y' \\ (=_x, v) : (x,y) - (x,y') & \text{if } v : y - y' \\ (u, =_y) : (x,y) - (x',y) & \text{if } u : x - x' \end{cases}$

  We suppose here that for all x, the symbol $=_x$ is "fresh".

In fact, $\otimes$ can be extended into a functor as follows. Suppose that $f : X \to Y$ and $g : Z \to T$ are two graph morphisms. We build the graph morphism: $f \otimes g : X \otimes Z \to Y \otimes T$ by:

- $(f \otimes g)(x,z) = (f(x), g(z))$,
- - $(f \otimes g)((p,q) : (x,z) - (x',z')) = (f(p), g(q))$,
  - $(f \otimes g)((=_x, q) : (x,z) - (x,z')) = (=_{f(x)}, g(q))$,
  - $(f \otimes g)((p, =_z) : (x,z) - (x',z)) = (f(p), =_{g(z)})$.

Proposition 1.1.

(i) $\mathbf{I} \otimes X \simeq X \simeq X \otimes \mathbf{I}$,

(ii) $X \otimes Y \simeq Y \otimes X$,

(iii) $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$.

These isomorphisms, obviously natural in all variables, are exactly the usual isos between the underlying (cartesian products of) sets. As a consequence, the tensor $\otimes$ makes the category of multigraphs a symmetric monoidal category.

Definition 1.4. Given two graphs X and Y, we define $X \multimap Y$ to be:

- $|X \multimap Y| = \mathcal{M}(X, Y)$.
- An edge between $\mathbf{u}$ and $\mathbf{v}$ in $X \multimap Y$ is a pair $\mathbf{p} = (\mathbf{p}^+, \mathbf{p}^\star)$ of functions such that for all $k, k' \in X$, $q : k - k' \in X$, we have: $\mathbf{p}^+_k : \mathbf{u}_k - \mathbf{v}_k \in Y$ and $\mathbf{p}^\star_q : \mathbf{u}_k - \mathbf{v}_{k'}$. If we represent the edges as arrows:

$$\begin{tikzcd}
\mathbf{u}_{k'} \arrow[r, "\mathbf{p}^+_{k'}"] & \mathbf{v}_{k'} \\
\mathbf{u}_k \arrow[u, "\mathbf{u}_q"] \arrow[ur, "\mathbf{p}^\star_q"] \arrow[r, "\mathbf{p}^+_k"'] & \mathbf{v}_k \arrow[u, "\mathbf{v}_q"']
\end{tikzcd}$$

We often use bold fonts for elements in $X \multimap Y$, in vector style. In that case, we note the arguments as indices.

Proposition 1.2.

(i) $E \multimap (D \multimap X) \simeq (E \otimes D) \multimap X$, the isomorphism being natural in all variables.

(ii) $\mathbf{I} \multimap X \simeq X$,

(iii) $\mathbf{0} \multimap X \simeq \mathbf{1}$.

The proof is just curryfication applied to multigraphs.

In other words we have proved that $(\mathcal{M}, \otimes, \multimap)$ is a symmetric monoidal closed category.

Recall the following well-known constructions. Given a graph $D \in \mathcal{M}$, the functor $D \multimap (-) : \mathcal{M} \to \mathcal{M}$ is:

- $X \mapsto (D \multimap X)$,
- given $f : X \to Y$, the morphism $D \multimap f : (D \multimap X) \to (D \multimap Y)$ is: $\mathbf{u} \mapsto f \circ \mathbf{u}$.

Proposition 1.3. $D \multimap (-)$ restricts to skeletal graphs, pre-orders and orders.

Proof. It is a direct consequence of the definitions. $\square$

There is also a contravariant functor $(-) \multimap D : \mathcal{M}^{op} \to \mathcal{M}$:

- $X \mapsto (X \multimap D)$,
- given $f : X \to Y$, the morphism $f \multimap D : (Y \multimap D) \to (X \multimap D)$ is: $\mathbf{u} \mapsto \mathbf{u} \circ f$.

1.2 Constructions in multigraphs

Definition 1.5. The category $\mathcal{M}$ has push-outs. In particular, given a multigraph G and a vertex $k \in G$, the pushout of the diagram $\mathbf{2} \xleftarrow{1} \mathbf{I} \xrightarrow{k} G$, is the graph $(\bullet \to_k G)$ composed by obtained by adding to G a new vertex $(\bullet)$ and a new arrow called k from $\bullet$ to $k \in |G|$. Or:

- $|(\bullet \to_k G)| = |G| + \{\bullet\}$,
- $R_{(\bullet \to_k G)} = R_G + \{k\}$, the elements in R_G have the domain and the codomain they had in G, $dom(k) = \bullet$, $codom(k) = k$.

It is easy to see that given two graph morphisms $h : \mathbf{2} \to H$ and $g : G \to H$ such that $g \circ k = h \circ 1$, there is a unique graph morphism f such that:

$$\begin{tikzcd}
 & G \arrow[dr, hook, "G"] \arrow[drr, bend left, "g"] & & \\
\mathbf{I} \arrow[ur, "k"] \arrow[dr, "1"'] & & (\bullet \to_k G) \arrow[r, "f"] & H \\
 & \mathbf{2} \arrow[ur, "\left\{\begin{smallmatrix} 0 \mapsto \bullet \\ 1 \mapsto k \end{smallmatrix}\right\}"] \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

Definition 1.6. Given a diagram $D \xleftarrow{l} K \xrightarrow{r} E$, we construct the graph $[DKE]$ (we forget to denote the two morphism l and r for the simplicity of notation, but they belong to the definition of $[DKE]$).

- $|[DKE]| = |D| + |E|$, the disjoint sum of $|D|$ and $|E|$,
- $R_{[DKE]} = R_D + R_E + |K| + R_K$. The domain and the codomain are defined as follows. If $p \in R_D$ or $p \in R_E$, p has the domain and the codomain it had in D or E; if $p \in |K|$, $dom(p) = l(p)$, $codom(p) = r(p)$; if $p \in R_K$, $dom(p) = dom_D(l(p))$, $codom(p) = codom_E(r(p))$.

In fact, the reason why we define $[DKE]$ is that it is the colimit of the diagram:

$$\begin{tikzcd}
 & K \arrow[dl, "l"'] \arrow[dr, "K\otimes 0"] & & K \arrow[dl, "K\otimes 1"'] \arrow[dr, "r"] & \\
D & & K \otimes \mathbf{2} & & E
\end{tikzcd}$$

It can also be seen as a weighted colimit [Kel82].

