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► **To cite this version:**

Hervé Brönnimann, Olivier Devillers, Vida Dujmovic, Hazel Everett, Marc Glisse, et al.. On the Number of Lines Tangent to Four Convex Polyhedra. 14th Canadian Conference on Computational Geometry - CCCG'02, 2002, Lethbridge, Canada, 2002. <inria-00099449>

**HAL Id: inria-00099449**

**<https://hal.inria.fr/inria-00099449>**

Submitted on 15 Dec 2009

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# On the Number of Lines Tangent to Four Convex Polyhedra\*

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June 28, 2002

## Abstract

We prove that, under a certain general position assumption, the number of lines tangent to four bounded disjoint convex polyhedra in  $\mathbb{R}^3$  with a total of  $n$  edges is  $O(n^2)$ . Under the same assumption, we show that a set of  $k$  bounded disjoint convex polyhedra has at most  $O(n^2 k^2)$  lines, possibly occluded, that are tangent to four of these polyhedra.

## 1 Introduction

The number of visibility events in a scene determines the complexity of many visibility-related problems arising in computer graphics, such as radiosity computation or hidden surface removal. Informally, a visibility event corresponds to a combinatorial change in the view of a moving observer; such an event occurs when a viewing direction becomes tangent to some objects. Typically, a line in  $\mathbb{R}^3$  can be tangent to up to four objects, unless they are in some kind of degenerate position. A key step in estimating the number of visibility events is to solve the following geometric problem:

Given  $k$  objects in  $\mathbb{R}^3$ , determine how many lines are tangent to 4 of them.

For  $k$  triangles, it is easy to see that the worst-case bound is  $\Omega(k^4)$ . In the case of 4 convex polygons with  $n$  vertices in total, the paper of Teller and Hohmeyer [4] proves, implicitly, that the number of tangents is  $O(n^2)$ .

This paper studies the case where the objects are convex, bounded, and disjoint polyhedra under some general position assumption. We first prove that 4 such objects, with a total of  $n$  edges, have  $O(n^2)$  common

tangent lines. We then extend this result, and show that  $k$  such polyhedra, with a total of  $n$  edges, have  $O(k^2 n^2)$  lines tangent to 4 of them, possibly intersecting the others.

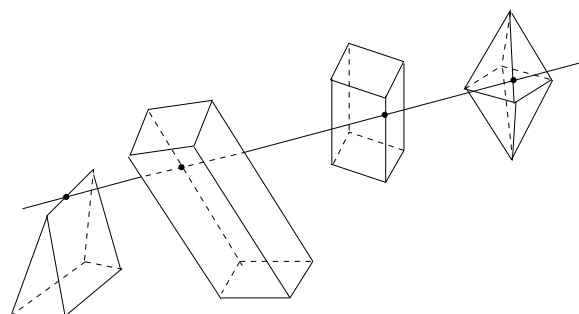


Figure 1: A line tangent to four convex polyhedra.

In order to define our notion of general position for polyhedra, we first define this notion for sets of lines. The following well-known fact is essential here and throughout the paper.

**Fact 1** Four lines in  $\mathbb{R}^3$  admit either 0, 1, 2, or an infinite number of transversals, i.e., lines that intersect each of the four given lines (see e.g. [2, p. 164], [3]).

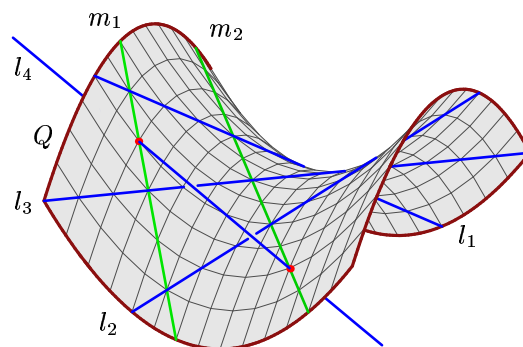


Figure 2: Two lines  $m_1$  and  $m_2$  meeting the four lines  $l_1, l_2, l_3$ , and  $l_4$  (figure taken from [6]).

