Resultant over the residual of a complete intersection

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Abstract

In this article, we study the residual resultant which is the necessary and sufficient condition for a polynomial system F to have a solution in the residual of a variety, defined here by a complete intersection G. We show that it corresponds to an irreducible divisor and give an explicit formula for its degree in the coefficients of each polynomial. Using the resolution of the ideal $(F : G)$ and computing its regularity, we give a method for computing the residual resultant using a matrix which involves a Macaulay and a Bezout part. In particular, we show that this resultant is the gcd of all the maximal minors of this matrix. We illustrate our approach for the residual of points and end by some explicit examples.

1 Introduction

Projection is one of the more used operation in Effective Algebraic Geometry [Eis94], [CLO92]. The resultant is a tool to perform it and has many applications in this domain. It leads to efficient methods for solving polynomial equations, based on matrix formulations [EM99]. Such techniques allow a control of the computations and can be used with approximate coefficients (which is important in many applications), taking into account the continuity of the problem in a neighborhood of the input data. Unfortunately, they apply only for systems which are generic for the considered resultant formulation. The problems encountered in practice are not always generic, but usually we know the extra-component which makes them degenerate and that we want to remove [Fau93], [RR95], [Mou96]. The aim of this paper is to extend the theory and applicability of resultants to such systems which are *generic cases of degenerate situations.*

Resultant theory is concerned with the study of polynomial equations depending on parameters of the general form

$$\mathbf{f_c} := \begin{cases} f_0(\mathbf{t}) = \sum_{j=0}^{s_0} c_{0,j}\, \kappa_{0,j}(\mathbf{t}) \\ \quad\vdots \\ f_m(\mathbf{t}) = \sum_{j=0}^{s_m} c_{m,j}\, \kappa_{m,j}(\mathbf{t}) \end{cases}$$

where $\mathbf{c} = (c_{i,j})$ are parameters, $\mathbf{t}$ is a point of an open subset U of the affine space $\mathbb{A}^m$ over a field $\mathbb{K}$, and the $\kappa_{i,j}$ are polynomials in the variables $\mathbf{t} = (t_1, \ldots, t_m)$. The aim consists in computing necessary and sufficient condition(s) on the parameters $\mathbf{c}$ such that the system $\mathbf{f_c}$ has *a solution*. A first difficulty is to specify what is meant by *a solution*.

In the classical case, $(\kappa_{i,j})_{j=0\ldots s_i}$ is the set of all monomials in $t_1, \ldots, t_m$ of degree at most d_i (d_i is a positive integer), $U = \mathbb{A}^m$. This condition is given by the projective resultant [Mac02], [Jou91]. It vanishes if and only if the system of homogenized equations has a solution in the projective space $\mathbb{P}^m$ over $\mathbb{K}$.

In the toric context, the open subset U is $(\mathbb{K} - \{0\})^m$ and the $\kappa_{i,j}$ are (Laurent) monomials in $t_1^{\pm 1}, \ldots, t_m^{\pm 1}$. This yields to the notion of toric resultant, which is a condition on $\mathbf{c}$ such that the system $\mathbf{f_c}$ "homogenized" in a convenient way has a solution in the corresponding toric variety [GKZ94], [Stu93], [CE93], [Cox95].

A resultant over a unirational algebraic variety is constructed in [BEM00]: If X is a projective variety parameterized by a map σ defined on an open subset $U \subset \mathbb{A}^m$, and $\psi_{i,j}$ are homogeneous polynomials such that $\kappa_{i,j} = \psi_{i,j} \circ \sigma$. The existence of an irreducible resultant polynomial Res_X in $\mathbf{c}$ is shown, under some minimal conditions. This resultant satisfies $\mathrm{Res}_X(\mathbf{f_c}) = 0$ if and only if the system $\mathbf{f_c}$ has a solution in the following sense: There exists $x \in X$ such that $(x, \mathbf{c}) \in \overline{W}$, where $W = \{(x, \mathbf{c}) \in \sigma(U) \times \mathbb{P}^{s_0} \times \cdots \times \mathbb{P}^{s_m} : \sum_{j=0}^{s_i} c_{i,j}\, \psi_{i,j}(x) = 0 \text{ for } i = 0 \ldots m\}$.

