On the Algebra of Structural Contexts

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Abstract. We discuss a general way of defining contexts in linear logic, based on the observation that linear universal algebra can be symmetrized by assigning an additional variable to represent the output of a term. We give two approaches to this, a syntactical one based on a new, reversible notion of term, and an algebraic one based on a simple generalization of typed operads. We relate these to each other and to known examples of logical systems, and show new examples, in particular discussing the relationship between intuitionistic and classical systems. We then present a general framework for extracting deductive systems from a given theory of contexts, and prove that all these systems have cut-elimination by the means of a generic argument.

Keywords: algebraic theories, combinatorial species, cut-elimination, display logic, linear logic, multi-categories, operads, sequent calculus, structural contexts, substructural logics, universal algebra.

1 Introduction

In this paper we introduce a general theory of multiplicative substructural logical systems by using and extending one of the most crucial ideas behind linear logic, namely that linearity brings symmetry, that is, reversibility between input an output. At some point in time logicians became aware that the nature of a logical system is very much dependent on the structure that "binds" formulas in a context. Before that time a context was just one (or two) lists of formulas and the "structural" information amounted so some operations (e.g., exchange, weakening) that could be effected on these lists. It is probable that the first time that a context had to be considered as a set-with-structure was with Lambek's non-associative calculus of 1961 [37], where it was a binary tree structure, with the antecedent formulas as leaves and the conclusion as root.

This point that contexts are sets-with-structure is best illustrated if a sufficient class of examples is given to show the diversity of structural contexts and contextural rules; this is what is done is G. Restall's textbook [64], and in M. Moortgat's handbook survey [53], which is oriented towards linguistic applications. But what we are looking for is a theory of theories of contexts, a general, abstract factory of logics. The present approaches to the problem of a general, systematic definition of contexts and their associated logics have followed the lead of N. Belnap's Display Logic, originating in [5], and perfected by several authors [31, 72]; the state of the art is probably in [23, 24]. Before we attempt a comparison with this work, let us state that we think it is profitable to view the construction of a logical calculus as a two-step process:

– The definition of a theory of contexts $\mathscr{T}$
– The extraction of a choice of connectors which is representative of $\mathscr{T}$.

We emphasize that once the connectors are chosen, then their introduction rules in the sequent calculus should be *determined* by $\mathscr{T}$.

With this in mind, we claim that the main conceptual (as opposed to technical) novelty in our approach is the emphasis on *structure* as opposed to *presentation* of structure. We think this distinction is very important and has never been made explicit in previous accounts of display logic: display calculi abound with structural connectives, and rewriting rules between them, i.e., structural rules. They are present for technical reasons, namely, to make possible the "pointing out" (or display, or focusing) of a chosen formula of the context. We claim that the structural rules which can be "undone", or reversed, do not change the structure, and so should not belong to the logic proper. They are simply artifacts of the presentation, necessary for calculations, but they tend to hide the real objects of interest. In our approach the objects of interest are equivalence classes of context presentations, and the only structural rules that are left are entropies, those rules which entail loss of information.

In this paper

– We use abstract algebra to give a completely general description of what it means to be a theory of *linear* contexts. Thus these theories become algebraic structures, which we call structads, very close in nature to groups and rings, and to be studied by similar means. In particular, structural connectors should be seen as their *generators*. Structads have well-known ancestors, in logic (Lambek's multi- and polycategories) as well as in topology (May's operads) and in combinatorics (Joyal's species). Thus their study should benefit from an already extant corpus of literature. Moreover, once an algebraic structure has been clearly identified, examples of it pop up in unexpected places, as some of our examples should illustrate: there are a lot of structads in nature, each one with associated logical calculi.
– The linearity constraint allows the definition of a special kind of syntactical object, a term with an additional variable for its output. Thus outputs/values do not differ in nature from inputs/arguments; if we want we can completely erase that difference, as is done in the standard, one-sided-sequent version of linear logic. But we can also keep that distinction by the mean of polarities, and moreover get as many outputs as we want for a term, for two-sided "classical-like" systems.
– This algebraic-axiomatic approach extends to the second step of the process, in the sense that the idea of a "choice of connectors" is given an algebraic formulation. In our approach the difference between a structural connective and a logical connective is that the latter contains the additional information of what its output variable (conclusion) is. Once this choice is made, the extraction of a sequent calculus, which benefits from a cut-elimination theorem, is completely automatic. A given theory of contexts will determine to some degree the arities of the connectors associated to it, but there is no upper or lower bound on arities when ranging over

all possible theories of contexts. There is no essential distinction between binary or unary (modal) or ternary or n-ary connectors. They are all treated uniformly as far as their introduction rules and behavior with respect to cut-elimination is concerned.

Our work is very much in the tradition of linear logic for other reasons than the ones we have already mentioned. First the fact that we use Gentzen's sequent calculus for formalizing all logical systems presented here does not make us forget that a sequent proof is only a *presentation* of a proof object, — a proof net — which naturally quotients many sequent proofs which differ only because of the tyranny of ordinary syntax. As a matter of fact this work originated in an attempt to give a general conceptual umbrella that would cover the author's and Ph. de Groote's classical non-associative calculus [17], Abrusci and Ruet's two-tensor classical system [1, 68] and Q. Puite and R. Moot's two-sided multimodal systems [62], and the systems presented in Puite's thesis [61]. All these logics benefi t from a theory of proof nets, with correctness criterion. We claim that the work presented here extends to a general theory of proof nets, with a generic correctness criterion; but this claim will not be substantiated here. Another aspect of linear logic which is very important to our work is the distinction between multiplicatives and additives: what we present is entirely multiplicative; the rules for traditional additives are automatic in our setting, since they correspond to the ordinary universal properties of product and coproduct; the generalized additives introduced in [22] are to be considered in a sequel.

The recent, independent work of J.-M. Andreoli [4] also presents a theory of contexts for linear logic, with a system of connectives (that does not have an ordinary syntactical presentation) that includes the generalized additives and a system of exponentials. It is also work which needs to be compared with ours, and for which this task is not trivial, given the differences of viewpoints. At this point we can only make the observation that we are certain that there is a close relationship between a set of varieties as axiomatized in [4] and our own concept of a structad over $\mathbb{C}$; the latter being probably more general than the former.

Now that we have said what this paper is about, we would like to say something on what it is *not* about. As we have said above, our theory is general but strictly linear-multiplicative. Thus, for instance, the deduction rules given in the books [53, 64] that happen to be non linear are automatically beyond the scope of this article. For the same reasons, the work on bunched logics [58, 63], which is certainly about structuring contexts, but uses non-linear rules in an essential way and cannot be thought of as a particular case of the present paper.

We can summarize our approach by the means of a (slightly reductive) slogan: the logic of structads is algebra freely doubled up on itself. That is, the correspondence between an algebraic system (a structad) and the logical system which is extracted from it amounts to a "doubling up" of every chosen algebraic operation (more precisely: operation – choice of conclusion pair) into two connectives, a positive (tensor-like) and a negative (par-like) one. This doubling up can also be seen as the standard left introduction/right introduction duality when our systems happen to be two-sided. When we say that this doubling up operation is donne freely we mean that the tensor and the par world do not interact at all, from the point of view of ordinary algebra: no equation is added.

There is one example in the purely multiplicative realm for which our approach is bound not to apply, and it is Retoré's pomset logic [65]. Here the self-duality of the non-commutative connector $<$ shows that here this "doubling up" is not done freely, since this connector obviously incorporates both tensor-nature and par-nature. But then our approach takes the sequent calculus as the reference for the syntax of deductive systems. It has been known for a decade that finding a sequent calculus for pomset logic (which benefits from a theory of proof nets, and thus does have some kind of syntax associated to it) is a very difficult problem, making people wonder whether it had a solution. This very problem has led to the invention of a new approach for defining deductive sytems called the calculus of structures [25, 26, 9]. It has been shown [70] that the one system in the calculus of structures which is conjectured to be equivalent to pomset logic [25] provably cannot be formalized by the means of a sequent calculus. Such a situation makes it natural to consider the extension of the theory of structads so as to make it applicable to the calculus of structures, a problem on which we are working.

This paper is about the algebraic nature of syntactic calculi, more precisely sequent calculi. The standard approach to the algebra of linear logic uses the theory of monoidal categories, but we have taken pains to make the present work completely independent from this kind of categorical algebra, much more oriented towards semantcs, and postponed these questions to a subsequent paper [34] (which has already appeared, due to the vagaries of the editorial process). There is a class of multiplicative linear calculi that are founded on a specific algebraic property of monoidal categories, namely the law of linear distributivity (formerly weak distributivity) [13, 8, 11, 12]. Linear distributivity can be summarized as a relation between a tensor and its associated par, but this relation does not contradict our concept of "freely doubling up" since it automatically holds in any categorical model of a structad-derived system where we can test this law. The extent to which structads cover linearly distributive logics is as follows. It is easy to show (for a hint see [34]) that commutative multiplicative linear distributive logic can be defined via a structad. In the non-commutative cases, the fact that the cut rule in these logics splits into four cases mean that an ordinary structad cannot be used. This can be remedied in two ways, by either expanding the theory of contexts (and thus the logic) to one that uses a single ordinary cut (and thus an ordinary structad), or generalizing the notion of structad so that structadic composition is restricted, in the very same way that a category can be considered to be a monoid with restricted composition. This was already mentioned in [34], perhaps too briefly.

There are two reasons why this paper is rather long:

- Since we introduce a few rather abstract concepts we give many examples to help the reader. Some of these examples are familiar and some of them are new cases that are suggested by the algebraic framework, illustrating its power.
- Although the whole of the paper is in the spirit of abstract algebra, we have taken pains to make it self-contained. In particular we assume no familiarity with category theory; the reader should be aware that some of the proofs in Section 4 are actually special cases of much more general results.

2 Linear order-enriched algebraic theories

Before we begin let us recall some very standard definitions:

A *preordered set* $(X, \leq)$ is a set X equipped with a binary relation which is reflexive and transitive. An *ordered set*, or *poset* is a preorder satisfying Anti-symmetry: $x \leq y \wedge y \leq x \;\Rightarrow\; x = y$. A *monotone map* $f\colon X \to Y$ between (pre)orders is a function that preserves the (pre)order structure: $x \leq y \;\Rightarrow\; f(x) \leq f(y)$ for all $x, y \in X$. A monotone map f is an *embedding* if it is injective and in addition reflects the order: $f(x) \leq f(y) \Rightarrow x \leq y$. An *isomorphism* of posets is an embedding f which is bijective; this condition is equivalent to saying that it has an inverse, in other words that there is a (necessarily unique) $f^{-1}\colon Y \to X$ with $f(f^{-1}(y)) = y, f^{-1}(f(x)) = x$ for all $x \in X, y \in Y$. The *opposite* X^{op} of a poset $(X, \leq)$ is the poset $(X, \sqsubseteq)$ with $x \sqsubseteq y$ iff $y \leq x$. A poset is said to be *discrete* if $x \leq y \Rightarrow x = y$; a set determines a unique discrete poset and vice-versa, which allows us to imagine that the universe (or class) of sets is embedded in the universe of posets. Given a preorder $(X, \leq)$, the relation $x \leq y$ and $y \leq x$ is an equivalence relation E. When we take the quotient X/E by that equivalence relation (the set of equivalence classes), we get a poset $(X/E, \leq)$ given by $A \leq B$ iff $(\forall_{x \in A, y \in B})\, x \leq y$, and the definition obtained by replacing the universal quantifier by an existential one is equivalent.

Definition 1. *A* polarity structure *is a pair* $(\mathbb{P}, (-)^{\perp})$*, where* $\mathbb{P}$ *is a set and* $\mathsf{p} \mapsto \mathsf{p}^{\perp}$ *an involution on* $\mathbb{P}$*, i.e, a function such that* $(\mathsf{p}^{\perp})^{\perp} = \mathsf{p}$*. Given two polarity structures* $\mathbb{P}, \mathbb{P}'$ *a* map*, or* morphism *is a function* $F\colon \mathbb{P} \to \mathbb{P}'$ *which respects the involution, i.e. such that* $F(\mathsf{p}^{\perp}) = F(\mathsf{p})^{\perp}$ *for all* $\mathsf{p} \in \mathbb{P}$*.*

The two most important cases in this paper are $\mathbb{C} = \{\mathsf{c}\}$, with (obviously) $\mathsf{c}^{\perp} = \mathsf{c}$, which we call the Basic One-sided case, and $\mathbb{B} = \{\bullet, \circ\}$, with $\bullet^{\perp} = \circ$, which we call the Basic Two-sided case, where $\bullet$ means "left side of a sequent" and $\circ$ "right side".

But we will see other examples. The elements of a polarity structure act as sorts, or types, but in such a way that we can make the distinction, if we want, of a sort used in an input and a sort used in an output.

We will have variables for any sort/polarity, and declare it by superscripts:
$x^{\bullet}, y^{\mathsf{c}}, g^{\circ}(x, y)\ldots$ to simplify life, we start with a set of unsorted variables
$\mathscr{V} = \{x, y, z, x_1, x_2 \ldots\}$, and for every sort p that we happen to come across, there will be a sorted copy $\{x^{\mathsf{p}}, y^{\mathsf{p}}, z^{\mathsf{p}}, x_1^{\mathsf{p}}, x_2^{\mathsf{p}} \ldots\}$ of every variable with that sort. This way we can define an *algebraic context* (as opposed to structural) in $\mathbb{P}$ to be a pair (Γ, Pol), where $\Gamma \subseteq \mathscr{V}$ is a finite set of variables, and $\mathrm{Pol}\colon \Gamma \to \mathbb{P}$ a function that gives their polarities. Another way of looking at a context is as a finite subset $\Gamma \subseteq \mathscr{V} \times \mathbb{P}$ which is the graph of a partial function. We denote the set of all algebraic contexts in $\mathbb{P}$ by $\mathrm{Ctx}(\mathbb{P})$.

Given a map $F\colon \mathbb{P} \to \mathbb{P}'$ of polarity structures it determines a map $\mathrm{Ctx}(\mathbb{P}) \to \mathrm{Ctx}(\mathbb{P}')$ between all respective contexts, denoted by $\Gamma \mapsto F\Gamma$ and defined by

$$F\{x_1^{\mathsf{p}}, \ldots x_n^{\mathsf{p}_n}\} = \{x_1^{F\mathsf{p}_1}, \ldots, x_n^{F\mathsf{p}_n}\}\,.$$

Since we are at the meeting point of algebra and proof theory, our notation deliberately draws from both traditions; for example, we will write Γ, Δ or $\Gamma + \Delta$ for the result of taking the disjoing union (sum) of two contexts, or Γ, x or $\Gamma + \{x\}$ or $\Gamma + x$ for the result of adding a new variable to a context, and $\Gamma - \{x\}$ or $\Gamma - x$ for the result of removing a variable from a context (in this last case it will always be assumed that $x \in \Gamma$ to start with).

Let polarity structure $\mathbb{P}$ be given, and let Σ be a signature, i.e. a set of function symbols, each with a given sort; if $f \in \Sigma$ then its sort is described by $f^{\mathsf{p}_0}(x_1^{\mathsf{p}_1}, x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n})$, for $\mathsf{p}_i \in \mathbb{P}$. We say a signature over $\mathbb{P}$ is *disjoint* if whenever a polarity p appears in a function symbol (e.g. p is one of the $\mathsf{p}_i, 0 \leq i \leq n$ above) then it is guaranteed that $\mathsf{p}^\perp$ does not appear anywhere in any function symbol of the signature. This seems to make the presence of $\mathsf{p}^\perp$ useless, but we will show this is not the case at all. We say a signature is *ultrabasic* if its set of polarities is $\mathbb{B} = \{\bullet, \circ\}$ and every symbol is sorted by $f^\bullet(x_1^\bullet, \ldots, x_n^\bullet)$. Ultrabasic theories are obviously disjoint.

Terms are constructed the usual way, taking account that sorts/polarities must be respected, but we are interested only in linear terms: a variable has the right to appear only once. We denote by $\mathrm{Term}(\Sigma)$ the set of (linear!) terms we have obtained.

In general, instead of equality judgments, we have order judgements, i.e. expressions of the form

$$x_1, \ldots, x_n \vdash s \leq t \tag{1}$$

where $\{x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n}\} \in \mathrm{Ctx}(\mathbb{P})$, the terms s, t have the *same* polarity, and $x_1, \ldots, x_n$ are *exactly* the variables that appear in *both* s, t. The judgement $x_1, \ldots, x_n \vdash s = t$ is to be considered as an abbreviation for the pair of judgements

$$x_1, \ldots, x_n \vdash s \leq t \quad x_1, \ldots, x_n \vdash t \leq s\,.$$

This way we are doing a slight generalization of linear universal algebra, where the terms are not quotiented under an equivalence (congruence) relation, but under a preorder. The reason for this is the possible presence of entropy rules (also called structural rewriting rules): the judgement (1) above means that we are allowed the inference rule that replaces the structural context structure s (or more pedantically the one determined by the equivalence class of term s) by the structure t.

The rules of inference are just as expected: Reflexivity

$$\frac{}{\Gamma \vdash t \leq t}$$

for any linear t in context Γ, Transitivity

$$\frac{\Gamma \vdash r \leq s \quad \Gamma \vdash s \leq t}{\Gamma \vdash r \leq t}$$

and Congruence

$$\frac{\Gamma, x \vdash s \leq s' \quad \Delta \vdash t \leq t}{\Gamma, \Delta \vdash s[x \backslash t] \leq s'[x \backslash t'].}$$

We denote by $\mathrm{OrdJ}(\Sigma)$ the set of all order judgements between terms of $\mathrm{Term}(\Sigma)$.

Definition 2. *A* linear order-enriched theory $\mathscr{T}$ *is given by a triple* $\mathscr{T} = (\mathbb{P}, \Sigma, \mathrm{Thm})$, *where* $\mathbb{P}$ *is a set of sorts/polarities,* Σ *is a signature on* $\mathbb{P}$ *and* $\mathrm{Thm} \subseteq \mathrm{OrdJ}(\Sigma)$ *a set of of order judgments (called the* theorems *of* $\mathscr{T}$*) which is closed under the deduction rules. Naturally, often in practice* Thm *is described by a subset* $\mathrm{Ax} \subseteq \mathrm{Thm}$ *of axioms, that generate* Thm. *We say* $\mathscr{T}$ *is disjoint (resp. ultrabasic) if its signature is disjoint (ultrabasic). If* Γ *is a set of variables, then* $\left(\widetilde{\mathscr{T}}(\Gamma, \mathsf{p}), \lesssim\right)$ *is the pre-ordered set whose elements are all the linear terms of polarity* p *whose variables are exactly in* Γ, *with the preorder structure determined by the order judgements in* Thm. *The relation* $s \approx t$ *denotes the conjunction of* $s \lesssim t, t \lesssim s$. *We take* $\left(\mathscr{T}(\Gamma, \mathsf{p}), \leq\right)$ *to be the quotient of* $\widetilde{\mathscr{T}}(\Gamma, \mathsf{p})$ *under the usual preorder-to-order equivalence* $\approx$.

The first examples will be ultrabasic theories:

Example 1. Let $\mathscr{T}_{\mathrm{Ass}}$ have a single binary symbol $(-)*(-)$ and the associativity axiom

$$x, y, z \vdash (x * y) * z = x * (y * z).$$

This is the theory of semigroups, and it is the theory of contexts for the standard (associative) Lambek calculus $\mathbf{L}$ [36].

Example 2. Let $\mathscr{T}_{\mathrm{Comm}}$ have a single binary symbol $(-) \cdot (-)$ and the associativity and commutativity axioms:

$$\begin{array}{lll} x, y, z & \vdash & (x \cdot y) \cdot z = x \cdot (y \cdot z). \\ x, y & \vdash & x \cdot y = y \cdot x \end{array}$$

Example 3. Let $\mathscr{T}_{\mathrm{deGr}}$ [16] have the symbols $*, \cdot$, with the same equations as above, and the additional entropy law

$$x, y \vdash x \cdot y \leq x * y\,.$$

Example 4. Let $\mathscr{T}_{\mathrm{S-P}}$ have the same symbols as above, but in addition the axioms [15] :

$$\begin{array}{ll} x, y, z & \vdash (x * y) \cdot z \leq x * (y \cdot z) \\ x, y, z & \vdash x \cdot (y * z) \leq (x \cdot y) * z \\ x, y, z, w & \vdash (x * y) \cdot (z * w) \leq (x \cdot z) * (y \cdot w)\,. \end{array}$$

Example 5. To all the theories above we can add a constant 1, which is a unit for all the operations, and thus get $\mathscr{T}_{\mathrm{Ass+Unit}}, \mathscr{T}_{\mathrm{Comm+Unit}}\ \ldots$ all these theories share the property that every closed term is provably equal to 1. Naturally this corresponds to the idea that the only contextual structure on the one-formula sequent asserts the truth or that formula. From the point of view of algebra, nothing prevents having more classes of closed terms. This should not be dismissed first hand as uninteresting from the point of view of logic; the existence of different orders of truth may give some syntactic embodiement to the idea of possible worlds.

Example 6. Let $\mathscr{T}_{\mathrm{Mor}}$ have three binary operations $\star, \star', *$ such that $*$ obeys associativity and in addition the unique law $x, y, x \vdash (x \star y) \star' z = x * y * z$. This example, along with similar ones can be found in [56] where the associated logics are used to model syntactical phenomena like median extraction.

The table on p. 26 in [64] gives a list of possible axioms that can be put on a theory built with a single binary operation (denoted $-;-$). In [53] there is a family of binary symbols, denoted $(-,-)^i$, where the indices are used to distinguish between the different symbols, and several examples of interactions between two (different or equal) symbols are given; in addition unary operators are presented for modalities; many papers (here is only a sampling) [55, 28, 54] give examples of linguistic applications of these techniques. We emphasize that the present paper only deals with the cases in these works where the axioms are linear.

Another example of a systematic treatment of axioms between several binary and unary function symbols is in [18].

Later in this paper we will give a natural example of a ternary symbol.

We have to think of a term $\Gamma \vdash t$ as a structure that has been given to the set Γ. In some of the examples above, the structure in question can be given a more familiar aspect. For $\mathscr{T}_{\mathrm{Ass}}$, it should be obvious that a term t as above is simply a word on the alphabet Γ which is nonrepeating (no letter of Γ appears twice), and such that every variable of Γ appears in t; in other words the structure on Γ is a *total ordering* of the variables. For $\mathscr{T}_{\mathrm{S-P}}$ the structure in question is also an ordering, but a less constrained one known as a series-parallel order [15, 52, 71]. For $\mathscr{T}_{\mathrm{Comm}}$, the structure in question is... no structure at all, or more correctly just that of a *set* of variables.

As we have said a judgement of the form $\Gamma \vdash s \leq t$ means that the replacement of s by t is a valid rule in any logic derived from $\mathscr{T}$. Sometimes this is called structural rewriting, sometimes entropy, but the use of (pre)-orders comes from the fact that there is no point in knowing more about *how* the rewriting was done, i.e., the actual sequence of rewriting steps; they have nothing to do with the nature of proofs in ordinary (i.e., one-dimensional) logic. This is an insight that the author got from conversations with R. Moot and Q. Puite.

An algebraically-minded person will contend that the family of posets

$$\big(\mathscr{T}(\Gamma,\mathsf{p}),\leq\big)_{\Gamma,\mathsf{p}}$$

where Γ ranges over the set of all possible $\mathbb{P}$-sorted contexts and p all sorts in $\mathbb{P}$, *is* the order-sorted theory $\mathscr{T}$, in other words, it is the mathematical object we should focus our attention on. The idea of treating first-order theories as algebraic objects, not very different from groups and fields, dates from Lawvere's seminal thesis [40], which inaugurated, among other things, of the field of categorical model theory [30].