1.3 Universal Algebra on $\mathcal{M}$

Let D be a set. We can think of D as an arity, and given another set X, of an *operation of arity D on X* as a function $f\colon X^D \to X$. Usually D is finite, but category theorists have known for years that this needs not be so, and that there are interesting examples where D has to be infinite. So we can do the same in $\mathcal{M}$. If D, X are now *graphs*, we can define an *operation of arity D* as a graph morphism $D \multimap X \to X$. But this definition is too restrictive. The problem is when we want to compose operations. In sets, if we have an E-ary operation $g\colon X^E \to X$ and associate to it an E-ary function symbol, all we need to plug something in it is an E-indexed family of D-ary operations $(f_e\colon X^D \to X)_{e\in E}$; for some arity D; then it is trivial to construct the composite $g \circ \langle f_e\rangle_e\colon X^D \to X$. But the construction of E-uples of operations is not as simple in graphs. It involves the enriched universal property associated to the *cotensor* [Kel82] and the construction of (the equivalent to) the E-uple $\langle f_e\rangle_{e\in E}$ involves the internal (i.e., arrow) structure of the graph E. It is thus natural and simpler to define an *operation of arity D and co-arity E on X* as a graph morphism $D \multimap X \to E \multimap D$. Then composition of operations is trivial. Furthermore, given a graph morphism $k\colon D \to E$ we think of $k \multimap X\colon E \multimap X \to D \multimap X$ as a generalised *projection*. This allows us to recover the components of an E-uple (when this makes sense, say when E is discrete), and also to implement the substitutions of variables necessary in universal algebra.

The theories we will present will have equations, but they will be restricted in the following way: they will only be of the form $f \circ g = g' \circ f'$, where f, f' are defined operations (operation symbols in the theory) and g, g' projections as above. This kind of equation is invisible in ordinary universal algebra. This allows us to conjecture that such theories are the enriched equivalent to absolutely free theories, but more work needs to be done on this.

This is all we will need on the theory of enriched universal algebra. The reader who wants to know more (in particular about the relationship with monads) should consult the very general paper by Kelly and Power [KP93].

Remark 1.1. A consequence of proposition 1.3 is that universal algebra can be done in the same way within the category of skeletal graphs or orders.

1.4 Constructive orders

Definition 1.7. A *constructive order* is a triple $(G, \mathtt{ref}(-), \mathtt{tr}(-,-))$ where G is a multigraph, $\mathtt{ref}(-)$: $|G| \to R_G$ and $\mathtt{tr}(-,-)$, a partial function $R_G \times R_G \to R_G$. Moreover, $\mathtt{ref}(-)$ and $\mathtt{tr}(-,-)$ verify:

- $\mathtt{ref}(-)$ is a *proof of reflexivity*: for all $b \in |G|$, $\mathtt{ref}(b) : b - b \in G$.
- There is also *transitivity*. Given $p : a - b \in G$ and $q : b - c \in G$, $\mathtt{tr}(p, q) : a - c \in G$.

In other words, $(G, \mathtt{ref}(-), \mathtt{tr}(-,-))$ is a category without the associativity and unit laws.

Remark 1.2. Note first that a constructive order which is a skeletal multigraph is an ordinary pre-order. As a consequence, if $(G, \mathtt{ref}_G(-), \mathtt{tr}_G(-,-))$ is a constructive order, $\check{G}$ is a preorder.

2 Building constructive orders

We propose here a particular construction on multigraphs.

Definition 2.1. A *Ψ-theory* is given by:

(i) a strictly ordered set $(I, \prec)$ called the set of *indices*. Furthermore, for all $\iota \in I$, we have an operator symbol Ψ_ι whose *arity* is a multigraph D_ι.

(ii) a subset $J \subset I$, up-closed under $\prec$, and for all $\jmath \in J$, a set $\Delta_\jmath \subset D_\jmath$. If $k \in \Delta_\jmath$, the operator symbol $\Psi_\jmath$ is said to be *inflationary* with respect to k. We will give a formal definition of inflationarity shortly.

(iii) for each pair $(\iota \prec \jmath)$, a diagram $D_\iota \xleftarrow{l_{\iota,\jmath}} K_{\iota,\jmath} \xrightarrow{r_{\iota,\jmath}} D_\jmath$. When it is clear which is the pair $(\iota \prec \jmath)$, we will simply write l and r instead of $l_{\iota,\jmath}$ and $r_{\iota,\jmath}$.

We suppose furthermore that:

(a) For all $\iota \prec \jmath \prec \kappa$, there are two graph morphisms $l_{\iota,\jmath,\kappa}$, $r_{\iota,\jmath,\kappa}$ such that $l_{\iota,\kappa} = l_{\iota,\jmath} \circ l_{\iota,\jmath,\kappa}$ and $r_{\iota,\kappa} = r_{\jmath,\kappa} \circ r_{\iota,\jmath,\kappa}$. Moreover, there is an arrow $p_{\iota,\jmath,\kappa} : r_{\iota,\jmath} \circ l_{\iota,\jmath,\kappa} - l_{\jmath,\kappa} \circ r_{\iota,\jmath,\kappa} \in K_{\iota,\kappa} \multimap D_\jmath$.

$$\begin{tikzcd}
& & K_{\iota,\kappa} \arrow[dl, dotted, "l_{\iota,\jmath,\kappa}"'] \arrow[dr, dotted, "r_{\iota,\jmath,\kappa}"] & & \\
& K_{\iota,\jmath} \arrow[dl, "l_{\iota,\jmath}"'] \arrow[dr, "r_{\iota,\jmath}"] & \Rightarrow & K_{\jmath,\kappa} \arrow[dl, "l_{\jmath,\kappa}"'] \arrow[dr, "r_{\jmath,\kappa}"] & \\
D_\iota & & D_\jmath & & D_\kappa
\end{tikzcd}$$

(b) For all $\iota \prec \jmath$, with $\iota \in J$ and all $k \in \Delta_\iota$, there is a vertex $k_{\iota,\jmath,k} \in K_{\iota,\jmath}$ and an arrow $p_{\iota,\jmath,k} : k - l(k_{\iota,\jmath,k})$. Moreover, $r(k_{\iota,\jmath,k}) \in \Delta_\jmath$. Observe that the hypothesis that J is closed by $\prec$ is used here.

$$\begin{tikzcd}
& \mathbf{I} \arrow[dl, "k"'] \arrow[dr, dotted, "k_{\iota,\jmath,k}"] & & \\
D_\iota & & K_{\iota,\jmath} \arrow[ll, "l"] \arrow[r, "r"] & D_\jmath
\end{tikzcd}$$