In this paper, we focus on *residual resultants*, which correspond to the following situation: Let $g_1, \ldots, g_n$ be homogeneous polynomials of degree $k_1, \ldots, k_n$ in $R = \mathbb{K}[x_0, \ldots, x_n]$. Let $f_0, \ldots, f_m \in R$ be generic homogeneous polynomials of degree $d_0, \ldots, d_m$ in the ideal $G = (g_1, \ldots, g_n)$ and F the ideal that they generate. We have

$$\mathbf{f_c} := \begin{cases} f_0(x) = \sum_{i=1}^{n} h_{i,0}(x)\, g_i(x) \\ \quad\vdots \\ f_m(x) = \sum_{i=1}^{n} h_{i,m}(x)\, g_i(x) \end{cases} \tag{1}$$

where $h_{i,j} = \sum_{|\alpha|=d_j-k_i} c_{\alpha}^{i,j}\, \mathbf{x}^{\alpha}$ is the generic homogeneous polynomial of degree $d_j - k_i$. We denote by H the matrix $(h_{i,j})_{1\le i\le n, 0\le j\le m}$ so that $F = G\, H$. We are looking for the condition(s) on the coefficients $\mathbf{c} = (c_{\alpha}^{i,j})$ of $h_{i,j}$ such that $\mathbf{f_c}$ has a solution "outside" the variety $V(G)$ defined by G.

In the next section, we extend the condition given in [GKZ94], for the existence of the resultant of global sections of $m+1$ invertible sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_m$ on a projective variety X of dimension m. We show that the associated divisor is reduced and give its degree in terms of the first Chern class of $\mathcal{L}_0, \ldots, \mathcal{L}_m$. We use this generalization to construct the residual resultant of G when $n \le m+1$. It is the necessary and sufficient condition on $\mathbf{c}$ for the system $\mathbf{f_c}$ to have a solution on the blowing-up $\tilde{X}$ of $\mathbb{P}^m$ along $V(G)$. We give an explicit formula for its degree in the coefficients of the polynomials f_i.

After these geometrical considerations (sections 2 and 3), we move in sections 4 and 5 to algebra and effective computations. Using the resolution of the ideal $(F : G)$ due to [BKM90] (in the case of a regular sequence $g_1, \ldots g_n$), and computing the Castelnuovo-Mumford regularity of this ideal, we construct a matrix, whose maximal nonzero minors are multiples of the residual resultant. This matrix combines a Macaulay part (i.e. monomial multiples of the polynomials $f_0, \ldots, f_m$) with a "Bezout" one (i.e. the $n \times n$ minors of the matrix H). Its size is smaller than the Macaulay matrix of the usual projective resultant. We prove that the residual resultant is exactly the greatest common divisor of all the maximal minors of this matrix and give another characterization of its vanishing in terms of saturations of ideals F and G. Finally, we propose an algorithm based on Gröbner basis computations to construct a maximal nondegenerate minor of this matrix which is of minimal degree in the coefficients of a fixed f_i. We detail our approach in the case of residual points and end with some examples.

Hereafter we will use the following notations: $\mathbb{K}$ is an algebraically closed field, R is the polynomial ring $\mathbb{K}[x_0, \ldots, x_m]$, $\mathbb{P}^m$ the projective variety over $\mathbb{K}$. Generally, if V is a vector space, $\mathbb{P}(V)$ will be the projective space defined by V. Let I be a homogeneous ideal. The variety defined by I in $\mathbb{P}^m$ is $V(I)$. If

$\nu \in \mathbb{N}$, $I_{[\nu]}$ will be the part of I in degree ν. For any $\alpha = (\alpha_0, \ldots, \alpha_m) \in \mathbb{N}^{m+1}$, $\mathbf{x}^\alpha = x_0^{\alpha_0} \ldots x_m^{\alpha_m}$, and $|\alpha| = \alpha_0 + \cdots + \alpha_m$. For any subset $S \subset R$, $\langle S \rangle$ denotes the $\mathbb{K}$-vector subspace of R generated by S.