To give an idea why this is true, consider three pairwise skew lines  $l_1, l_2$ , and  $l_3$  (see Figure 2). It is well

\*This research was initiated at the McGill-INRIA Workshop on Computational Geometry in Computer Graphics, February 9-15, 2002, co-organized by H. Everett, S. Lazard, and S. Whitesides, and held at the Bellairs Research Institute of McGill University. This research was supported by INRIA, NSERC, FCAR and NSF.

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known [2, p. 15] that they always lie on a degree-two ruled surface  $\mathcal{Q}$ , namely a hyperbolic paraboloid or a hyperboloid of one sheet. In either case, there are two infinite families of pairwise skew lines that generate  $\mathcal{Q}$ . One family contains  $l_1, l_2$ , and  $l_3$ , and the other consists of all lines intersecting each of  $l_1, l_2$ , and  $l_3$ . In general, a fourth line  $l_4$  meets  $\mathcal{Q}$  in at most two points, and each of these points lies on exactly one line of each family. In particular they each lie on exactly one line of the second family ( $m_1$  and  $m_2$  in Figure 2). Thus there are, in general, at most two lines,  $m_1$  and  $m_2$ , that meet each of  $l_1, l_2, l_3$ , and  $l_4$ . If  $l_4$  lies on  $\mathcal{Q}$ , or if the lines  $l_1, l_2$ , and  $l_3$  are not pairwise skew, there may be infinitely many lines meeting these four.

In the context of the study of common tangents to polyhedra, we wish to isolate as a degenerate case the situation of polyhedra having infinitely many common tangents. Thus we say that four lines are in *general position* if they admit at most 2 transversals, and that a set of convex polyhedra is in *general position* if every four edges, no two of which belong to the same polyhedron, are contained in four lines in general position.

To appreciate the impact of this general position assumption for convex polyhedra, we give a characterization of the degenerate sets of four lines. We omit here the proof, which is elementary. Four lines are not in general position if and only if (i) they belong to the same family of generators of a hyperbolic paraboloid or a hyperboloid of one sheet, (ii) two of the lines are coplanar, and the two other lines intersect in that plane, (iii) three of the lines are coplanar and the fourth line intersects that plane, or (iv) at least three of the four lines are concurrent. It follows from this characterization that sets of random polyhedra, defined as the convex hulls of random points, are in general position with probability one. On the other hand, boxes lying on the floor are not in general position.

Our main result is an upper bound on the number of lines tangent to four convex polyhedra in  $\mathbb{R}^3$ .

**Theorem 2** *Four bounded, disjoint convex polyhedra in general position in  $\mathbb{R}^3$ , with  $n$  edges in total, have  $O(n^2)$  common tangent lines. If one of the polyhedra has constant size, this bound improves to  $O(n)$ .*

When the number  $k$  of polyhedra is greater than 4, a direct application of the above theorem yields an upper bound of  $O(n^2 k^4)$ , which can be improved as stated in the following theorem.

**Theorem 3** *Given  $k$  bounded, disjoint convex polyhedra in general position in  $\mathbb{R}^3$ , with  $n$  edges in total, the number of lines tangent to four of the polyhedra is  $O(n^2 k^2)$ . (The tangents may be occluded by some of the  $k - 4$  other polyhedra.)*

## 2 Proof of Theorem 2

We prove in this section the following proposition which directly yields Theorem 2.

**Proposition 4** *Consider four bounded, disjoint convex polyhedra in general position in  $\mathbb{R}^3$ . Let  $P, Q$ , and  $R$  be three of these polyhedra, having  $p, q$ , and  $r$  edges, respectively, and let  $e$  be an edge of the fourth polyhedron. The number of lines intersecting  $e$  and tangent to  $P, Q$ , and  $R$  is  $O(p + q + r)$ .*

The proof proceeds as follows. We sweep the space with a plane rotating about the line  $l$  containing  $e$ . We define some critical events occurring during the sweep. We show that at and between two consecutive such events there are at most a constant number of tangent lines, and that the number of critical events is  $O(p + q + r)$ .