The structure we are dealing with does not have a single carrier (underlying) set, but a family of these, namely the $\big(\mathscr{T}(\Gamma,\mathsf{p})\big)_{\Gamma,\mathsf{p}}$. Thus the algebra associated to a linear order-enriched theory is a multi-sorted algebra. The structure that these carrier sets are naturally equipped with consists of:

– the order structure $\big(\mathscr{T}(\Gamma,\mathsf{p}),\leq\big)$ for every Γ,p.
– The operations associated with renaming/substitution of variables: for every polarity-respecting bijection $\sigma\colon\Gamma\to\Delta$ between contexts, we have an action $\sigma\colon\mathscr{T}(\Gamma,\mathsf{p})\to\mathscr{T}(\Delta,\mathsf{p})$ obtained by renaming

$$\sigma(t) = t[x_1\backslash\sigma(x_1),\ldots,x_n\backslash\sigma(x_n)] \in \mathscr{T}(\Delta,\mathsf{p})\,.$$

– The operations associated with substitution of terms: for every $s \in \mathscr{T}(\Gamma, \mathsf{p}), t \in \mathscr{T}(\Delta, \mathsf{q})$ and $x^{\mathsf{p}} \in \Delta$ we can construct the composite $t \circ_x s \in \mathscr{T}(\Delta - \{x\} \cup \Gamma, \mathsf{q})$, by the means of substitution: $t \circ_x s = t[x \backslash s]$, provided the new context is a disjoint union; if it is not, just rename to satisfaction.

These operations, and some associated constants naturally satisfy some equations and relations, that we will fully describe in section 4 (some readers will have already guessed that we are dealing with poset-enriched multicategories, the kind with permutations). The most important fact about the algebraic approach to model-theoretical logic is that it allows the easy description of the interpretation of a theory into another as a structure-preserving homomorphism between the associated algebraic structures. But first:

Definition 3. *Let $\mathscr{T} = (\mathbb{P}, \Sigma, \mathrm{Thm})$ and $\mathscr{T}' = (\mathbb{P}', \Sigma', \mathrm{Thm}')$ be linear order-enriched theories. An* interpretation *(or* map*) $F \colon \mathscr{T} \to \mathscr{T}'$ is given by:*

- *a map of polarity structures $F \colon \mathbb{P} \to \mathbb{P}'$. Given a context $\Gamma = x_1^{\mathsf{p}_1}, x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n}$ we write $F\Gamma$ for the context $F\Gamma = x_1^{F\mathsf{p}_1}, x_2^{F\mathsf{p}_2}, \ldots, x_n^{F\mathsf{p}_n}$,*
- *for every function symbol $f \in \Sigma$, given a choice $\Gamma \vdash f^{\mathsf{p}}$ of context , we have a term Ff with $F\Gamma \vdash (Ff)^{F\mathsf{p}}$ in $\mathscr{T}'$. The exact choice of the variables in Γ is obviously not important, since given any two contexts $x_1, x_2 \ldots, x_n \vdash f(x_1, \ldots, x_n)$ and $y_1, y_2 \ldots, y_n \vdash f(y_1, \ldots, y_n)$ there is a* unique *polarity-respecting bijection $x_i \leftrightarrow y_i$ that will map one atomic term to the other. Once we have this, we can define Ft for any term $\Delta \vdash t$ in any context of $\mathscr{T}$. This is done in the usual fashion: structural induction on t.*
- *Finally F is required to obey the rule that for any theorem $\Gamma \vdash s \leq t$ of $\mathscr{T}$ we have to have that $F\Gamma \vdash Fs \leq Ft$ is a theorem of $\mathscr{T}'$.*

This last stipulation ensures that for every Γ, p, we get preorder maps $\widetilde{F}_{\Gamma,p} \colon \widetilde{\mathscr{T}}(\Gamma, \mathsf{p}) \to \widetilde{\mathscr{T}}'(F\Gamma, F\mathsf{p})$ which collapse to monotone functions $F_{\Gamma,\mathsf{p}} \colon \mathscr{T}(\Gamma, \mathsf{p}) \to \mathscr{T}'(F\Gamma, F\mathsf{p})$. So an interpretation of theories defines a family of maps between the carrier sets. We will see that this family of maps respects the algebraic structure associated with theories.

In this first list of examples the map $F \colon \mathbb{P} \to \mathbb{P}'$ is always identity.

Example 7. There is an obvious interpretation $B \colon \mathscr{T}_{\mathrm{Ass}} \to \mathscr{T}_{\mathrm{Comm}}$ which is identity on polarities, defined by stating $B(x, y \vdash x * y) = x, y \vdash x \cdot y$. The action on a general term is: go from the list (non-repeating word) of variables to the associated set it determines.

Example 8. There is also an "endomap" $R \colon \mathscr{T}_{\mathrm{Ass}} \to \mathscr{T}_{\mathrm{Ass}}$ defined by

$$R\big(x, y \vdash x * y\big) \;=\; x, y \vdash y * x.$$

So what it does in general is turn a word of variables into its reverse.

Example 9. In addition to the obvious embeddings $\mathscr{T}_{\mathrm{Ass}} \to \mathscr{T}_{\mathrm{deGr}}$ and $\mathscr{T}_{\mathrm{Comm}} \to \mathscr{T}_{\mathrm{deGr}}$, and the "inclusion of order structures" $\mathscr{T}_{\mathrm{deGr}} \to \mathscr{T}_{\mathrm{S-P}}$ there is an interpretation $\mathscr{T}_{\mathrm{DeGr}} \to \mathscr{T}_{\mathrm{Comm}}$ that collapses the two operations into $(-) \cdot (-)$ and the order relation into the identity.

Example 10. Let $\mathscr{T}_{\text{FreeBin}}$ have a unique binary symbol $\star$ and no axioms, while $\mathscr{T}_{K-M}$ is the associative theory $\mathscr{T}_{\text{Ass}}$ augmented with one unary operator K and no additional axiom. There is a map $\mathscr{T}_{\text{FreeBin}} \to \mathscr{T}_{\text{K}-\text{M}}$ that sends $x, y \vdash x \star y$ to $x, y \vdash K(x * y)$. The point of this map *is that it is an embedding* (every $\mathscr{T}_{\text{FreeBin}}(\Gamma, \bullet) \to \mathscr{T}_{\text{K}-\text{M}}(\Gamma, \bullet)$ is an injective function), thus allowing [32] a conservative translation between the associated logics. In that paper more such results can be found. Notice that the property of being an embedding is preserved if we add the stipulation that K is idempotent: $x \vdash K(K(x)) = K(x)$.

3 Reversibility

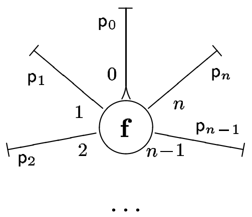
We come to the main technical tool of this paper, a syntax for which terms have an additional variable at their output. An ordinary term can always be seen as a tree, whose nodes are decorated with function symbols. Such a tree has a predetermined root: the "output" of the outermost function symbol, in other words the only end which is not decorated by a variable. What we are interested in is trees *all* whose roots are decorated with a variable. It would be possible to present these objects using the language of graphs and trees, but it is more practical to use the standard technology of terms as a basis and construct our trees by adding a new constructor.

Let $(\mathbb{P}, (-)^{\perp})$ be a polarity structure.

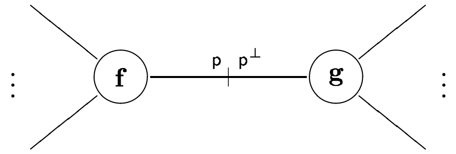
Definition 4. *A* tree signature *over* $\mathbb{P}$ *is a triple* $(\mathbf{\Sigma}, \mathrm{Val}, \mathrm{Pol})$, *which we tend to abbreviate by* $\mathbf{\Sigma}$, *where*

- $\mathbf{\Sigma}$ *is a set of node symbols*
- $\mathrm{Val}\colon \mathbf{\Sigma} \to \mathbb{N}^{+}$, *is a function, the* valence,
- Pol *is a function that to* $\mathbf{f} \in \mathbf{\Sigma}$ *and* $n \in \{0, 1, \ldots, \mathrm{Val}(\mathbf{f}) - 1\}$ *assigns a polarity* $\mathrm{Pol}(\mathbf{f}, n) \in \mathbb{P}$.

So every node symbol $\mathbf{f}$ of valence $n + 1$ should be seen



as an atom with $n + 1$ wires (or ports) attached to it, numbered from 0 to n. Each port has its polarity $\mathsf{p}_i = \mathrm{Pol}(\mathbf{f}, n)$. At this point the "orientation tab" $\succ\!\!-$ has no other function than to distinguish the zero port; when we know where that port is our graphical convention is that the others are numbered in a counterclockwise fashion. So our intention is to connect tree symbols via their ports, while respecting the polarity stipulations:



a consequence of this is that the following

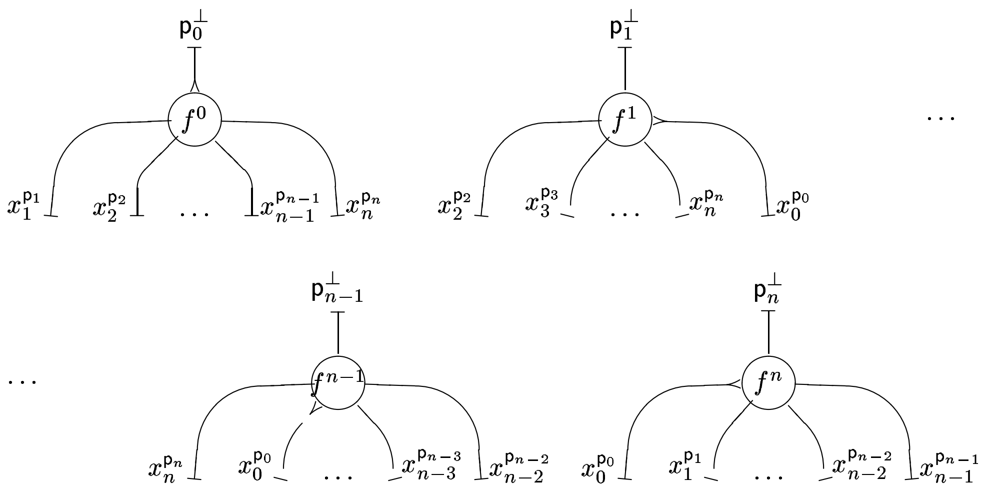
will always mean exactly the same thing. In other words, the involution operation on polarities corresponds to inverting the direction you a looking at a wire; when $\mathsf{p} = \mathsf{p}^\perp$ this direction is unimportant.

Given a tree signature $\mathbf{\Sigma}$ as above its *derived ordinary signature* $\mathrm{OrdSign}(\mathbf{\Sigma})$ is obtained by taking, for every symbol $\mathbf{f} \in \mathbf{\Sigma}$ of valence $n+1$, a set of $n+1$ ordinary function symbols of arity n, which we denote $(f^i)_{0 \leq i \leq n}$, with the following sort assignments:

$$f^{0\,\mathsf{p}_0^\perp}(x_1^{\mathsf{p}_1}, x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n}),\, f^{1\,\mathsf{p}_1^\perp}(x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n}, x_0^{\mathsf{p}_0}),\, f^{2\,\mathsf{p}_2^\perp}(x_3^{\mathsf{p}_3}, \ldots, x_n^{\mathsf{p}_n}, x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}), \ldots$$

$$\ldots, f^{n\,\mathsf{p}_n^\perp}(x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}, \ldots, x_{n-1}^{\mathsf{p}_{n-1}})\,.$$

so we see that the ordinary signature is obtained by expanding the tree signature with the additional information “right now I am looking down port n”.



and that the way ordinary terms are sorted is affected by this point of view: the polarities in the ordinary syntax are the ones that are on the viewer’s side of the wires, so the “output” gets inverted but not the “inputs”.

But there is more to it. Let us add a new binary symbol $\langle -, - \rangle$, and define a *reversible (or one-sided)*[1] *term judgement* with context Γ to be an expression of the form

$$\Gamma \vdash_{\mathsf{r}} \langle t_1, t_2 \rangle$$

[1] what is one sided is is not the judgement but the term.

where $t_1^{\mathsf{p}}, t_2^{\mathsf{p}^\perp}$ are terms in $\mathrm{OrdSign}(\mathbf{\Sigma})$ that have *opposite* polarity, and such that there is a partition $\Gamma = \Gamma_1 + \Gamma_2$ into two disjoint sets of variables, with

$$\Gamma_1 \vdash t_1$$
$$\Gamma_2 \vdash t_2$$

term judgments in $\mathrm{Term}(\mathrm{OrdSign}(\mathbf{\Sigma}))$.

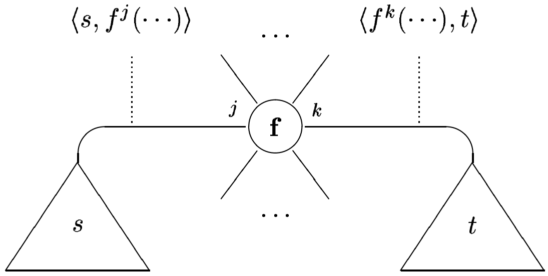
Reversible terms are actually equivalence classes of such expressions, since they are quotiented under the equivalence relation $\equiv$, defined by the (reflexive symmetric, transitive closure) of the rule of Adjunction:

$$\Gamma \vdash_{\mathsf{r}} \left\langle f^k(\ldots,_j s, \ldots), t \right\rangle \equiv \left\langle s, f^j(\ldots,_k t, \ldots) \right\rangle \tag{2}$$

where something like $f^k(\ldots,_j s, \ldots)$ means that the term s appears at the j-th place in the atomic term $f^k(\ldots)$, along with rule Perm:

$$\Gamma \vdash_{\mathsf{r}} \langle t_1, t_2 \rangle \equiv \langle t_2, t_1 \rangle \, .$$

This last rule will never be mentioned, being so innocuous. We write $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ for a reversible term, when there is no need to go into more details.



We see that a reversible term $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$, i.e. a $\equiv$-class, is a tree, whose nodes are labeled with node symbols, and where the polarity discipline for connecting nodes is followed, as described above; in addition every port of a node which is not connected to another node is labeled with a single variable in Γ. Every representative of the $\equiv$-class of $\mathbf{t}$, i.e, every $\langle s, r\rangle$ with $\Gamma \vdash_{\mathsf{r}} \mathbf{t} \equiv \langle s, r\rangle$ corresponds to choosing a wire of $\mathbf{t}$ and "pinching" it, turning the tree into a pair of terms.

Remark 1. The name Adjunction comes from the formal, syntactical aspect, since the rule is just like an adjunction between matrices or operators. But it is actually a more primitive concept than an adjunction (called residuation by many logicians) in category theory, in the sense that the latter adjunctions involve *two* distinct operators. These "real" adjunctions will appear once we define logical calculi, and their existence will be due to the formal Adjunction defined above. But at the present moment we are only dealing with *a single* reversible operator and this formal adjunction is simply a relation between two ordinary symbols that represent *that very same* reversible operator.

Proposition 1 (Generalized Adjunction). *Let $\Gamma \vdash_{\mathsf{r}} \langle t, y\rangle \equiv \langle x, t'\rangle$ be obtained by a sequence of Adjunctions. Then given any ordinary terms $\Delta \vdash r, \Theta \vdash s$ of the right polarities, we have*

$$\Delta, \Theta, \Gamma - \{x, y\} \vdash_{\mathsf{r}} \langle t[x\backslash r], s\rangle \equiv \langle r, t'[y\backslash s]\rangle .$$

The result is obtained by applying the "same" sequences of Adjunctions on the larger terms obtained by substituting r and s; more fastidiously, if $\Gamma \vdash_{\mathsf{r}} \mathbf{t}_1 \equiv \mathbf{t}_2 \equiv \ldots \equiv \mathbf{t}_m$ is a sequence of Adjunctions (or Perms) such that $\langle t, y\rangle = \mathbf{t}_1$ and $\langle x, t'\rangle = \mathbf{t}_m$ then the sequence

$$\Delta, \Theta, \Gamma - \{x, y\} \vdash_{\mathsf{r}} \mathbf{t}_1[x\backslash r, y\backslash s] \equiv \mathbf{t}_2[x\backslash r, y\backslash s] \equiv \cdots \equiv \mathbf{t}_m[x\backslash r, y\backslash s]$$

will give the desired result.

Definition 5. *Let $\langle s_0, t_0\rangle \equiv \langle s_1, t_1\rangle \equiv \cdots \langle s_n, t_n\rangle$ be a sequence of n Adjunctions. To each pair $\langle s_i, t_i\rangle \equiv \langle s_{i+1}, t_{i+1}\rangle$ assign the letter* **r** *if t_i is a subterm of t_{i+1} (in other words, if it applied in the order given in Equation (2), and the letter* **l** *if s_i is a subterm of s_{i+1}, i.e. if it is applied in the reverse order. Then the sequence is said to be in* normal form *if every application is an* **r***-Adjunction, or every application if an* **l***-Adjunction, or if $n = 0$.*

So a sequence is in normal form if it always the same side that gets larger.

Proposition 2. *Every pair $\Gamma \vdash_{\mathsf{r}} \langle s, t\rangle \equiv \langle s', t'\rangle$ can be deduced by a normal sequence of Adjunctions.*

Proof. If we have a two-adjunction sequence $\langle s_0, t_0\rangle \equiv \langle s_1, t_1\rangle \equiv \langle s_2, t_2\rangle$ where there is a change of direction, for example where the first Adjunction is **r** and the second one **l** then necessarily it looks like (the order of writing the subterms has nothing to do with the numbers associated to them)

$$\langle f^i(\ldots,_j s, \ldots,_k r \ldots), t\rangle \equiv \langle s, f^j(\ldots,_i t, \ldots,_k r, \ldots)\rangle \equiv \langle f^k(\ldots,_i t, \ldots,_j s \ldots), r\rangle ,$$

and two things may happen:

– $i = k$, so $r = t$ in which case it can be replaced by the zero-Adjunction sequence $\langle f^i(\ldots,_j s, \ldots), t\rangle$.
– $i \neq k$ in which case it can be replaced by the single **r**-Adjunction $\langle f^i(\ldots,_k r, \ldots), t\rangle \equiv \langle r, f^k(\ldots,_i t, \ldots)\rangle$

In the same way, if a two-adjunction sequence is an **l** followed by a **r** it can be replaced either by a single **l**-adjunction or by an empty sequence. So we can repeat these reduction steps as long as they are possible, and we know that if the result is not empty, then the Adjunctions will all have the same type, which will coincide with the type of the first Adjunction of the original sequence.

The following result is obvious from a geometrical point of view… but nothing less than tricky if we demand a proof by induction.

Theorem 1. *Let $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ be a reversible term. Then for any $x \in \Gamma$ there is a unique ordinary term $\Gamma - \{x\} \vdash \mathbf{t}^{[x]}$ with $\Gamma \vdash_{\mathsf{r}} \langle \mathbf{t}^{[x]}, x\rangle \equiv \mathbf{t}$.*

Proof. We proceed by induction on the number of atoms of $\mathbf{t} \equiv \langle t_1, t_2\rangle$, which is the sum of the number of atoms of t_1, t_2 (in other words, $\langle -, -\rangle$ is not considered to be an atom).

So choose k and assume that the result has been proved for every $m < n$, and let $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ be a reversible term with k symbols, and $x \in \Gamma$. Notice that this precludes the possibility that Γ be empty, i.e., that $\mathbf{t} = \langle c, c'\rangle$ for two closed terms c, c'. Then there are two possibilities: One of them is that $k = 1$, in which case we can rename the context Γ as $\Gamma = \{x_0, x_1, \ldots, x_n\}$ and there is a node symbol $\mathbf{f} \in \mathbf{\Sigma}$ such that

$$x^0, \ldots, x^n \vdash_{\mathsf{r}} \mathbf{t} \equiv \mathbf{f} \equiv \langle f^0, x^0\rangle \equiv \langle f^1, x_1\rangle \equiv \cdots \equiv \langle f^n, x_n\rangle$$

and given that $x = x_i$ for some i we take $\mathbf{t}^{[x]}$ to be $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \vdash f^i$. Uniqueness is quite trivial.

The other possibility is that $k > 1$, so $\mathbf{t}$ has at least two symbols. We can ensure that $\Gamma \vdash_{\mathsf{r}} \mathbf{t} \equiv \langle t_1, t_2\rangle$ where both t_1, t_2 have more than one symbol, i.e. neither is a variable. If it is not the case, just apply Adjunction once and transfer one symbol to the side which is a variable. So we get $\Gamma = \Gamma_1 + \Gamma_2$ with $\Gamma_i \vdash t_i, (i = 1, 2)$, with both t_i having fewer than k symbols. Without loss of generality we can assume that $x \in \Gamma_2$. Defining the term $\mathbf{s}$ by $\Gamma_2, y \vdash_{\mathsf{r}} \langle y, t_2\rangle$ we can apply the induction hypothesis and get ordinary terms $\mathbf{s}^{[x]}, \mathbf{s}^{[y]}$ with

$$\Gamma_2, y \vdash_{\mathsf{r}} \langle \mathbf{s}^{[x]}, x\rangle \equiv \langle y, t_2\rangle \equiv \langle y, s^{[y]}\rangle .$$

By induction hypothesis we have $\Gamma_2 \vdash t_2 = \mathbf{s}^{[y]}$, and then, by applying Generalized Adjunction we get

$$\Gamma_1, \Gamma_2 \vdash_{\mathsf{r}} \langle \mathbf{s}^{[x]}[y \backslash t_1], x\rangle \equiv \langle t_1, \mathbf{s}^{[y]}\rangle \equiv \langle t_1, t_2\rangle$$

so the desired $\mathbf{t}^{[x]}$ is obtained by taking $\mathbf{t}^{[x]} = \mathbf{s}^{[x]}[y \backslash t_1]$. The uniqueness is obtained by using the normal form theorem: if $\langle t^{[x]}, x\rangle \equiv \langle t', x\rangle$ then the sequence of Adjunctions that defines this equation can be normalized, and we can only end with a no-Adjunction sequence, i.e. syntactical identity between $t^{[x]}, t'$.

Reversible terms can be composed/substituted. If $\Gamma + x^{\mathsf{p}} \vdash_{\mathsf{r}} \mathbf{t}$ and $\Delta + z^{\mathsf{p}^{\perp}} \vdash_{\mathsf{r}} \mathbf{s}$ are reversible terms, we define

$$\Gamma + \Delta \vdash_{\mathsf{r}} \mathbf{t} \;{}_{x:z}\; \mathbf{s}$$

as

$$\Gamma + \Delta \vdash_{\mathsf{r}} \langle \mathbf{t}^{[x]}, \mathbf{s}^{[z]}\rangle .$$

Obviously we have $\mathbf{t} \;{}_{x:z}\; \mathbf{s} = \mathbf{s} \;{}_{z:x}\; \mathbf{t}$, and for $y \in \Gamma + \Delta$

$$(\mathbf{t} \;{}_{x:z}\; \mathbf{s})^{[y]} = \begin{cases} \mathbf{s}^{[y]}[z \backslash \mathbf{t}^{[x]}] \text{ if } y \in \Delta \\ \mathbf{t}^{[y]}[x \backslash \mathbf{s}^{[z]}] \text{ if } y \in \Gamma \end{cases} . \tag{3}$$

Using this fact we immediately get the associativity of reversible substitution, which results from the associativity of ordinary substitution:

Proposition 3. *Let* $\mathbf{r}, \mathbf{s}, \mathbf{t}$ *be reversible terms such that* $\mathbf{r}\ {}_{w:y}\ \mathbf{s}$ *and* $\mathbf{s}\ {}_{z:x}\ \mathbf{t}$ *are defined and* $y \neq z$*. Then we have*

$$(\mathbf{r}\ {}_{w:y}\ \mathbf{s})\ {}_{z:x}\ \mathbf{t} = \mathbf{r}\ {}_{w:y}\ (\mathbf{s}\ {}_{z:x}\ \mathbf{t})\,.$$

Remark 2. So another way to define a reversible term $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ with an n-variable context $\Gamma = x_0, x_1, \ldots, x_{n-1}$ would be as a set $\{t_0, \ldots, t_{n-1}\}$ of terms in the ordinary signature, with $\Gamma - \{x_i\} \vdash t_i$, these t_i being obtained as $t_i = \mathbf{t}^{[x_i]}$. The Adjunction rule can then be replaced by the rule given by Equation 3. This approach seems at first to be slightly less flexible, since it does not allow valence zero, i.e. the possibility of Γ being empty. But notice that these "more-than-constants" (terms of arity –1) do not interact with anything once they have been formed; they are inside a black hole. Therefore they are not useful for anything, they just are a byproduct of the formalism.