2 Resultant over a projective variety

In order to define the resultant over an irreducible projective variety X (and to control its degree), instead of polynomials we consider global sections of invertible sheaves. We will recall briefly some facts on sheaves to fix the notations that we will use (for more details see [Har77]). A sheaf $\mathcal{L}$ of rings on X is given by a collection of rings $\mathcal{L}(U)$ parameterized by the open subsets U of X with gluing conditions. For instance, the sheaf of rings $\mathcal{O}_X$ is locally the ring of regular functions. A global section of $\mathcal{L}$ is an element of $\mathcal{L}(X)$. The set of global sections is denoted by $H^0(X, \mathcal{L})$. The set of global sections of the sheaf $\mathcal{O}_{\mathbb{P}^m}(d)$ is the vector space of homogeneous polynomials of degree d in $m+1$ variables. When $\mathcal{L}$ is a coherent sheaf and $\mathbb{K}$ is a field, $H^0(X, \mathcal{L})$ is a finite dimensional vector space over $\mathbb{K}$ (see [Har77][Theorem 5.19 p. 122]). This will be satisfied for all the sheaves that we will consider below. For any $f \in H^0(X, \mathcal{L})$, $X(f) = \{x \in X : f(x) = 0\}$ is the zero set defined by f.

A sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules is a collection $\mathcal{L}(U)$ of $\mathcal{O}_X(U)$-modules with compatibility conditions. The sheaf $\mathcal{L}$ is invertible if locally it is a $\mathcal{O}_X$-module of rank 1. We recall that an invertible sheaf is coherent.

We say that $\mathcal{L}$ is generated by a subset $\{s_1, \ldots, s_l\}$ of its global sections if for every $x \in X$, the germs $s_{1,x}, \ldots, s_{l,x}$ generate the stalk $\mathcal{L}_x$ over $\mathcal{O}_x$ (i.e. the direct limit of $\mathcal{L}(U)$ when U ranges over the open subsets of X containing x).

We recall that the Chow ring of X is the class of cycles modulo rational equivalence (see [Ful84]). The sum represents the union of varieties and the product the intersection. The global sections of an invertible sheaf $\mathcal{L}$ define the same class in the Chow ring of X. This class, denoted by $c_1(\mathcal{L})$ and called the first Chern class, is the divisor associated to $\mathcal{L}$ (see [Har77]). The degree $\int_X Z$ of a 0-cycle Z (or cycle of dimension 0) in the Chow ring of X counts the points in Z with their multiplicities.

Let X be an irreducible projective variety of dimension m over an algebraically closed field $\mathbb{K}$, and consider $m+1$ invertible sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_m$ on X. Let V_i be a vector subspace of $H^0(X, \mathcal{L}_i)$ for $i = 0 \ldots m$. We assume that V_i is very ample on a nonempty open subset U of X. This means that the map

$$\begin{aligned} \Gamma : U &\longrightarrow \mathbb{P}(V_i^*) \\ x &\mapsto \Gamma(x) = \{f \in V_i : f(x) = 0\} \end{aligned}$$

is an embedding (ie. injective and with non-zero differential everywhere, see [GrHa78] p.180).

The following result is a generalization of propositions 3.1 and 3.3 of chapter 3 in [GKZ94]. It will be useful for the construction of residual resultants.

