Interpolation problem with curvature constraints

Hafsa Deddi, Hazel Everett, Sylvain Lazard

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Abstract

We address the problem of controlling the curvature of a Bézier curve interpolating a given set of data. More precisely, given two points M and N, two directions $\vec{u}$ and $\vec{v}$, and a constant k, we would like to find two quadratic Bézier curves Γ_1 and Γ_2 joined with continuity G^1 and interpolating the two points M and N, such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$ respectively, the curvature is everywhere upper bounded by k, and some evaluating function, the length of the resulting curve for example, is minimized.

In order to solve this problem, we first need to determine the maximum curvature of quadratic Bézier curves. This problem was solved by Sapidis and Frey in 1992. Here we present a simpler formula that has an elegant geometric interpretation in terms of distances and areas determined by the control points. We then use this formula to solve several problems. In particular, we solve the variant of the curvature control problem in which Γ_1 and Γ_2 are joined with continuity C^1, where the length α between the first two control points of Γ_1 is equal to the length between the last two control points of Γ_2, and where α is the evaluating function to be minimized. We also study the variant where we require a continuity G^2, instead of C^1, at the junction point. Finally, given two endpoints of a quadratic Bézier curve Γ, we characterize the locus of control points such that the maximum curvature of Γ is prescribed.

1 Introduction

An important problem in CAGD is the construction of curves interpolating given sets of data that also satisfy constraints on their curvature. Such curves are visually pleasing and are said to be "fair" [1, 2]. Fair curves are also important in the design of highways, railways and trajectories of mobile robots (see [10] and [7]). In these applications, curvature continuous curves with bounded curvature are desirable. Constructing fair curves has been the subject of recent research; see, for example, [4, 5, 6, 8] for results about constraining the curvature at the endpoints, and [3, 9] for results about monotonicity of curvature.

In this paper we consider the problem of controlling the curvature along the whole length of a Bézier curve interpolating a given set of data. More precisely, given two points M and N, two directions $\vec{u}$ and $\vec{v}$, and a constant k, we want to find two quadratic Bézier curves Γ_1 and Γ_2 joined with continuity G^1 and interpolating the two points M and N, such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$ respectively, the curvature is everywhere upper bounded by k, and

*This work was started while the first two authors were at Université du Québec à Montréal, and the last author was at McGill University.

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some evaluating function, the length of the resulting curve for example, is minimized. We call this problem the *curvature control problem.*

In order to solve this problem, we first need to determine the maximum curvature of quadratic Bézier curves, that is, to find an exact formula in terms of the control points. Note that, for our problem, it is not sufficient to compute the maximum curvature of a particular Bézier curve using numerical methods. Note also that a quadratic Bézier curve is a parabola and, although it presents no special difficulties to compute the maximum curvature of a parabola in terms of the coefficients of its implicit equation, what we require is a formula in terms of the control points.

In [9], Sapidis and Frey give a formula for finding the maximum curvature for quadratic Bézier curves. In Section 2, we recall these results and present a simpler formula that has an elegant geometric interpretation in terms of distances and areas determined by the control points. We then use this formula to solve several problems. We first characterize the locus of points p_1 such that, with two given points p_0 and p_2 and a positive constant k, the quadratic Bézier curve with control points p_0, p_1 and p_2 has maximum curvature k. We then solve some variants of the curvature control problem. Definitions and motivations for these variants are presented in Section 4.1. We solve in Section 4.2 the version of the curvature control problem where Γ_1 and Γ_2 are joined with continuity C^1, where the length α between the two first control points of Γ_1 is equal to the length between the two last control points of Γ_2, and where α is the evaluating function to be minimized. In Section 4.3, we prove that if we require in the previous variant a continuity G^2 instead of C^1 at the junction point, then there exist non-degenerate data for which there is no solution to the curvature control problem. However, if a solution exists, we show how it can be computed.

Throughout the paper, curvature refers to non-signed curvature, unless otherwise indicated. We denote by $\|pq\|$ the distance between points p and q, and by " $\times$ " and " $\cdot$ " the outer and inner products, respectively, between two vectors.

2 Maximum curvature of quadratic Bézier curves

Let Γ be a quadratic Bézier curve with control points p_0, p_1 and p_2 (see Figure 1). Recall that Γ is defined for every t in $[0,1]$ by

$$\Gamma(t) = (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2.$$

Let $\mathcal{A}$ be the area of the control triangle $p_0p_1p_2$ and m be the midpoint of the segment p_0p_2. We assume that Γ does not degenerate into a line segment, i.e., p_0, p_1 and p_2 are not collinear.

In this section, we prove the following theorem:

Theorem 2.1 *The maximum curvature of a quadratic Bézier curve* Γ *is either equal to* $\|p_1m\|^3/\mathcal{A}^2$ *if* p_1 *lies strictly outside the two disks of diameter* p_0m *and* mp_2*, or is equal to* $\max\{\kappa_0, \kappa_1\}$ *where* $\kappa_0 = \mathcal{A}/\|p_0p_1\|^3$ *and* $\kappa_1 = \mathcal{A}/\|p_1p_2\|^3$ *are the curvature of* $\Gamma(t)$ *at the endpoints* $\Gamma(0)$ *and* $\Gamma(1)$*.*

Before proving Theorem 2.1, we recall the result by Sapidis and Frey [9] characterizing quadratic Bézier curves with monotone curvature.

Theorem 2.2 ([9]) *The quadratic Bézier curve* Γ *has monotone curvature if and only if one of the angles* $\angle(p_0p_1m)$ *and* $\angle(mp_1p_2)$ *is equal to or larger than* $\frac{\pi}{2}$*. In other words,* Γ *has monotone curvature if and only if* p_1 *lies on or inside one of the two circles having as diameter* p_0m *and* mp_2*.*

Sapidis and Frey also present in [9] the following expressions for the maximum curvature of quadratic Bézier curves. When the curvature is not monotone along Γ, then its maximum curvature

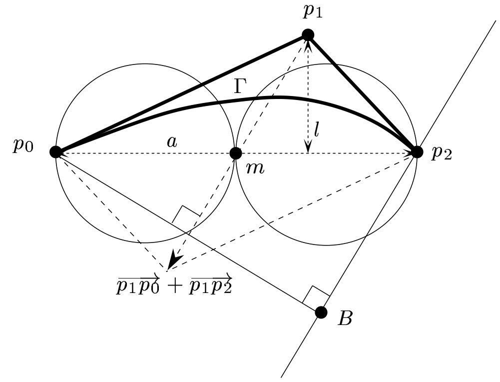

Figure 1: The quadratic Bézier curve has monotone curvature if and only if p_1 lies on or inside one of two circles.

is $4al/\|p_0B\|^3$, where (see Figure 1), a is the distance between p_0 and p_2, l is the distance between p_1 and the line joining p_0 and p_2, and $\|p_0B\|$ is the distance between p_0 and the line passing through p_2 and directed by $\overrightarrow{p_1p_0} + \overrightarrow{p_1p_2}$. When the curvature is monotone along Γ, its maximum is reached at one endpoint p_0 or p_2 of the curve, and is equal to $\frac{al}{2\|p_0p_1\|^3}$ or $\frac{al}{2\|p_1p_2\|^3}$ respectively.