Let  $\Pi_t, t \in [0, \pi]$ , denote the parameterized sweep plane such that  $\Pi_t$  contains the line  $l$  for all  $t$ , and  $\Pi_0 = \Pi_\pi$ . Each plane  $\Pi_t$  intersects the three polyhedra  $P, Q$ , and  $R$  in three, possibly degenerate or empty, disjoint convex polygons,  $P_t, Q_t$ , and  $R_t$  (see Figure 3).

A line in plane  $\Pi_t$  that is tangent to  $P$  is either tangent to  $P_t$  or properly intersects  $P_t$ , in which case  $P_t$  must be a face or an edge of  $P$  lying in  $\Pi_t$ , implying  $\Pi_t$  is tangent to  $P$ . We thus have the following simple lemma.

**Lemma 5** *Any line tangent to  $P, Q$ , and  $R$  that intersects  $e$  is contained in some plane  $\Pi_t$  such that either  $\Pi_t$  is tangent to  $P, Q$ , or  $R$ , or the line is tangent to all three polygons  $P_t, Q_t$ , and  $R_t$  in that plane  $\Pi_t$ .*

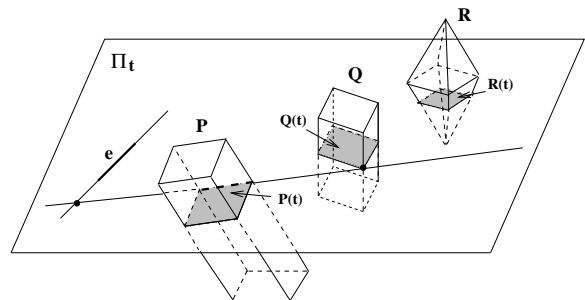


Figure 3: Plane  $\Pi_t$  contains edge  $e$  and intersects polyhedra  $P, Q$ , and  $R$  in polygons  $P_t, Q_t$ , and  $R_t$ , respectively. In this figure  $\Pi_t$  is in critical  $F$ -position.

The sweep planes  $\Pi_t$  that are tangent to  $P, Q$  or  $R$  are called *tangent planes*.

**Lemma 6** *There are at most  $24$  lines tangent to  $P, Q$ , and  $R$  and intersecting  $e$  that are distinct from  $l$  and lie in the tangent planes.*

**Proof:** Let  $\Pi^*$  denote a tangent plane. It intersects  $P, Q$ , and  $R$  in three possibly degenerate convex polygons,  $P^*, Q^*$ , and  $R^*$ . At most one of these, say  $P^*$ , is precisely an edge or a face of one of the polyhedra; otherwise, one could choose three coplanar edges from distinct polyhedra (edge  $e$  and two others), leading to a contradiction of the general position assumption. Thus a line in  $\Pi^*$  that is tangent to  $P, Q$ , and  $R$  has to be tangent to at least two polygons,  $Q^*$  and  $R^*$ . Thus  $\Pi^*$  contains at most four lines tangent to  $P, Q$  and  $R$ .

We now count the number of relevant tangent planes. If the intersection of  $\Pi^*$  with  $P$  is an edge or a vertex of  $P$  lying on  $l$ , then a line in  $\Pi^*$  that is tangent to  $P$  and intersects  $e$  is necessarily  $l$ , since  $e \cap P = \emptyset$ . Thus we need only to consider tangent planes that do not intersect  $P$  along  $l$ . At most two such sweep planes are tangent to  $P$ . A similar argument applies to  $Q$  and  $R$ . Thus there are at most six such tangent planes in total, which implies the result.  $\square$

We now count the number of lines tangent to all three polygons  $P_t, Q_t$ , and  $R_t$ , for  $t$  ranging over  $[0, \pi]$ . The number of such tangents is equal to the number of times a bitangent to the polygons  $P_t$  and  $Q_t$  coincides with a bitangent to the polygons  $Q_t$  and  $R_t$ , for  $t \in [0, \pi]$ . (A bitangent is a line tangent to two polygons; its contact points with polygon vertices are called support vertices.)