Proposition 1 *Suppose that each V_i generates the sheaf $\mathcal{L}_i$ on X and that V_i is very ample on a nonempty open subset U of X. Then there exists an irreducible polynomial on $\prod_{i=0}^m V_i$, denoted by $\mathrm{Res}_{V_0,\ldots,V_m}$ and called the $(V_0,\ldots,V_m)$-resultant, which satisfies*

$$\mathrm{Res}_{V_0,\ldots,V_m}(f_0,\ldots,f_m) = 0 \quad \Longleftrightarrow \quad \exists\, x \in X : f_0(x) = \cdots = f_m(x) = 0.$$

Moreover, $\mathrm{Res}_{V_0,\ldots,V_m}$ is homogeneous in the coefficients of each f_i, and of degree

$$\int_X \prod_{j\neq i} c_1(\mathcal{L}_j). \tag{2}$$

PROOF. We consider the incidence variety

$$W = \left\{ (x, f_0, \ldots, f_m) \in X \times \prod_{i=0}^m \mathbb{P}(V_i) : f_0(x) = \cdots = f_m(x) = 0 \right\} \subset X \times \prod_{i=0}^m \mathbb{P}(V_i)$$

and the natural projections

$$X \xleftarrow{\pi_1} W \xrightarrow{\pi_2} \prod_{i=0}^m \mathbb{P}(V_i).$$

Since V_i generates $\mathcal{L}_i$ (for any x, there exists one section in V_i which does not vanish at x), the fiber $\pi_1^{-1}(x)$ of any point $x \in X$ is the product of $m+1$ hyperplanes. As X is an irreducible projective variety, we deduce by the fiber theorem [Sha74] that W is an irreducible projective variety of dimension

$$\dim W = \sum_{i=0}^m \dim V_i - 2.$$

We denote by $Z = \pi_{2_*}(W)$ the projection of W in the sense of cycles (i.e. taking into account the multiplicity in the projection, see [Ful84] I.1.4). The support of Z is an irreducible variety. Consider the fibers of π_2. Clearly, $\pi_2^{-1}(f_0,\ldots,f_m)$ is in correspondence with the set of common zeros of $f_0,\ldots,f_m$ on X. Since each V_i is very ample on a dense open subset of X, we are going to show that, for generic $(f_0, \ldots, f_m) \in Z$, this set of common zeros is just one point. Indeed, as X is a variety over the field $\mathbb{K}$, the locus of its singular points has codimension at least one ([Har77], II.8.16). So X is a disjoint union of a dense open subset U^o of nonsingular points, on which each V_i is very ample, and its complement F of codimension at least one. We choose f_0 such that

i) for any irreducible component of F, there exists x with $f_0(x) \neq 0$ so that $F \cap X(f_0)$ is of codimension 2,

ii) $U^o \cap X(f_0)$ is smooth and nonempty (because V_0 is very ample).
We repeat this construction for f_1: On each irreducible component of $F \cap X(f_0)$ there exists x such that $f_1(x) \neq 0$ (which implies that $F \cap X(f_0) \cap X(f_1)$ is of codimension 3) and $U^o \cap X(f_0) \cap X(f_1)$ is smooth and nonempty. Similarly, we choose $f_2, \ldots, f_{m-1}$ so that $X(f_0) \cap \ldots \cap X(f_{m-1})$ is included in U^o (because $F \cap X(f_0) \cap \ldots \cap X(f_{m-1})$ is of codimension $m+1$, i.e. empty). Moreover it is smooth and nonempty of dimension 0. Finally we choose f_m such that f_m vanishes only at one of the smooth points of $X(f_0) \cap \ldots \cap X(f_{m-1})$. This implies that the map $\pi_2 : W \longrightarrow Z$ is a birational isomorphism and hence again by the fiber theorem

$$\dim\ Z = \dim\ W = \sum_{i=0}^{m} \dim\ V_i - 2.$$

The degree of π_2 is 1, therefore Z is a reduced divisor of X. We define $\mathrm{Res}_{V_0,\ldots,V_m}$ to be the canonical section of the invertible sheaf associated to Z. It can be seen as an irreducible polynomial on $\prod_{i=0}^m V_i$ which vanishes exactly on Z. The homogeneity of $\mathrm{Res}_{V_0,\ldots,V_m}$ comes from the fact that if we multiply the coefficients of f_i by a nonzero constant factor, we do not change the zero locus $X(f_0) \cap \ldots \cap X(f_m)$.