Definition 2.2. A Ψ-*algebra*, is a 4-tuple $(G, \Psi, \mathtt{infl}, \mathtt{comp})$ where G is a multigraph, Ψ, `infl` and `comp` are three families such that:

- For each $\iota \in J$ and $k \in \Delta_\iota$, the diagram commutes:

$$\begin{tikzcd}
D_\iota \multimap G \arrow[rrr, "\Psi_\iota"] & & & G \\
(\bullet \to_k D_\iota) \multimap G \arrow[u, "D_\iota \multimap G"] \arrow[rrr, "\mathtt{infl}_{\iota,k}"] \arrow[dr, "\bullet \multimap G"'] & & & \mathbf{2} \multimap G \arrow[u, "1 \multimap G"'] \arrow[dl, "0 \multimap G"] \\
& G \arrow[r, phantom, "="] & G &
\end{tikzcd}$$

  This diagram express the inflationarity of the operator Ψ_ι relatively to k. Roughly, it means that for all $\mathbf{u} : D_\iota \multimap G$, we have: $\mathbf{u}_k - \Psi_\iota \mathbf{u}$.

- For each $\iota \prec \jmath$, the diagram commutes:

$$\begin{tikzcd}
D_\jmath \multimap G \arrow[r, "\Psi_\jmath"] & G \\
{[D_\iota K_{\iota,\jmath} D_\jmath]} \multimap G \arrow[u, "D_\jmath \multimap G"] \arrow[d, "D_\iota \multimap G"] \arrow[r, "\mathtt{comp}_{\iota,\jmath}"] & \mathbf{2} \multimap G \arrow[u, "1 \multimap G"'] \arrow[d, "0 \multimap G"'] \\
D_\iota \multimap G \arrow[r, "\Psi_\iota"] & G
\end{tikzcd}$$

  This diagram allows comparisions between the operators Ψ_ι and $\Psi_\jmath$ whenever two vertices are comparable on some "subvectors" of their domains of definitions given by the $K_{\iota,\jmath}$.

A homomorphism of Ψ-algebra $(G, \Psi^G, \mathtt{infl}^G, \mathtt{comp}^G) \to (H, \Psi^H, \mathtt{infl}^H, \mathtt{comp}^H)$ is a morphism of graph $f : G \to H$ which respects the operations Ψ, `infl` and `comp`.

Remark 2.1. Because of proposition 1.3, the notion of Ψ-algebra restricts to skeletal graphs or orders.

2.1 A bit of syntax

The algebraic constructions we will do on graphs need a syntactical representation, and we have chosen to copy the practices of formal systems, like the sequent calculus, with the use of axioms and deduction rules, that are combined to give (potentially infinite) proof trees. So to every graph one assigns a type symbol, and to the vertices and arrows of the graphs there are constants (corresponding to the elements of a given graph) and variables.

Some deduction rules belong to the "pure calculus", being independent of the choice of Ψ-theory.

$$\frac{\mathbf{u} \in D \multimap X \quad p : k - k' \in D}{\mathbf{u}_p : \mathbf{u}_k - \mathbf{u}_{k'} \in X} \qquad \frac{\mathbf{u} : D \multimap X \quad \mathbf{v} : D \multimap X \quad \forall k \in D \vdash q_k^+ : \mathbf{u}_k - \mathbf{v}_k \quad \forall p : k - k' \in D \vdash q_p^\star : \mathbf{u}_k - \mathbf{v}_{k'}}{\lambda p.q : \mathbf{u} - \mathbf{v} : D \multimap X}$$

By $\forall p : k - k' \in D \vdash \mathbf{u}_k - \mathbf{v}_{k'}$, we mean that for all $p : k - k' \in D$ there is a tree which has the conclusion $q_p^\star : \mathbf{u}_k - \mathbf{v}_{k'}$. So, trees may have an infinite branching but each branch has to be finite.

In the second rule, we use $\lambda p.q$ to denote the pair of functions $(p^+, p^\star)$ in functor style.

2.2 Definition of the free Ψ-algebra

Given a multigraph B, we intend to build the free Ψ-algebra generated by B. For that, we define the multigraph $U = F(B)$ by the rules:

$$\frac{b \in B}{\Phi b \in U} \qquad \frac{\mathbf{u} : D_\iota \multimap U}{\Psi_\iota \mathbf{u} \in U}$$

The first rule is the universal embedding for vertices, the second one introduces an operator of arity D_ι, namely Ψ_ι.

The arrows are:

$$\frac{p : b - c \in B}{\Phi p : \Phi b - \Phi c \in U} \qquad \frac{p : \mathbf{u} - \mathbf{v} \in D_\iota \multimap U}{\Psi_\iota p : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v} \in U}\ (\iota \in I)$$

$$\frac{\mathbf{u} : D_\iota \multimap U \quad p : x - \mathbf{u}_k \in U}{\mathtt{infl}(p, \mathbf{u}, \iota, k) : x - \Psi_\iota \mathbf{u} \in U}\ (\iota \in J, k \in \Delta_\iota) \qquad \frac{\mathbf{u} : D_\iota \multimap U \quad \mathbf{v} : D_\jmath \multimap U \quad p : \mathbf{u} \circ l - \mathbf{v} \circ r \in K_{\iota,\jmath} \multimap U}{\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{v}}\ (\iota \prec \jmath)$$

The first rule is the universal embedding of arrows of B, the second rule is for the monotonicity of the operators Ψ_ι, the third deals with inflationarity and the fourth with comparisions.

The *form* of a vertex or an arrow is the last rule of its definition.

Theorem 2.1. U is the free Ψ-algebra generated by B. That is, given an other Ψ-algebra $(X, \Psi^X, \mathtt{infl}, \mathtt{comp})$ and a graph morphism $f : B \to X$, we have to prove the existence (and uniqueness) of a morphism of graph h such that the diagram commute:

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "\Phi"'] & X \\ & U \arrow[u, dotted, "h"'] \end{tikzcd}$$