The transformation of an atom with n ports into n function symbols for the sake of notation is a systematization of the procedure used for interaction nets [33, 43], where a single atom may have more than one function symbol associated to it, each one ccorresponding to an "interesting" port.

In addition, the idea of having a variable for the "output port" of a term as well as the input ports is also present in [14], where term languages for a class of *-automomous categories is presented. There too, substitution (or rather Cut) is presented as a binding operator, although the same variable is used on both sides of the cut, i.e. what we write as $t\ {}_{x:z}\ s$ is written there as $t \cdot_x s$. Thus, this notation is quite close to ours, and it also has an ancestor in the "binding" notation for interaction nets [33, 43]. The equivalent of our operator $\langle -, - \rangle$ is simply written as a comma (i.e., $-, -$) in [17]. In [8] a notation very close to our $\langle -, - \langle$ is used for describing proof net graphs.

The idea of treating the output of a term on the same footing as the inputs is also implicit in another body of work, namely the relational frame semantics inaugurated (to the best of our knowledge) in [66, 67] and developed by many researchers (for a textbook introduction see [64]). In this kind of semantics, an n-ary connector is modeled by an $n+1$-ary relation on a set (more generally by an up-closed subset of a product of $n+1$ posets).

We now can extend linear order-enriched theories to reversible terms. A reversible order judgement will be an expression $\Gamma \vdash \mathbf{s} \leq \mathbf{t}$, where $\mathbf{s}, \mathbf{t}$ are reversible terms. The rules of inference are easily modified for reversibility: Reflexivity and Transitivity are:

$$\frac{}{\Gamma \vdash \mathbf{t} \leq \mathbf{t}} \qquad \frac{\Gamma \vdash \mathbf{r} \leq \mathbf{s} \quad \Gamma \vdash \mathbf{s} \leq \mathbf{t}}{\Gamma \vdash \mathbf{r} \leq \mathbf{t}}$$

and Congruence can be stated as

$$\frac{\Delta + z \vdash_{\mathsf{r}} \mathbf{s}_1 \leq \mathbf{s}_2 \qquad \Gamma + x \vdash_{\mathsf{r}} \mathbf{t}_1 \leq \mathbf{t}_2}{\Delta + \Gamma\ \vdash_{\mathsf{r}}\ \mathbf{s}_1\ {}_{z:x}\ \mathbf{t}_1\ \leq\ \mathbf{s}_2\ {}_{z:x}\ \mathbf{t}_2\,.}$$

Remark 3. It should be clear that if we forbid the premises of a Congruence application from being of the form $x, y \vdash_{\mathsf{r}} \langle x, y \rangle \leq \langle x, y \rangle$, we get a system which is as powerful, and on which we can do induction (with this restriction it is impossible to have an infinite upwards proof search branch).

Definition 6. *A* reversible (linear, order-enriched) theory $\mathscr{T}$ *is a triple* $(\mathbb{P}, \mathbf{\Sigma}, \mathrm{RevThm})$*, where* $(\mathbb{P}, \mathbf{\Sigma})$ *is a tree signature, and* RevThm*, the set of* reversible theorems*, is a set of reversible term judgements closed under the inference rules above.*

As usual, all we need to present a reversible theory is a set of reversible axioms that generate the theorems. Notice that because of Remark 2 a reversible theory is *almost* a particular kind of ordinary linear order-enriched theory, with some constraints added to the shape of the set of symbols, and with the requirement that the preorder on terms $\lesssim$ defined by the theorems has to contain a particular reflexive, symmetric and transitive relation, given by

$$\Gamma \vdash s \sim t \text{ iff } \Gamma \vdash_{\mathsf{r}} \langle s, x\rangle \equiv \langle t, y\rangle,$$

which is defined from the signature and nothing else. The only reason that a reversible theory is not exactly a kind of linear order-enriched theory is that the relation $\sim$ is defined between terms that are *not necessarily of the same sort*.

Given a reversible theory $\mathscr{T} = (\mathbb{P}, \mathbf{\Sigma}, \mathrm{RevThm})$ and a context Γ we denote by $(\widetilde{\mathscr{T}}\langle\Gamma\rangle, \lesssim)$ the set of reversible terms in context Γ, with the preorder $\mathbf{s} \lesssim \mathbf{t}$ when $\Gamma \vdash_{\mathsf{r}} \mathbf{s} \leq \mathbf{t}$, and by $(\mathscr{T}\langle\Gamma\rangle, \leq)$ the poset obtained by quotienting that preorder.

Definition 7. *Let* $\mathscr{T} = (\mathbb{P}, \Sigma, \mathrm{Thm})$ *be an ordinary linear order-enriched theory. Its* universal reversible extension $\langle\mathscr{T}\rangle$ *is the reversible theory* $(\mathbb{P}, \mathbf{\Sigma}, \mathrm{RevThm})$ *obtained by:*

- *For every symbol* $f^{\mathsf{p}_0}(x_1^{\mathsf{p}_1}, x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n}) \in \Sigma$ *of arity* n*, we have one node symbol* $\mathbf{f} \in \mathbf{\Sigma}$ *of valence* $n+1$*, with polarity* $\mathrm{Pol}(\mathbf{f}, i) = \mathsf{p}_i$ *when* $1 \leq i \leq n$*, and* $\mathrm{Pol}(\mathbf{f}, 0) = \mathsf{p}_0^{\perp}$*. Thus, after taking the derived ordinary signature* $\mathrm{OrdSign}(\mathbf{\Sigma})$ *of that tree signature, we get that every* f *in the original* Σ *of arity* n *has become* $n+1$ *symbols* $f^{0\,\mathsf{p}_0}, f^{1\,\mathsf{p}_1^{\perp}}, \ldots, f^{n\,\mathsf{p}_n^{\perp}}$ *of the same arity. Since* f^0 *is sorted exactly as* f*, every* Σ*-term* t *has a corresponding* ϕt *in the derived ordinary signature, obtained by replacing every symbol* $g \in \Sigma$ *of* t *by its corresponding* g^0 *in* $\mathrm{OrdSign}(\mathbf{\Sigma})$.
- *For every theorem* $\Gamma \vdash_{\mathsf{r}} s \leq t$ *of* $\mathscr{T}$ *we have the axiom* $\Gamma, x \vdash_{\mathsf{r}} \langle \phi s, x\rangle \leq \langle \phi t, x\rangle$ *in* $\langle\mathscr{T}\rangle$.

It should be obvious that the map ϕ induces, for every $\Gamma, x^{\mathsf{p}^{\perp}}$ a monotone map $\phi_{\Gamma,x}\colon \mathscr{T}(\Gamma, \mathsf{p}) \to \langle\mathscr{T}\rangle\langle\Gamma, x\rangle$, that maps t to $\langle\phi t, x\rangle$. We say that a reversible term $\mathbf{t}$ is *in the image of* ϕ if there is $t \in \mathscr{T}$ with $\Gamma \vdash_{\mathsf{r}} \mathbf{t} \equiv \langle\phi t, x\rangle$. The terms in the image of ϕ are the reversible terms where all the orientation tabs of their atoms point in the same direction. We say a reversible judgement $\Gamma \vdash_{\mathsf{r}} \mathbf{s} \leq \mathbf{t}$ is *in the image of* ϕ if there is $x \in \Gamma$ and is a judgement (not necessarily a theorem) $\Gamma - x \vdash s \leq t$ with $\Gamma \vdash_{\mathsf{r}} \langle s, x\rangle \equiv \mathbf{s}, \langle t, x\rangle \equiv \mathbf{t}$.

Theorem 2 (Conservativity). *The map* $\phi_{\Gamma,x}$ *is always an embedding of posets, and if* $\mathscr{T}$ *is a disjoint theory it is an isomorphism.*

The first statement follows obviously from the following, stronger

Lemma 1. *Given a theorem* $\Gamma, x \vdash \mathbf{s} \leq \mathbf{t}$ *in* $\langle\mathscr{T}\rangle$*, if one of* $\mathbf{s}, \mathbf{t}$ *is the* ϕ*-image of a term of* $\mathscr{T}$ *in context* Γ*, then the judgement* $\mathbf{s} \leq \mathbf{t}$ *is the* ϕ*-image of a judgement* $\Gamma \vdash s \leq t$ *of* $\mathscr{T}$ *which is moreover a theorem in* $\mathscr{T}$.

Proof. This is done by induction on the proof of $\mathbf{s} \leq \mathbf{t}$(cf. Remark3 .

- If the last rule is a Reflexivity axiom, the result is trivial to obtain.
- If the last rule is an application of Transitivity, giving $\mathbf{r} \leq \mathbf{t}$ from the premises $\mathbf{r} \leq \mathbf{s}$ and $\mathbf{s} \leq \mathbf{t}$, assume that $\mathbf{r} \equiv \langle \phi r, x \rangle$ is in the image of ϕ. Then by using the induction hypothesis on the fi rst premiss and the fact that $\mathbf{r}$ belongs to that premiss, we get that it is of the form $\langle \phi r, x \rangle \leq \langle \phi s, x \rangle$, with $r \leq s$ a theorem of $\mathscr{T}$. This shows that $\mathbf{s}$ is in the image of ϕ, and we can apply the induction hypothesis on the second premiss, getting that if is of the form $\langle \phi s, x \rangle \leq \langle \phi t, x \rangle$, for $s \leq t$ a theorem of $\mathscr{T}$. But then $\Gamma \vdash_{\mathsf{r}} r \leq t$ is a theorem of $\mathscr{T}$ because of ordinary Transitivity, and $\mathbf{r} \leq \mathbf{s}$ is the ϕ-image of that theorem. The other case ($\mathbf{t}$ in the image of ϕ) is obtained the same way.
- If the last rule is an application of Congruence

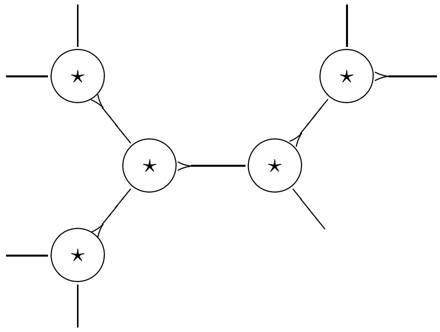
$$\frac{\Delta + z \vdash_{\mathsf{r}} \mathbf{s}_1 \leq \mathbf{s}_2 \qquad \Gamma + x \vdash_{\mathsf{r}} \mathbf{t}_1 \leq \mathbf{t}_2}{\Delta + \Gamma \ \vdash_{\mathsf{r}} \ \mathbf{s}_1 \ {}_{z:x}\, \mathbf{t}_1 \ \leq \ \mathbf{s}_2 \ {}_{z:x}\, \mathbf{t}_2} \quad ,$$

  let $\mathbf{r} = \mathbf{s} \ {}_{z:x}\, \mathbf{t}$ be one of $\mathbf{s}_i \ {}_{z:x}\, \mathbf{t}_i$ $(i = 1, 2)$ and assume $\mathbf{r} = \langle \phi r, y \rangle$ is in the image of ϕ. Because of the symmetry of the formula we can assume without loss of generality that y is a variable of $\mathbf{t}$, i.e., $y \in \Gamma$. Then, the defi nition of reversible substitution being $\mathbf{r} \equiv \langle \phi r, y \rangle = \langle \mathbf{t}^{[y]}[x \backslash \mathbf{s}^{[z]}], y \rangle$ (equation 3), we get from the uniqueness part of Theorem 1 that $\phi r =\equiv \mathbf{t}^{[y]}[x \backslash \mathbf{s}^{[z]}]$, and since both $\mathbf{t}^{[y]}, \mathbf{s}^{[z]}$ are subterms of one which is in the image of ϕ, they are themselves in the image of ϕ because all their atomic symbols have superscript zero. So we can apply induction on the premisses and get that the left premiss is of the form $\Delta + z \vdash_{\mathsf{r}} \langle \phi s_1, z \rangle \leq \langle \phi s_2, z \rangle$ and the right one of the form $\Gamma + x \vdash_{\mathsf{r}} \langle \phi t_1, y \rangle \leq \langle \phi t_2, y \rangle$, with $\Delta \vdash s_1 \leq s_2$ and $\Gamma + x - y \vdash t_1 \leq t_2$ theorems of $\mathscr{T}$. Then we can apply ordinary Congruence on these terms and get $\Delta + \Gamma - y \vdash t_1[x \backslash s_1] \leq t_2[x \backslash s_2]$ and since obviously $\langle \phi t_i[x \backslash \phi s_i], y \rangle \equiv \mathbf{s}_i \ {}_{x:z}\, \mathbf{t}_i$ for both $i = 1, 2$ we obtain the desired result.
- If the last rule of the deduction is an axiom (i.e. the image by ϕ of a theorem of $\mathscr{T}$), the the result holds by defi nition.

We still have to prove the second statement, which amounts to saying that $\phi_{\Gamma,x}$ is always surjective when $\mathscr{T}$ is disjoint. But if $\mathscr{T}$ is disjoint, a reversible term $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ will always have a single variable $x^{\mathsf{p}^{\perp}}$, where p is a polarity that appears in the signature of $\mathscr{T}$. This is very easy to show: we know it is the case for atomics, and is proved in the general case by a trivial induction. Then obviously the term t of the ordinary signature such that $\Gamma \vdash_{\mathsf{r}} \mathbf{t} \equiv \langle t, x \rangle$ will be of the form $\phi t'$.

One conclusion that can be drawn from this result is that the world of reversible terms has more expressive power than the ordinary world of terms (as will be amply shown below), but it is not *too* powerful, since the process of universal reversible extension turns a disjoint theory into a reversible one that does not have more terms/structures and theorems, only more ways of writing the same objects.

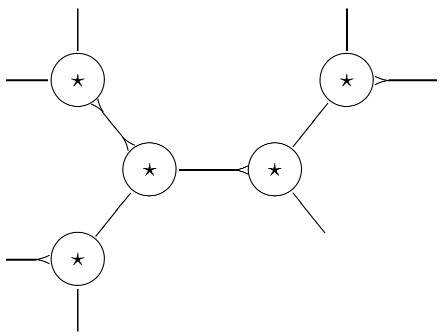
Example 11. The Conservativity result tells us that any ultrabasic theory $\mathscr{T}$, like the ones we have seen above, can be turned into a reversible one without changing anything except the presentation of the objects. For example, let $\mathscr{T}_{\mathrm{FreeBin}}$ be the ultrabasic theory with a single binary operation $(-)\star(-)$, and nothing else. Here a term with n variables can be seen as a rooted binary tree with n leaves. If we take its universal reversible extension, the ordinary signature associated to it will now have three binary symbols $\star^0, \star^1, \star^2$, with the expected equations $\langle x \star^0 y, z\rangle \equiv \langle y \star^1 z, x\rangle \equiv \langle z \star^2 x, y\rangle$, and the reversible terms will differ in that they will have one additional variable for output. But this variable will be distinguished by being the only one with polarity $\circ$. A term like $(x\star y)\star z$ in $\mathscr{T}_{\mathrm{FreeBin}}$ will now be represented by $\langle (x\star^0 y)\star^0 z, w\rangle$, with $x^\bullet, y^\bullet, z^\bullet, w^\circ$, but also by $\langle w\star^2 (x\star^0 y), z\rangle$, $\langle (z\star^1 w)\star^2 x, y\rangle$, and son on, but Conservativity tells us that in the end there will not be more reversible terms, i.e. $\equiv$-classes, than in the original theory, since any reversible term is $\equiv$-equivalent to one where only the symbol $\star^0$ appears. The geometric meaning ot this is that the trees have a natural root assignment because of the polarity, in other words “all orientation tabs have to point in the same direction, that is, towards the root”.



This is the theory of contexts for Lambek's original non-associative calculus $\mathbf{NL}$ [37].

Example 12. Suppose now that we change only one thing: we decide there is a unique polarity c. We get a reversible theory $\langle \mathscr{T}^{\mathsf{c}}_{\mathrm{FreeBin}}\rangle$ that has more reversible terms (e.g., $\langle (x\star^0 y)\star^2 z, w\rangle$ could not be constructed before), because the polarity discipline is relaxed.

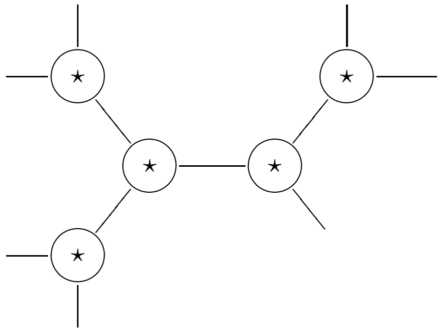
A term now is still a binary tree, but since the “orientation tabs” of their nodes can point in any direction there is no way of assigning a root to an arbitrarily chosen term without making an arbitrary choice.



Example 13. To $\langle \mathscr{T}^{\mathsf{c}}_{\mathrm{FreeBin}}\rangle$ we can add the axiom Cyc given by $\langle x\star^0 y, z\rangle = \langle x\star^1 y, z\rangle$. We can then deduce

$$\langle x \star^2 y, z\rangle \equiv \langle z \star^1 x, y\rangle = \langle z \star^0 x, y\rangle \equiv \langle x \star^1 y, z\rangle$$

So we get that $\langle x \star^i y, z\rangle$ are equal for $i = 0, 1, 2$ which allows us to drop these superscripts. In other words the "orientation tab" has disappeared, and we have the theory of non-rooted binary *planar* trees, which is the theory of contexts for the system **NCLL** of non-associative Lambek calculus, discussed in [17] and also in [61]. There is still some rigidity in these trees because they have to be thought of as embedded in the plane and they are still somewhat oriented.



We know that every reversible term $\Gamma \vdash_{\mathsf{r}} \mathbf{t}$ is of the form $\Gamma \vdash_{\mathsf{r}} \langle t', z\rangle$ for t' a term built with the three symbols of $\langle \mathscr{T}\rangle$, but in addition we can prove that $\Gamma \vdash_{\mathsf{r}} \langle t', z\rangle = \langle t'', z\rangle$, where t'' is obtained from t' by replacing every occurence of $\star^1$ and $\star^2$ by $\star^0$. So we get that the composite map

$$\langle \mathscr{T}_{\mathrm{FreeBin}}\rangle\langle \Gamma\rangle \longrightarrow \langle \mathscr{T}^{\mathsf{c}}_{\mathrm{FreeBin}}\rangle\langle \Gamma\rangle \longrightarrow \langle \mathscr{T}^{\mathsf{c}}_{\mathrm{FreeBin+Cyc}}\rangle\langle \Gamma\rangle$$

is a bijection of sets (here the posets are discrete since all the axioms are equalities), the first map being the inclusion of the "rootable" trees into the larger set of "unrootable" ones. So the quotienting introduced by the Cyc axiom has the effect of "cancelling" all these new terms that were added by the switch from ultrabasic to $\mathbb{C}$-sorted.

Example 14. To the example above we can add the axiom $\langle x \star y, z\rangle = \langle y \star x, z\rangle$, in which case we get the theory of non-rooted binary trees without any orientation, i.e. for which all possible permutations of the ports of an atom are valid rules, (in other words, do not change the context structure). This is the theory of "mobiles", which is discussed in [61].

Example 15. Now look at $\mathscr{T}_{\mathrm{Ass}}$, the theory of semigroups (one associative binary operator $*$). The universal reversible extension has for signature $\{*^0, *^1, *^2\}$, and Conservativity tells us that any reversible term will have the normal form $\langle x_1 *^0 x_2 *^0 \ldots *^0 x_n, z\rangle$, i.e. a string of x_i and a z to mark the output. If we now replace the set of polarities by $\mathbb{C}$, we get the theory $\mathscr{T}^{\mathsf{c}}_{\mathrm{Ass}}$, where there are many more reversible terms. It is not hard to find a geometric representation for these, and we will leave this as an exercise.

The axiom Cyc above can also be added to the theory, and the same argument gives us $\langle x *^0 y, z\rangle = \langle x *^1 y, z\rangle = \langle x *^2 y, z\rangle$. From this it is easy to prove that in general a reversible term has the form $\Gamma \vdash_{\mathsf{r}} \langle a, b\rangle$, where $a, b \in \Gamma^*$ are words such that every variable of Γ appears exactly once in the pair, subject to the equations

$$\langle a * b, c\rangle = \langle a, b * c\rangle = \langle c * a, b\rangle \; ,$$

in other words where the reversible term $\langle a, b\rangle$ is the equivalence class of the word $a * b$ modulo all cyclic permutations (see Example 22). These objects are sometimes called cyclic words. Here also the composite map:

$$\langle \mathscr{T}_{\mathrm{Ass}}\rangle\langle \Gamma\rangle \longrightarrow \langle \mathscr{T}^{\mathsf{c}}_{\mathrm{Ass}}\rangle\langle \Gamma\rangle \longrightarrow \langle \mathscr{T}^{\mathsf{c}}_{\mathrm{Ass+Cyc}}\rangle\langle \Gamma\rangle$$

is always bijective (a more proper, although more cumbersome, notation is $\langle \mathscr{T}^{\mathsf{c}}_{\mathrm{Ass}}\rangle_{+\mathrm{Cyc}}$), since it is the correspondence $a \leftrightarrow \langle a, z\rangle$, where $a \in \Gamma^*$ is a non-repeating word and z is a new variable: if you pick out a single letter from a cyclic word the other letters form an ordinary word.

This is the theory of contexts for Yetter's cyclic linear logic [73].

Example 16. There are interesting theories that fall between $\langle \mathscr{T}^{\mathsf{c}}_{\mathrm{Ass}}\rangle$ and $\langle \mathscr{T}^{\mathsf{c}}_{\mathrm{Ass+Cyc}}\rangle$. A simple example is the one axiomatized by

$$\langle (x *^0 y) *^1 z, w\rangle = \langle x, (y *^0 z) *^1 w\rangle\,.$$

A complete study of all such reversible theories is an interesting problem, but beyond this fi rst paper.

Remark 4. In Display Logic the function of reversible terms is taken up by what is known as display postulates. But as their name implies, display postulate have a logical rôle as well as an algebraic one, while at this stage we are still purely in the algebraic realm. In our approach the logical function of display postulates is taken up by the orientation function Or defi ned in the last section.

The reader may think that this distinction is mere hair splitting, since the expressive power of our approach is pretty much equivalent to that of Display logics. But this separation of the algebraic and logical functions greatly increases our abilities to understand these systems. For example the examples at the end of this paper all seem to be new systems. Also, the author has found structads to be a great tool for investigating the relationship between a non-involutive negation and non-commutativity (as in [2])

4 The Algebraic Approach

There is some amount of independence between this section an the one that follows, and we do not think that it is necessary to master everything here to be able to understand the fi nal chapter of this paper.