Now we compute the degree α_i of $\mathrm{Res}_{V_0,\ldots,V_m}$ with respect to the coefficients of f_i. Let $Y = \prod_{i=0}^m \mathbb{P}(V_i)$, and consider the following fiber square (we refer the reader to [Ful84]):

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \int_Y} \\ X & \xrightarrow{\int_X} & \mathrm{Spec}(\mathbb{K}). \end{array}$$

The canonical section of the invertible sheaf of $\mathcal{O}_{X \times Y}$-modules

$$\mathcal{W} = \bigoplus_{i=0}^{m} {\pi_1}^*(\mathcal{L}_i) \otimes {\pi_2}^*\Big(\mathcal{O}_{\mathbb{P}(V_i)}(1)\Big)$$

vanishes exactly on W. Hence, denoting by h_i the generator of the Chow ring of $\mathbb{P}(V_i)$, the class of $\mathrm{Res}_{V_0,\ldots,V_m}$ in the Chow ring of Y is

$$\pi_{2*}\Big(c_{m+1}(\mathcal{W})\Big) = \pi_{2*}\left(\prod_{i=0}^{m} c_1\Big({\pi_1}^*(\mathcal{L}_i)\otimes{\pi_2}^*(\mathcal{O}_{\mathbb{P}(V_i)}(1))\Big)\right) = \sum_{i=0}^{m}\alpha_i\cdot h_i\ ,$$

and so the degree α_i of $\mathrm{Res}_{V_0,\ldots,V_m}$ in f_i is

$$\alpha_i = \int_Y \pi_{2*}\left(\prod_{i=0}^{m} c_1\Big({\pi_1}^*(\mathcal{L}_i)\otimes{\pi_2}^*(\mathcal{O}_{\mathbb{P}(V_i)}(1))\Big)\right)\cdot\prod_{j\neq i} h_j^{\dim V_j-1}\cdot h_i^{\dim V_i-2}.$$

We denote $H_i = \pi_2^*(h_i)$ and $L_i = \pi_1^*\Big(c_1(\mathcal{L}_i)\Big)$. As c_1 commutes with π_1^* and π_2^*, the projection formula gives

$$\alpha_i = \int_Y \pi_{2*}\left(\prod_{i=0}^{m}(L_i{+}H_i)\cdot\prod_{j\neq i}H_j^{\dim V_j-1}\cdot H_i^{\dim V_i-2}\right) = \int_Y \pi_{2*}\left(\prod_{j\neq i}L_j\cdot\prod_{i=0}^{m}H_i^{\dim V_i-1}\right).$$

Again by the projection formula

$$\alpha_i = \int_Y \pi_{2*}\left(\prod_{j\neq i}L_j\right)\cdot\prod_{i=0}^{m}h_i^{\dim V_i-1} = \int_Y \pi_{2*}\left(\prod_{j\neq i}L_j\right),$$

and finally

$$\alpha_i = \int_Y \pi_{2*}{\pi_1}^*\left(\prod_{j\neq i}c_1(\mathcal{L}_j)\right) = \int_X \pi_{1*}{\pi_1}^*\left(\prod_{j\neq i}c_1(\mathcal{L}_j)\right) = \int_X \prod_{j\neq i}c_1(\mathcal{L}_j).$$