Polygons  $P_t, Q_t$ , and  $R_t$  have vertices determined by the intersections of  $\Pi_t$  with the edges of the polyhedra. As the sweep plane rotates, these polygons deform. The edges of the polyhedra  $P, Q$ , and  $R$  intersected by  $\Pi_t$  do not change except when  $\Pi_t$  passes through a vertex of a polyhedron. However the set of polyhedron edges containing the support vertices of a bitangent to two polygons in  $\Pi_t$  may change even though  $\Pi_t$  does not pass through a vertex of a polyhedron (see Figures 3 and 4).

We define a set of *critical positions* of the plane  $\Pi_t$  such that in the open interval between consecutive critical positions, the sets of polyhedron edges containing the support vertices of the bitangents to polygons  $P_t$  and  $Q_t$ , and the support vertices of the bitangents to  $Q_t$  and  $R_t$ , do not change. We define two types of critical positions: *critical V-positions* and *critical F-positions*.

Plane  $\Pi_t$  is at a *critical V-position* if it goes through a vertex of  $P, Q$ , or  $R$ . As the plane rotates between two consecutive critical V-positions, the vertices of the polygons  $P_t, Q_t$ , and  $R_t$  stay on the same polyhedron edges.

Plane  $\Pi_t$  is at a *critical F-position* relative to polyhedron  $P$  if it contains a line distinct from  $l$  that

- (i) goes through some face  $f$  of  $P$ ,
- (ii) is tangent to one of the polygons  $Q \cap \Psi$  or  $R \cap \Psi$ , where  $\Psi$  is the plane determined by  $f$ , and

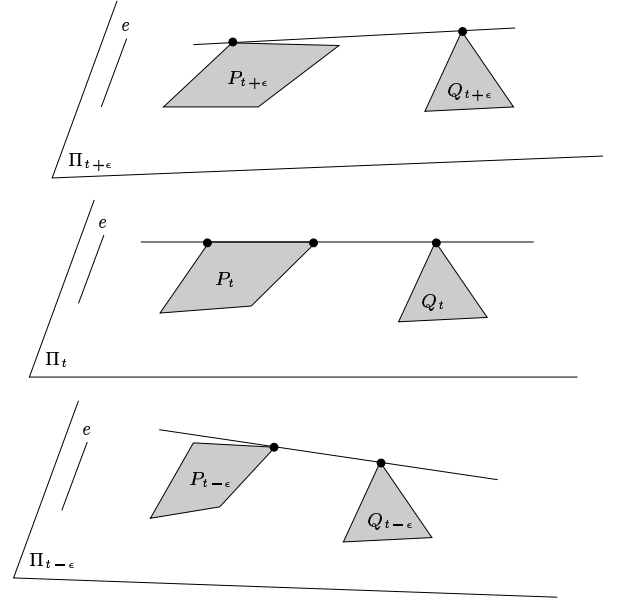


Figure 4: The plane  $\Pi_t$  is at a critical F-position. The planes  $\Pi_{t-\epsilon}$  and  $\Pi_{t+\epsilon}$  show the situation shortly before and after the critical F-position. The set of polyhedron edges containing the support vertices of the bitangent changes at  $\Pi_t$ .

(iii) goes through  $\Psi \cap l$  if this intersection is a point, and otherwise is parallel to  $l$ .

Critical F-positions relative to  $Q$  and  $R$  are defined similarly. A plane  $\Pi_t$  is at a *critical F-position* if it is at a critical F-position relative to polyhedron  $P, Q$  or  $R$ . A *critical position* is a critical V-position or a critical F-position. We now prove in the following lemma that the critical positions have the desired property.

**Lemma 7** *In the open interval between consecutive critical positions, the sets of polyhedron edges containing the support vertices of the bitangents to any two of the polygons  $P_t, Q_t$ , and  $R_t$  do not change.*

**Proof:** Consider a plane  $\Pi_{t^*}$  such that the set of polyhedron edges containing the support vertices of a bitangent  $m$  to two polygons, say without loss of generality  $P_{t^*}$  and  $Q_{t^*}$ , changes for  $t$  in any neighborhood of  $t^*$ . We prove that  $\Pi_{t^*}$  is necessarily at a critical position.