Proof. We build h by induction on the construction of U,

- $h(\Phi b) = fb$,
- $h(\Psi_\iota \mathbf{u}) = \Psi_\iota^X(\lambda k : D_\iota.h(\mathbf{u}_k), \lambda q : k - k'.h(\mathbf{u}_q))$,
- $h(\Phi p : \Phi b - \Phi c) = fp : fb - fc$,
- $h(\Psi_\iota \mathbf{p} : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}) = \Psi_\iota^X(\lambda k.h(\mathbf{p}_k^+), \lambda q : k - k'.h(\mathbf{p}_q^\star)) : \Psi_\iota^X h(\mathbf{u}) - \Psi_\iota^X h(\mathbf{v})$,
- $h(\mathtt{infl}(p, \mathbf{u}, \iota, k))$. By induction, we construct $h(\mathbf{u}) : D_\iota \multimap X$ and $h(p) : h(x) - h(\mathbf{u}_k)$. We observe that the diagram commutes:

$$\begin{tikzcd}
& \mathbf{I} \arrow[dl, "k"'] \arrow[dr, "1"] & \\
D_\iota \arrow[dr, "h(\mathbf{u})"'] & & \mathbf{2} \arrow[dl, "h(p)"] \\
& X &
\end{tikzcd}$$

As $(\bullet \to_k D_\iota)$ is a pushout, there is a graph morphism $\mathbf{h} : (\bullet \to_k D_\iota) \to X$ such that $\texttt{infl}_{\iota,k}(\mathbf{h}) : h(x) - h(\mathbf{u})$. We define $h(\texttt{infl}(p, \mathbf{u}, \iota, k)) = \texttt{infl}_{\iota,k}(\mathbf{h}) : h(x) - h(\mathbf{u})$

- $h(\texttt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath))$. By induction, we construct three morphisms $h(\mathbf{u})$, $h(\mathbf{v})$ and $h(p)$. These make the diagram commute.

$$\begin{tikzcd}
& K_{\iota,\jmath} \arrow[dl, "l_{\iota,\jmath}"'] \arrow[dr, "K_{\iota,\jmath}\otimes 0"] & & K_{\iota,\jmath} \arrow[dl, "K_{\iota,\jmath}\otimes 1"'] \arrow[dr, "r"] & \\
D_\iota \arrow[drr, "h(\mathbf{u})"'] & & K_{\iota,\jmath} \otimes \mathbf{2} \arrow[d, "h(p)"] & & D_\jmath \arrow[dll, "h(\mathbf{v})"] \\
& & X & &
\end{tikzcd}$$

This gives us a (unique) morphism $\mathbf{h} : [D_\iota K_{\iota,\jmath} D_\jmath] \multimap X$ such that $\texttt{comp}(\mathbf{h}, h(\mathbf{u}), h(\mathbf{v}), \iota, \jmath) : \Psi_\iota^X h(\mathbf{u}) - \Psi_\jmath^X h(\mathbf{v})$. We define $h(\texttt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath)) = \texttt{comp}(\mathbf{h}, h(\mathbf{u}), h(\mathbf{v}), \iota, \jmath) : \Psi_\iota^X h(\mathbf{u}) - \Psi_\jmath^X h(\mathbf{v})$.

□

2.3 Constructive order structure preservation

We study now the particular case where B is a constructive order. To emphasize this, we use $\leq$ instead of $-$.

Theorem 2.2. If $(B, \texttt{ref}(-), \texttt{tr}(-,-))$ is a constructive order, then $U = F(B)$ inherits its constructive order structure.

Proof. First, we construct a reflexivity function $\texttt{Ref}$ on U. It is done by induction on the definition of the vertex x,

- $x = \Phi b$. As B is an order, $\texttt{ref}(b) : b \leq b \in B$. So,

$$\dfrac{\texttt{ref}(b) : b \leq b \in B}{\Phi\texttt{ref}(b) : \Phi b - \Phi b \in U}$$

$\texttt{Ref}(\Phi b) = \Phi\texttt{ref}(b)$

- $x = \Psi_\iota \mathbf{u}$ with $\mathbf{u} : D_\iota \multimap U$. In that case, by induction hypothesis, $p : \mathbf{u}_k - \mathbf{u}_k$. We conclude with:

$$\dfrac{\dfrac{\dfrac{\text{induction}}{\texttt{Ref}(\mathbf{u}_k) : \mathbf{u}_k - \mathbf{u}_k} \qquad \dfrac{q : k - k' : D_\iota \quad \mathbf{u} : D_\iota \multimap U}{\mathbf{u}_q : \mathbf{u}_k - \mathbf{u}_{k'}}}{(\lambda k.\texttt{Ref}(\mathbf{u}_k), \lambda q.\mathbf{u}_q) : \mathbf{u} - \mathbf{u}}}{\Psi_\iota(\lambda k.\texttt{Ref}(\mathbf{u}_k), \lambda q.\mathbf{u}_q) : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{u}}$$

$\texttt{Ref}(\Psi_\iota \mathbf{u}) = \Psi_\iota(\lambda k : D_\iota.\texttt{Ref}(\mathbf{u}_k), \lambda q : k - k'.\mathbf{u}_q)$

Now, it is time to construct the transitivity function $\texttt{Tr}$. We use for that an induction on the pair $((p : x - y), (q : y - z))$. Note that the number of possible pairs is limited due to the fact that y appears in both part of the inequalities,

- If $\left(\dfrac{p : b \leq c \in B}{\Phi p : \Phi b - \Phi c \in U}, \dfrac{q : c \leq d \in B}{\Phi q : \Phi c - \Phi d \in U}\right)$

  As B is an order, we have $\mathtt{tr}(p,q) : b \leq d$.

  This is summarized by: $\mathtt{Tr}(\Phi p, \Phi q) = \Phi(\mathtt{tr}(p,q))$.

- If $\left(\dfrac{p : b \leq c \in B}{\Phi p : \Phi b - \Phi c \in U}, \dfrac{q : \Phi c - \mathbf{u}_k \in U}{\mathtt{infl}(q, \mathbf{u}, \iota, k) : \Phi c - \Psi_\iota \mathbf{u} \in U}\right)$

  We have: $\cfrac{\cfrac{\text{induction}}{\mathtt{Tr}(\Phi(p), q) : \Phi b - \mathbf{u}_k}}{\mathtt{infl}(\mathtt{Tr}(\Phi(p), q), \mathbf{u}, \iota, k) : \Phi b - \Psi_\iota \mathbf{u}}$.

  $\mathtt{Tr}(\Phi(p), \mathtt{infl}(q, \mathbf{u}, \iota, k)) = \mathtt{infl}(\mathtt{Tr}(\Phi(p), q), \mathbf{u}, \iota, k)$

- If $\left(\dfrac{p : \mathbf{u} - \mathbf{v}}{\Psi_\iota p : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}}, \dfrac{q : \mathbf{v} - \mathbf{w}}{\Psi_\iota q : \Psi_\iota \mathbf{v} - \Psi_\iota \mathbf{w}}\right)$.