In the present section we defi ne the concept of structad, which is a precise description of the algebraic structure possessed by a reversible theory. Thus every reversible theory gives rise to its canonical associated “concrete” structad. A reversible theory should be seen as a *presentation* of a structad, the same kind of presentation that can be given to a monoid: generators and relations. The generators are the elements of the tree signature and the relations the axioms of the theory. We will end up showing the natural converse, namely that from a given structad we can always extract a tree signature and a set of and a reversible theory on it, such that the concrete structad thus constructed will be isomorphic to the one we started with. The most important reason that we introduce abstract structads in addition to the concrete ones (which, arguably, have already been defined in the preceding section) is not the desire to align our approach with conventional abstract algebra. The real reason is that there are many examples of structads that do not come pre-equipped with a signature and an accompanying theory, as we will see.

We have already been using the fact that renaming commutes with repolarizing: in other words if $P\colon \mathbb{P} \to \mathbb{P}'$ is an involution-respecting map, and $\sigma\colon \Gamma \to \Delta$ a renaming in $\mathrm{Ctx}(\mathbb{P})$, then $P(\sigma(\Gamma)) = \sigma(P(\Gamma))$. One quick way of establishing this is to recall the remark that an algebraic context Γ can be seen as a finite set of pairs $\Gamma \subseteq \mathscr{V} \times \mathbb{P}$; then σ acts on the first components and P acts on the second components.

In the following definition, the presence of an involution on $\mathbb{P}$ is not needed.

Definition 8. *A* $\mathbb{P}$-sorted poset species *is given by the following:*

- *For every* $\Gamma \in \mathrm{Ctx}(\mathbb{P})$ *a poset* $\big(\mathbf{O}\langle\Gamma\rangle, \leq\big)$*. We call an element of* $\mathbf{O}\langle\Gamma\rangle$ *a* structure *on* Γ*.*
- *for every bijective* $\sigma\colon \Gamma \to \Delta$ *an isomorphism of posets* $\mathbf{O}\langle\sigma\rangle\colon \mathbf{O}\langle\Gamma\rangle \to \mathbf{O}\langle\Delta\rangle$*, that respects the composition and identities: for* $\rho\colon \Delta \to \Phi$ *we have* $\mathbf{O}\langle\rho \circ \sigma\rangle = \mathbf{O}\langle\rho\rangle \circ \mathbf{O}\langle\sigma\rangle$ *and* $\mathbf{O}\langle\mathrm{Id}_\Gamma\rangle = \mathrm{Id}_{\mathbf{O}\langle\Gamma\rangle}$*.*

This definition is a natural (in other words, not adventurous at all) generalization of the concept of $\mathbb{P}$-*sorted species of structure* [29, 6], due to A. Joyal. The added generality is due to the fact that we allow the sets of structures to be ordered and not necessarily finite. The category-inclined reader can reformulate this by saying that $\mathbf{O}$ is a functor from the groupoid[2] of $\mathbb{P}$-contexts and renamings to the category of posets and monotone functions.

Example 17. Let $\mathbb{P} = \mathbb{C}$. Given a set Γ we take $\mathbf{O}\langle\Gamma\rangle$ to be the set of all order (poset) structures on Γ. In other words, an element $\alpha \in \mathbf{O}\langle\Gamma\rangle$ is a subset $\alpha \subseteq \Gamma \times \Gamma$ that obeys Reflexivity, Transitivity and Antisymmetry. The order on $\mathbf{O}\langle\Gamma\rangle$ is the inclusion order, in other words given $\alpha, \beta \in \mathbf{O}\langle\Gamma\rangle$ we have $\alpha \leq \beta$ when $\alpha \subseteq \beta$. Given a bijection $\sigma\colon \Gamma \to \Delta$ and $\alpha \in \mathbf{O}\langle\Gamma\rangle$, we take $\mathbf{O}\langle\sigma\rangle(\alpha) \in \mathbf{O}\langle\Delta\rangle$ to be the binary relation β on Δ defined by

$$x \leq_\beta y \quad \text{if} \quad \sigma^{-1}(x) \leq_\alpha \sigma^{-1}(y)\,.$$

It is easy to see that this definition ensures that β is an order structure on Δ and that σ is an isomorphism of posets $\sigma\colon (\Gamma, \alpha) \to (\Delta, \beta)$. It is also easy to see that the map $\mathbf{O}\langle\sigma\rangle\colon \big(\mathbf{O}\langle\Gamma\rangle, \leq\big) \to \big(\mathbf{O}\langle\Delta\rangle, \leq\big)$ is also an isomorphism of posets, these two posets being quite distinct from the previous ones.

Naturally, instead of all order structures on Γ, we could have taken total (linear) orderings, or just plain preorders (i.e., removed Anti-symmetry), and obtained different species. This example illustrates well what a species is about: it is way of describing a type of structure that can be put on a finite set, by the means of “enumerating” all possible instances of it on all possible finite sets[3]. The concept of a mathematical structure

[2] Groupoid in the category-theoretical sense, of course.

[3] It should be clear that by “all” finite sets, we actually mean “enough finite sets to have a representative for every cardinality class”, in other words “at least one set per cardinality”. In

is very broad, and is probably the kind of idea that will never be pinned down exactly, since this may mean the freezing of progress in mathematics. For example many kinds of structures (like the three above) can be defined by the means of first order logic, which is already a very general set of techniques for defining structures. But first-order logic has well-known limitations. Species are based on the idea that if you know what a structure on a set Γ is, *you have to know how to transport it to another set Δ via a bijection*.

It often turns out that, given a type of structures, the set of all structures of that type that can be put on a chosen set can be naturally ordered, as is the case above. We have said that the importance of this ordering for us is that it allows the calculi derived from structads to have entropy rules.

We do not have time to say much about the theory of species, but we want to mention the following.

Definition 9. *Let $\mathbf{O}$ be a $\mathbb{C}$-sorted Joyal species, in other words it is a poset species as above, with the restriction that $\mathbf{O}\langle\Gamma\rangle$ is always a discretely ordered finite set. For every $n \in \mathbb{N}$ let o_n be the cardinal of the set $\mathbf{O}\langle\Gamma\rangle$ when Γ has n elements. It should be obvious that the value of o_n is independent of the choice of Γ. The* generating series *associated to $\mathbf{O}$ is the formal power series in one variable.*

$$\mathbf{O}(x) = \Sigma_{n=0}^{\infty} \frac{o_n}{n!} x^n$$

Species were invented as a foundational tool for enumerative combinatorics. Generating series have been around since at least Euler, and are used to count how many structures of a given type (traditionally: graphs, trees, permutations, partitions. . .) can be put on a finite set of cardinality n. A Joyal species $\mathbf{O}$ should be seen as a "beefed up" power series, an object which contains more information than its associated generating series. It turns out that a great number of the operations that can be done on power series (addition, multiplication, substitution, derivation. . .) can be lifted to the world of species and be given a direct combinatorial meaning. We find it obvious that problems of proof search complexity in substructural logics would benefit from the availability of this powerful combinatorial tool.

We have to broaden the idea of structure on a set to include the possibility of more than one sort. For example:

Example 18. Let now $\mathbb{P} = \mathbb{B}$. Obviously any $\mathbb{P}$-context Γ can be split into two sets $\Gamma = \Gamma^\bullet + \Gamma^\circ$, according to the polarity of its elements/variables. Given any context Γ we define $\mathbf{O}\langle\Gamma\rangle$ to be the set of functions $\Gamma^\bullet \to \Gamma^\circ$. Given a renaming of contexts

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this paper we have chosen our universe of finite sets to be all subsets of a given infinite set $\mathscr{V}$. We find this approach to be the most natural, given the intended meaning and the constructions we want to make. But the reader can also imagine that we are working in hereditaty finite sets, a very standard was of constructing a world of finite sets. Another approach, which is favored by practitioners of the theory of operads, is to keep a single isomorphism class for every finite cardinality. So in this case, for every natural number n we would have a single given context $\{x_1^n, x_2^n, \ldots, x_n^n\}$, and two contexts end up being disjoint if their cardinalities are distinct. All these approaches are equivalent in the end, but require more or less technical ingenuity if we want everything to be completely explicit.

$\sigma: \Gamma \to \Delta$, it naturally splits into two bijections $\sigma^{\mathsf{p}}: \Gamma^{\mathsf{p}} \to \Delta^{\mathsf{p}}, \mathsf{p} = \bullet, \circ$. Given $f \in \mathbf{O}\langle\Gamma\rangle$ we define $\mathbf{O}\langle\sigma\rangle(f)$ to be the function $\sigma^{\circ} \circ f \circ (\sigma^{\bullet})^{-1}$. We denote that species by $\mathbf{O}_{\mathrm{Funct}}$.

In other words, if a function between a pair of finite sets is thought of as a species, it is naturally a two-sorted species, one sort for the source and one for the target. This is another example of species that has many variations: we can take $\mathbf{O}_{\mathrm{Surj}}$ to be the subspecies that has only the surjective (onto) functions, or $\mathbf{O}_{\mathrm{Injec}}$, which has only the injective functions. . .

The generating series associated to multi-sorted Joyal species are power series in many variables, one for each sort/polarity.

In general, given a poset species $\mathbf{O}$, when we say a structure in $\mathbf{O}$ we mean a pair (Γ, α) with $\alpha \in \mathbf{O}\langle\Gamma\rangle$; we will also say things like "let $\alpha \in \mathbf{O}\langle\Gamma\rangle$ be an $\mathbf{O}$-structure" to mean the same thing.

Definition 10. *Let $\mathbf{O}$ be a poset species, and $\alpha \in \mathbf{O}\langle\Gamma\rangle, \alpha' \in \mathbf{O}\langle\Gamma'\rangle$ be two structures. An* isomorphism $\sigma: (\Gamma, \alpha) \to (\Gamma, \alpha')$ *is a renaming $\sigma: \Gamma \to \Gamma'$ such that $\mathbf{O}\langle\sigma\rangle(\alpha) = \alpha'$. An* automorphism *is an isomorphism $\sigma: (\Gamma, \alpha) \to (\Gamma, \alpha)$ from a structure to itself. We say two structure are* isomorphic *if there is an isomorphism σ between them.*

For example, we have already seen that if $\mathbf{O}$ is the species of posets, to say that two orderings α on Γ and α' on Γ' are isomorphic by the definition above iff they are isomorphic as finite posets. In general, if a species is defined by the means of first order logic, this purely formal notion of isomorphism will coincide with the one prescribed by logic. Naturally this definition works without the need for a specification language like first-order logic.

The following are proved by easy computations:

- the identity renaming on Γ is an automorphism for any (Γ, α).
- isomorphisms compose: if $\sigma: (\Gamma, \alpha) \to (\Gamma', \alpha')$ and $\sigma': (\Gamma', \alpha') \to (\Gamma'', \alpha'')$ are isomorphisms then $\sigma' \circ \sigma: (\Gamma, \alpha) \to (\Gamma', \alpha'')$ is an iso.
- If $\sigma: (\Gamma, \alpha) \to (\Gamma, \alpha')$ is an isomorphism then $\sigma^{-1}: (\Gamma, \alpha') \to (\Gamma, \alpha)$ is also an isomorphism.
- Being isomorphic is an equivalence relation on structures.

We can now introduce the main concept of this section

Definition 11. *A* structad *is a poset species $\mathbf{O}$ which in addition is equipped, for every pair $\Gamma + x^{\mathsf{p}}, \Delta + z^{\mathsf{p}^{\perp}}$ of contexts, with a composition operation,*

$$(-) \;{}_{x:z}\; (-) \;\; : \;\; \mathbf{O}\langle\Gamma + x\rangle \times \mathbf{O}\langle\Delta + z\rangle \to \mathbf{O}\langle\Gamma + \Delta\rangle$$

monotone in both variables, which is subject to three laws, described very soon. These laws are associativity, unit(s), and agreement with transport.

Before we state these laws, let us say that the strictest reading of the diagram above, the first one that will come to many readers, makes us assume that Γ and Δ are disjoint sets, but that x may belong to Δ and z to Γ. This is correct, but there is also a more relaxed way of seeing things, where Γ, Δ can be *any* two sets of variables. This approach may be more natural for some readers, but for the time being we will stick to the "strict" reading of the definition of composition. The first two laws are as follows:

– Associativity: $(\alpha \ {}_{x:z}\ \beta)\ {}_{y:w}\ \gamma = \alpha\ {}_{x:z}\ (\beta\ {}_{y:w}\ \gamma)$, whenever this is defined, which implies in particular that $y \neq z$ and both $\alpha\ {}_{x:z}\ \beta$ and $\beta\ {}_{y:w}\ \gamma$ are defined.
– Unit: for every polarity p and every two-variable context $x^{\mathsf{p}}, z^{\mathsf{p}^{\perp}}$ there is a structure $1_{x,z} = 1_{z,x} \in \mathbf{O}\langle x, z\rangle$ ("the x-z-wire", which naturally can also be denoted $\langle x, z\rangle$) which obeys the usual identity law: for $\beta \in \mathbf{O}\langle \Delta + z^{\mathsf{p}^{\perp}}\rangle$ and $\alpha \in \mathbf{O}\langle \Gamma + x^{\mathsf{p}}\rangle$ we have $\beta\ {}_{z:x}\ 1_{x,z} = \beta$ and $\alpha\ {}_{x:z}\ 1_{x,z} = \alpha$. The very-well-known argument for monoids applies here to show that $1_{x,z}$ is uniquely defined.

Let us introduce an improvement in notation that will make the agreement law look very natural. Let there be $\sigma\colon \Gamma \to \Gamma'$ and $\tau\colon \Delta \to \Delta'$ along with $\alpha \in \mathbf{O}\langle \Gamma\rangle, x^{\mathsf{p}} \in \Gamma$ and $\beta \in \mathbf{O}\langle \Delta\rangle, z^{\mathsf{p}^{\perp}} \in \Delta$. Let us denote by $\Gamma\ {}_{x}\dagger_{z}\ \Delta$ the set $\Gamma - \{x\} + \Delta - \{z\}$. So, in accordance with our "strict reading", this assumes that $\Gamma \cap \Delta \subseteq \{x, z\}$. If, in addition, we have $\Gamma' \cap \Delta' \subseteq \{\sigma x, \tau z\}$, we can define

$$\sigma\ {}_{x}\dagger_{z}\ \tau : \ \Gamma\ {}_{x}\dagger_{z}\ \Delta \xrightarrow{\sim} \Gamma'\ {}_{\sigma x}\dagger_{\tau z}\ \Delta'$$

in the obvious way. The agreement law is the requirement that

$$\mathbf{O}\langle \sigma\ {}_{x}\dagger_{z}\ \tau\rangle\big(\alpha\ {}_{x:z}\ \beta\big) = \big(\mathbf{O}\langle\sigma\rangle\alpha\big)\ {}_{\sigma x:\tau z}\ \big(\mathbf{O}\langle\tau\rangle\beta\big)\ .$$

Naturally this has to hold whenever we can transport composable structures α, β along compatible renamings σ, τ.

Now that the three rules are stated, they can be combined in interesting ways, that permit the "relaxed" reading we have alluded to. The following discussion can be omitted if the reader is perfectly satisfied with the way we have defined things; it is the outcome of conversations with people who were puzzled by remarks like (just before Definition 3) "if the union of contexts is not disjoint just rename to satisfaction".

We have defined an algebraic structure, but its operations can only be applied when some constraints are respected, and these constraints do not look very algebraic themselves. Let $\alpha \in \mathbf{O}\langle\Gamma\rangle$ and $\beta \in \mathbf{O}\langle\Delta\rangle$ and $x \in \Gamma, y \in \Delta$ be given. If we want to construct $\alpha\ {}_{x:y}\ \beta$, a necessary (and natural) condition is that x, y have opposite polarities, but there is also this need that the only variables common to Γ and Δ be x, y. Suppose this were not the case. We could construct $\alpha\ {}_{x:y}\ \beta$ (or something absolutely equivalent) if we renamed one of the contexts, or both: we can always find sets Γ', Δ' along with bijections $\sigma\colon \Gamma \to \Gamma'$ and $\rho\colon \Delta \to \Delta'$ such that $\Gamma' \cap \Delta' = \emptyset$, (this is stricter than what we have asked in the definition, but very easy to achieve) $\sigma x = x$ and $\rho y = y$. Then we could transport α along σ, getting $\alpha' \in \mathbf{O}\langle\Gamma'\rangle$ and β along ρ, getting $\beta' \in \mathbf{O}\langle\Delta'\rangle$, and in this new context the structure $\alpha'\ {}_{x:y}\ \beta'$ would be legal. As a matter of fact we wouldn't even have to force $\sigma x = x, \sigma y = y$, since $\alpha'\ {}_{\sigma x:\rho x}\ \beta'$ would do just the same as an alternative definition of the composite, but for the simplicity's sake we will not follow this added possibility of generality.

The reader is allowed to think that if we decide to loosen things up this way (if you can't compose two things in a context, define their composite in a suitably chosen pair of isomorphic contexts, where composition is allowed) then anything goes and the operation of composition has become some kind of unstructured, soft camembert-like affair.

But this is not true. Such a seemingly “loose” composition is just as strict as the one we started with. The reason why: let $\sigma' : \mathit{\Gamma} \to \mathit{\Gamma}''$ and $\rho' : \mathit{\Delta} \to \mathit{\Delta}''$ be two new isomorphisms, such that $\mathit{\Gamma}'' \cap \mathit{\Delta}'' = \emptyset$ and $\sigma' x = x$ and $\rho' y = y$. Then again we can transport α, β along σ', ρ' and get $\alpha'' \in \mathbf{O}\langle \mathit{\Gamma}'' \rangle$ and $\beta'' \in \mathbf{O}\langle \mathit{\Delta}'' \rangle$. We obviously can construct $\alpha''\ {}_{x}{:}_{y}\ \beta''$ in the strictest sense. We now have two alternative choices for the composite $\alpha\ {}_{x}{:}_{y}\ \beta$, and we claim they are linked by a very important property: there is a canonical isomorphism

$$\tau : \mathit{\Gamma}'\ {}_{x}\dagger_{y}\ \mathit{\Delta}' \to \mathit{\Gamma}''\ {}_{x}\dagger_{y}\ \mathit{\Delta}''$$

obtained by taking $\tau = (\sigma' \circ \sigma^{-1})\ {}_{x}\dagger_{y}\ (\rho' \circ \rho^{-1})$ and it has the property that

$$\mathbf{O}\langle \tau \rangle (\alpha'\ {}_{x}{:}_{y}\ \beta') = \alpha''\ {}_{x}{:}_{y}\ \beta''\ .$$

The proof is an immediate consequence of the property of agreement with transport.

So we conclude that we can define $\alpha\ {}_{x}{:}_{y}\ \beta$ whenever we want, provided we change to suitable contexts *and keep track of the change of context*, that is, keep record of the isomorphisms we have used to change contexts and transport things there. There is a canonical way of relating the composite defined in one choice of context with the one defined with another choice, and it is a *uniquely defined* isomorphism of structures.

We can apply this idea to more complex cases. For example let $\alpha \in \mathbf{O}\langle \mathit{\Gamma}_\alpha \rangle$, $\beta \in \mathbf{O}\langle \mathit{\Gamma}_\beta \rangle$ and $\gamma \in \mathbf{O}\langle \mathit{\Gamma}_\gamma \rangle$ be given. If we want to construct $(\alpha\ {}_{x}{:}_{y}\ \beta)\ {}_{z}{:}_{w}\ \gamma$, the conditions that really matter are that $x \in \mathit{\Gamma}_\alpha, y, z \in \mathit{\Gamma}_\beta, w \in \mathit{\Gamma}_\gamma$ and $y \neq z$, along with the obvious polarity conditions. We can always find a large context $\mathit{\Delta}$ and three polarity-respecting embeddings $\mathit{\Gamma}_\alpha \to \mathit{\Delta}$, $\mathit{\Gamma}_\beta \to \mathit{\Delta}$ and $\mathit{\Gamma}_\gamma \to \mathit{\Delta}$, such that $\mathit{\Delta}$ is the union of the images of the three embeddings, and that the construction of $(\alpha\ {}_{x}{:}_{y}\ \beta)\ {}_{z}{:}_{w}\ \gamma$ in that context is correctly defined according to the strictest standard. Moreover if we find another context $\mathit{\Delta}'$, along with embeddings of the same kind as above, there will be a *unique* bijection $\mathit{\Delta} \to \mathit{\Delta}'$ that will respect the embeddings. Then the associativity law can be stated in any suitable context, whether $\mathit{\Delta}$ or $\mathit{\Delta}'$ or whatever. The canonical isomorphisms allow the coherent transport not only of structures, but of *equations* between structures.

In the same way we can use transport to get a more general form of the unit law: given $\alpha \in \mathbf{O}\langle \mathit{\Gamma} + x \rangle$ then for any variable y^{p} not in $\mathit{\Gamma}$, we have $\alpha\ {}_{x}{:}_{z}\ 1_{x,y} = \mathbf{O}\langle \sigma \rangle \alpha$, where σ is the unique bijection of contexts $\mathit{\Gamma} + x \to \mathit{\Gamma} + y$ which is the identity on $\mathit{\Gamma}$. This is the unit law, expressed in such a way that the variable y in the wire $1_{x,y}$ which is not cancelled by composition does not have to appear in the context of α.

The main conclusion the reader should draw from this discussion is that when we write a complex expression, there are obvious constraints that the variables that are not explicitly mentioned have to obey with respect to the ones that are explicitly mentioned. But these constraints do not have to be stated in details, and if a clash occurs, it can always be repaired by the means of tranport. What matters is not the exact *names* of the variables, but the way they are distributed with respect to the ports that they name. One way of saying things is that the exact naming of variables is the burden of the “system”; getting new variables is a result of a system call, and it is the system’s job of ensuring that these variables do not clash with the ones we have before; the formal machinery of renaming and transport tells us that this can always be done correctly and that the reader, who uses a higher level language, can forget about these details.

Definition 12. *A structad* $\mathbf{O}$ *is said to be* discrete *if* $\mathbf{O}\langle\Gamma\rangle$ *is discrete for every* Γ. *Given any structad* $\mathbf{O}$ *its* opposite $\mathbf{O}^{\mathrm{op}}$ *is defined by* $\mathbf{O}^{\mathrm{op}}\langle\Gamma\rangle = \big(\mathbf{O}\langle\Gamma\rangle\big)^{\mathrm{op}}$.

Example 19. It is not clear how to endow the species $\mathbf{O}_{\mathrm{Poset}}$ of orderings (Example 17) with a composition operation; but we will see that this poset species has a nice one-sided structad completion in the form of $\mathbf{O}_{\mathrm{OrdVar}}$ (Example 27).

Example 20. The discrete species $\mathbf{O}_{\mathrm{Funct}}$ of functions has a natural structad structure. Assume for ease of notation that $\mathbf{O} = \mathbf{O}_{\mathrm{Funct}}$. Let $f \in \mathbf{O}\langle\Gamma + x^\circ\rangle$ and $g \in \mathbf{O}\langle\Delta + y^\bullet\rangle$. By definition $f\ {}_{x}{:}_{y}\ g$ should be a function $h\colon \Gamma^\bullet + \Delta^\bullet \to \Gamma^\circ + \Delta^\circ$. It is given by

$$h(z^\bullet) = \begin{cases} f(z) \text{ if } z \in \Gamma \text{ and } f(z) \neq x \\ g(y) \text{ if } z \in \Gamma \text{ and } f(z) = x \\ g(z) \text{ if } z \in \Delta. \end{cases}$$

The unique structure on a set of the form $\{x^\bullet, y^\circ\}$ can easily be seen to be a unit for composition. We will leave it to the reader to check the rest of the structad axioms.