We are now ready to prove Theorem 2.1. Note that the area $\mathcal{A}$ of the control triangle $p_0p_1p_2$ is equal to $al/2$. Thus, in order to prove Theorem 2.1, based on the results by Sapidis and Frey, it suffices to prove that $8\mathcal{A}/\|p_0B\|^3 = \|p_1m\|^3/\mathcal{A}^2$ or $2\mathcal{A} = \|p_1m\|.\|p_0B\|$. For completeness, we show how our result is derived from Theorem 2.2.

We assume first that p_1 lies strictly outside the two disks of diameter p_0m and mp_2. Thus, the curvature $\kappa(t)$, $t \in [0,1]$, of the quadratic Bézier curve Γ is not monotone by Theorem 2.2. It follows that the maximum curvature of Γ is obtained when the derivative of $\kappa(t)$ is zero.

The first and second derivatives of the Bézier curve Γ are

$$\begin{aligned} \Gamma'(t) &= 2((1-t)(p_1-p_0) + t(p_2-p_1)) \\ &= 2(p_1-p_0) + 2t(p_2 - 2p_1 + p_0) \qquad (1) \\ \Gamma''(t) &= 2(p_2 - 2p_1 + p_0). \qquad (2) \end{aligned}$$

The curvature of Γ at $\Gamma(t)$ is thus, for any $t \in [0,1]$,

$$\kappa(t) = \frac{|\Gamma'(t) \times \Gamma''(t)|}{\|\Gamma'(t)\|^3} = \frac{|4(p_1-p_0)\times(p_2-2p_1+p_0)|}{\|\Gamma'(t)\|^3} = \frac{|4(p_1-p_0)\times(p_2-p_1)|}{\|\Gamma'(t)\|^3},$$

giving

$$\kappa(t) = \frac{8\mathcal{A}}{\|\Gamma'(t)\|^3}, \qquad (3)$$

where $\mathcal{A} = |(p_1-p_0)\times(p_2-p_1)|/2$ is the area of the control triangle $p_0p_1p_2$. The derivative of $\kappa(t)$ is

$$\kappa'(t) = \frac{-24\mathcal{A}(\|\Gamma'(t)\|)'}{\|\Gamma'(t)\|^4} = \frac{-12\mathcal{A}(\|\Gamma'(t)\|^2)'}{\|\Gamma'(t)\|^5}.$$

Since we assumed that the Bézier curve Γ is not degenerate, p_0, p_1 and p_2 are not collinear and thus $\mathcal{A} \neq 0$. Thus, $\kappa'(t) = 0$ if and only if $(\|\Gamma'(t)\|^2)' = 0$, or alternatively, $\Gamma'(t)\cdot\Gamma''(t) = 0$. Using

Equations 1 and 2, we get

$$\begin{aligned}\Gamma'(t)\cdot\Gamma''(t) &= 4[(p_2-2p_1+p_0)t+(p_1-p_0)]\cdot[p_2-2p_1+p_0]\\ &= 4(\alpha t-\beta)\end{aligned}$$

where $\alpha = \|p_2-2p_1+p_0\|^2$ and $\beta = -(p_1-p_0)\cdot(p_2-2p_1+p_0)$.

Thus, the derivative of the curvature $\kappa(t)$ vanishes if and only if $t=\tau=\beta/\alpha$. Note that τ is in $(0,1)$ because the curvature of Γ is not monotone by assumption. Therefore, the maximum curvature along Γ is obtained for $t=\tau$.

Lemma 2.3 $\|\Gamma'(\tau)\| = \dfrac{2\mathcal{A}}{\|p_1m\|}$.

Proof: By Equation 1, the square of the first derivative of $\Gamma(t)$ at τ is

$$\begin{aligned}\|\Gamma'(\tau)\|^2 &= 4[(p_2-2p_1+p_0)\tau+(p_1-p_0)]^2 = 4(\alpha\tau^2-2\tau\beta+\|p_0p_1\|^2)\\ &= 4(\alpha\frac{\beta^2}{\alpha^2}-2\frac{\beta}{\alpha}\beta+\|p_0p_1\|^2) = \frac{4}{\alpha}(\alpha\|p_0p_1\|^2-\beta^2),\end{aligned}$$

where, as before, $\alpha = \|p_2-2p_1+p_0\|^2$ and $\beta=-(p_1-p_0)\cdot(p_2-2p_1+p_0)$. Since $p_2-2p_1+p_0 = \overrightarrow{p_1p_0}+\overrightarrow{p_1p_2} = 2\overrightarrow{p_1m}$, we get $\alpha = 4\|p_1m\|^2$, $\beta = -2\overrightarrow{p_0p_1}\cdot\overrightarrow{p_1m}$, and thus

$$\|\Gamma'(\tau)\|^2 = \frac{1}{\|p_1m\|^2}(4\|p_1m\|^2\|p_0p_1\|^2 - 4(\overrightarrow{p_0p_1}\cdot\overrightarrow{p_1m})^2).$$

It follows from the canonical equation $(U\times V)^2+(U\cdot V)^2 = U^2V^2$, for any two vectors U, V, that

$$\|\Gamma'(\tau)\|^2 = \frac{4(\overrightarrow{p_0p_1}\times\overrightarrow{p_1m})^2}{\|p_1m\|^2}.$$

Now, $|\overrightarrow{p_0p_1}\times\overrightarrow{p_1m}|$ is equal to $\mathcal{A}$, the area of the control triangle $p_0p_1p_2$. Indeed, $\overrightarrow{p_1m} = (\overrightarrow{p_1p_0}+\overrightarrow{p_1p_2})/2$ and thus $|\overrightarrow{p_0p_1}\times\overrightarrow{p_1m}| = |\overrightarrow{p_0p_1}\times\overrightarrow{p_1p_2}|/2 = \mathcal{A}$. Thus, $\|\Gamma'(\tau)\|^2 = 4\mathcal{A}^2/\|p_1m\|^2$ which yields the result. □

The expression of $\kappa_{max} = \kappa(\tau)$ now follows easily. By Lemma 2.3, $\|\Gamma'(\tau)\|^3 = 8\mathcal{A}^3/\|p_1m\|^3$. Thus, Equation 3 gives

$$\kappa(\tau) = \frac{\|p_1m\|^3}{\mathcal{A}^2}.$$

That ends the proof of Theorem 2.1 when p_1 lies strictly outside the two disks of diameter p_0m and mp_2.

