



Optimality of the parameterization of quadrics and their intersections

Daniel Lazard

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Optimality of the Parameterization of Quadrics and their Intersections

Daniel Lazard

In collaboration with

Sylvain Lazard, Sylvain Petitjean
and Laurent Dupont

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means here optimality in term of
the number of radicals involved

Optimality

Quadrics of rank 1 or 2

Gauss-Jordan reduction leads to equations :

$$x^2$$

Rank 1 :

Rational parameterization = no radicals

Clearly optimal

$$x^2 - ay^2$$

Rank 2 :

Rational parameterization in $\mathbb{Q}(\sqrt{a})$

Optimal : There are regular rational points iff \sqrt{a} is rational.

Quadratics of rank 3

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2$$

Parameterization :

$$x = 2uv, y = \frac{b}{u^2 - av^2}, z = \frac{\sqrt{bc}}{u^2 + av^2}, t = w$$

\iff At most one square root

There exists a parameterization rational over \mathbb{Q}

\iff The conic $ax^2 + by^2 - cz^2$ has a rational point over \mathbb{Q}

\iff Gauss-Jordan gives equation $x^2 + y^2 - z^2$ when starting from this

point

Example where a **square root** is needed :

$$x^2 + y^2 - 3z^2 \text{ has no rational point (Proof over } \mathbb{Z}/4\mathbb{Z})$$

Quadratics of signature (2,2)

Gauss-Jordan reduction leads to equation :

$$ax^2 + by^2 - cz^2 - dt^2$$

Parameterization :

$$x = \frac{uv+av'v'}{uv+bu'v'}, y = \frac{b}{uv-bu'v'}, z = \frac{\sqrt{ac}}{uv'-au'v'}, t = \frac{\sqrt{bd}}{uv+bu'v'}$$

2 square roots

$\delta := abcd$ = discriminant of the quadric.

Invariant **up to a square** by change of coordinates

We claim :

$\sqrt{\delta}$ always needed

Another square root is needed **iff** the quadric has no rational point.

Quadratics of signature (2,2) (continued)

- \mathcal{Q} admits a rational parameterization over \mathbb{K} , linear in one parameter \Leftrightarrow
- \mathcal{Q} contains a line which is rational over \mathbb{K} \Leftrightarrow
- \mathcal{Q} has a rational point over \mathbb{K} and δ is a square in \mathbb{K} \Leftrightarrow
- \mathcal{Q} has equation $x^2 + y^2 - z^2 - t^2$, in some rational frame over \mathbb{K} \Leftrightarrow
- \Leftrightarrow The field of a rational parameterization contains $\mathbb{Q}(\sqrt{\delta})$
- \mathcal{Q} has a rational point over \mathbb{Q} \Leftrightarrow \mathcal{Q} has a rational point over \mathbb{Q} and equation $x^2 + y^2 - z^2 - \delta t^2$ in some rational frame.

$$x^2 + y^2 - 3z^2 - 11t^2$$

has no rational point over \mathbb{Q}

and no rational parameterization over $\mathbb{Q}(\sqrt{33})$

Proof over $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$

Intersections of quadrics \mathcal{P} and \mathcal{Q} quadrics

The intersection is contained in any quadric of the pencil $\lambda\mathcal{P} + \mu\mathcal{Q}$

There is exactly one quadric of the pencil passing through a point outside of the intersection (**linear equation to solve**)

If the intersection is not empty nor singular

(i.e. $\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has no multiple (projective) root), the pencil contains a quadric of signature $(2,2)$ with a rational point

Parameterization through this quadric involves $\sqrt{2}$:

A constant one, $\sqrt{\delta}$, the discriminant of the quadric
One depending on the parameter, $\sqrt{\Delta}$

If the intersection is non singular, $\sqrt{\Delta}$ is needed

Putting the parameterization of the quadric in \mathcal{P} gives an equation

between the parameters.

Isomorphism of the intersection

with a curve in $\mathbb{P}_1 \times \mathbb{P}_1$, the spaces of parameters

De-homogenizing and re-homogenizing :

bi-rational equivalence with a curve in \mathbb{P}_2

of degree 4 with 2 ordinary singular points at infinity.