  We have: $\cfrac{\cfrac{\text{induction}}{\mathtt{Tr}(p, q) : \mathbf{u} - \mathbf{w}}}{\Psi_\iota \mathtt{Tr}(p, q) : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{w}}$.

  $\mathtt{Tr}(\Psi_\iota p, \Psi_\iota q) = \Psi_\iota \mathtt{Tr}(p, q)$

- If $\left(\dfrac{p : \mathbf{u} - \mathbf{v}}{\Psi_\iota p : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}}, \dfrac{q : \Psi_\iota \mathbf{v} - \mathbf{w}_k}{\mathtt{infl}(q, \mathbf{w}, \iota', k) : \Psi_\iota \mathbf{v} - \Psi_{\iota'} \mathbf{w}}\right)$.

  We have: $\cfrac{\cfrac{\text{induction}}{\mathtt{Tr}(\Psi_\iota p, q) : \Psi_\iota \mathbf{u} - \mathbf{w}_k}}{\mathtt{infl}(\mathtt{Tr}(\Psi_\iota p, q), \mathbf{w}, \iota', k') : \Psi_\iota \mathbf{u} - \Psi_{\iota'} \mathbf{w}}$.

  $\mathtt{Tr}(\Psi_\iota p, \mathtt{infl}(q, \mathbf{w}, \iota, k)) = \mathtt{infl}(\mathtt{Tr}(\Psi_\iota p, q), \mathbf{w}, \iota', k')$

- If $\left(\dfrac{p : \mathbf{u} - \mathbf{v}}{\Psi_\iota p : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}}, \dfrac{q : \mathbf{v} \circ l - \mathbf{w} \circ r}{\mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \iota, \jmath) : \Psi_\iota \mathbf{v} - \Psi_\jmath \mathbf{w}}\right)$.

  We have:

  $$\cfrac{\cfrac{\cfrac{p : \mathbf{u} - \mathbf{v}}{p \circ l : \mathbf{u} \circ l - \mathbf{v} \circ l} \quad q : \mathbf{v} \circ l - \mathbf{w} \circ r}{\mathtt{Tr}(p \circ l, q) : \mathbf{u} \circ l - \mathbf{w} \circ r}}{\mathtt{comp}(\mathtt{Tr}(p \circ l, q), \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{w}}$$

  $\mathtt{Tr}(\Psi_\iota p, \mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \iota, \jmath)) = \mathtt{comp}(\mathtt{Tr}(p \circ l, q), \mathbf{u}, \mathbf{w}, \iota, \jmath)$.

- If $\left(\dfrac{p : x - \mathbf{v}_k}{\mathtt{infl}(p, \mathbf{v}, \iota, k) : x - \Psi_\iota \mathbf{v}}, \dfrac{q : \mathbf{v} - \mathbf{w}}{\Psi_\iota q : \Psi_\iota \mathbf{v} - \Psi_\iota \mathbf{w}}\right)$.

  We have: $\cfrac{\cfrac{\text{induction}}{\mathtt{Tr}(p, q_k) : x - \mathbf{w}_k}}{\mathtt{infl}(\mathtt{Tr}(p, q_k), \mathbf{w}, \iota, k) : x - \Psi_\iota \mathbf{w}}$.

  $\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), \Psi_\iota q) = \mathtt{infl}(\mathtt{Tr}(p, q_k), \mathbf{w}, \iota, k)$

- If $\left(\dfrac{p : x - \mathbf{v}_k}{\mathtt{infl}(p, \mathbf{v}, \iota, k) : x - \Psi_\iota \mathbf{v}}, \dfrac{q : \Psi_\iota \mathbf{v} - \mathbf{w}_{k'}}{\mathtt{infl}(q, \mathbf{w}, \iota', k') : \Psi_\iota \mathbf{v} - \Psi_{\iota'} \mathbf{w}}\right)$.

  We have: $\cfrac{\cfrac{\text{induction}}{\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), q) : x - \mathbf{w}_{k'}}}{\mathtt{infl}(\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), q), \mathbf{w}, \iota', k') : x - \Psi_{\iota'} \mathbf{w}}$.

  $\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), \mathtt{infl}(q, \mathbf{w}, \iota', k')) = \mathtt{infl}(\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), q), \mathbf{w}, \iota', k')$

- If $(\dfrac{p : x - \mathbf{v}_k}{\mathtt{infl}(p, \mathbf{v}, \iota, k) : x - \Psi_\iota \mathbf{v}}, \dfrac{q : \mathbf{v} \circ l - \mathbf{w} \circ r}{\mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \iota, \jmath) : \Psi_\iota \mathbf{v} - \Psi_\jmath \mathbf{w}})$.

We have:
$$\dfrac{\dfrac{\dfrac{p : x - \mathbf{v}_k \quad \dfrac{p_{\iota,\jmath,k} : k - l(k_{\iota,\jmath,k})}{\mathbf{v}_{p_{\iota,\jmath,k}} : \mathbf{v}_k - \mathbf{v}_{l(k_{\iota,\jmath,k})}}}{\mathtt{Tr}(p, \mathbf{v}_{p_{\iota,\jmath,k}}) : x - \mathbf{v}_{l(k_{\iota,\jmath,k})}} \quad \dfrac{q : \mathbf{v} \circ l - \mathbf{w} \circ r}{q_{k_{\iota,\jmath,k}} : \mathbf{v}_{lk_{\iota,\jmath,k}} - \mathbf{w}_{rk_{\iota,\jmath,k}}}}{\mathtt{Tr}(\mathtt{Tr}(p, \mathbf{v}_{p_{\iota,\jmath,k}}), q_{k_{\iota,\jmath,k}}) : x - \mathbf{w}_{rk_{\iota,\jmath,k}}}}{\mathtt{infl}(\mathtt{Tr}(\mathtt{Tr}(p, \mathbf{v}_{p_{\iota,\jmath,k}}), q_{k_{\iota,\jmath,k}}), \mathbf{w}, \jmath, rk_{\iota,\jmath,k}) : x - \Psi_\jmath \mathbf{w}}$$
$\mathtt{Tr}(\mathtt{infl}(p, \mathbf{v}, \iota, k), \mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \iota, \jmath)) = \mathtt{infl}(\mathtt{Tr}(\mathtt{Tr}(p, \mathbf{v}_{p_{\iota,\jmath,k}}), q_{k_{\iota,\jmath,k}}), \mathbf{w}, \jmath, rk_{\iota,\jmath,k})$

- If $(\dfrac{p : \mathbf{u} \circ l - \mathbf{v} \circ r}{\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{v}}, \dfrac{q : \mathbf{v} - \mathbf{w}}{\Psi_\jmath q : \Psi_\jmath \mathbf{v} - \Psi_\jmath \mathbf{w}})$.