Example 21. Let $\mathscr{T} = (\mathbb{P}, \mathbf{\Sigma}, \mathrm{RevThm})$ be a reversible order-enriched theory. It is now only a formality to show that it defines a structad $\mathbf{O}$, where, given a $\mathbb{P}$-context Γ, we take $\mathbf{O}\langle\Gamma\rangle = \mathscr{T}\langle\Gamma\rangle$. We will let the reader check the details, only pointing out that the bimonotonicity of composition has for syntactical counterpart the Congruence rule. By tradition it is natural to call it the *term model structad* of $\mathscr{T}$. Since we keep our habit of using subscripts to distinguish particular cases, we can transfer the subscripts for the theories we already have defined to the corresponding structads. For example, the structad associated to the reversible theory $\langle\mathscr{T}_{\mathrm{FreeBin}}\rangle$ (Example 11) will be denoted $\mathbf{O}_{\mathrm{FreeBin}}$, the one associated to $\langle\mathscr{T}^{\mathsf{c}}_{\mathrm{FreeBin}}\rangle$ (Example 12) will be denoted $\mathbf{O}^{\mathsf{c}}_{\mathrm{FreeBin}}$... but it will also be useful notational practice (when no subscripts are available) to identify a reversible theory with the operad it determines, e.g. to consider that $\langle\mathscr{T}\rangle$ is a structad. Notice that to say that two terms are isomorphic in the term model structad is equivalent to saying that they are provably equal modulo a variable substitution.

Example 22. Let $\mathbb{P} = \mathbb{C}$ and for $\Gamma \in \mathrm{Ctx}(\mathbb{P})$ let $\mathbf{O}\langle\Gamma\rangle$ be the set of all cyclic permutations over Γ. We recall that a cyclic permutation over a finite set is a bijective function $\alpha\colon \Gamma \to \Gamma$ such that for any pair of elements $x, y \in \Gamma$ there exists an n such that $y = \alpha^n(x)$. If $\sigma\colon \Gamma \to \Gamma'$ is a renaming bijection, with $\alpha \in \mathbf{O}\langle\Gamma\rangle$ and $x \in \Gamma'$ we define $\mathbf{O}\langle\sigma\rangle(\alpha)$ by

$$\mathbf{O}\langle\sigma\rangle(\alpha) = \sigma \circ \alpha \circ \sigma^{-1}\,,$$

which is the natural way (i.e., transposition) of transporting a cyclic permutation from set to set. Let us take the usual notation for cyclic permutations: the expression $\alpha = (a_0 a_1 \cdots a_n)$ means that $\alpha(a_i) = a_{i+1}, \alpha(a_n) = a_0$. Given contexts Γ, Δ and cyclic permutations $\alpha = (a_0 a_1 \cdots a_n x) \in \mathbf{O}\langle\Gamma\rangle$ and $\beta = (y b_0 b_1 \cdots b_m) \in \mathbf{O}\langle\Delta\rangle$, we define $\alpha\ {}_{x}{:}_{y}\ \beta$ as the cycle $(a_0 a_1 \cdots a_n b_0 b_1 \cdots b_m)$. Notice that composing a cycle of length n with a cycle of length m gives a cycle of length $n + m - 2$; in particular if $m = 1$ (we have a fixpoint) then we get a cycle of length $n - 1$. We denote this structad by $\mathbf{O}_{\mathrm{CycPerm}}$.

Example 23. As a matter of fact this composition is defined if α, β are *any* two permutations, so we get another, larger structad $\mathbf{O}_{\mathrm{Perm}}$ that way, whose value on Γ is the set of all permutations of Γ. In general a permutation can be written as a set $(A^0)(A^1)\cdots(A^m)$ of cycles, where every A^i is an abbreviation for a word $A^i = a_0^i a_1^i \cdots a_{n_i}^i$. Given $\alpha = (A^0)(A^1)\cdots(A^m x)$ and $\beta = (yB^0)(B^1)\cdots(B^p)$ then their composite is defined by

$$\alpha \,{}_{x:y}\, \beta = (A^0)(A^1)\cdots(A^m B^0)(B^1)\cdots(B^p)\,.$$

Checking the axioms of a structad is very easy.

Example 24. In the two structads above, we can add the restriction that a permutation (or cyclic permutation) cannot be allowed to have fixpoints. This gives us "substructads" $\mathbf{O}_{\mathrm{Perm-Fix}}, \mathbf{O}_{\mathrm{CycPerm-Fix}}$ of the ones previously defined, (this rather intuitive concept will be formalized below). Then it is natural to assume that $\mathbf{O}\langle\emptyset\rangle = \emptyset$ for both these structads.

Example 25. Let $\mathbf{O}_{\mathrm{Part}}$ be the structad that associates to every Γ the set of all partitions (or equivalently, the set of all equivalence relations) on Γ. It has a natural ordering, if we say that $\alpha \leq \beta$ if β is a finer partition than α (or, equivalently, that the equivalence relation associated to α contains the one associated to β. Given α on $\Gamma + x$ and β on $\Delta + y$ we define $\alpha \,{}_{x:y}\, \beta$ as the partition on $\Gamma + \Delta$ that fuses the class of x together with the class of y and leaves the rest undisturbed. The axioms of a structad are very easy to verify.

Example 26. An *operad* is a structad $\mathbf{O}$ over $\mathbb{B}$ such that every context Γ with $\mathbf{O}\langle\Gamma\rangle$ nonempy has a unique $x^\circ \in \Gamma$. So we already have seen quite a few operads: every term model structad associated to the universal reversible extension of an ultrabasic theory is an operad.

Operads were introduced by P. J. May in the context of homotopy theory [48]. They have recently been the subject of much interest from geometers and physicists [45, 44]. Our definition of an operad as a special kind of structad is actually a bit simpler than the standard definition, which does not take the output port into account. As a matter of fact, in [20] the concept of a *cyclic operad* is defined, as an ordinary operad with additional structure; this definition turns out to be equivalent to that of a structad over $\mathbb{C}$. Our operads are enriched in posets; May's operads were enriched in topological spaces, and abelian-like structures like graded modules are a very standard way to enrich a discrete operad. A. Joyal has shown how to define operads in the context of the theory of species, that is, by the means of the rich set of operations that are available in there, this is can be found in [69]. Structads over $\mathbb{C}$, like that of total cyclic orders, are first ones that were found in the context of linear logic. This is because the "natural" calculi associated with proof nets are one-sided; such structads are visible in filigree in the work of Andreoli [4]. But the recent work of Puite [61] gives theories of proof nets for two-sided calculi, and defines two-sided structads. There is a standard way of turning a $\mathbb{C}$-structad $\mathbf{O}$ into a two-sided one; one simply takes $\mathbf{O} \times \mathbb{B}^\sharp$, where $\times$ is the categorical product of structads, and $\mathbb{B}^\sharp$ is defined in Example 28. The experienced reader can check what this means, but there is also a quickie definition: if $\mathbf{O}$ is any structad on any polarity structure $\mathbb{P}$, then $\mathbf{O} \times \mathbb{B}^\sharp$ is the structad $\mathbf{O}'$ over $\mathbb{P} \times \mathbb{B}^\sharp$ such that a structure $\mathbf{O}'\langle\Gamma\rangle$ is a pair (α, f) where $\alpha \in \mathbf{O}\langle\Gamma\rangle$ and f is a function $f\colon \Gamma \to \mathbb{B}$. The structad used in Puite's work on two-sided cyclic linear logic is exactly $\mathbf{O}_{\mathrm{TotCycOrd}} \times \mathbb{B}^\sharp$.

Example 27. Let Γ be a (finite) set. Given a ternary relation α on Γ we define the following properties, that α may or may not have:

Cyclicity $\alpha(x, y, z) \;\Rightarrow\; \alpha(y, z, x)$
Anti-reflexivity $\alpha(x, x, y) \;\Rightarrow\;$ False
Transitivity $\alpha(x, y, z), \alpha(z, t, x) \;\Rightarrow\; \alpha(y, z, t)$
Spread $\alpha(x, y, z) \;\Rightarrow\; \forall w\ \alpha(w, y, z) \vee \alpha(x, w, z) \vee \alpha(x, y, w)$
Totality x, y, z distinct $\;\Rightarrow\; \alpha(x, y, z) \vee \alpha(x, z, y)$.

Let us show that Totality together with the first three axioms imply Spread. Assume that these four axioms hold for α, and that $\alpha(x, y, z)$ is true. Let $w \in \Gamma$. Because of Antireflexivity we know that x, y, z are distinct. Then either

– w is one of x, y, z and then $\alpha(w, y, z) \vee \alpha(x, w, z) \vee \alpha(x, y, w)$ holds trivially.
– or w is a new, distinct element. Then since x, y, w are distinct, we get $\alpha(x, y, w) \vee \alpha(x, w, y)$. If the first case is true, we have proved Spread. If the second case is true we also know that $\alpha(y, z, x)$ by Cyclicity and and applications of Transitivity:

$$\alpha(x, w, y), \alpha(y, z, x) \;\Rightarrow\; \alpha(w, y, z)$$

shows that α is Spreading.

A ternary relation α on Γ is said to be a *cyclic order* [57] if it is Cyclic, Anti-reflexive and Transitive; it is called an *order variety* [68] if it in addition it is Spreading, and it is called a *total cyclic order* if it is Total.

Proposition 4. *Let α, β be two ternary relations over the sets $\Gamma + \{x\}$, $\Delta + \{y\}$ respectively. Let us define $\alpha\ {}_{x:y}\ \beta$ as the ternary relation γ on $\Gamma + \Delta$, where in what follows the variables a_i, a denote elements of Γ and the variables b_i, b stand for elements of Δ:*

$$\begin{array}{ll} \gamma(a_1, a_2, a_3) \text{ if } \alpha(a_1, a_2, a_3) & \gamma(b_1, b_2, b_3) \text{ if } \beta(b_1, b_2, b_3) \\ \gamma(a_1, a_2, b) \text{ if } \alpha(a_1, a_2, x) & \gamma(b_1, b_2, a) \text{ if } \beta(b_1, b_2, y)\,. \end{array}$$

If both α, β are a cyclic orders, (resp. order varieties), (resp. total cyclic orders) then $\alpha\ {}_{x:y}\ \beta$ is a cyclic order (resp. order variety) (resp. total cyclic order).

The proof is a mechanical case analysis.

From this we can define three structads, $\mathbf{O}_{\mathrm{CycOrd}}, \mathbf{O}_{\mathrm{OrdVar}}, \mathbf{O}_{\mathrm{TotCycOrd}}$ by taking $\mathbf{O}\langle\Gamma\rangle$ to be the corresponding set of ternary relations over Γ. Notice that the first two of these are not discrete: we define $\alpha \leq \beta$ if $\alpha \subseteq \beta$ as ternary relations, i.e. as subsets of $\Gamma \times \Gamma \times \Gamma$. It is easy to see that the elements of $\mathbf{O}_{\mathrm{OrdVar}}$ that are maximal for this order are the total cyclic orders, and so we get that $\mathbf{O}_{\mathrm{TotCycOrd}}$ is a discrete structad with the induced ordering.

Let $\alpha \in \mathbf{O}_{\mathrm{OrdVar}}\langle\Gamma + \{z\}\rangle$. Define the binary relation $x <_{\alpha,z} y$ on Γ by:

$$x <_{\alpha,z} y \quad \text{if} \quad \alpha(x, y, z).$$

It is easy to show (or see [68]) that $x <_{\alpha,z} y$ is a strict (antireflexive) order on Γ. What is very interesting is that the converse is true, in a certain sense: in [68] it is shown how to complete any order structure $<$ on Γ to a cyclic order α on $\Gamma + z$, in such a way that $x <_{\alpha,z} y$ iff $x < y$. This is why an order variety should be seen as a "cyclic one-point completion" of an order structure… and where the name order variety comes from.

Given a total cyclic order α on Γ, and $z \in \Gamma$, the axiom of Totality implies that the order $<_{\alpha,z}$ is total. Thus it has a smallest element, call it $\alpha(z)$. One can then show that the function $z \mapsto \alpha(z)$ is a cyclic permutation; conversely a cyclic permutation always gives rise to a total cyclic order: say $\alpha(x, y, z)$ whenever the cyclic permutation can be written as $(axbyczd)$, where a, b, c, d are nonrepeating words of Γ. Because of this certain authors will define a cyclic order as what we call a total cyclic order. The observation that cyclic orders can be profitably studied from the point of view of Joyal species appears in [68].

Example 28. One special class of structad is constituted by those where $\mathbf{O}\langle\Gamma\rangle$ is either empty or the one-element set, whatever the context Γ. Let us call such structads *flat* structads. Naturally there are two extremes: let $(\mathbb{P}, (-)^\perp)$ be given. There is a "minimum" (or "initial") $\mathbb{P}$-structad $\mathbb{P}^\flat$ such that

$$\mathbb{P}^\flat\langle\Gamma\rangle = \begin{cases} \{*\} \text{ if } \Gamma \text{ is of the form } \{x^{\mathsf{p}}, y^{\mathsf{p}^\perp}\} \\ \emptyset \text{ otherwise} \end{cases}$$

and a "maximum" (or "terminal") one which we denote by $\mathbb{P}^\sharp$ where $\mathbb{P}^\sharp\langle\Gamma\rangle = \{*\}$ always. It would be interesting, given an arbitrary $(\mathbb{P}, (-)^\perp)$, to have a general description of everything that fits between these two extremes, in other words to describe all the flat structads over $\mathbb{P}$. For the time being we can observe that if $\mathbf{O}$ is flat, the fact that a context $\Gamma = \{x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n}\}$ is inhabited or not only depends on the multiset (bag) $\mathsf{p}_0 + \mathsf{p}_1 + \ldots + \mathsf{p}_n$ of polarities it determines. So a flat structad over $\mathbb{P}$ is entirely determined by a set S of multisets in $\mathbb{P}$, subject to the conditions

$$\mathsf{p} + \mathsf{p}^\perp \in S \;\text{ for all } \mathsf{p} \in \mathbb{P}, \quad \text{and} \quad m + \mathsf{p}, n + \mathsf{p}^\perp \in S \Rightarrow m + n \in S\,.$$

Let us identify flat structad structures over $\mathbb{P}$ with these sets of multisets. This gives us an ordering on the set of flat structad structures, determined by inclusion. The least element of that ordering is $\mathbb{P}^\flat = \{\mathsf{p} + \mathsf{p}^\perp \mid \mathsf{p} \in \mathbb{P}\}$ and the largest $\mathbb{P}^\sharp = \mathrm{Multiset}(\mathbb{P})$. Flat structads are the theories of contexts for commutative forms of linear logic.

One very important case of flat structad is $\mathbf{J}$, given by

$$\mathbf{J}\langle\Gamma\rangle = \begin{cases} \{*\} \text{ if } \Gamma \text{ contains exactly one } x^\circ \\ \;\emptyset \quad \text{otherwise}\,. \end{cases}$$

It is the "theory of (linear) intuitionistic contexts". It has a big brother $\mathbf{C} \supseteq \mathbf{J}$:

$$\mathbf{C}\langle\Gamma\rangle = \begin{cases} \{*\} \text{ if } \Gamma \text{ contains at most one } x^\circ \\ \;\emptyset \quad \text{otherwise}\,. \end{cases}$$

This is the theory of polarities/contexts for Girard's calculus LC for classical logic [21].

Remark 5. The definition of an operad ensures that if $\mathbf{O}$ is an operad, then $\mathbf{O}\langle\emptyset\rangle$ is guaranteed to be empty. Things can be quite different when the polarity structure over which a structad is defined contains fixpoints for the involution. For example suppose that $\mathbf{O}$ is defined over $\mathbb{C}$ and that $a \in \mathbf{O}\langle x\rangle$. Then we get $a \;{}_{x}{:}_{y}\; a \in \mathbf{O}\langle\emptyset\rangle$. As we have said in Remark 2 this is not really problematic since elements of $\mathbf{O}\langle\emptyset\rangle$ should more be seen as artifacts of the formalism than as entities that carry actual information. We already have seen the case with $\mathbf{O}_{\mathrm{CycPerm}}$ and $\mathbf{O}_{\mathrm{Perm}}$: composing the fixpoint with itself gives us the empty permutation, (see Remark 2).

Definition 13. *Given a pair $\mathbf{O}, \mathbf{O}'$ of structads, the first one over $\mathbb{P}$ and the second one over $\mathbb{P}'$ we define a* map *(or* homomorphism*, or plain* morphism*) $F\colon \mathbf{O} \to \mathbf{O}'$ to be*

- *a map of polarity structures $\mathrm{Pol}(F)\colon \mathbb{P} \to \mathbb{P}'$. We say that $\mathrm{Pol}(F)$ is the* restriction of $F\colon \mathbf{O} \to \mathbf{O}'$ to the polarities. *If $f\colon \mathbb{P} \to \mathbb{P}'$ is an involution-respecting map we say F* is above f *if $\mathrm{Pol}(F) = f$. The function $\mathrm{Pol}(F)\colon \mathbb{P} \to \mathbb{P}'$ induces (recall the discussion just below 1) a function $\Gamma \mapsto F\Gamma : \mathrm{Ctx}(\mathbb{P}) \to \mathrm{Ctx}(\mathbb{P}')$; notice that the notation is simplified, to avoid syntactical excesses. In addition, for every renaming $\sigma\colon \Gamma \to \Delta$ we get a renaming $F\sigma\colon F\Gamma \longrightarrow F\Delta$, whose definition should be obvious.*
- *for every $\Gamma \in \mathrm{Ctx}(\mathbb{P})$ a monotone map $F_\Gamma\colon \mathbf{O}\langle\Gamma\rangle \to \mathbf{O}'\langle F\Gamma\rangle$, such that for every renaming $\sigma\colon \Gamma \to \Delta$ we have that*

$$\begin{tikzcd} \mathbf{O}\langle\Gamma\rangle \arrow[r, "\mathbf{O}\langle\sigma\rangle"] \arrow[d, "F_\Gamma"'] & \mathbf{O}\langle\Delta\rangle \arrow[d, "F_\Delta"] \\ \mathbf{O}'\langle F\Gamma\rangle \arrow[r, "\mathbf{O}'\langle F\sigma\rangle"'] & \mathbf{O}'\langle F\Delta\rangle \end{tikzcd}$$

  commutes, in other words $F_\Delta \circ \mathbf{O}\langle\sigma\rangle = \mathbf{O}'\langle F\sigma\rangle \circ F_\Gamma$.
- *Given $\alpha \in \mathbf{O}\langle\Gamma + x^{\mathsf{p}}\rangle$ and $\beta \in \mathbf{O}\langle\Delta + z^{\mathsf{p}^\perp}\rangle$ we have*

$$F_{\Gamma+\Delta}(\alpha \;{}_{x}{:}_{z}\; \beta) = F_{\Gamma+x}(\alpha) \;{}_{x}{:}_{z}\; F_{\Delta+z}(\beta)\,.$$

  Notice that the polarities of x, z differ in the left and the right sides. In addition we require that F respects wires: $F_{\{x,z\}}(1_{x,z}) = 1_{x,z}$ for any x, z of opposite polarities.

In standard categorical terminology, in addition to $\mathrm{Pol}(F)\colon \mathbb{P} \to \mathbb{P}'$ a map is defined by a natural transformation $\mathbf{O} \to \mathbf{O}' \circ F$ that respects the composition operation, where $F\colon \mathrm{Ctx}(\mathbb{P}) \to \mathrm{Ctx}(\mathbb{P}')$ is the "repolarize" functor of the groupoids of contexts.

Since a map of structads is first and foremost determined by a class of monotone functions, all possible qualifiers for monotone functions can be transfered to maps of structads. In other words we can say a map F of structads is $\{$injective, surjective, bijective, an embedding…$\}$ when, for every context Γ in the source contexts, the map F_Γ is $\{$injective, surjective, bijective, an embedding…$\}$. We can also say what a substructad $\mathbf{O} \subseteq \mathbf{O}'$ is: it is a map $\mathbf{O} \to \mathbf{O}'$ of structads above identity such that for every context Γ the monotone $\mathbf{O}\langle\Gamma\rangle \to \mathbf{O}'\langle\Gamma\rangle$ is an inclusion of posets; in this paper we will suppose that in particular it is an embedding, in other words that the order on $\mathbf{O}\langle\Gamma\rangle$ is induced by the one on $\mathbf{O}'\langle\Gamma\rangle$. Notice that there is an "orthogonal" concept, when the sets $\mathbf{O}'\langle\Gamma\rangle, \mathbf{O}\langle\Gamma\rangle$ are identical, but the left order is a suborder structure of the right one.

Proposition 5. *Given structads $\mathbf{O}$ above $\mathbb{P}$, $\mathbf{O}'$ above $\mathbb{P}'$ and $\mathbf{O}''$ above $\mathbb{P}''$, along with maps $F\colon \mathbf{O} \to \mathbf{O}'$ and $G\colon \mathbf{O}' \to \mathbf{O}''$ the composite $G \circ F$ defined by $\mathrm{Pol}(G \circ F) = \mathrm{Pol}(G) \circ \mathrm{Pol}(F)$ and $(G \circ F)_\Gamma = G_{F\Gamma} \circ F_\Gamma$ is a map of structads. Furthermore every structad $\mathbf{O}$ has an identity map $\mathrm{Id}_{\mathbf{O}}$, which is above identity and whose value on every context is the identity map.*

The proof is totally straightforward. So we get that yes, structads and maps form a category. In other words we have the following properties, very easy to check: we always have $H \circ (G \circ F) = (H \circ G) \circ F$ (given $H\colon \mathbf{O}'' \to \mathbf{O}'''$) and $F \circ \mathrm{Id}_{\mathbf{O}} = \mathrm{Id}_{\mathbf{O}'} \circ F = F$.

Notice also that if $F, G\colon \mathbf{O} \to \mathbf{O}'$ are structad maps such that $\mathrm{Pol}(F) = \mathrm{Pol}(G)$, then they are comparable for the structures order; that is, given F, G as above we say that $F \leq G$ whenever, for any context Γ and $\alpha \in \mathbf{O}\langle\Gamma\rangle$, we have that $F_\Gamma(\alpha) \leq G_\Gamma(\alpha)$, or in other words, we have that $F_\Gamma \leq G_\Gamma$ for the pointwise order on monotone maps.

Proposition 6. *Let $F, F'\colon \mathbf{O} \to \mathbf{O}'$ be structad maps above the same map of polarity structures such that $F \leq F'$. Then given any $G\colon \mathbf{O}' \to \mathbf{O}''$ and $H\colon \mathbf{O}''' \to \mathbf{O}$ we have $F \circ H \leq F' \circ H$ and $G \circ F \leq G \circ F'$.*

Proof. This will be left to the reader.

There is one kind of map which deserves special treatment:

Definition 14. *a map $F\colon \mathbf{O} \to \mathbf{O}'$ is said to be an* isomorphism *if*

- $\mathrm{Pol}(F)\colon \mathbb{P} \to \mathbb{P}'$ *is bijective (and therefore has an involution-respecting inverse* $\mathrm{Pol}(F)^{-1}$*)*
- *for every $\Gamma \in \mathrm{Ctx}(\mathbb{P})$ the function $F_\Gamma\colon \mathbf{O}\langle\Gamma\rangle \to \mathbf{O}'\langle F\Gamma\rangle$ is an isomorphism of posets.*

The reader who is new at this kind of game should check that this defi nition is equivalent to saying that F has an inverse morphism F^{-1}, i.e. such that $F \circ F^{-1}$ and $F^{-1} \circ F$ are the identity structad maps.

So it is time for examples. The fi rst ones will be above identity.

Example 29. We already have given a few examples of inclusion maps of substructads, e.g. $\mathbf{O}_{\mathrm{CycPerm}} \subseteq \mathbf{O}_{\mathrm{Perm}}, \mathbf{O}_{\mathrm{TotCycOrd}} \subseteq \mathbf{O}_{\mathrm{OrdVar}} \ldots$

Example 30. Any map of ordinary order-enriched theory $\mathscr{T} \to \mathscr{T}'$ (see Examples 7–10) gives rise to a map of the operads obtained from the universal reversible extension. All the examples we have given are disjoint theories, but this is not really necessary.