Non singular intersection has **genus 1**

⇐ No rational parameterization over \mathbb{C}

$\sqrt{\delta}$ is not needed
iff

a surface of degree 8 has a rational point

If $\sqrt{\delta}$ is not needed,

the intersection contains pair of conjugate points of degree 2
the quadric passing through a point on a rational line
defined by such a pair has a square discriminant

\Leftrightarrow The surface

$$z^2 = \det(\mathcal{Q}(x, y, 1, 0) \mathcal{P} - \mathcal{P}(x, y, 1, 0) \mathcal{Q})$$

has a rational point.

This surface has no rational point in the case :

$$\begin{aligned} \mathcal{P} &= 5y^2 + 6xy + 2z^2 - t^2 + 6zt; \\ \mathcal{Q} &= 3x^2 + y^2 - z^2 - t^2 \end{aligned}$$

Singular intersections

0 is a multiple root of $\det(\mathcal{F} + \lambda \mathcal{Q})$

$\Leftrightarrow \mathcal{F}$ has rank ≤ 2 (pair of planes) **or**

\mathcal{F} is a cone and all quadrics of the pencil are tangent at its vertex

\Leftrightarrow The intersection has a **singular (complex) point** which is also a

singular point of \mathcal{F}

Up to a real projective transformation,

there are **28 singular intersections** corresponding to **46 pencils**.

Algorithm sketched by **Sylvain Lazard**

managed through the number of multiple roots of $\det(\lambda P + \mu Q)$



If the intersection does not consist in 4 lines

at most 2 \surd for parameterizing all components

at most 1 \surd may be unnecessary

it may be removed by finding a rational point on a conic

If the intersection consists in 4 non coplanar lines (skew quadrilateral)

at most 2 \surd for each line

possibility of 3 \surd for all lines together

always optimal for each line and for all lines together.

In the case of 4 concurrent lines

general degree 4 equation

Singular quartic

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has a single multiple root

Parameterize the cone corresponding to this root : 1 or 0 \checkmark

Plug it in the equation of \mathcal{P}

\Leftarrow equation which is linear in one of the parameter.

One \checkmark , zero if one finds a rational point on the cone

Cubic an line

$\det(\lambda\mathcal{P} + \mu\mathcal{Q})$ has 2 double roots, real or not

Parameterize through a (2, 2) quartic : no \checkmark

\Leftarrow Equation in the parameters factors (GCD computation) in

a univariate factor (the line)

and a factor linear in that variable (the cubic)

Totally rational parameterization

TWO CONICS

$\det(\lambda P + \mu Q)$ has 1 (rational) double root

The corresponding quadric is a pair of planes

Compute these planes : 1 or 0 $\sqrt{\quad}$

Parameterize through a (2, 2) quartic and plug the parameters in the equations of the planes : 1 or 0 $\sqrt{\quad}$

\iff Linear equations in the parameters

First $\sqrt{\quad}$ is needed if the planes are not rational

Second $\sqrt{\quad}$ may be avoided for each conic with a rational point on the field of the planes

Conic and 2 lines

$\det(\lambda P + \mu Q)$ has 2 (rational) double roots

The corresponding quadrics are a cone and a pair of planes

Parameterize through a (2, 2) quadric : one $\sqrt{\quad}$ iff the lines are irrational

\implies Equation in the parameters factors (GCD computation) in

2 univariate factors (the lines)

and a bilinear one (the conic)

At most 1 $\sqrt{\quad}$

It may be avoided for the conic iff it has a rational point

4 lines

$\det(\lambda P + \mu Q)$ has 2 double roots

The corresponding quadrics are pairs of planes

If the roots are rational

Compute the planes and their intersection : 0, 1 or 2 $\sqrt{\delta}$, unavoidable

Else

Parameterize through a $(2, 2)$ quadric : one $\sqrt{\delta}$

\implies Equation in the parameters factors (GCD computation)

in 2 univariate factors of degree 2.

Each linear factor parameterizes one line with at most 2 $\sqrt{\delta}$

(all together 3 $\sqrt{\delta}$)

If the roots of $\det(\lambda P + \mu Q)$ generates the same field as $\sqrt{\delta}$

Plug the parameterization of the quadric in each plan of a pair :

2 $\sqrt{\delta}$ all together

Always optimal