When p_1 lies inside one of these disks, Sapidis and Frey (see Theorem 2.2) proved that the curvature of the quadratic Bézier curve Γ is monotone. The maximum curvature is thus the curvature at one endpoint $\Gamma(0)$ or $\Gamma(1)$. Our expression of the curvature at these points comes directly from the result by Sapidis and Frey mentioned above. These expressions also come directly from a straightforward computation, which we present here for completeness.

Equation 1 gives $\Gamma'(0) = 2(p_1-p_0)$ and $\Gamma'(1) = 2(p_2-p_1)$. It then follows from Equation 3 that

$$\kappa(0) = \frac{\mathcal{A}}{\|p_0p_1\|^3} \text{ and } \kappa(1) = \frac{\mathcal{A}}{\|p_1p_2\|^3}.$$

3 Quadratic Bézier curves with prescribed maximum curvature

The goal of this section is to show how, in an interactive curve design context, to help the user control the maximum curvature of quadratic Bézier curves by moving the control points.

Given a positive constant k and two points p_0 and p_2, we show how to compute the locus of points p_1 such that the maximum curvature of a quadratic Bézier curve Γ, with control points p_0, p_1 and p_2, is equal to k. In an interactive curve design context, the idea will then be to draw these curves for some sample values of k (see Figure 2). The user can then (i) for a given p_1, obtain immediately an approximation of the maximum curvature of Γ by checking its proximity to the displayed curves, or, (ii) for a given k, choose p_1 so that Γ has maximum curvature k by dragging p_1 along the corresponding curve.

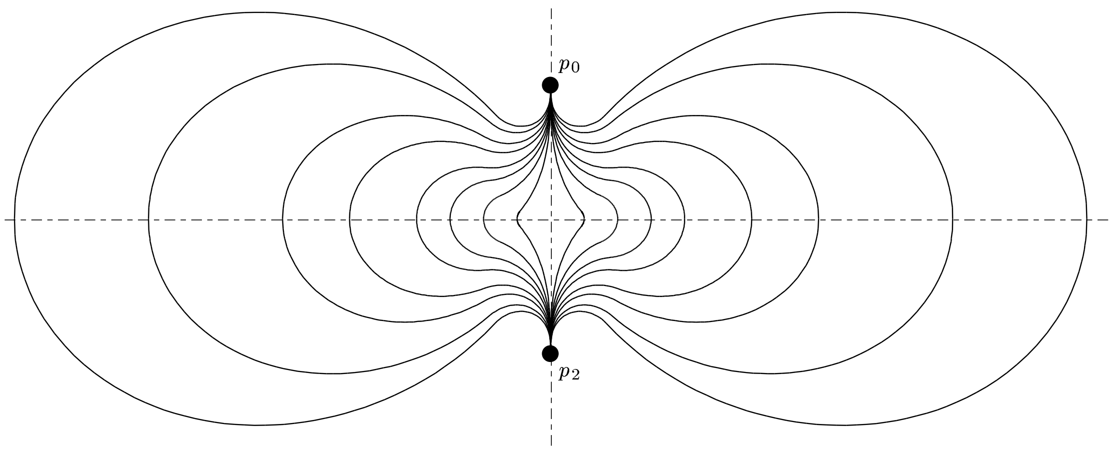

Figure 2: Sample of curves, loci of p_1, for which the maximum curvature of Γ is constant.

We assume for the sake of simplicity that the coordinates of p_0, p_1 and p_2 are $(0, d)$, (x, y) and $(0, -d)$, respectively, in an orthonormal frame $(O, \vec{\imath}, \vec{\jmath})$.

Theorem 3.1 *The locus of points p_1 such that the maximum curvature of Γ is k is the union of the curves*

$$\begin{array}{l} \{(x, \pm\sqrt{(kd^2x^2)^{2/3} - x^2}) \mid -kd^2 \leqslant x < -\sqrt{d\tilde{y} - \tilde{y}^2}\}, \\ \{(x, \pm\sqrt{(kd^2x^2)^{2/3} - x^2}) \mid +\sqrt{d\tilde{y} - \tilde{y}^2} < x \leqslant +kd^2\}, \\ \{(x, +d - \sqrt{(dx/k)^{2/3} - x^2}) \mid -\sqrt{d\tilde{y} - \tilde{y}^2} \leqslant x \leqslant \sqrt{d\tilde{y} - \tilde{y}^2}\} \text{ and} \\ \{(x, -d + \sqrt{(dx/k)^{2/3} - x^2}) \mid -\sqrt{d\tilde{y} - \tilde{y}^2} \leqslant x \leqslant \sqrt{d\tilde{y} - \tilde{y}^2}\}, \end{array}$$

where $\tilde{y} = \dfrac{1 + 2k^2d^2 - \sqrt{1 + 4k^2d^2}}{2k^2d}$.

Proof: Note first that the midpoint of p_0p_2 is O. However, for consistency with Section 2, we denote by m the midpoint of p_0p_2. Let D_0 and D_2 be the two closed disks of diameter p_0m and mp_2, respectively, and C_0 and C_2 be their respective boundaries. Let $\mathcal{A}$ denote the area of the control triangle $p_0p_1p_2$.