We consider first the case  $m \neq l$ . If  $m$  goes through a vertex of  $P$  or  $Q$ , then  $\Pi_{t^*}$  is at a critical V-position. Suppose now that  $m$  does not go through a vertex of  $P$  or  $Q$ . Then the bitangent  $m$

- (a) contains an edge of  $P_{t^*}$  or  $Q_{t^*}$ , say  $P_{t^*}$ ,
- (b) is tangent to  $Q_{t^*}$ , and
- (c) intersects (or is parallel to) the line  $l$ , since  $l$  also lies in  $\Pi_{t^*}$ .

It follows that  $m$

- (a') goes through a face of  $P$ ,

(b') is tangent to  $Q$ , and

(c') intersects (or is parallel to) the line  $l$ .

Let  $\Psi$  denote the plane containing the face of  $P$  in question. Condition (b') implies that  $m$  is either tangent to the polygon  $Q \cap \Psi$  or properly intersects it, in which case  $Q \cap \Psi$  is a face or an edge of  $Q$ . If  $m$  is tangent to the polygon  $Q \cap \Psi$ , then  $\Pi_{t^*}$  is at critical F-position by definition. Suppose now that  $m$  properly intersects  $Q \cap \Psi$ . In  $\Psi$ , there are infinitely many lines  $\chi$  satisfying conditions (a'), (b') and (c') and intersecting  $Q \cap \Psi$ . Each of the lines  $\chi$  lies in a sweep plane  $\Pi_t$  and thus has the form  $\chi_t = \Psi \cap \Pi_t$ . Note that  $m = \chi_{t^*}$ .

Line  $l$  does not lie in  $\Psi$  because otherwise  $e$ , an edge of  $P$ , and an edge of  $Q$  would be coplanar, leading to a contradiction of the general position assumption. Thus  $l$  intersects  $\Psi$  in at most one point and, by (c'), all the lines  $\chi_t$  pass through this point (or are parallel to  $l$ ). Thus the lines  $\chi_t$  form in  $\Psi$  a double-wedge or a strip bounded by two extremal lines. It follows that the lines  $\chi_t$  are defined for  $t$  in an interval, say  $[t_0, t_1]$ . Each of the two extremal lines,  $\chi_{t_0}$  and  $\chi_{t_1}$ , goes through a vertex of  $\Psi \cap P$  or  $\Psi \cap Q$  that is also a vertex of  $P$  or  $Q$ . Thus the planes  $\Pi_{t_0}$  and  $\Pi_{t_1}$  are at critical V-positions.

Note that  $t^* \in [t_0, t_1]$  since  $m$  is one of the lines  $\chi_t$ . If  $t^* = t_0$  or  $t_1$ , then  $\Pi_{t^*}$  is at a critical position. Otherwise  $t^* \in (t_0, t_1)$ , and as  $t$  ranges over any small enough neighborhood of  $t^*$ , the set of polyhedron edges containing the support vertices of the bitangent  $\chi_t$  does not change, although  $\chi_t$  has more than two support vertices. This contradicts the definition of  $\Pi_{t^*}$ .

Finally, we consider the case where  $m = l$ . Then  $l$  is a bitangent to  $P_t$  and  $Q_t$  in all planes  $\Pi_t$  and its support vertices always stay on the same polyhedron edges, for all  $\Pi_t$ . Thus the set of polyhedron edges containing the support vertices of the bitangent  $m = l$  does not change for any  $t$ . This again contradicts the definition of  $\Pi_{t^*}$ .  $\square$

**Lemma 8** *There are at most 16 lines tangent to all three polygons  $P_t$ ,  $Q_t$ , and  $R_t$  in planes  $\Pi_t$  ranging over the open interval between two consecutive critical positions.*

**Proof:** In any plane  $\Pi_t$ , for each of the at most four bitangents between  $P_t$  and  $Q_t$ , at most two bitangents between  $Q_t$  and  $R_t$  can share the same support vertices on  $Q_t$ . Each of those at most eight pairs of bitangents corresponds to a quadruple of polyhedron edges which admits at most two transversals, by Fact 1 and the general position assumption. This concludes the proof since by Lemma 7, in the open interval between consecutive critical positions, the support vertices of the bitangents to polygons  $P_t$  and  $Q_t$ , and of the bitangents to  $Q_t$  and  $R_t$ , stay on the same polyhedron edges.  $\square$