Example 31. The reader can show that the three structads

$\mathbf{O}_{\mathrm{CycPerm}}$ (Example 22), $\mathbf{O}_{\mathrm{TotCycOrd}}$ (Example 27), $\mathbf{O}^{\mathfrak{c}}_{\mathrm{Ass+Cyc+Unit}}$ (Example 15)

are isomorphic, and all this above identity. Each of these is also equipped with an involution $R\colon \mathbf{O} \to \mathbf{O}, R \circ R = \mathrm{Id}_{\mathbf{O}}$, given by reversing the cyclic order/permutation.

Example 32. There is a morphism $\mathbf{O}_{\text{Perm}} \to \mathbf{O}_{\text{Part}}$ between the structad of permutations (Example 23) and the structad of partitions (Example 25). If α is a permutation of the set Γ then its sets of cycles determines a partition of Γ. Notice that the source structad is discrete while the target is not.

Example 33. If $\mathbb{P}$ is given, and $\mathbf{O}, \mathbf{O}'$ are flat structads over $\mathbb{P}$, then to have $\mathbf{O} \subseteq \mathbf{O}'$ by the definition given in Example 28 is equivalent to having a (necessarily unique) map $\mathbf{O} \to \mathbf{O}'$. A map either exists or not because all the sets of structures are either empty or singletons.

Example 34. Recall that $\mathbb{P}^\sharp$ is the "largest" flat structad over $\mathbb{P}$ (Example 28). It is easy to see that, given any structad $\mathbf{O}$ over $\mathbb{P}$, there is a unique map $\mathbf{O} \to \mathbb{P}^\sharp$ above identity, since every $\mathbb{P}^\sharp\langle\Gamma\rangle$ is the one-element set. This map determines a flat structad $\mathbf{M}$ which is its "direct image": for every Γ we have

$$\mathbf{M}\langle\Gamma\rangle = \begin{cases} \{*\} & \text{if } \mathbf{O}\langle\Gamma\rangle \text{ is nonempty} \\ \emptyset & \text{otherwise}\,. \end{cases}$$

and we get a diagram $\mathbf{O} \to \mathbf{M} \to \mathbb{P}^\sharp$, where the first map is surjective. This is the "flattening" of the structad we have started with, its "universal commutative collapse". We can give an interesting example of this direct image construction: the direct image of $\mathbf{O}_{\text{Func}} \to \mathbb{B}^\sharp$ is the flat structad

$$\mathbf{M}\langle\Gamma\rangle = \begin{cases} \{*\} & \text{if } \Gamma^\bullet \text{ nonempty implies } \Gamma^\circ \text{ nonempty} \\ \emptyset & \text{otherwise}\,. \end{cases}$$

This is a new regime of polarities, which can be compared naturally with $\mathbf{J}$ and $\mathbf{C}$: we get $\mathbf{J} \subseteq \mathbf{M}$ but $\mathbf{C} \not\subseteq \mathbf{M}$. So the logics associated to the structad $\mathbf{O}_{\text{Funct}}$ are too general to be intuitionistic, but too restricted to be fully two-sided classical.

Example 35. Given any ultrabasic theory $\mathscr{T}$, if we decide to replace its polarity structure $\mathbb{B}$ by $\mathbb{C}$, as in Examples 12,15, we will get a morphism $\langle\mathscr{T}\rangle \to \langle\mathscr{T}^{\mathsf{c}}\rangle$.

Example 36. Notice that from Example 15 we have seen that the map $\langle\mathscr{T}_{\text{Ass}}\rangle \to \langle\mathscr{T}^{\mathsf{c}}_{\text{Ass}}\rangle_{+\text{Cyc}}$ is bijective, but it is not an isomorphism because of the change of polarity structure.

Example 37. Notice that the flat structad $\mathbf{J}$ is isomorphic to $\langle\mathscr{T}_{\text{Comm+Unit}}\rangle$.

Example 38. Take a version of linear logic that has a theory of proof nets; to fix things let us assume it is multiplicative cyclic linear logic (without constants), whose theory of proof nets is discussed in [35, 3, 49–51]. The salient feature of cyclic linear logic [73] is that the formulas in a sequent are arranged in a total cyclic order. Let $\mathbb{P}$ be the set of formulas for that logic, with the involution being negation as it is ordinarily defined in one-sided linear logic. Define a structad $\mathbf{O}$ as follows: if Γ is a set of polarized (i.e., typed) variables, then an element $\alpha \in \mathbf{O}\langle\Gamma\rangle$ is a pair (c, N) where c is a total cyclic order on Γ and N a *cutless* proof net structure on the sequent given by the formulas and that total cyclic order. Given $\alpha \in \mathbf{O}\langle\Gamma, x^A\rangle$ and $\beta \in \mathbf{O}\langle\Delta, x^{A^\perp}\rangle$ then $\alpha \;{}_{x}{:}_{y}\; \beta$ is obtained by doing cut elimination on α, β via the pair of formulas $A, A^\perp$. The reason this defines a structad is because cut elimination in proof nets is associative (e.g., [7]). So the conclusion is that there is an obvious "forgetful" map $\mathbf{O} \to \mathbf{O}_{\mathrm{TotCycOrd}}$, that keeps only the total cyclic structure of the context, i.e. that sends (c, N) to c. This shows clearly that the notion of polarity is very general, and that a polarity structure can be a big set, with additional structure.

Example 39. Let $\mathbf{O}$ be a structad. We define a *two-sided structure* over $\mathbf{O}$ to be a morphism of structads $S\colon \mathbf{O} \to \mathbb{B}^\sharp$. The map S determines a map $\mathrm{Pol}(S)\colon \mathbb{P} \to \mathbb{B}$ of polarity structures, and the converse is true: once we know $\mathrm{Pol}(S)$, the fact that $\mathbb{B}^\sharp\langle\Gamma\rangle$ is always a singleton entirely determines the maps $\mathbf{O}\langle\Gamma\rangle \to \mathbb{B}^\sharp\langle S\Gamma\rangle$. So a two-sided structure is simply the choice for every polarity $\mathsf{p} \in \mathbb{P}$ of a side $S(\mathsf{p}) \in \{\circ, \bullet\}$, such that $S(\mathsf{p}^\perp) = (S(\mathsf{p}))^\perp$. We can read this as saying that S tells us, for every structure $\alpha \in \mathbf{O}\langle\Gamma\rangle$, if $x \in \Gamma$ is to the left $(\bullet)$ or to the right $(\circ)$ of the turnstile. Naturally this precludes $\mathbb{P}$ from having any fixpoints for the involution. The same way we can define an *intuitionistic structure* for $\mathbf{O}$ to be a map $\mathbf{O} \to \mathbf{J}$. This time, not only do we know what's left or right, but we are assured that there is always a unique thing at the right.

A (poset-enriched) *typed operad* (also called a *colored operad* or a *multicategory with permutations*) is a pair $(\mathbf{O}, S)$ where $\mathbf{O}$ is a structad and S an intuitionistic structure. What we are doing is defining a very standard concept by the means of our new technology. In other words, Let $O \subseteq \mathbb{P}$ (the set of types, or colors, or objects) be the set $O = S^{-1}(\{\bullet\})$, ie., all polarities p such that[4] $S(\mathsf{p}) = \bullet$. Then if $\alpha \in \mathbf{O}\langle\Gamma\rangle$, the map S tells us that there is a unique $z^{\mathsf{p}^\perp} \in \Gamma$ whose polarity is not in O. So if $\Gamma - z = \{x_1^{\mathsf{p}_1}, \ldots x_n^{\mathsf{p}_n}\}$, we can view α as an ordinary term judgement $x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n} \vdash \alpha^{\mathsf{p}}$. Because of the polarities a permutation of Γ has to leave z fixed. This observation is at the root of the traditional definition of typed operads.

So traditionally linear algebraic theories have been associated with the pair: ordinary term/typed operad, where the left side represents syntax and the right side algebra. We argue that the pair: reversible term/structad is the more fundamental concept, being technically simpler and more general. Moreover we have just seen that the traditional pair can be recovered by the means of a simple map of structads.

Example 40. A *multicategory* is a pair $(\mathbf{O}, S)$ where $\mathbf{O}$ is a structad and S a map $\mathbf{O} \to \mathbf{O}_{\mathrm{Ass+Unit}}$. The target structad being an operad, $\mathbf{O}$ inherits an intuitionistic structure, and a multicategory is a typed operad in the sense above. But, given $\alpha \in \Gamma$ as above, the intuitionistic sequent $x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n} \vdash \alpha^{\mathsf{p}}$ that it defines is such that the set of variables has a *total order structure* on it in addition. So it is only natural to number/write these variables in that order. Multicategories were defined by J. Lambek more than thirty years ago [38, 39] as the algebraic counterpart of substructural deductive systems that were around at the time. He also remarked that the set of deductions in a context-free grammar has a natural multicategory structure. The study of typed operads and multicategories is in full expansion [19] (for an up-to-date survey, see [42]), one direction of research being the "quest for higher order", which is of great interest for homotopy theory and mathematical physics. Higher-order concepts have recently been

[4] it would be more in keeping with tradition to use $O = S^{-1}(\{\circ\})$ instead, but this is consistent with our convention of putting the variables "inside" the contexts.

given a proof-theoretical meaning by G.-F. Mascari [47], and they were proposed as a general framework for the foundations of mathematics by Makkai [46]. Another related area of pursuit has been axiomatization in a general categorical framework, which originated thirty years ago [10] and has been twice rediscovered independently recently [41, 27].

Let $\mathbf{\Sigma}$ be a tree signature over polarity structure $\mathbb{P}$. We write $\mathbf{F}(\mathbf{\Sigma})$ to denote the free structad generated by $\mathbf{\Sigma}$, i.e. the structad whose structures are the reversible terms over $\mathbf{\Sigma}$, without any additional (in)equations.

The following result is the standard way (known as a "universal property") of expressing by the means of algebra the fact that reversible term structads are free, i.e. without constraints/axioms.

Theorem 3. *Let $\mathbf{\Sigma}$ be a tree signature over $\mathbb{P}$, and $\mathbf{O}$ a structad over $\mathbb{P}$. For every $\mathbf{f} \in \mathbf{\Sigma}$ suppose we have chosen a pair $(\Gamma_{\mathbf{f}}, \alpha_{\mathbf{f}})$, where $\alpha_{\mathbf{f}} \in \mathbf{O}\langle\Gamma_{\mathbf{f}}\rangle$ and $\Gamma_{\mathbf{f}}$ is a context of the form $\Gamma_{\mathbf{f}} = \{x_0^{\mathrm{Pol}(\mathbf{f},0)}, x_1^{\mathrm{Pol}(\mathbf{f},1)}, \ldots, x_n^{\mathrm{Pol}(\mathbf{f},n)}\}$. Then there is a unique morphism $F: \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$ above identity that maps every $\mathbf{f} \in \mathbf{\Sigma}$ to $\alpha_{\mathbf{f}}$. Furthermore, if $(\beta_{\mathbf{f}})_{\mathbf{f}\in\mathbf{\Sigma}}$ is another family $\beta_{\mathbf{f}} \in \Gamma_{\mathbf{f}}$ such that $\alpha_{\mathbf{f}} \leq \beta_{\mathbf{f}}$ for every $\mathbf{f}$, then the unique $G: \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$ that maps every $\mathbf{f}$ to $\beta_{\mathbf{f}}$ is such that $F \leq G$.*

Proof. Let us rephrase the phrase "mapping every $\mathbf{f}$ to $\alpha_{\mathbf{f}}$" into a more precise, if perhaps pedantic way. The defining property of F is that for every context $\Gamma_{\mathbf{f}}$ of the form above, the unique element $\mathbf{f}' \in \mathbf{F}(\mathbf{\Sigma})\langle\Gamma_{\mathbf{f}}\rangle$ which is obtained by putting the variables into the atomic symbol $\mathbf{f}$ according to the port numbering, is mapped to $\alpha_{\mathbf{f}}$, i.e. $F_\Gamma(\mathbf{f}') = \alpha_{\mathbf{f}}$.

The construction of F is by induction on the number of symbols of $\mathbf{\Sigma}$ on the reversible terms $\mathbf{t}$ that inhabit $\mathbf{F}(\mathbf{\Sigma})$. Suppose $\mathbf{t}$ is given, in context Γ. If it has a single symbol, there is a unique $\mathbf{f} \in \mathbf{\Sigma}$ and a unique renaming $\sigma: \Gamma_{\mathbf{f}} \to \Gamma$ respecting the port numbering, and this forces the value of $F(\mathbf{t}) = \mathbf{O}\langle\sigma\rangle(\alpha_{\mathbf{f}})$. If it has more than one symbol, we can write $\mathbf{t} = \mathbf{s} \,{}_{x:y}\, \mathbf{r}$ where $\mathbf{s}, \mathbf{t}$ have strictly fewer symbols, and this forces $F(\mathbf{t}) = F(\mathbf{s}) \,{}_{x:y}\, F(\mathbf{r})$. Uniqueness is proved as follows: if $\mathbf{t} = \mathbf{s}' \,{}_{z:w}\, \mathbf{r}'$ then due to the tree structure of reversible terms there either will be

- a decomposition of the form $\mathbf{t} = \mathbf{s}' \,{}_{z:w}\, \mathbf{t}' \,{}_{x:y}\, \mathbf{r}$ with $\mathbf{s} = \mathbf{s}' \,{}_{z:w}\, \mathbf{t}'$ and $\mathbf{r}' = \mathbf{t}' \,{}_{x:y}\, \mathbf{r}$
- or a decomposition of the form $\mathbf{t} = \mathbf{s} \,{}_{x:y}\, \mathbf{t}'' \,{}_{z:w}\, \mathbf{r}'$ with $\mathbf{s}' = \mathbf{s} \,{}_{x:y}\, \mathbf{t}''$ and $\mathbf{r} = \mathbf{t}' \,{}_{z:w}\, \mathbf{r}'$.

This is proved by choosing a variable $v \in \Gamma$ and using pattern-matching on the terms $\mathbf{t}^{[v]}, \mathbf{r}^{[v]}, \mathbf{s}^{[v]}$, etc. If $\mathbf{t}'$ or $\mathbf{t}''$ is empty, there is nothing to prove. If $\mathbf{t}'$ or $\mathbf{t}''$ is nonempty then we can use induction hypothesis on the smaller words to show that $F(\mathbf{t}) = F(\mathbf{s}) \,{}_{x:y}\, F(\mathbf{r}) = F(\mathbf{s}') \,{}_{z:w}\, F(\mathbf{t}')$ and we get that F is uniquely defined. The proof of the second part, that $F \leq G$ is done with the same induction steps, using the Congruence rule.

Definition 15. *With everything as above, if the family $(\Gamma_{\mathbf{f}}, \alpha_{\mathbf{f}})_{\mathbf{f}}$ above is such that F is surjective we say that $(\Gamma_{\mathbf{f}}, \alpha_{\mathbf{f}})_{\mathbf{f}}$ is* a family (or set) of generators *for* $\mathbf{O}$.

Let $F: \mathbf{O} \to \mathbf{O}'$ be any map of structads, and for simplicity let us assume F is above identity on the polarities. Let $\alpha \in \mathbf{O}\langle\Gamma\rangle$ be a structure in $\mathbf{O}$ and σ be an isomorphism $\sigma: (\Gamma, F_\Gamma(\alpha)) \to (\Delta, \beta)$. The definition of a map forces

$$F_\Delta\big(\mathbf{O}\langle\sigma\rangle(\alpha)\big) = \mathbf{O}'\langle F\sigma\rangle\big(F_\Gamma(\alpha)\big) = \mathbf{O}'\langle\sigma\rangle\big(F_\Gamma(\alpha)\big) = \beta\,.$$

In other words, if a structure is in the image of F, then any isomorphic structure is also in the image of F. So a family of generators can be chosen in such a way that two isomorphic structures in it are necessarily equal, which eliminates quite many redundancies.

Lemma 2. *Every structad* $\mathbf{O}$ *has a set of generators.*

Proof. We can always use the "brute force" method and choose one representative for every isomorphism class of structure, ranging over *all* the structures (Γ, α) in $\mathbf{O}$.

We can immediately make use of the two results just above. Let $\mathscr{T} = (\mathbb{P}, \mathbf{\Sigma}, \mathrm{RevAx})$ be a reversible order-enriched theory. For every axiom $A \in \mathrm{RevAx}$ there is a unique number n_A which is the number of variables needed to express it, and the axiom A is of the form

$$\Gamma_A = x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}, \ldots, x_{n_A}^{\mathsf{p}_{n_A}} \vdash_{\mathsf{r}} \mathbf{s}_{A,1} \leq \mathbf{s}_{A,2}\,,$$

where $\mathbf{s}_{A,1}, \mathbf{s}_{A,2} \in \mathbf{F}(\mathbf{\Sigma})\langle \Gamma_A \rangle$. Let $\mathbf{\Sigma}'$ be a new signature, obtained by taking, for every $A \in \mathrm{RevAx}$ a *unique* $\mathbf{g}_A \in \mathbf{\Sigma}'$ with valence n_A and sort

$$x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}, \ldots, x_{n_A}^{\mathsf{p}_{n_A}} \vdash_{\mathsf{r}} \mathbf{g}_A\,.$$

The previous theorem allows us to define two maps $F_1, F_2 \colon \mathbf{F}(\mathbf{\Sigma}') \to \mathbf{F}(\mathbf{\Sigma})$, simply by specifying that $F_{i,\Gamma_A}(\mathbf{g}_A) = \mathbf{s}_{A,i}$ for $i = 1, 2$. Let now $\mathbf{O}$ be the structad associated to the theory $\mathscr{T}$. There is a map $G\colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$ that sends every atomic $f \in \mathbf{\Sigma}$ to its equivalence class in $\mathscr{T}$, and thus every $t \in \mathbf{O}\langle \Gamma \rangle$ is also mapped to its equivalence class in $\mathscr{T}$. For every axiom $A \in \mathrm{RevAx}$ we do have that $\Gamma_A \vdash_{\mathsf{r}} \mathbf{s}_{A,1} \leq \mathbf{s}_{A,2}$ in $\mathbf{O}\langle \Gamma \rangle_A$. Therefore since $F_{i,\Gamma_A}(\mathbf{g}_A) = \mathbf{s}_{A,i}$ the preceding theorem tells us that $G \circ F_1 \leq G \circ F_2$.

Theorem 4. *The map* $G\colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$ *has the following universal property: for any structad* $\mathbf{O}'$ *and any map* $H\colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}'$ *above identity such that* $H \circ F_1 \leq H \circ F_2$ *there is a* unique $K\colon \mathbf{O} \to \mathbf{O}'$ *with* $K \circ G = H$.

Before we prove this we need the following

Lemma 3. *Let* $\Gamma \vdash_{\mathsf{r}} \mathbf{s} \leq \mathbf{t}$ *be a theorem of* $\mathscr{T}$. *Then* $H(\mathbf{s}) \leq H(\mathbf{t})$ *in* $\mathbf{O}'\langle \Gamma \rangle$.

Proof. This is just a simple induction on the proof of $\Gamma \vdash_{\mathsf{r}} \mathbf{s} \leq \mathbf{t}$.

The proof of the theorem proper can begin. Let $\alpha \in \mathbf{O}\langle \Gamma \rangle$ be given. Since α is an equivalence class of reversible terms in $\mathbf{F}(\mathbf{\Sigma})$ we can choose $\mathbf{t} \in \mathbf{F}(\mathbf{\Sigma})\langle \Gamma \rangle$ with $G(\mathbf{t}) = \alpha$. Therefore $H(\mathbf{t})$ is a likely candidate for $K(\alpha)$. Suppose $\alpha' \leq \alpha$ and $\mathbf{t}' \in \mathbf{F}(\mathbf{\Sigma})\langle \Gamma \rangle$ is a representative of α'. By definition we have $\Gamma \vdash_{\mathsf{r}} \mathbf{t}' \leq \mathbf{t}$, by the Lemma above we get $H(\mathbf{t}') \leq H(\mathbf{t})$ and this shows:

- That K is uniquely defined, because $\alpha' = \alpha$ iff $\Gamma \vdash_{\mathsf{r}} \alpha \leq \alpha'$ and $\Gamma \vdash_{\mathsf{r}} \alpha' \leq \alpha$ in $\mathscr{T}$.
- That K is monotone.

There is a little more to check to show that K respects all the properties of a map of structads.

Notice that to have a pair of parallel maps $F_1, F_2 \colon \mathbf{F}(\mathbf{\Sigma}') \to \mathbf{F}(\mathbf{\Sigma})$ above identity between free structads is *absolutely equivalent* to having a reversible theory with signature $\mathbf{\Sigma}$: we have shown how to go from the latter to the former, and the reverse direction is just as simple: every $\mathbf{g} \in \mathbf{\Sigma}'$ can be assimilated to an axiom, given by $\Gamma_{\mathbf{g}} \vdash_{\mathsf{r}} F_1(\mathbf{g}) \leq F_2(\mathbf{g})$.

Definition 16. *Let* $\mathbf{O}$ *be a structad. A* presentation *for* $\mathbf{O}$ *is a pair* $F_1, F_2 \colon \mathbf{F}(\mathbf{\Sigma}') \to \mathbf{F}(\mathbf{\Sigma})$ *of parallel arrows between free structads such that* $\mathbf{O}$ *is isomorphic to the term model of the theory defined by* $F_1, F_2, \mathbf{O}, \mathbf{O}'$.

In other words a presentation is the algebraic way of defining the concept of a reversible theory inside the world of structads.