We assume first that p_1 lies outside the two disks D_0 and D_2. Then, by Theorem 2.1, the maximum curvature of Γ is equal to k if and only if $\|p_1m\|^6/\mathcal{A}^4 = k^2$. Since $\|p_1m\|^2 = x^2 + y^2$

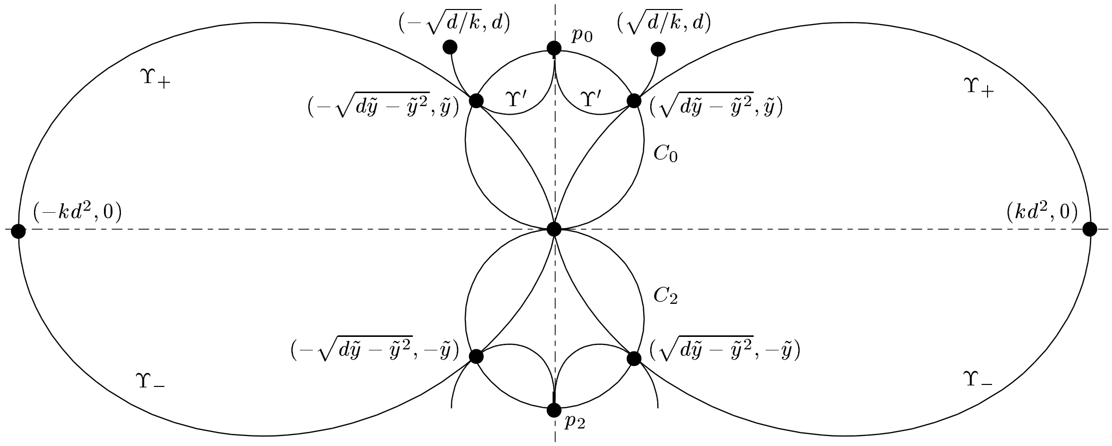

Figure 3: Example of curves $\Upsilon_\pm$, Υ', C_0 and C_2.

and the area $\mathcal{A}$ of the triangle $p_0p_1p_2$ is equal to half the length of the base, $p_0p_2 = 2d$, times the height, $|x|$, that is $\mathcal{A} = |dx|$, we get that the maximum curvature of Γ is equal to k if and only if

$$(x^2+y^2)^3 = k^2(dx)^4 \iff y = \pm\sqrt{(kd^2x^2)^{2/3} - x^2}.$$

Let Υ be the curve of equation $(x^2+y^2)^3 = k^2(dx)^4$. This curve is the union of the two curves Υ_+ and Υ_- defined respectively as the set points $(x, +\sqrt{(kd^2x^2)^{2/3} - x^2})$ and $(x, -\sqrt{(kd^2x^2)^{2/3} - x^2})$ where x varies in $[-kd^2, kd^2]$ (see Figure 3).

In order to determine the range of x on which the curve Υ lies outside $D_0 \cup D_2$, we compute the intersection between Υ, of equation $(x^2+y^2)^3 = k^2(dx)^4$, and the circles C_0 and C_2 of equations $x^2+y^2 = \epsilon dy$, $\epsilon = \pm 1$. By replacing, in $(x^2+y^2)^3 = k^2(dx)^4$, x^2+y^2 and x^4 by ϵdy and $(\epsilon dy - y^2)^2$, we get

$$\begin{aligned}
(\epsilon dy)^3 = k^2d^4(\epsilon dy - y^2)^2 &\iff y^2(k^2d(\epsilon d - y)^2 - \epsilon y) = 0 \\
&\iff y^2(k^2dy^2 - \epsilon(1+2k^2d^2)y + k^2d^3) = 0 \\
&\implies y \in \{0, \frac{\epsilon(1+2k^2d^2) \pm \sqrt{1+4k^2d^2}}{2k^2d}\}.
\end{aligned}$$

The two solutions $\pm\frac{(1+2k^2d^2)+\sqrt{1+4k^2d^2}}{2k^2d}$ are equal to $\pm(d + \frac{1+\sqrt{1+4k^2d^2}}{2k^2d})$ which do not belong to the interval $[-d, d]$. It follows that these solutions do not correspond to an intersection between Υ and the circles C_0 and C_2. Thus, an intersection can only occur when $y \in \{0, \epsilon\tilde{y}\}$, where $\tilde{y} = \frac{1+2k^2d^2-\sqrt{1+4k^2d^2}}{2k^2d}$. The intersection points satisfy the equation $x^2+y^2 = \epsilon dy$, therefore any intersection must occur at one of the five points $(0,0)$ and $(\pm\sqrt{d\tilde{y}-\tilde{y}^2}, \epsilon\tilde{y})$, where $\epsilon = \pm 1$ specifies on which circle C_0 or C_2 the intersection occurs (see Figure 3). We now distinguish between Υ_+ and Υ_-. Since Υ_+ lies on or above the x-axis, Υ_+ and $C_0 \cup C_2$ can only intersect at $(0,0)$ and $(\pm\sqrt{d\tilde{y}-\tilde{y}^2}, \tilde{y})$. We get by computing the derivative of $y = \sqrt{(kd^2x^2)^{2/3} - x^2}$ with respect to x that the tangent line to Υ_+ at $O = (0,0)$ is vertical, and that Υ_+ belongs to D_0 in a neighborhood of O. Thus, Υ_+ and $C_0 \cup C_2$ can properly intersect only at the two points $(\pm\sqrt{d\tilde{y}-\tilde{y}^2}, \tilde{y})$. It then follows from the fact that the two endpoints $(\pm kd^2, 0)$ of Υ_+ lie outside $D_0 \cup D_2$, that Υ_+ lies outside $D_0 \cup D_2$ if and only if x is in the range $[-kd^2, -\sqrt{d\tilde{y}-\tilde{y}^2}) \cup (\sqrt{d\tilde{y}-\tilde{y}^2}, kd^2]$. A similar proof holds for Υ_-, thus the parts of Υ that lie outside D_0 and D_2 are the two curves

$$\begin{array}{l}\{(x, \pm\sqrt{(kd^2x^2)^{2/3} - x^2}) \mid -kd^2 \leqslant x < -(d\tilde{y} - \tilde{y}^2)\} \text{ and} \\ \{(x, \pm\sqrt{(kd^2x^2)^{2/3} - x^2}) \mid +(d\tilde{y} - \tilde{y}^2) < x \leqslant +kd^2\},\end{array}$$

where $\tilde{y} = \dfrac{1 + 2k^2d^2 - \sqrt{1 + 4k^2d^2}}{2k^2d}$.

We now assume that p_1 lies inside one of the two disks D_0 and D_2. By Theorem 2.1, the maximum curvature of Γ is $\max(\kappa_0, \kappa_1)$ where $\kappa_0 = \mathcal{A}/\|p_0p_1\|^3$ and $\kappa_1 = \mathcal{A}/\|p_1p_2\|^3$ are the curvatures of Γ at $\Gamma(0)$ and $\Gamma(1)$ respectively. Note first that $\kappa_0 \geqslant \kappa_1$ if and only if $\|p_0p_1\| \leqslant \|p_1p_2\|$, that is, if and only if, p_1 lies inside D_0.