We have:
$$\dfrac{\dfrac{q : \mathbf{u} \circ l - \mathbf{v} \circ r \quad \dfrac{q : \mathbf{v} - \mathbf{w}}{q \circ r : \mathbf{v} \circ r - \mathbf{w} \circ r}}{\mathtt{Tr}(p, q \circ r) : \mathbf{u} \circ l - \mathbf{w} \circ r}}{\mathtt{comp}(\mathtt{Tr}(p \circ l, q), \mathbf{u}, \mathbf{w}, \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{w}}$$
$\mathtt{Tr}(\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, l, r), \Psi_\jmath q) = \mathtt{comp}(\mathtt{Tr}(p \circ l, q), \mathbf{u}, \mathbf{w}, \iota, \jmath)$

- If $(\dfrac{p : \mathbf{u} \circ l - \mathbf{v} \circ r}{\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{v}}, \dfrac{q : \Psi_\jmath \mathbf{v} - \mathbf{w}_{k'}}{\mathtt{infl}(q, \mathbf{w}, \iota', k') : \Psi_\jmath \mathbf{v} - \Psi_{\iota'} \mathbf{w}})$.

We have: $\dfrac{\dfrac{\text{induction}}{\mathtt{Tr}(\Psi_\iota p, q) : \Psi_\iota \mathbf{u} - \mathbf{w}_{k'}}}{\mathtt{infl}(\mathtt{Tr}(\Psi_\iota p, q), \mathbf{w}, \iota', k') : \Psi_\iota \mathbf{u} - \Psi_{\iota'} \mathbf{w}}$.

$\mathtt{Tr}(\Psi_\iota p, \mathtt{infl}(q, \mathbf{w}, \iota, k)) = \mathtt{infl}(\mathtt{Tr}(\Psi_\iota p, q), \mathbf{w}, \iota', k')$

- If $(\dfrac{p : \mathbf{u} \circ l_{\iota,\jmath} - \mathbf{v} \circ r_{\iota,\jmath}}{\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath) : \Psi_\iota \mathbf{u} - \Psi_\jmath \mathbf{v}}, \dfrac{q : \mathbf{v} \circ l_{\jmath,\kappa} - \mathbf{w} \circ r_{\jmath,\kappa}}{\mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \jmath, \kappa) : \Psi_\jmath \mathbf{v} - \Psi_\kappa \mathbf{w}})$.

We have :
$$\dfrac{\dfrac{\dfrac{\dfrac{p : \mathbf{u} \circ l_{\iota,\jmath} - \mathbf{v} \circ r_{\iota,\jmath}}{pl_{\iota,\jmath,\kappa} : \mathbf{u}_{l_{\iota,\jmath} l_{\iota,\jmath,\kappa}} - \mathbf{v}_{r_{\iota,\jmath} l_{\iota,\jmath,\kappa}}} \quad \dfrac{p_{\iota,\jmath,\kappa} : r_{\iota,\jmath} l_{\iota,\jmath,\kappa} - l_{\jmath,\kappa} r_{\iota,\jmath,\kappa}}{\mathbf{v} p_{\iota,\jmath,\kappa} : \mathbf{v}_{r_{\iota,\jmath} l_{\iota,\jmath,\kappa}} - \mathbf{v}_{l_{\jmath,\kappa} r_{\iota,\jmath,\kappa}}}}{\mathtt{Tr}(pl_{\iota,\jmath,\kappa}, \mathbf{v} p_{\iota,\jmath,\kappa}) : \mathbf{u}_{l_{\iota,\jmath} l_{\iota,\jmath,\kappa}} - \mathbf{v}_{l_{\jmath,\kappa} r_{\iota,\jmath,\kappa}}} \quad \dfrac{q : \mathbf{v} \circ l_{\jmath,\kappa} - \mathbf{w} \circ r_{\jmath,\kappa}}{qr_{\iota,\jmath,\kappa} : \mathbf{v}_{l_{\jmath,\kappa} r_{\iota,\jmath,\kappa}} - \mathbf{w}_{r_{\jmath,\kappa} r_{\iota,\jmath,\kappa}}}}{\mathtt{Tr}(\mathtt{Tr}(pl_{\iota,\jmath,\kappa}, \mathbf{v} p_{\iota,\jmath,\kappa}), qr_{\iota,\jmath,\kappa}) : \mathbf{u}_{l_{\iota,\jmath} l_{\iota,\jmath,\kappa}} - \mathbf{w}_{r_{\jmath,\kappa} r_{\iota,\jmath,\kappa}}}}{\mathtt{comp}(\mathtt{Tr}(\mathtt{Tr}(pl_{\iota,\jmath,\kappa}, \mathbf{v} p_{\iota,\jmath,\kappa}), qr_{\iota,\jmath,\kappa}), \mathbf{u}, \mathbf{w}, \iota, \kappa) : \Psi_\iota \mathbf{u} - \Psi_\kappa \mathbf{w}}$$
$\mathtt{Tr}(\mathtt{comp}(p, \mathbf{u}, \mathbf{v}, \iota, \jmath), \mathtt{comp}(q, \mathbf{v}, \mathbf{w}, \jmath, \kappa)) = \mathtt{comp}(\mathtt{Tr}(\mathtt{Tr}(pl_{\iota,\jmath,\kappa}, \mathbf{v} p_{\iota,\jmath,\kappa}), qr_{\iota,\jmath,\kappa}), \mathbf{u}, \mathbf{w}\iota, \kappa)$

□

3 Applications, examples

We present now some examples of the construction presented before.

Example 3.1. If we consider the particular case where we take a unique inflationary unary constructor, we get a constructive order which represent the ordinal ω. More formally, take $I = J = \{\bullet\}$, with $D_\bullet = \Delta_\bullet = \mathbf{I}$. Here, we note by s the operator $\Psi_\bullet$, and the rule of inflationarity can be reduced to: $\dfrac{p : x - y}{\mathtt{infl}(p) : x - sy}$. We take $B = \mathbf{1}$, a unique element 0 which is reflexive. Now, an ordinal $n \in \omega$ is represented by:

$$\overline{n} = \underbrace{s(\cdots s(0) \cdots)}_{n \text{ times } s}$$

As intended, an arrow $p : \overline{n} - \overline{m}$ represents a proof that $n \leq m$. Indeed, one can observe that there is an arrow $p : n - m$ iff $\overline{n} \leq \overline{m}$. Simply, take

$$\underbrace{\texttt{infl}(\cdots(\texttt{infl}}_{m-n \text{ times } \texttt{infl}}(\underbrace{s(\cdots(s(s(1_0)))\cdots)}_{n \text{ times } s}\cdots) : \overline{n} - \overline{m}$$

One may note that there are in fact more than one proof of $n \leq m$. Indeed, one can exchange the use of the rules s and `infl` in the proof above. As a result, there are $\binom{m}{n}$ proofs that $n \leq m$.