Theorem 5. *Every structad admits a presentation, i.e, every structad is isomorphic to the term model structad of a reversible theory.*

Proof. Choose a set of generators for $\mathbf{O}$, getting a signature $\mathbf{\Sigma}$ and a surjective map $H \colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$. Let now $\mathbf{Q}$ be the structad defined by: a structure $\mathbf{s} \in \mathbf{Q}\langle \Gamma \rangle$ is a pair $(\mathbf{s}_1, \mathbf{s}_2)$ of reversible terms in $\mathbf{F}(\mathbf{\Sigma})\langle \Gamma \rangle$ such that $H_\Gamma(\mathbf{s}_1) \leq H_\Gamma(\mathbf{s}_2)$ in $\mathbf{O}\langle \Gamma \rangle$. If $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2) \in \mathbf{Q}\langle \Gamma + x \rangle$ and $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2) \in \mathbf{Q}\langle \Gamma + z \rangle$ we define $\mathbf{s}\,{}_{x:z}\mathbf{t}$ as $(\mathbf{s}_1\,{}_{x:z}\mathbf{t}_1, \mathbf{s}_1\,{}_{x:z}\mathbf{t}_2)$; it is very easy to show that $H(\mathbf{s}_1\,{}_{x:z}\,\mathbf{t}_1) \leq H(\mathbf{s}_2\,{}_{x:z}\,\mathbf{t}_2)$ in $\mathbf{O}\langle \Gamma + \Delta \rangle$. The rest of the proof that this is a structad will be left to the reader, as well as the fact that there are two maps of structads $K_1, K_2 \colon \mathbf{Q} \to \mathbf{F}(\mathbf{\Sigma})$, defined by $K_{i\Gamma}(\mathbf{s}_1, \mathbf{s}_2) = \mathbf{s}_i$. Now choose a set of generators for $\mathbf{Q}$, getting a new tree signature $\mathbf{\Sigma}'$ and a map $F \colon \mathbf{F}(\mathbf{\Sigma}') \to \mathbf{Q}$. If we define $F_1 = K_1 \circ F$ and $F_2 = K_2 \circ F$, we get a pair of maps $F_1, F_2 \colon \mathbf{F}(\mathbf{\Sigma}') \to \mathbf{F}(\mathbf{\Sigma})$ between free structads, thus defining a reversible linear order-enriched theory. Let $\mathbf{O}'$ be the term model structad of that theory, and $G \colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}'$ the universal map of Theorem 4. By definition we have that $H \circ K_1 \leq H \circ K_2$ and thus (Proposition 6) $H \circ F_1 = H \circ F \circ K_1 \leq H \circ F \circ K_2 = H \circ F_2$. Therefore by the universal property of Theorem 4 there is a map of structads $L \colon \mathbf{O}' \to \mathbf{O}$ with $L \circ G = H$.

$$\begin{tikzcd}
 & & & \mathbf{O}' \arrow[dd, dotted, "L"] \\
\mathbf{F}(\mathbf{\Sigma}') \arrow[r, "F"] & \mathbf{Q} \arrow[r, shift left, "K_1"] \arrow[r, shift right, "K_2"'] & \mathbf{F}(\mathbf{\Sigma}) \arrow[ur, "G"] \arrow[dr, "H"'] & \\
 & & & \mathbf{O}
\end{tikzcd}$$

If we show that L is an isomorphism of structads we have proved our claim. This map is necessarily surjective, since H is by definition and $L \circ G = H$. So all is left to prove is that if $\alpha \leq \beta$ are structures in $\mathbf{O}\langle \Gamma \rangle$ then there are $\alpha' \leq \beta'$ in $\mathbf{O}'\langle \Gamma \rangle$ with $L(\alpha') = \alpha, L(\beta') = \beta$; this will show monotonicity and injectivity in one fell swoop. Choose $\mathbf{s}, \mathbf{t} \in \mathbf{F}(\mathbf{\Sigma})$ such that $H(\mathbf{s}) = \alpha, H(\mathbf{t}) = \beta$. By definition we have $(\mathbf{s}, \mathbf{t}) \in \mathbf{Q}$. Since F is surjective there is $\mathbf{r} \in \mathbf{F}(\mathbf{\Sigma}')$ such that $F(\mathbf{r}) = (\mathbf{s}, \mathbf{t})$. We know that a generator $\mathbf{g} \in \mathbf{\Sigma}'$ in context Δ is interpreted as an axiom of the theory that constructs $\mathbf{O}'$, namely $\Delta \vdash_{\mathsf{r}} F(K_1(\mathbf{g})) \leq F(K_2(\mathbf{g}))$. By induction on the atoms of $\mathbf{r}$, we will get $\Gamma \vdash_{\mathsf{r}} F(K_1(\mathbf{r})) \leq F(K_2(\mathbf{r}))$, in other words $\Gamma \vdash_{\mathsf{r}} \mathbf{s} \leq \mathbf{t}$. Then by taking $\alpha' = G(\mathbf{s}), \beta' = G(\mathbf{t})$ we have proved our claim.

The moral of the story is that the two ways of seeing structads, the "logical-syntactical" and the "algebraic-semantical", are complementary and partially interchangeable. Some structads we encountered, like the ones associated with order varieties, were first observed in what mathematicians call "the real world". Other structads, like the ones that interest linguists, are built from syntax, with generators and relations. But it is very profitable to have access to both ways of seeing things. As a matter of fact, from now on we will freely mix the notations we have kept separate before, when we wanted to distinguish the "logical-syntactical" objects from the "algebraic-semantical" ones, so we now allow ourselves to write things like $\alpha \;{}_{x:y}\; \mathbf{t}$ and $\alpha^{[x]}$.

Naturally, for most logical applications we will like to our structads to be presented by a small (i.e. finite or recursive, or maybe recursively enumerable) set of generators, and to be decidable. It is well-known that a good way, perhaps the best, to give a decision procedure for an algebraic object defined by sets and generators, is to describe that object by the means of semantics. For example, a group presentation gets much more meaningful when it is interpreted as the set of permutations of some set-with-structure.

Example 41. Let $\mathbf{O}_{\mathrm{Perm}}$ be the structad of all permutations. It is easy to show (cf. Example 31) that the three-element cyclic permutation (xyz) generates the substructad $\mathbf{O}_{\mathrm{CycPerm-Fix}}$ of all cyclic permutations over sets of more than two elements (the 2-cycle does not need to be counted as a generator, since it is the identity). If we add the fixpoint (x), we get $\mathbf{O}_{\mathrm{CycPerm}}$, all cyclic permutations over all finite sets. If we add the two-fixpoint permutation $(x)(y)$, let us show that with these three we can generate the whole $\mathbf{O}_{\mathrm{Perm}}$. For $n \geq 1$ let χ_n denote the n-transposition permutation, in other words $\chi_n = (x_1x'_1)(x_2x'_2)\cdots(x_nx'_n)$. We have that $\chi_2 = (x_1x'_1z) \;{}_{z:y}\; (yx_2x'_2)$, that χ_1 is identity, and that $\chi_{n+1} = \big((\chi_{n-1})(x_nz)\big) \;{}_{z:y}\; \big((yx'_n)(x_{n+1}x'_{n+1})\big)$, where the first permutation is obviously isomorphic to χ_n and the second one to χ_2. So all the χ_n are generated by $(x)(y)$ and the three-cycle. Let now $\alpha = (A^1)\ldots(A^n)$ be an arbitrary permutation, with its decomposition in cycles given by the (A^i). Since every cycle is nonempty, by renaming we can ensure that A_i contains the variable x_i but not the variable x'_i. If B^i is the cycle obtained by $A^i[x_i\backslash x'_i]$ it is easy to see that

$$\alpha \quad = \quad \Big(\ldots\big((\chi_n \;{}_{x_1:x'_1}\; B^1) \;{}_{x_2:x'_2}\; B^2\big)\ldots\Big) \;{}_{x_n:x'_n}\; B^n\,.$$

A corollary of this is that $(x)(y)$ and the three-cycle generate all permutations without fixpoints, wich cannot be obtained without the one-element fixpoint.

Notice that χ_2 generates the substructad of all involutions without fixpoints, and $\{\chi_2,(x)\}$ the substructad of all involutions.

The interested reader can try to find the relations that hold between the generators of $\mathbf{O}_{\mathrm{Perm}}$, which is not very hard.

Example 42. Let $\mathbf{O}_{\mathrm{Funct}}$ be the structad of all functions (Example 20), so a structure X is a function $X\colon \Gamma \to \Delta$, living in context $X \in \mathbf{O}_{\mathrm{Func}}\langle \Gamma + \Delta\rangle$, with everything in Γ of polarity $\bullet$, everything in Δ of polarity $\circ$. Define the following (we do not bother to name the elements/variables of the sets, only their polarities):

$$\begin{array}{lll} - S & \text{the function} & \{\bullet,\bullet\} \to \{\circ\} \\ - B & \text{the bijection} & \{\bullet,\bullet\} \to \{\circ,\circ\} \\ - I & \text{the function} & \{\bullet\} \to \{\circ,\circ\} \\ - E & \text{the function} & \emptyset \to \{\circ\} \end{array}$$

When we say "the" function, what we mean is that any choice of a function satisfying the specification above will do, since any two choices will necessarily be isomorphic structures. Then it is easy to see that

- The substructad of $\mathbf{O}_{\mathrm{Funct}}$ generated by B is the structad of all bijections between nonempty sets.
- The substructad generated by B, I is the structad of all injective functions between nonempty sets.
- The substructad generated by B, S is the structad of all surjective functions.
- The substructad generated by B, S, I is the structad of all functions with nonempty domain.
- The substructad generaded by B, S, I, E is the structad of all functions with nonempty codomain.

And if we want the empty function to be part of our structad, we have to add a structure/generator in $\mathbf{O}\langle\emptyset\rangle$.

Most structads of interest turn out to have "natural" sets of generators.

Definition 17. *Let* $\mathbf{O}$ *be a structad. A* prime structure *is a structure* $\alpha \in \mathbf{O}\langle\Gamma\rangle$ *which is not identity, and such that* $\alpha = \beta \,{}_{x:y}\, \gamma$ *implies that* $\beta = \alpha$ *or* $\gamma = \alpha$*. A structad is said to be* prime-generated *if its prime structures form a set of generators.*

Example 43. It is very easy to see that every one of $(xyx), (x), (x)(y)$ in $\mathbf{O}_{\mathrm{Perm}}$ and B, I, S, E in $\mathbf{O}_{\mathrm{Funct}}$ in the two examples above is prime, and so that the structads involved are prime-generated.

Example 44. We have also seen that $(xy)(zw)$ generates the substructad $\mathbf{O}_{\mathrm{Invol-Fix}} \subseteq \mathbf{O}_{\mathrm{Perm}}$ of involutions without fixpoints. In that structad it is prime, while it is obviously not in $\mathbf{O}_{\mathrm{Perm}}$.

Example 45. All free structads are prime-generated, the primes corresponding exactly to the elements of the signature.

Example 46. Here is an example of a structad which is not prime-generated: take $\mathbb{B}$ as the set of polarities. Let G be a group, with composition denoted multiplicatively. Let $\mathbf{O}$ be the structad defined as

$$\mathbf{O}\langle\Gamma\rangle = \begin{cases} G \text{ if } \Gamma \text{ is of the form } \{x^\bullet, y^\circ\}. \\ \emptyset \text{ otherwise.} \end{cases}$$

Composition is defined by the group law: $g \,{}_{x^\circ:y^\bullet}\, h = gh$. Then since $g = (g^2) \,{}_{x:y}\, (g^{-1})$ the primeness of g would imply that either $g = g^2$, in other words $g = 1$, or $g = g^{-1}$, in other words $g^2 = 1$. But if 1 is prime the group is necessarily trivial, so this forces g to have order 2. We will let the reader show that in that case g can indeed be prime but the group G has to be the two-element group, which leaves an ample selection of structads that are not prime-generated.

Example 47. Except for $\mathbf{O}_{\mathrm{TotCycOrd}}$ which is isomorphic to $\mathbf{O}_{\mathrm{CycPerm}}$, we do not know much about the prime structures in the structads associated to cyclic orders and cyclic order varieties. We conjecture that all these structads are prime-generated, but determining the set of all these primes is an open problem. This is the reason that prompted the invention of *series-parallel order varieties* [68], which profitably can be seen as the substructad of $\mathbf{O}_{\mathrm{OrdVar}}$ generated by the three-element cycle β, (i.e. such that $\alpha(x, y, z)$ holds in the notation of Example 27), the three-element discrete (empty) order varity γ, and the unique order variety structure δ one the one-element set. There is still an open problem in the much smaller world of series-parallel order varieties: if the axiom $x, y, z \vdash_{\mathsf{r}} \beta \leq \gamma$ sufficient to axiomatize that structad?

5 Formal systems and cut elimination

In this section we will show how to build a multiplicative linear propositional deductive system out of a given structad, and some additional information having to do with the choice of connectives. But before we begin we will give a summary of the correspondence between structads and reversible theories, in the hope that sine readers will be able to read this section without fully mastering the previous one.

Structad	$\longleftrightarrow$	Reversible theory
Structure in a structad	$\longleftrightarrow$	Reversible term
$\mathbf{O}\langle\Gamma\rangle$	$\longleftrightarrow$	The set of reversible terms in context Γ
Map of structads	$\longleftrightarrow$	Interpretation of a reversible theory into another
The free structad $\mathbf{F}(\mathbf{\Sigma})$ over a tree signature $\mathbf{\Sigma}$	$\longleftrightarrow$	the theory generated by $\mathbf{\Sigma}$
A map $\mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$	$\longleftrightarrow$	A choice for every symbol in $\mathbf{\Sigma}$ of a term of $\mathbf{O}$ of the right type
The structad $\mathbf{J}$	$\longleftrightarrow$	The associative-commutative theory with unit

Interestingly nothing in the work requires that the structads/theories involved be decidable, although the reader can add that assumption, which is quite normal for a deductive system. We use the sequent calculus, which is by far the most popular approach to defining substructural logics, although there are examples of the use of natural deduction [59, 60] in the non-commutative world.

Given two polarity structures $\mathbb{P}, \mathbb{P}'$ it is quite obvious that their Cartesian product $\mathbb{P} \times \mathbb{P}'$ is also a polarity structure, defining the involution by $(\mathsf{pq})^{\perp} = (\mathsf{p}^{\perp}\mathsf{q}^{\perp})$. Also, the projections $\mathbb{P} \times \mathbb{P}' \to \mathbb{P}$, $\mathbb{P} \times \mathbb{P}' \to \mathbb{P}'$, obviously respect the involution.

Let $\mathbf{O}$ be a structad on the polarity structure $\mathbb{P}$ and $\alpha \in \mathbf{O}\langle\Gamma\rangle$ a structure. Suppose we want to associate a logical connector to α. One thing which we certainly have to do is to decide which port of α (equivalently, which $z \in \Gamma$) is the conclusion (or principal port, in the standard terminology), which will then immediately mark all the other ports/variables as premises. In order to do this we introduce a new set of polarities $\mathbb{U} = \{\mathsf{u}, \mathsf{d}\}$, with $\mathsf{u}^{\perp} = \mathsf{d}$, along with the flat structad $\mathbf{P} \subseteq \mathbb{U}^{\sharp}$ defined by

$$\mathbf{P}\langle\Gamma\rangle = \begin{cases} \{*\} & \text{if } \Gamma \text{ contains exactly one } x^{\mathsf{d}} \\ \emptyset & \text{otherwise} . \end{cases}$$

So $\mathbf{P}$ and $\mathbf{J}$ are isomorphic, with the correspondence $\circ \leftrightarrow \mathsf{d}$, $\bullet \leftrightarrow \mathsf{u}$. The reason for the distinction is that these two structads are used in different ways; $\mathbf{J}$ is used for the *proofs* of the formal system, and is needed only if we want that system to be intuitionistic, while $\mathbf{P}$ is used for the *types* of the system and cannot be done without.

So the natural polarity structure for a primitive f associated to the structure α above will be $\mathbb{P} \times \mathbb{U}$.

Definition 18. *Let $\mathbb{P}$ be a system of polarities. A* framework $\mathscr{F}$ *above $\mathbb{P}$ is a diagram of structads*

$$\begin{array}{ccc} \mathbf{F}(\mathbf{\Sigma}) & \xrightarrow{\mathrm{Or}} & \mathbf{P} \\ {\scriptstyle P}\downarrow & & \\ \mathbf{O} & & \end{array}$$

where $\mathbf{O}$ is called the structad of contexts, *$\mathbf{\Sigma}$ is a tree signature over $\mathbb{P} \times \mathbb{U}$ called the* choice of primitives, *and P, Or structad maps. We require that P, Or be above the projections $\mathbb{P} \times \mathbb{U} \to \mathbb{P}$ and $\mathbb{P} \times \mathbb{U} \to \mathbb{U}$ respectively. We say that a framework* strongly covers $\mathbf{O}$ *if $P_{\Gamma} \colon \mathbf{F}(\mathbf{\Sigma})\langle\Gamma\rangle \to \mathbf{O}\langle\Gamma\rangle$ is surjective whenever $\Gamma \neq \emptyset$; let us call this very slightly weakened form of surjectivity* quasi surjectivity.

So a framework can be defined more pedantically as a quadruple $(\mathbf{O}, \mathbf{\Sigma}, P, \mathrm{Or})$.

We have to think of Or as a function which assigns to every primitive $f \in \mathbf{\Sigma}$ its *orientation*, i.e., its choice of conclusion port. From now on until said otherwise we assume we have a strongly covering framework $\mathscr{F}$, with the same notation as above for its constituents.

So because of Or a primitive $f \in \mathbf{\Sigma}$ of valence $n+1$, i.e.

$$x_0^{(\mathsf{p}_0\mathsf{d})}, x_1^{(\mathsf{p}_1\mathsf{u})}, x_2^{(\mathsf{p}_2\mathsf{u})}, \ldots, x_n^{(\mathsf{p}_n\mathsf{u})} \vdash_{\mathsf{r}} f \tag{4}$$

can be seen as an ordinary n-ary function symbol, and we can dispense with the $\mathbb{U}$ information, so this can be abbreviated to an ordinary term:

$$x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n} \vdash f^{\mathsf{p}_0^{\perp}}(x_1, \ldots, x_n) \ . \tag{5}$$

We will denote its projection $P(f)$ on $\mathbf{O}$ by f^*. But now there is no intuitionistic structure left in general and we have to write

$$x_0^{\mathsf{p}_0}, x_1^{\mathsf{p}_1}, x_2^{\mathsf{p}_2}, \ldots, x_n^{\mathsf{p}_n} \vdash_{\mathsf{r}} f^* \tag{6}$$

So the free structad $\mathbf{F}(\mathbf{\Sigma})$ has a split personality. Its structures have to be reversible terms because it is mapped to $\mathbf{O}$. But because of Or they can also be seen as an ordinary terms. This is what makes the best choice of notation for its terms/structures (e.g. boldface vs. ordinary vs. Greek) rather problematic.

A primitive is not a connective yet. To each primitive we have to associate a *pair* of connectives, i.e. a positive-negative, or tensor-par pair. Given $f \in \mathbf{\Sigma}$ of arity n its associated positive $\otimes_f$ will always have an n-ary introduction rule while its associated negative $\parr_f$ will always have a unary rule. So we get two new tree signatures $\mathbf{\Sigma}_\parr, \mathbf{\Sigma}_\otimes$ over $\mathbb{P} \times \mathbb{U}$, with the same cardinality and valences as $\mathbf{\Sigma}$, and the following polarities:

- The signature $\mathbf{\Sigma}_\parr$ is isomorphic to $\mathbf{\Sigma}$. In other words for any $f \in \mathbf{\Sigma}$ the associated $\parr_f \in \mathbf{\Sigma}_\parr$ has exactly the same polarities as f, so we can reuse the presentation of (4,5), e.g.
$$x_1^{\mathsf{p}_1}, \ldots, x_n^{\mathsf{p}_n} \vdash \parr_f^{\mathsf{p}_0^\perp}(x_1, \ldots, x_n) \ ; \tag{7}$$
this allows us to define the equivalent of P on the pars, so we get $\parr_f^* = f^*$, as in 6.
- For every $f \in \mathbf{\Sigma}$, also assumed to be as in (4,5) we associate a $\otimes_f \in \mathbf{\Sigma}_\otimes$, with the following signature:
$$x_1^{\mathsf{p}_1^\perp}, \ldots, x_n^{\mathsf{p}_n^\perp} \vdash \otimes_f^{\mathsf{p}_0}(x_n, x_{n-1}, \ldots, x_1) \ . \tag{8}$$
In other words the $\mathbb{P}$-polarities are the *reverse* of those of f and $\parr_f$, the $\mathbb{U}$ polarities are the *same*, and the order of the operands of $\otimes_f$ is the reverse of that of $\parr_f$. This last stipulation is purely a writing convention, whose usefulness is well known when dealing with non-commutative calculi.

As we did for context variables, we will assume a set of unpolarized type variables; they will be denoted by letters like $X, X', Y, Z, X_i, \ldots$ and we assume that to every type variable X there exists a negated version $X^\perp$. So our set of type variables looks like $\mathrm{TypVar} + \mathrm{TypVar}^\perp$. Then we take the set of *atomic types* to be $\mathrm{At} = (\mathrm{TypVar} + \mathrm{TypVar}^\perp) \times \mathbb{P}$. This set is equipped with an involution, given by $(X^{\mathsf{p}})^\perp = X^{\perp \mathsf{p}^\perp}, (X^{\perp \mathsf{p}})^\perp = X^{\mathsf{p}^\perp}$. Notice that an alternative approach, more in line with what we have done with the context variables, would be to define $\mathrm{At} = \mathrm{TypVar} \times \mathbb{P}$, with $(X^{\mathsf{p}})^\perp = X^{\mathsf{p}^\perp}$, but this would create the possibility of type variables that are their own negation, e.g. $(X^{\mathsf{c}})^\perp = X^{\mathsf{c}}$; there is nothing wrong with this from a formal point of view, the problem is more that it clashes with tradition, people not being very enthusiastic about a negation operation that does nothing.

As we have seen in Example 38 the types are to be seen as an expansion of the polarity structure $\mathbb{P}$, and form a polarity structure by themselves: the set $\mathrm{Types}(\mathscr{F})$ of types associated to a framework $\mathscr{F}$ is given by the closed ordinary terms associated to free structad $\mathbf{F}(\mathbf{\Sigma}_\parr + \mathbf{\Sigma}_\otimes + \mathrm{At})$. There is an obvious map $P\colon \mathrm{Types} \to \mathbb{P}$ that assigns the polarity of the last connector/variable *as an ordinary term* (7,8), and that we also denote as $P(A) = A^*$, along with an involution on Types defined as usual as

$$\begin{aligned} \otimes_f(A_1, \ldots, A_n)^\perp &= \parr_f(A_n^\perp, \ldots, A_1^\perp) \\ \parr_f(A_1, \ldots, A_n)^\perp &= \otimes_f(A_n^\perp, \ldots, A_1^\perp), \end{aligned}$$

and by the just-defined involution on type variables. So we get that P is a map of polarity structures.

The definition of a framework is very general, and its real advantage is not that generality, but its simplicity: it is all which is needed to define the sequent calculus and prove cut-elimination. The frameworks we will encounter in practice will be more specific. But we contend that the concept of a framework can profitably be made *more* general, by replacing $\mathbf{F}(\mathbf{\Sigma})$ by a structad which is not necessarily free; we will not pursue that line of thought in this paper.

To see what frameworks are about, let us look at them from the point of view of the structad $\mathbf{O}$. If $\alpha \in \mathbf{O}\langle \Gamma \rangle$ is an $\mathbf{O}$-structure, and $t \in \Delta$ a structure/term in $\mathbf{F}(\mathbf{\Sigma})$ such that α and $P(t)$ are isomorphic, we can always transport t to a $t' \in \Delta'$ with $P(t') = \alpha$. So we can arrange that the symbols $f \in \mathbf{\Sigma}$ such that $P(f)$ is isomorphic to α have their chosen contexts such that $P(t) = \alpha$ actually, and we can define the subset $P^{-1}(\alpha) \cap \mathbf{\Sigma}$ of primitives that are mapped to α. Notice that if Δ, Δ' are $\mathbb{P} \times \mathbb{U}$-contexts such that $P\Delta = P\Delta' = \Gamma$, their *unsorted* variables will coincide with those of Γ, but they do not have to be identical contexts, because of the presence of $\mathbb{U}$-information. All we know is that there will be unique $x \in \Delta, x' \in \Delta'$ with $x^{(\mathsf{pd})}, x'^{(\mathsf{p'd})}$, the ports of α which are considered to be the conclusion of the connector $\parr_f, \parr_{f'}$ respectively. Suppose that $x = x'$ (forcing $\Delta = \Delta'$), then we will see below that the connectors $\parr_f, \parr_{f'}$ and $\otimes_f, \otimes_{f'}$ have the exact same introduction rules, so they will be provably equivalent in the deductive system assigned to $\mathscr{F}$, in other words this particular $\mathscr{F}$ has redundancies and one of f, f' can be removed without any real changes.

So we can make a stab at removing redundancies.

Definition 19. *Assume A is a set of generators for $\mathbf{O}$ that does not contain isomorphic pairs (see Definition 15), and let $\mathbf{\Sigma}'$ be a tree signature which has a single symbol $\mathbf{f}_\alpha$ for every $\alpha \in A$ (with the same context, need we say). An A-*regular framework *is a framework $\mathscr{F}$ such that its signature $\mathbf{\Sigma}$, seen as an ordinary term signature, is a subset of the derived signature of $\mathbf{\Sigma}'$.*

Since A is a set of generators, it is a necessary condition that every α be represented by a symbol in $\mathbf{\Sigma}$, or else P will not be quasi-surjective. But this is not enough to ensure the quasi-surjectivity of P, as we will see. The meaning of the strong covering condition is that it ensures that $\mathbf{O}$ is really the natural structad associated with $\mathscr{F}$'s logic. In other words, if P is not quadi-surjective, we have the possibility that some structure $\alpha \in \mathbf{O}$ does not appear in the sequent calculus associated to $\mathscr{F}$, meaning that we could have used a smaller structad and generated the same logic.