Suppose first that p_1 lies inside the disk D_0. Then, the maximum curvature $\kappa_0 = \mathcal{A}/\|p_0p_1\|^3$ of Γ is equal to k if and only if $\mathcal{A}^2 = k^2\|p_0p_1\|^6$, that is,

$$\begin{aligned}(dx)^2 = k^2[x^2 + (y-d)^2]^3 \iff & (y-d)^2 = (dx/k)^{2/3} - x^2 \\ \iff & y - d = -\sqrt{(dx/k)^{2/3} - x^2} \text{ since } p_1 \in D_0 \\ \iff & y = d - \sqrt{(dx/k)^{2/3} - x^2}.\end{aligned}$$

Let Υ' be the curve defined by the points $(x, d - \sqrt{(dx/k)^{2/3} - x^2})$, $x \in [-\sqrt{d/k}, \sqrt{d/k}]$ (see Figure 3). Similarly as before, in order to determine the range of x for which Υ' lies inside D_0, we compute the intersection between Υ' and the circle C_0 of equation $x^2 + y^2 = dy$. We replace, in the equation $(dx)^2 = k^2[x^2 + (y-d)^2]^3$, x^2 by $dy - y^2$ and get

$$\begin{aligned}d^2(dy - y^2) = k^2[(dy - y^2) + (y-d)^2]^3 \iff & d^2y(d-y) = k^2[(d-y)(d-y+y)]^3 \\ \iff & (d-y)[k^2d(d-y)^2 - y] = 0 \\ \iff & (d-y)(k^2dy^2 - (1+2k^2d^2)y + k^2d^3) = 0 \\ \implies & y \in \{d, \frac{(1+2k^2d^2) \pm \sqrt{1+4k^2d^2}}{2k^2d}\}.\end{aligned}$$

As before, the solution $\frac{(1+2k^2d^2)+\sqrt{1+4k^2d^2}}{2k^2d}$ is not in $[0, d]$ and thus does not correspond to an intersection between Υ' and C_0. As before, Υ' and C_0 can only intersect at $(0, d)$ and $(\pm\sqrt{d\tilde{y} - \tilde{y}^2}, \tilde{y})$ where $\tilde{y} = \frac{1+2k^2d^2-\sqrt{1+4k^2d^2}}{2k^2d}$. We also get by computing the derivative of $y = d - \sqrt{(dx/k)^{2/3} - x^2}$ with respect to x that the tangent line to Υ' at $(0, d)$ is vertical and that Υ' belongs to D_0 in a neighborhood of $(0, d)$. It follows that Υ' belongs to D_0 in the range $x \in [-\sqrt{d\tilde{y} - \tilde{y}^2}, +\sqrt{d\tilde{y} - \tilde{y}^2}]$. We get that Υ' does not belong to D_0 when x is not in that range because the two endpoints $(\pm\sqrt{d/k}, d)$ of Υ' clearly do not belong to D_0. Thus, Υ' lies inside D_0 if and only if x is in the range $[-\sqrt{d\tilde{y} - \tilde{y}^2}, +\sqrt{d\tilde{y} - \tilde{y}^2}]$. In other words, the locus of points $p_1 \in D_0$ for which the maximum curvature of Γ is k is the curve

$$\{(x, d - \sqrt{(dx/k)^{2/3} - x^2} \mid -\sqrt{d\tilde{y} - \tilde{y}^2} \leqslant x \leqslant \sqrt{d\tilde{y} - \tilde{y}^2}\}.$$

A similar proof holds when p_1 lies inside D_2 which yields the result. $\square$

4 Controlling the curvature of a piecewise quadratic Bézier curve

4.1 Preliminaries

Let Γ_1 and Γ_2 denote two quadratic Bézier curves with control points (p_0, p_1, p_2) and (q_0, q_1, q_2) respectively, and let Γ denote the concatenation of Γ_1 and Γ_2.

The general curvature control problem we address is: *Given two points M and N, two unit vectors $\vec{u}$ and $\vec{v}$, and a constant k, we would like to find two quadratic Bézier curves Γ_1 and Γ_2 joined with continuity G^1 (at $p_2 = q_0$), interpolating the two points M and N (at p_0 and q_2 respectively), such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$ respectively, the curvature is everywhere upper bounded by k, and some evaluating function is minimized.*

In the sequel, we consider without loss of generality $k = 1$; for any $k \neq 0$, we can obtain an equivalent problem where $k = 1$ by scaling the plane.

The curves Γ_1 and Γ_2 are connected (at $p_2 = q_0$) with continuity G^1 if and only if there exists $\mu \in (0, 1)$ such that $p_2 = q_0 = \mu p_1 + (1 - \mu) q_1$. The curve Γ interpolates M and N, such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$, respectively, if and only if $p_0 = M$, $q_2 = N$ and there exists α and β positive real numbers such that $p_1 - p_0 = \alpha\vec{u}$ and $q_2 - q_1 = \beta\vec{v}$ (see Figure 4).

One way to solve the general curvature control problem is to

1. find the set of $(\alpha, \beta, \mu) \in (0, +\infty)^2 \times (0, 1)$ on which the curvature of Γ is everywhere smaller or equal to 1, and then,

2. find a value (α, β, μ) in that set for which the evaluating function is minimized.

In general, this is a non-linear optimization problem with non-linear constraints, and thus, cannot necessarily be solved quickly and accurately. Clearly, the difficulty depends on the complexity of the set of feasible solutions and on the evaluating function that is to be minimized. Here we consider simplifying assumptions. First, we require a continuity C^1 at the junction point between the two curves Γ_1 and Γ_2. This fixes μ to $1/2$ and reduces the number of variables to two. To bring the number of variables down to one, we arbitrarily consider $\alpha = \beta$. We then choose as evaluating function the length α. By minimizing α, we ensure that all the control points p_1, $p_2 = q_0$ and q_1 remain close to the the points M and N we want to interpolate; in other words, by minimizing α, we expect that the length of the resulting curve Γ will not be too far from its minimum. With these further assumptions, we solve (in Section 4.2) the given interpolation and minimization problem, except for the degenerate case when $\vec{u}$ and $\vec{v}$ are parallel, for which we prove that a solution does not necessarily exist.

In Section 4.3, we also consider $\alpha = \beta$, but we require a continuity G^2 (instead of C^1) at the junction point between the two curves Γ_1 and Γ_2. In other words, we require the signed curvature to be continuous on Γ. The variables are then reduced to (α, μ) but the constraint that the continuity is G^2 links these two variables, and thus the problem is actually one-dimensional. We prove in Section 4.3 that this set of additional constraints is too restrictive in the sense that there exists non-degenerate data $(M, N, \vec{u}, \vec{v})$ that cannot be interpolated. However, if a solution exists, we show how it can be computed.