**Lemma 9** *In any plane  $\Pi_t$ , there are at most 4 lines tangent to all three polygons  $P_t$ ,  $Q_t$ , and  $R_t$ .*

**Proof:** The lemma is obvious since any tangent to all three polygons is also a tangent to two of them.  $\square$

**Lemma 10** *There are at most  $5(p + q + r)$  critical positions.*

**Proof:** The number of critical V-positions is at most the total number of vertices of  $P$ ,  $Q$ , and  $R$ , and hence is less than  $p + q + r$ , the total number of edges of  $P$ ,  $Q$ , and  $R$ .

We now count the number of critical F-positions. A plane  $\Psi$  supporting a face of  $P$  contains at most four lines that go through point  $\Psi \cap l$ , or are parallel to  $l$ , and that are tangent to  $Q \cap \Psi$  or  $R \cap \Psi$ . Thus a face of  $P$  generates at most four critical F-positions. A similar argument holds for the faces of  $Q$  and  $R$ .

Thus the total number of critical F-positions for a given edge  $e$  is at most four times the number of faces of  $P$ ,  $Q$ , and  $R$ , and thus is at most  $4(p + q + r)$ , since the number of faces of a polyhedron is at most the number of its edges.  $\square$

We can now conclude the proof of Proposition 4. By Lemma 10, there are at most  $5(p + q + r)$  critical positions. Thus, by Lemmas 8 and 9, there are at most  $(16 + 4)5(p + q + r)$  lines tangent to all three polygons  $P_t$ ,  $Q_t$ , and  $R_t$ , as  $t$  ranges over  $[0, \pi]$ . On the other hand, by Lemma 6, there are at most 24 lines tangent to  $P$ ,  $Q$ , and  $R$  that lie in the tangent planes. Thus, by Lemma 5, there are at most  $100(p + q + r) + 24$  lines intersecting  $e$  and tangent to  $P$ ,  $Q$ , and  $R$ .

### 3 Proof of Theorem 3

Consider  $k$  polyhedra  $P_1, \dots, P_k$ , and let  $n_i$  denote the number of edges of  $P_i$ . Choose an edge  $e$ , and let  $P_j$ ,  $P_l$ , and  $P_m$  be distinct polyhedra not containing edge  $e$ . From Proposition 4, we know that the number of tangents to  $P_j$ ,  $P_l$ , and  $P_m$  intersecting  $e$  is no more than  $C(n_j + n_l + n_m)$ , where  $C$  is some constant. We sum, over all edges  $e$ , the number of tangents intersecting  $e$ . There are  $n$  edges in the scene, so the number  $T$  of tangents to four polyhedra satisfies

$$T \leq n \sum_{j < l < m} C(n_j + n_l + n_m).$$

Since each  $n_i$ ,  $1 \leq i \leq k$ , appears  $\binom{k-1}{2}$  times in the sum, it follows that

$$T \leq C n \sum_{1 \leq i \leq k} n_i \binom{k-1}{2} = C n^2 \binom{k-1}{2}$$

so  $T$  is  $O(n^2 k^2)$  as claimed.

## 4 Discussion and open problems

We have presented bounds on the number of lines tangent to four polyhedra. The proofs are inspired by a method which was, to our knowledge, first used in [1] (see also Schifffenbauer's survey [5]).

We believe our results generalize to polyhedra that are not pairwise disjoint, and that our proofs can easily be transformed into an  $O(n^2 \log n)$  time algorithm for computing the lines tangent to four polyhedra. Furthermore, we have constructions that give matching lower bounds for our upper bounds. The final version of this paper should contain these extensions.

We conclude with the following open problem. The bound of Theorem 3 is only for lines tangent to four polyhedra among  $k$ . Ideally, we would like to bound the number of tangent lines such that the shortest line segment spanning all four points of contact does not intersect any other polyhedron. Is it possible to get a better bound for this kind of non-occluded visibility event?

**Acknowledgements.** We thank the other workshop participants for stimulating conversations and for creating a productive atmosphere.

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