So, if it is seen as a tree signature, $\mathbf{\Sigma}$ will be the universal reversible extension of a subset of the ordinary signature associated to tree signature $\mathbf{\Sigma}'$.

Example 48. There is always a largest A-regular framework, obtained by taking the full ordinary derived signature. In particular, if $\mathbf{O}$ has a "canonical" set of generators (for example if it is prime-generated), then this full framework is "the" logic associated to the structad $\mathbf{O}$. But the concept of a (strongly covering) framework gives a precise, algebraic definition of a concept of fragment of a "full" logic, and so allow us to do a systematic search for all these fragments.

Example 49. Let $\mathbf{O}$ be $\mathbf{O}^{\mathsf{c}}_{\mathrm{FreeBin}}$, in other words the free structad over c with one generator of valence three (Example 12); choosing a context $x_0, x_1, x_2 \vdash_{\mathsf{r}} \alpha$ for this generator, let us recall the reversible term notation associated with the derived signature:

$$x_0, x_1, x_2 \vdash_{\mathsf{r}} \alpha = \langle x_0 \star^2 x_1, x_2 \rangle \equiv \langle x_1 \star^0 x_2, x_0 \rangle \equiv \langle x_2 \star^1 x_0, x_1 \rangle$$

We claim that the only possible $\{\alpha\}$-regular framework is the one with the full derived signature, $\mathbf{\Sigma} = \{\star^0, \star^1, \star^2\}$. This is because of what we have already remarked in 12, that the orientation tabs of an arbitrary structure in $\mathbf{O}^{\mathsf{c}}_{\mathrm{FreeBin}}$ can point in all directions. In other words, let $\mathbf{\Sigma}$ be a signature that makes $P\colon \mathbf{F}(\mathbf{\Sigma}) \to \mathbf{O}$ an $\{\alpha\}$-regular framework. Let $\Delta \vdash_{\mathsf{r}} \beta$ be an arbitrary structure in $\mathbf{O}_{\mathrm{FreeBin}}$. By assumption there is an ordinary term $\Delta' - \{x\} \vdash t$ in $\mathbf{F}(\mathbf{\Sigma})$ with $\Delta \vdash_{\mathsf{r}} t^* = \beta$ (here Δ' only differs from Δ by the polarities of its variables). In other words $\Delta \vdash_{\mathsf{r}} \beta = \langle t, x \rangle$. So t is actually $\beta^{[x]}$. But we know that because β has been arbitrarily chosen, any of the symbols $\star^0, \star^1, \star^2$ can appear in t. So $\mathbf{\Sigma}$ has to contain all three.

So the conclusion is that the logic associated to $\mathbf{O}_{\mathrm{FreeBin}}$ has to have three binary pars and three binary tensors.

This argument applies when $\mathbf{O}$ is any free structad over $\mathbb{C}$. In other words in general such a structad can have a unique regular framework associated to it.

Example 50. Let now $\mathbf{O}$ be $\mathbf{O}^{\mathsf{c}}_{\mathrm{FreeBin+Cyc}}$ (Example 13). This being a quotient of the previous example, it has a single generator, and we can use the same notation as before. But this time we have

$$x_0, x_1, x_2 \vdash_{\mathsf{r}} \langle x_1 \star^0 x_2, x^0 \rangle = \langle x_1 \star^1 x_2, x^0 \rangle = \langle x_1 \star^2 x_2, x^0 \rangle$$

so in order to make P surjective we have to choose only one of $\star^0, \star^1, \star^2$ to define $\mathbf{\Sigma}$. More than one symbol will be redundant. The logic defined by this $\mathscr{F}$ is the one described in [17, 61].

Example 51. Let $\mathbf{O}$ be equipped with an intuitionistic structure $S\colon \mathbf{O} \to \mathbf{J}$. Then there is an additional "canonical" regular framework structure associated to a generating set A: the one where, for every $\alpha \in A$ a single primitive is chosen for $\mathbf{\Sigma}$, the one whose d-port corresponds to α's $\circ$-port. The logics associated to these frameworks are the "tensor-only" fragments of intuitionistic calculi, which are studied by category theorists, because models of them abound in nature.

Example 52. Let $\mathbf{O} = \mathbf{O}_{\mathrm{Ass}}$, the ordinary (intuitionistic) theory of a associativity (semigroups). We already know two frameworks associated to it: one is the full one, whose logic is the original Lambek calculus $\mathbf{L}$: remember that since we do not have a unit, all contexts must have at least two variables; since one of these has to have polarity $\circ$ (i.e. be the conclusion), we will have a calculus where the left part of sequents cannot be empty, just like the one Lambek defined for linguistic purposes. The other framework we already know is the tensor-only fragment as described above. But there are other sub-frameworks of the full one. If $*^0, *^1, *^2$ are the operators of the ordinary signature, with, as usual

$$x^\bullet, y^\bullet, z^\circ \vdash_{\mathsf{r}} \langle x *^0 y, z \rangle \equiv \langle x, y *^1 z \rangle \equiv \langle y, z *^2 x \rangle,$$

(we have used rule Perm to makes things more in line with practice). We know that every structure can be written as

$$x_1^\bullet, x_2^\bullet, \ldots, x_{n-1}^\bullet, z^\circ \vdash_{\mathsf{r}} \left\langle\, x_1 *^0 x_2 *^0 \cdots *^0 x_{n-1}, z \,\right\rangle$$

where the bracketing can be dropped because of associativity. But this can be rewritten as (by repeated use of the first equation just above):

$$x_1^\bullet, x_2^\bullet, \ldots, x_{n-1}^\bullet, z^\circ \vdash_{\mathsf{r}} \left\langle\, x_1, x_2 *^1 \left(\cdots x_{n-2} *^1 (x_{n-1} *^1 z)\cdots\right) \,\right\rangle$$

wich shows that $*^1$ is enough by itself to make a regular framework since every structure can be presented using only that primitive; the logic generated this way is the $/$-only fragment of the Lambek calculus.[5] The same way, by choosing $*^2$ as the only primitive we can get the $\backslash$-only fragment, and any nonempty subset of $\{\otimes, /, \backslash\}$ will give rise to a framework.

To all the examples we have just given we can add a unit; our logic will then have a pair of constants, which by tradition can be denoted $\mathbf{1}$ (positive-tensor) and $\perp$ (negative-par). Notice that we can use such a structad with units with a language that does not have the constants; this is standard practice. This mean that the concept of strongly covering framework is a little too strong to catch all conceivable fragments of logic. It can be weakened slightly but more conditions have to be added to the structad $\mathbf{O}$.

Remark 6. Suppose $\mathbf{O}$ is equipped with a two-sided structure $S\colon \mathbf{O} \to \mathbb{B}^\sharp$, not necessarily an intuitionistic one. Then we can use a more traditional notation for the connectives and types, and have, say a “real” implication symbol. We still want an involution $(-)^\perp$ because we want a polarity structure $\mathbb{T}$ and a map of polarity structures $\mathbb{T} \to \mathbb{P}$ to the set of polarities of $\mathbf{O}$, but we can work in such a way that this involution will not have to be defined by structural induction on the set of types; instead, for every primitive $f \in \mathbf{\Sigma}$ assign a single connective $*_f$ instead of the two, and define the set of types as being built from these connectives, using only the set TypVar of positive type variables (no negavariables anymore). Now define $\mathbb{T}$ by $\mathbb{T} = \mathrm{TypVar} \times \mathbb{B}$, and the involution by $(A\mathsf{p})^\perp = (A\mathsf{p}^\perp)$. There is a correspondence with the $\otimes - \invamp$ set of types (which is more general) that the reader can work out. Notice that if a negation is present in the system it will have to be defined via a logical symbol, with the standard introduction rules of a two-sided system.

Definition 20. *A (logical)* sequent *is an expression of the form*

$$\vdash_\alpha \; x_0\colon A_0, \ldots, x_n\colon A_n$$

where $A_0, \ldots, A_n$ are types, for every $0 \le i \le n$, x_i is an algebraic context variable of polarity $\mathsf{p}_i = A_i^*$, and α is an element of $\mathbf{O}\langle x_0, \ldots, x_n\rangle$, in other words $x_0, \ldots, x_n \vdash_{\mathsf{r}} \alpha$.

So α is the context structure on the formulas $A_0, \ldots, A_n$. This approach allows us to put the formulas A_i in any order we want, since the structural information is contained in α and the assignment of variables.

[5] If we use the most frequent way of writing its introduction rule, but some people would say it is the $\backslash$-only fragment.

We will abbreviate such a context as $\vdash_\alpha \ \Gamma$, and use the notation Γ^* for the restriction to its context variables, i.e. $\Gamma^* \vdash_{\mathsf{r}} \alpha$.

Given a sequent $\vdash_\alpha \ \Gamma$ we use the standard notation $\Vdash_\alpha \ \Gamma$ to represent the assertion that $\vdash_\alpha \ \Gamma$ is provable. We define the sequent calculus[6] associated to our framework by:

$$\frac{}{\Vdash_{\langle x,y\rangle} \ x^{\mathsf{p}}\colon X^{\mathsf{p}}, y^{\mathsf{p}^\perp}\colon X^{\perp\mathsf{p}^\perp}}\,\mathrm{Ax} \qquad \frac{\Vdash_{\langle \alpha^{[x]}, f^*(x_1,\ldots,x_n)\rangle} \ \Gamma\,,\, x_1\colon A_1, \ldots, x_n\colon A_n}{\Vdash_\alpha \ \Gamma\,,\, x\colon \parr_f(A_1,\ldots,A_n)}\,\parr_f$$

$$\frac{\Vdash_{\alpha_1} \ \Gamma_1\,,\, x_1\colon A_1 \quad \cdots \quad \Vdash_{\alpha_n} \ \Gamma_n\,,\, x_n\colon A_n}{\Vdash_{\langle y, f^*(\alpha_1^{[x1]}, \ldots, \alpha_n^{[xn]})\rangle} \ y\colon \otimes_f(A_n, A_{n-1}, \ldots, A_1), \Gamma_1, \ldots, \Gamma_n}\,\otimes_f$$

$$\frac{\Vdash_\alpha \ \Gamma \qquad \Gamma^* \vdash_{\mathsf{r}} \alpha \le \beta}{\Vdash_\beta \ \Gamma}\,\mathrm{Entropy} \qquad \frac{\Vdash_\alpha \ \Gamma, x\colon A \qquad \Vdash_\beta \ \Delta, y\colon A^\perp}{\Vdash_{\alpha x:y\beta} \ \Gamma, \Delta}\,\mathrm{Cut}$$

Example 53. Sometimes the structural structad is simple enough for an approximation of traditional notation to be used. For example let $\mathbf{O}$ be the permutation structad $\mathbf{O}_{\mathrm{Perm}}$. It has three canonical generators (Example 41), namely $(xyz), (x)(y)$ and (x). Since (xyz) obeys the axiom of Cyclicity, because of Example 50 we know that it can be assigned a single pair of binary connectives, call them $\otimes, \parr$. By the same kind of argument (stability of the structure under the permutation that exchanges x, y), the two-fixpoint permutation needs only a single pair of unary connectives to be fully represented in the logic, call them $\diamondsuit, \Box$, $\diamondsuit$ being the positive one (unary tensor). A permutation on a finite set (decorated with formulas) is a structure that can easily be represented on a single line, by the means of the sum-of-cycles notation, as we already have done extensively. For legibility we will add commas inside cycles, and get things like $\vdash \ (A, B)\,(C, A, B)\,(C)$, which would more traditionally be written as something resembling $\vdash \ A, B\,;\, C, A, B\,;\, C$. Let the letters Γ, Δ denote lists of formulas, so (Γ) is the associated cyclic permutation, while Ψ, Θ denote lists of cyclic lists, i.e. permutations that can have more than one cycle. What is usually called structural rules (apart from Axioms and Cut) correspond to transformations on the *notation* that do not change the structure: so in our case we have

$$\frac{\Vdash \ \Theta\,(\Gamma, A)\,\Psi}{\Vdash \ \Theta\,(A, \Gamma)\,\Psi}\,\mathrm{Cyc} \qquad \frac{\Vdash \ \Theta\,(\Gamma)\,(\Delta)\,\Psi}{\Vdash \ \Theta\,(\Delta)\,(\Gamma)\,\Psi}\,\mathrm{Exch}$$

$$\frac{}{\Vdash \ (A, A^\perp)}\,\mathrm{Ax} \qquad \frac{\Vdash \ \Theta\,(\Gamma, A) \qquad \Vdash \ (A^\perp, \Delta)\,\Psi}{\Vdash \ \Theta\,(\Gamma, \Delta)\,\Psi}\,\mathrm{Cut}$$

[6] We have chosen a calculus where the only axioms involve atomic types; it is not hard to show the identity is derivable for every formula.

In this notation the introduction rules are:

$$\frac{\vdash\ \Theta\,(\Gamma, A) \qquad \vdash\ (B, \Delta)\,\Psi}{\vdash\ \Theta\,(\Gamma, A \otimes B, \Delta)\,\Psi}\otimes \qquad \frac{\vdash\ \Theta\,(\Gamma, A, B, \Delta)\,\Psi}{\vdash\ \Theta\,(\Gamma, A \parr B, \Delta)\,\Psi}\parr$$

$$\frac{\vdash\ \Theta\,(\Gamma, A)}{\vdash\ \Theta\,(\Gamma)\,(\Diamond A)}\Diamond \qquad \frac{\vdash\ \Theta\,(\Gamma)\,(A)}{\vdash\ \Theta\,(\Gamma, \Box A)}\Box$$

$$\frac{}{\vdash\ (\mathbf{1})}\mathbf{1} \qquad \frac{\vdash\ \Theta\,(\Gamma)}{\vdash\ \Theta\,(\Gamma, \bot)}\bot$$

Example 54. The example above needs only minor changes to apply to the structad $\mathbf{O}_{\mathrm{Part}}$ of partitions. In this case the cyclical exchange rule Cyc has to be replaced by the stronger, ordinary Exchange rule inside the "cycles" (which are now the classes of the partition):

$$\frac{\vdash\ \Theta\,(\Gamma, A, B, \Delta)\,\Psi}{\vdash\ \Theta\,(\Gamma, B, A, \Delta)\,\Psi}\mathrm{InnExch}$$

and the presence of an order on structures has to be expressed by an entropy rule

$$\frac{\vdash\ \Theta\,(\Gamma, \Delta)\,\Psi}{\vdash\ \Theta\,(\Gamma)\,(\Delta)\,\Psi}\mathrm{Entrop}$$

Example 55. If we now take the structad $\mathbf{O}_{\mathrm{Inv-Fix}}$ of involutions without fixpoints, we know it has a single generator $(xy)(zw)$. This means ternary tensors and pars, and it does not take long to see that a single tensor and a single par will make for a full framework. A sequent looks as above, except that all the cycles are of the form (A, B), with only two formulas. The introduction rules are

$$\frac{\vdash\ \Theta\,(A, A') \qquad \vdash\ \Psi\,(B, B') \qquad \vdash\ \Phi\,(C, C')}{\vdash\ \Theta\,\Psi\,\Phi\,(A', B')\,(\otimes(A, B, C), C')}\otimes \qquad \frac{\vdash\ \Theta\,(A, B)\,(C, D)}{\vdash\ \Theta\,(\parr(A, B, C), D)}\parr$$

5.1 Cut-Elimination

We can state and prove the expected result

Theorem 6. *Any proof can be tranformed into one that does not use the Cut rule.*

The proof of this depends on the following

Lemma 4. *Let P be a proof in the sequent calculus that contains*

$$\frac{\dfrac{\vdash_{\alpha_1}\ \Gamma_1 \quad \cdots \quad \vdash_{\alpha_n}\ \Gamma_n}{\vdash_\alpha\ \Gamma, x{:}\,A}R \qquad \dfrac{\cdots}{\vdash_\delta\ y{:}\,A^\perp,\ \Delta}}{\vdash_{\alpha x:y\delta}\ \Gamma, \Delta}\mathrm{Cut}$$

a rule application R followed by a Cut, such that R does not introduce the root (outermost) symbol of A. Then rule R can pushed down so as to be applied after the Cut.

Proof. The proof is a simple case analysis

- R cannot be an Axiom, since then the formula A would necessarily be an atomic type, contradicting our assumption.
- R is an application of Entropy, so $n = 1$ and the left branch of the proof has the form

$$\dfrac{\dfrac{\cdots}{\vdash_\beta \ \Gamma, A}}{\vdash_\alpha \ \Gamma, x\colon A}\,R$$

with $\Gamma^* \vdash_{\mathsf{r}} \beta = \alpha_1 \leq \alpha$, so P can be replaced by

$$\dfrac{\dfrac{\dfrac{\cdots}{\vdash_\beta \ \Gamma, x\colon A} \qquad \dfrac{}{\vdash_\delta \ y\colon A^\perp,\ \Delta}}{\vdash_{\beta x:y\delta} \ \Gamma, \Delta}\,\text{Cut}}{\vdash_{\alpha x:y\delta} \ \Gamma, \Delta}\,\text{Entrop}$$

the last deduction being valid since $\Gamma^*, \Delta^* \vdash_{\mathsf{r}} \beta\, x{:}y\, \delta \leq \alpha\, x{:}y\, \delta$ because of Congruence.
- R is a $\wp_f$-rule, so looks like

$$\dfrac{\dfrac{\cdots}{\vdash_{\langle \alpha^{[z]}, f^*(z_1,\ldots,z_n)\rangle} \ z_1\colon B_1, \ldots, z_n\colon B_n, \Gamma', A}}{\vdash_\alpha \ z\colon \wp_f(B_1, \ldots, B_n), \Gamma', A}\,R$$

and we can do

$$\dfrac{\dfrac{\vdash_{\langle \alpha^{[z]}, f^*(z_1,\ldots,z_n)\rangle} \ z_1\colon B_1, \ldots, z_n\colon B_n, \Gamma', A \qquad \vdash_\delta \ A^\perp, \Delta}{\vdash_{\langle \alpha^{[z]}, f^*(z_1,\ldots,z_n)\rangle x:y\,\delta} \ z_1\colon B_1, \ldots, z_n\colon B_n, \Gamma', \Delta}\,\text{Cut}}{\vdash_{\alpha x:y\,\delta} \ z\colon \wp_f(B_1, \ldots, B_n), \Gamma', \Delta}\,\wp_f$$

the conclusion is identical because

$$\langle \alpha^{[z]}, f^*(z_1, \ldots, z_n)\rangle x{:}y\delta = \langle \alpha^{[z]} x{:}y\delta, f^*(z_1, \ldots, z_n)\rangle = \langle (\alpha x{:}y\delta)^{[z]}, f^*(z_1, \ldots, z_n)\rangle\,.$$
- R is a $\otimes_f$-rule and so looks like

$$\dfrac{\vdash_{\alpha_1} \ x_1\colon C_1\,,\ \Gamma_1 \qquad \cdots \qquad \vdash_{\alpha_n} \ x_n\colon C_n\,,\ \Gamma_n}{\vdash_{\langle z, f^*(\alpha_1^{[x_1]}, \ldots, \alpha_n^{[x_n]})\rangle} \ z\colon \otimes_f(C_n, C_{n-1}, \ldots, C_1), \Gamma_1, \ldots, \Gamma_n}\,\otimes_f$$

with $\Gamma^* \vdash_{\mathsf{r}} \alpha = \langle z, f^*(\alpha_1^{[x_1]}, \ldots, \alpha_n^{[x_n]})\rangle$. It follows that there is $i \leq n$ such that $\vdash_{\alpha_i} \ x_i\colon C_i\,,\ \Gamma_i$ is actually of the form $\vdash_{\alpha_i} \ x_i\colon C_i\,,\ \Gamma'\,,\ x\colon A$. Then we can do

$$\dfrac{\vdash_{\alpha_1} \ x_1\colon C_1\,,\ \Gamma_1 \quad \cdots \quad \dfrac{\vdash_{\alpha_i} \ x_i\colon C_i, \Gamma', x\colon A \qquad \vdash_\delta \ A^\perp, \Delta}{\vdash_{\alpha_i x:y\delta} \ x_i\colon C_i\,,\ \Gamma'\,,\ \Delta} \quad \cdots \ \vdash_{\alpha_n} \ x_n\colon C_n\,,\ \Gamma_n}{\vdash_{\alpha x:y\delta} \ z\colon \otimes_f(C_n, C_{n-1}, \ldots, C_1), \Gamma_1, \ldots, \Gamma', \ldots, \Gamma_n, \Delta}$$

and this completes the proof of the lemma

It is now quite easy to prove the theorem. Given a proof with a Cut, we know from the above that we can always rearrange that proof so that both formulas involved in the Cut have had their outermost symbols introduced right above them. Then there are only two possibilities: the first one is when both formulas are atomic, so we are cutting one axiom against itself: keep only one copy of that axiom.

The other possibility is when a Par is cut against a Tensor. So we have

$$\dfrac{\dfrac{\vdash_{\langle\alpha^{[x]},f^*(x_1,\ldots,x_n)\rangle} \mathit{\Gamma}, A_1,\ldots,A_n}{\vdash_{\langle\alpha^{[x]},x\rangle} \mathit{\Gamma}, x{:}\parr_f(A_1,\ldots,A_n)} \qquad \dfrac{\vdash_{\alpha_n} y_n{:}A_n^{\perp},\mathit{\Delta}_n \quad \ldots \quad \vdash_{\alpha_1} y_1{:}A_1^{\perp},\mathit{\Delta}_1}{\vdash_{\langle y,f^*(\alpha_1^{[y_1]},\ldots,\alpha_n^{[y_n]})\rangle} y{:}\otimes_f(A_n^{\perp},\ldots,A_1^{\perp}),\mathit{\Delta}_n,\ldots,\mathit{\Delta}_1}}{\vdash_{\langle\alpha^{[x]},f^*(\alpha_1^{[y_1]},\ldots,\alpha_n^{[y_n]})\rangle} \mathit{\Gamma},\mathit{\Delta}_n,\ldots,\mathit{\Delta}_1}$$

and we can replace by

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\langle\alpha^{[x]},f^*(x_1,\ldots,x_n)\rangle} \mathit{\Gamma}, A_1,\ldots,A_n \qquad \vdash_{\alpha_n} y_n{:}A_n^{\perp},\mathit{\Delta}_n}{\vdash_{\langle\alpha^{[x]},f^*(x_1,\ldots,\alpha_n^{[y_n]})\rangle} \mathit{\Gamma},\mathit{\Delta}_n,A_1,\ldots,A_{n-1}} \qquad \vdash_{\alpha_{n-1}} y_{n-1}{:}A_{n-1}^{\perp},\mathit{\Delta}_{n-1}}{\vdash_{\langle\alpha^{[x]},f^*(x_1,\ldots,\alpha_{n-1}^{[y_{n-1}]},\alpha_{n-1}^{[y_n]})\rangle} \mathit{\Gamma},\mathit{\Delta}_n,\mathit{\Delta}_{n-1},A_1,\ldots,A_{n-2} \qquad \cdots}}{\cdots}}{\vdash_{\langle\alpha^{[x]},f^*(\alpha_1^{[y_1]},\ldots,\alpha_{n-1}^{[y_{n-1}]},\alpha_n^{[y_n]})\rangle} \mathit{\Gamma},\mathit{\Delta}_n,\ldots,\mathit{\Delta}_1}$$

since we are in a linear system there is no need for a complex induction to prove that this process terminates.

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