4.2 Curvature control problem with C^1 continuity

We consider here the following variant of the curvature control problem: *Given two points M and N, and two unit vectors $\vec{u}$ and $\vec{v}$, we want to find two quadratic Bézier curves Γ_1 and Γ_2 joined with continuity C^1 (at $p_2 = q_0$), interpolating the two points M and N (at p_0 and q_2 respectively), such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$ respectively, the maximum curvature of the two curves is smaller or equal to 1, the distances $\alpha = \|p_0p_1\|$ and $\beta = \|q_1q_2\|$ are equal, and such that α is minimized.* See Figure 4.

We show in this section how to solve this problem for non-degenerate data, that is when $\vec{u}$ and $\vec{v}$ are not collinear. When $\vec{u}$ and $\vec{v}$ are collinear, we show that there is not necessarily a solution.

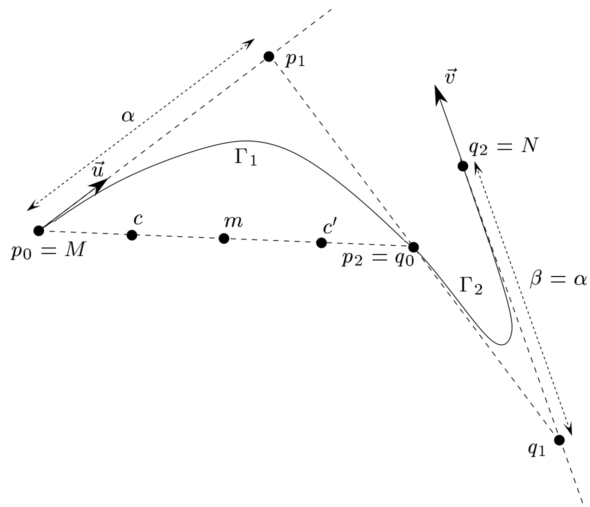

Figure 4: Curvature control problem with continuity C^1 and $\alpha = \beta$.

As we said in Section 4.1, this problem is equivalent to finding the smallest $\alpha \in (0, +\infty)$ such that the curvature of Γ_1 and Γ_2 is everywhere smaller or equal to 1, where

$$p_0 = M,\ q_2 = N,\ p_1 = p_0 + \alpha\vec{u},\ q_1 = q_2 - \alpha\vec{v} \text{ and } p_2 = q_0 = (p_1 + q_1)/2.$$

We show how we compute the smallest $\alpha \in (0, +\infty)$ such that the curvature of Γ_1 is everywhere smaller or equal to 1. Computing the smallest $\alpha \in (0, +\infty)$, for Γ_2 can be done similarly. We then return the curve Γ defined by the biggest of those two α.

First, for any value $\alpha \in (0, +\infty)$, we need to determine an expression for the maximum curvature of Γ_1. By Theorem 2.1, it remains to determine whether the maximum curvature of Γ_1 is given by the maximum curvature $\kappa_{max}(\Gamma_1)$ of the parabola supporting Γ_1, or by $\kappa_0(\Gamma_1)$ or $\kappa_1(\Gamma_1)$, the curvature of Γ_1 at its endpoints $\Gamma_1(0)$ or $\Gamma_1(1)$, respectively. Thus, for any value $\alpha \in (0, +\infty)$, we want to decide whether p_1 belongs to one of the disks of diameter p_0m and mp_2 where m is the midpoint of p_0p_2 (see Figure 4). Let c and c' be the respective centers of these disks and R be their radius. In order to determine whether p_1 belongs to one of these disks, we compute and compare R^2 with the distances $\|p_1c\|^2$ and $\|p_1c'\|^2$.

Since p_1 and q_1 are linear in α, and $p_2 = (p_1 + q_1)/2$, $m = (p_0 + p_2)/2$, $c = (p_0 + m)/2$, and $c' = (m + p_2)/2$, we have that $(c - p_0)^2$, $(c - p_1)^2$ and $(c' - p_1)^2$ are of degree 2 in α. Thus, $R^2 < \|p_1c\|^2$ and $R^2 < \|p_1c'\|^2$ are inequalities of degree at most 2 in α (namely $\alpha > \frac{16\vec{u}\cdot\overrightarrow{p_0q_2}}{(7\vec{u}+\vec{v})^2-(\vec{u}-\vec{v})^2}$ and $\alpha^2[(5\vec{u} + 3\vec{v})^2 - (\vec{u} - \vec{v})^2] - 2\alpha(16\vec{u} + 8\vec{v}) \cdot \overrightarrow{p_0q_2} + 8\|p_0q_2\|^2 > 0$). By solving these equations, we get a partition of $(0, +\infty)$ into two sets of intervals $\mathcal{I}$ and $\mathcal{I}'$ such that the maximum curvature of Γ_1 is given by $\kappa_{max}(\Gamma_1)$ for any $\alpha \in \mathcal{I}$, and by $\max(\kappa_0(\Gamma_1), \kappa_1(\Gamma_1))$ for any $\alpha \in \mathcal{I}'$.

With $\mathcal{A}(p_0p_1p_2)$ denoting the area of the control triangle $p_0p_1p_2$, we get by Theorem 2.1, when p_0, p_1 and p_2, are not collinear,

$$\kappa_{max}(\Gamma_1)^2 = \frac{\|p_1m\|^6}{\mathcal{A}(p_0p_1p_2)^4},\ \kappa_0(\Gamma_1)^2 = \frac{\mathcal{A}(p_0p_1p_2)^2}{\|p_0p_1\|^6} \text{ and } \kappa_1(\Gamma_1)^2 = \frac{\mathcal{A}(p_0p_1p_2)^2}{\|p_1p_2\|^6}.$$

A straightforward computation gives

$$\overrightarrow{p_1m} = \frac{\overrightarrow{p_0q_2} + \alpha(-3\vec{u} - \vec{v})}{4},\ \overrightarrow{p_0p_1} = \alpha\vec{u} \text{ and } \overrightarrow{p_1p_2} = \frac{\overrightarrow{p_0q_2} - \alpha(\vec{u} + \vec{v})}{2}.$$