Example 3.2. Tree ordinals are due to Dennis-Jones and Wainer [DJW83]; it is the set obtained by the three rules:

$$\frac{}{0 \in \Omega} \qquad \frac{\alpha \in \Omega}{s\alpha \in \Omega} \qquad \frac{f : \mathbb{N} \to \Omega}{f \in \Omega}$$

The order on tree ordinal is defined as the transitive closure of the smallest relation given by:

$$\frac{}{0 \leq \alpha}\ (\alpha \in \Omega) \qquad \frac{}{\alpha \leq s\alpha}\ (\alpha \in \Omega) \qquad \frac{f : \mathbb{N} \to \Omega}{\forall n \quad f(n) \leq f}$$

We take $I = J = \{\bullet, \circ\}$, with $\prec$ empty. $D_\bullet = \Delta_\bullet = \mathbf{I}$ and $D_\circ = \Delta_\circ = N$. Finally, we take $B = \mathbf{1}$.

Example 3.3. A third example is given by words. Suppose that M is an alphabet. M^* denotes the set of words over M. The sub-word order noted $\trianglelefteq$ is the transitive closure of:

$$\frac{}{w \trianglelefteq w}\ (w \in M^*) \qquad \frac{}{w \trianglelefteq aw}\ (w \in M^*, \quad a \in M) \qquad \frac{w \trianglelefteq w'}{aw \trianglelefteq aw'}\ (a \in M)$$

$\trianglelefteq$ is the free order obtained by taking $|M|$ inflationary unary operators. More precisely, we take $I = J = M$, $\prec$ being the discrete order, $D_m = \Delta_m = \mathbf{I}$ for all $m \in M$. A word w in M^* is represented by a vertex $\overline{w} \in F(\mathbf{1})$:

- $\overline{\epsilon} = 0$ with ϵ being the empty word,
- $\overline{aw} = \Psi_a(\overline{w})$, for all $a \in M$.

Example 3.4. But we can go even further: suppose we are given a signature Σ. $\mathcal{T}$ denotes the set of terms over Σ. That is, the least set:

$$\frac{f \in \Sigma}{f \in \mathcal{T}} \qquad \frac{f \in \Sigma \quad t_1, \cdots, t_n \in \mathcal{T}}{f(t_1, \cdots, t_n) \in \mathcal{T}}$$

On which, we define the embedding order by being the transitive closure of:

$$\frac{}{t_k \trianglelefteq f(t_1, \cdots, t_n)}\ (k \leq n) \qquad \frac{}{f(t_1, \cdots, t_{i-1}, t_i, t_{i+1}, \cdots, t_n) \trianglelefteq f(t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n)}$$

We take $I = J = \Sigma \times \mathbb{N}$, $D_{(f,n)} = \Delta_{(f,n)} = n$. Let $(f, n) \prec (f', n')$ iff $f = f'$ and $n < n'$. Finally, $B = \mathbf{1}$. A term is represented by:

- $\overline{f} = \Psi_{(f,0)}(0)$,
- $\overline{f(t_1, \cdots, t_n)} = \Psi_{(f,n)}(\overline{t_1}, \cdots, \overline{t_n})$

4 Poset normalisation

We have proved that the structure of constructive order is preserved by the construction of the free Ψ-algebra. Unfortunately, given an order B, in general $F(B)$ is not an order; indeed, recall the example of ω. In spite of this, we have the corollary to Theorem 2.2:

Corollary 4.1. Given an ordered set B, then, its skeletal collapse $\check{F(B)}$ is a pre-order. Thus, its order collapse is trivially an order, where the collapse of a pre-order is simply the identification of all elements x, y such that $x - y$ and $y - x$.

But, is it the free Ψ-algebra obtained within the category of orders? We conjecture that it is the case. But for the moment, we restrict our investigations to the particular case where we suppose that the set of indices is discrete. That is, there is no pair $\imath \prec \jmath \in I$. This simplifies our task as the rule `comp` does not apply anymore. In fact, taking into account this rule should be part of some future work.

Secondly, we suppose that each set Δ_ι contains at most one element. A justification of this hypothesis is the following. The transitivity implies that if there is a proof p of the form say: $\mathtt{infl}(q, \mathbf{v}, \iota, k) : x - \Psi_\iota \mathbf{v}$ and there is an other element k' greater than k in Δ_ι (ie. there is an arrow: $p : k - k' \in \Delta_\iota$), then, there is a second proof $\mathtt{infl}(\mathtt{Tr}(q, \mathbf{v}_p), \mathbf{v}, \iota, k') : x - \Psi_\iota \mathbf{v}$. It is to prevent these multiplicities that we add the hypothesis that each set Δ_ι contains at most one element.

Theorem 4.1. Let a Ψ-theory satisfy the two hypotheses just above. Given a poset B, then the order collapse of $U = \check{F}(B)$ is the free Ψ-algebra in the category of posets. This means that the constructive order $F(B)$ and its attached induction principle does correspond intuitively to the free poset we have in mind.

The remainder is dedicated to the proof of the theorem. It is done by normalising the free Ψ-algebra in $\mathcal{M}$, and showing that it is exactly the free Ψ-algebra in orders.

Definition 4.1. An arrow $p \in U$ is said to be *regular*:

- if $p = \Phi q$,
- if $p = \mathtt{infl}(q, \mathbf{v}, \iota, k)$ and q is regular,
- if $p = \Psi_\iota q : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}$ and
  - for all $k \in D_\iota$, q_k^+ is regular,
  - for all $r : k - k' \in D_\iota$, q_r^* is regular,
  - there is no arrows r such that $r : \Psi_\iota \mathbf{u} - \mathbf{v}_k$, with $k \in \Delta_\iota$.

Proposition 4.1. If there is an arrow $p : x - y$, then, there is a unique regular arrow $\widehat{p} : x - y$.

Proof. We construct the hat arrows by induction on p.