Thus, $\mathcal{A}(p_0p_1p_2) = |\overrightarrow{p_0p_1} \times \overrightarrow{p_1p_2}|/2 = |\alpha\vec{u} \times \overrightarrow{p_0q_2} - \alpha^2\vec{u} \times \vec{v}|/4$ and

$$\kappa_{max}(\Gamma_1)^2 = \frac{(\alpha^2(3\vec{u}+\vec{v})^2 - 2\alpha(3\vec{u}+\vec{v})\cdot\overrightarrow{p_0q_2} + \|\overrightarrow{p_0q_2}\|^2)^3}{16(\alpha^2\vec{u}\times\vec{v} - \alpha\vec{u}\times\overrightarrow{p_0q_2})^4},$$

$$\kappa_0(\Gamma_1)^2 = \frac{(\alpha^2\vec{u}\times\vec{v} - \alpha\vec{u}\times\overrightarrow{p_0q_2})^2}{16\alpha^6} \text{ and } \kappa_1(\Gamma_1)^2 = \frac{4(\alpha^2\vec{u}\times\vec{v} - \alpha\vec{u}\times\overrightarrow{p_0q_2})^2}{(\alpha(\vec{u}+\vec{v}) - \overrightarrow{p_0q_2})^6}.$$

Thus, $\kappa_{max}(\Gamma_1)^2 \leqslant 1$, $\kappa_0(\Gamma_1)^2 \leqslant 1$ and $\kappa_1(\Gamma_1)^2 \leqslant 1$ reduce to inequalities in α of degree at most 8, 6 and 6 respectively. Finding the intervals of $\mathcal{I}$ and $\mathcal{I}'$ on which those inequalities are satisfied can therefore simply be done by computing the roots of the corresponding equations. More precisely, the smallest of (i) the smallest root of $\kappa_{max}(\Gamma_1)^2 = 1$ in $\mathcal{I}$, and (ii) the smallest root of $\kappa_0(\Gamma_1)^2 = 1$ and $\kappa_1(\Gamma_1)^2 = 1$ in $\mathcal{I}'$, is the smallest α for which the maximum curvature of Γ_1 is smaller or equal to 1. Such a solution exists when $\vec{u}\times\vec{v} \neq 0$ because the maximum curvature of Γ_1 goes from $+\infty$ to 0 since $\kappa_{max}(\Gamma_1)^2$, $\kappa_0(\Gamma_1)^2$ and $\kappa_1(\Gamma_1)^2$ tend to $+\infty$ when α tends to 0, and tend to 0 when α tends to $+\infty$.

We have shown that, when $\vec{u}\times\vec{v} \neq 0$, the smallest $\alpha \in (0,+\infty)$ such that the curvature of Γ_1 is everywhere smaller or equal to 1, and such that the control points p_0, p_1 and p_2 are not collinear, exists and we can compute it. Suppose now that there exists $\tilde{\alpha} \in (0,+\infty)$ such that p_0, p_1 and p_2 are collinear (see Figure 5). Assume furthermore that p_1 lies in between p_0 and p_2; otherwise, Γ_1 is not smooth and does not satisfy the constraint on the curvature. Since p_2 is the midpoint of p_1q_1, it follows that p_0, p_1, p_2 and q_1 are, in this order, on the line L passing through p_0 and directed by $\vec{u}$ (the line is necessarily directed by $\vec{u}$ because $p_1 \neq p_0$ belongs to that line). With $\vec{u}\times\vec{v} \neq 0$, q_2 does not belong to L. Thus, for $\alpha < \tilde{\alpha}$, the triangle $p_0p_1p_2$ is not flat but tends to a flat triangle, with flat vertex at p_1, as α tends to $\tilde{\alpha}$. Therefore, when α tends from below to $\tilde{\alpha}$, Γ_1 tends to a straight line segment, and the maximum curvature of Γ_1 tends to 0. Thus, there exists $\alpha < \tilde{\alpha}$ such that the maximum curvature of Γ_1 is smaller than 1. It follows that $\tilde{\alpha}$ is bigger than the smallest solution α we found previously. Therefore, when $\vec{u}\times\vec{v} \neq 0$, there is always an optimal solution with p_0, p_1 and p_2 not all collinear.

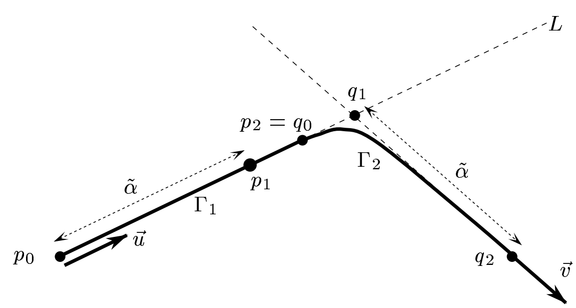

Figure 5: Case where p_0, p_1 and p_2 are collinear and consecutive.

We now show that, when $\vec{u}\times\vec{v} = 0$, there may not exist a solution. Assume for example that $\overrightarrow{p_0q_2}$ is not parallel to $\vec{u}$ and $\vec{v}$, and that $\vec{u}+\vec{v} = 0$ (a similar proof can be obtained when $\vec{u} = \vec{v}$). Then, when α tends to 0, $\kappa_0(\Gamma_1)$, $\kappa_1(\Gamma_1)$ and $\kappa_{max}(\Gamma_1)$ tend respectively to $+\infty$, 0 and $+\infty$. Similarly, when α tends to $+\infty$, they tend respectively to 0, $+\infty$ and $+\infty$. It follows that $\max(\kappa_0(\Gamma_1), \kappa_1(\Gamma_1))$ and $\kappa_{max}(\Gamma_1)$ tend to $+\infty$ when α tends to 0 and $+\infty$. In addition, $\kappa_0(\Gamma_1)$, $\kappa_1(\Gamma_1)$ and $\kappa_{max}(\Gamma_1)$ are never equal to 0 because then $\|p_1m\| = 0$ or $\mathcal{A}(p_0p_1p_2) = 0$ which would imply that p_0, p_1 and p_2 are collinear, which is impossible since the two rays starting at p_0 and q_2 with direction $\vec{u}$ and $-\vec{v}$ do not intersect. Thus, $\max(\kappa_0(\Gamma_1), \kappa_1(\Gamma_1))$ and $\kappa_{max}(\Gamma_1)$ are strictly greater than a positive constant for any $\alpha \in (0, +\infty)$, and, by scaling the plane, this constant can be scaled to a value greater than 1.

4.3 Curvature control problem with G^2 continuity

We consider here the following variant of the curvature control problem: *Given two points M and N, and two unit vectors $\vec{u}$ and $\vec{v}$, we want to find two quadratic Bézier curves Γ_1 and Γ_2 joined with continuity G^2 (at $p_2 = q_0$), interpolating the two points M and N (at p_0 and q_2 respectively), such that the tangent vectors at M and N have directions $\vec{u}$ and $\vec{v}$ respectively, the maximum curvature of the two curves is smaller or equal to 1, the distances $\alpha = \|p_0p_1\|$ and $\beta = \|q_1q_2\|$ are equal, and such that α is minimized.*

As we said in Section 4.1, the problem is equivalent to finding the smallest $\alpha \in (0, +\infty)$ such that Γ_1 and Γ_2 are connected G^2 and their curvature is everywhere smaller or equal to 1, where

$$p_0 = M,\ q_2 = N,\ p_1 = p_0 + \alpha\vec{u},\ q_1 = q_2 - \alpha\vec{v} \text{ and } \exists\mu \in (0,1) \mid p_2 = q_0 = \mu p_1 + (1-\mu)q_1.$$

The curves Γ_1 and Γ_2 are connected G^2 if and only if the two signed curvatures of Γ_1 and Γ_2 at p_2 are equal, that is, by Theorem 2.1,

$$\frac{\overrightarrow{p_0p_1} \times \overrightarrow{p_1p_2}}{2\|p_1p_2\|^3} = \frac{\overrightarrow{q_0q_1} \times \overrightarrow{q_1q_2}}{2\|q_0q_1\|^3},$$

when the triplets of points (p_0, p_1, p_2) and (q_0, q_1, q_2) are not collinear. We easily get that $\overrightarrow{p_1p_2} = (1-\mu)\overrightarrow{p_1q_1}$, $\overrightarrow{q_0q_1} = \mu\overrightarrow{p_1q_1}$, $\overrightarrow{p_1q_1} = \overrightarrow{p_0q_2} - \alpha(\vec{u}+\vec{v})$, $\overrightarrow{p_0p_1} = \alpha\vec{u}$ and $\overrightarrow{q_1q_2} = \alpha\vec{v}$. Thus, we get that Γ is G^2 if and only if

$$\frac{\alpha\vec{u} \times (1-\mu)(\overrightarrow{p_0q_2} - \alpha(\vec{u}+\vec{v}))}{(1-\mu)^3\|p_1q_1\|^3} = \frac{\mu(\overrightarrow{p_0q_2} - \alpha(\vec{u}+\vec{v})) \times \alpha\vec{v}}{\mu^3\|p_1q_1\|^3} \iff$$

$$\frac{\vec{u} \times \overrightarrow{p_0q_2} - \alpha\vec{u} \times \vec{v}}{(1-\mu)^2} = \frac{\overrightarrow{p_0q_2} \times \vec{v} - \alpha\vec{u} \times \vec{v}}{\mu^2} \iff$$

$$\mu^2 - 2\mu\Psi + \Psi = 0 \text{ where } \Psi = \frac{\overrightarrow{p_0q_2} \times \vec{v} - \alpha\vec{u} \times \vec{v}}{\overrightarrow{p_0q_2} \times (\vec{u}+\vec{v})} \text{ (if } \overrightarrow{p_0q_2} \times (\vec{u}+\vec{v}) \neq 0).$$

Standard calculations yield that the equation $\mu^2 - 2\mu\Psi + \Psi = 0$ admits a root in $(0,1)$ if and only if $\Psi \in (-1/3, 0)$. We can easily choose $p_0, q_2, \vec{u}$ and $\vec{v}$ such that $\Psi \notin (-1/3, 0)$. Indeed (see Figure 6), $\Psi > 0$ for any $\vec{u}$, $\vec{v}$ that are on the same side of $\overrightarrow{p_0q_2}$ (i.e., $\overrightarrow{p_0q_2} \times \vec{u}$ and $\overrightarrow{p_0q_2} \times \vec{v}$ have the same sign) and such that $\vec{v}$ lies in the small wedge defined by $\overrightarrow{p_0q_2}$ and $\vec{u}$ (i.e., $\vec{u} \times \vec{v}$ and $\overrightarrow{p_0q_2} \times \vec{v}$ have opposite signs). We thus proved that there is no solution to our curvature control problem for a set of non-degenerate choices of the parameters M, N, $\vec{u}$ and $\vec{v}$.

However, when a solution exists, it can be computed as in the previous section. Indeed, the curvature $\kappa_{max}(\Gamma_i)$, $i = 1, 2$, can be expressed as a ratio of a polynomial of degree 6 in α and μ over a polynomial of degree 8 in α and 4 in μ. As before, we need to find the smallest value of $\alpha > 0$, in the range where the maximum curvature of Γ_i is $\kappa_{max}(\Gamma_i)$, for which the ratio of polynomials is equal to 1. By multiplying by the denominator, the equation reduces to a polynomial equation

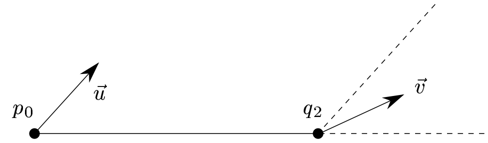

Figure 6: Example where $\Psi > 0$ for any $\alpha > 0$ ($\overrightarrow{p_0q_2} \times \vec{u} > 0$, $\overrightarrow{p_0q_2} \times \vec{v} > 0$ and $\vec{u} \times \vec{v} < 0$).

of degree 8 in α and 6 in μ. Since $\mu = -\Psi + \sqrt{\Psi^2 - \Psi}$ (the other root is negative), a polynomial of degree 6 in μ can be seen as a polynomial of degree 6 in the variables $\{\Psi, \sqrt{\Psi^2 - \Psi}\}$. The previous equation can thus be transformed into another one where $\sqrt{\Psi^2 - \Psi}$ is equal to a ratio of polynomials where the numerator is of degree 8 in α and 6 in Ψ and where the denominator is of degree 8 in α and 5 in Ψ. By squaring the equation, we get a polynomial equation of degree 16 in α and 12 in Ψ, that is an equation of degree 28 in α since Ψ is linear in α. As we said before, we cannot ensure that there exists a positive root of that equation in α. The same method can be applied to $\kappa_0(\Gamma_i)$ and $\kappa_1(\Gamma_i)$ and for determining the intervals on which the maximum curvature of Γ_i is equal $\kappa_{max}(\Gamma_i)$ or $\max(\kappa_0(\Gamma_i), \kappa_1(\Gamma_i))$.

5 Concluding remarks

It remains open to solve the curvature control problem when the length of the curve is to be minimized. Another interesting approach would be to determine how much longer than optimal our curves are. Also, we would like to consider the case when the data consist of more than two control points. Note also that, because of the high degree of the equations, it is not clear that the solutions presented in Sections 4.2 and 4.3 are usable in an interactive curve design context. This should be tested with an implementation.

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