- $p = \Phi q : \Phi b - \Phi b'$. We define $\widehat{p} = p$ and observe that p is regular. Suppose that there is another arrow $p' : \Phi b - \Phi b'$. Due to the form of Φb and $\Phi b'$, p' is necessarily of the form $\Phi q'$. As B is an order, we have $q' = q$.
- $p = \mathtt{infl}(q, \mathbf{v}, \iota, k) : x - \Psi_\iota \mathbf{v}$. We define $\widehat{p} = \mathtt{infl}(\widehat{q}, \mathbf{v}, \iota, k)$ which is regular. Is it unique? Suppose the contrary. So, there is an other morphism $p' : x - \Psi_\iota \mathbf{v}$. There are two case, whether $x = \Phi b$ or $x = \Psi_\iota \mathbf{u}$.
  - $x = \Phi b$. In that case, p' is necessarily of the form $\mathtt{infl}(q', \mathbf{v}, \iota, k)$; by induction, we observe that $q' = \widehat{q}$.
  - $x = \Psi_\iota \mathbf{u}$. To be regular, p' is necessarily of the form $\mathtt{infl}(q', \mathbf{v}, \iota, k)$, we use the same argument than above.
- $p = \Phi_\iota q : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}$. There are two cases,
  - There is an arrow r such that $r : \Psi_\iota \mathbf{u} - \mathbf{v}_k$. In that case, we define $\widehat{p} = \mathtt{infl}(\widehat{r}, \mathbf{v}, \iota, k)$. If there is an other morphism $p' : \Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}$, it is necessarily of the form `infl` or Ψ_ι. But a morphism of the form Ψ_ι would not be regular. So, we have by induction the unicity.
  - In the other case, we define simply $\widehat{\Psi_\iota q} = \Psi_\iota(\lambda k : D_\iota . \widehat{q_k^+}, \lambda r : k - k' . \widehat{q_r^*})$. To see that it is uniquely determined, one has to note that an other morphism with the conclusion $\Psi_\iota \mathbf{u} - \Psi_\iota \mathbf{v}$ can not be of the form `infl`. So, induction applies which gives the unicity.

$\square$

Definition 4.2. A vertex in $x \in U$ is said to be *regular*:

- if $x = \Phi b$,
- if $x = \Psi_\iota \mathbf{u}$ and
  - for all $k \in D_\iota$, $\mathbf{u}_k$ is regular,
  - for all $p : k - k' \in D_\iota$, $\mathbf{u}_p$ is regular.

Lemma 4.1. There is no arrow p such that $p : \Psi_\iota \mathbf{u} - \mathbf{u}_k$.

Proof. By induction on the construction of p. $\square$

Proposition 4.2. As we did for arrows, we can build for any vertex $x \in U$ a unique regular element $\overline{x}$ such that there is a regular arrow $p_x : x - \overline{x}$ and an arrow $q_x : \overline{x} - x$.

Proof. By induction on the construction of x.

- If $x = \Phi b$. We take $\overline{\Phi b} = \Phi b$. We have:
  1- It is regular and
  2- $\Phi 1_b : \Phi b - \overline{\Phi b}$
  3- $\Phi 1_b : \overline{\Phi b} - \Phi b$.
  4- Suppose that y is an other regular element with $p' : \Phi b - y$ and $q' : y - \Phi b$. Due to the form of Φb, q' is necessarily of the form $\Phi q : \Phi c - \Phi b$. Thus, it is also the case of $p' = \Phi p : \Phi b - \Phi c$. As B is an order, we get $b = c$.

- $x = \Psi_\iota \mathbf{u}$. By induction, we note that for all $k \in D_\iota$ and all $p : k - k' \in D_\iota$, we have: $\overline{\mathbf{u}_k} \xrightarrow{q_{\mathbf{u}_k}} \mathbf{u}_k \xrightarrow{\mathbf{u}_p} \mathbf{u}_{k'} \xrightarrow{p_{\mathbf{u}_{k'}}} \overline{\mathbf{u}_{k'}}$. So, there is a (unique) regular arrow:

$$\mathtt{Tr}(q_{\mathbf{u}_k}, \widehat{\mathtt{Tr}(\mathbf{u}}_p, p_{\mathbf{u}_{k'}})) : \overline{\mathbf{u}_k} - \overline{\mathbf{u}_{k'}}$$

  We define

$$\overline{\Psi_\iota \mathbf{u}} = \Psi_\iota(\lambda k.\overline{\mathbf{u}_k}, \lambda p : k - k'.\mathtt{Tr}(q_{\mathbf{u}_k}, \widehat{\mathtt{Tr}(\mathbf{u}}_p, p_{\mathbf{u}_{k'}})))$$

  1- It is regular and
  2- We note that for all $k \in D_\iota$, $p_{\mathbf{u}_k} : \mathbf{u}_k - \overline{\mathbf{u}_k}$ and for all $p : k - k'$: $\mathtt{Tr}(\mathbf{u}_p, p_{\mathbf{u}_{k'}}) : \mathbf{u}_k - \overline{\mathbf{u}_{k'}}$. So, $\Psi_\iota(\lambda k.p_{\mathbf{u}_k}, \lambda p : \widehat{k - k'}.\mathtt{Tr}(\mathbf{u}_p, p_{\mathbf{u}_{k'}})) : \Psi_\iota \mathbf{u} - \overline{\Psi_\iota \mathbf{u}}$
  3- In the same way, we get: $\Psi_\iota(\lambda k.q_{\mathbf{u}_k}, \lambda p : \widehat{k - k'}.\mathtt{Tr}(q_{\mathbf{u}_k}, \mathbf{u}_p)) : \overline{\Psi_\iota \mathbf{u}} - \Psi_\iota \mathbf{u}$
  4- For the unicity, we study the four cases depending on the form of an other pair $p : \Psi_\iota \mathbf{u} - y$, $q : y - \Psi_\iota \mathbf{u}$. If p and q are of the form Ψ_ι, we use induction. In the three other case, the proof use lemma 4.1.

$\square$

$\overline{(-)}$ can be extended into a morphism of graph. Suppose that $f : x - y$, we define $\overline{f}$ to be the unique regular morphism $\overline{x} - \overline{y}$.

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, shift left, "p_x"] & y \arrow[d, shift left, "p_y"] \\
\overline{x} \arrow[r, "\overline{f}"] \arrow[u, shift left, "q_x"] & \overline{y} \arrow[u, shift left, "q_y"]
\end{tikzcd}$$

Proposition 4.3. $\overline{(-)}$ is a normalisation procedure. That is: $\overline{\overline{(-)}} = \overline{(-)}$. Moreover, one may observe that it is an order, order which is in one-one correspondence with the free Ψ-algebra of orders. As a consequence is the theorem 4.1.

Proof. By straightforward induction. $\square$

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