□

Remark 2 The hypothesis that each V_i is very ample on a nonempty open subset of X is necessary to have Z reduced or in other words to have $\mathrm{Res}_{V_0,\ldots,V_m}$ irreducible with its degree in the coefficients of f_i given by the formula (2). Indeed, consider the following example: Set $X = \mathbb{P}^2$ and $V_0 = V_1 = V_2$ generated by x_0^2, x_1^2, x_2^2. This situation corresponds to a system of the form

$$\begin{cases} f_0(x) = c_{0,0}x_0^2 + c_{1,0}x_1^2 + c_{2,0}x_2^2 \\ f_1(x) = c_{0,1}x_0^2 + c_{1,1}x_1^2 + c_{2,1}x_2^2 \\ f_2(x) = c_{0,2}x_0^2 + c_{1,2}x_1^2 + c_{2,2}x_2^2 \end{cases}$$

We see easily that the condition for this system to have a solution is that the determinant of the matrix $(c_{i,j})$ vanishes. Hence it is of degree 1 in the coefficients of each f_i. Now if we compute the degree of the resultant given by the formula (2), we obtain $\alpha_i = 4$ (it is a classical projective resultant). The reason is that in this situation V_0, V_1, V_2 are not very ample. The solutions come by group of 4. In other words, we have $Z = 4\,Z'$, where Z' is the reduced divisor associated to the determinant of $(c_{i,j})$.

3 Residual resultant

We will use the notion of $(V_0, \ldots, V_m)$-resultant defined in proposition 1 to construct the residual resultant.

Let $X = \mathbb{P}^m = \mathrm{Proj}(\mathbb{K}[x_0, \ldots, x_m])$ be the projective space of dimension m over $\mathbb{K}$ and G be an ideal generated by n ($n \leq m+1$) homogeneous polynomials $g_1, \ldots, g_n$ of respective degree $k_1 \geq \cdots \geq k_n$. Let $\mathcal{G}$ be the coherent sheaf of ideals associated to G. We fix $m+1$ positive integers $d_0, \ldots, d_m$ such that $d_0 \geq \cdots \geq d_m \geq k_1 \geq \cdots \geq k_n$, and we consider the sheaves $\mathcal{G}(d_i) = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d_i)$ for $i = 0 \ldots m$. The vector space $V_i = H^0\Big(X, \mathcal{G}(d_i)\Big)$ is the set of homogeneous polynomials of degree d_i which are in the saturation of the ideal G. We denote by $\pi : \tilde{X} \to X$ the blow-up of X along the sheaf of ideals $\mathcal{G}$ (see [Har77]). The inverse image of the sheaf $\tilde{\mathcal{G}} = \pi^{-1}\mathcal{G} \cdot \mathcal{O}_{\tilde{X}}$ is an invertible sheaf on $\tilde{X}$. The sheaf $\tilde{\mathcal{G}} \otimes \pi^*\Big(\mathcal{O}_X(d_i)\Big)$ will be denoted by $\tilde{\mathcal{G}}_{d_i}$.

Proposition 3 *Suppose that $d_m \geq k_n + 1$. Then there exists an irreducible polynomial on $\prod_{i=0}^m V_i$, denoted by $\mathrm{Res}_{\mathcal{G}(d_0), \ldots, \mathcal{G}(d_m)}$ and called the $\Big(\mathcal{G}(d_0), \ldots, \mathcal{G}(d_m)\Big)$-residual resultant, which satisfies*

$$\mathrm{Res}_{\mathcal{G}(d_0), \ldots, \mathcal{G}(d_m)}(f_0, \ldots, f_m) = 0 \iff \exists\, x \in \tilde{X} : \pi^*(f_0)(x) = \cdots = \pi^*(f_m)(x) = 0,$$

where $\pi^*(f_i)$ is a global section of the invertible sheaf $\tilde{\mathcal{G}}_{d_i}$ on $\tilde{X}$. In particular, if there exists $x \in X \setminus V(G)$ such that $f_0(x) = \cdots = f_m(x) = 0$, then

$$\mathrm{Res}_{\mathcal{G}(d_0), \ldots, \mathcal{G}(d_m)}(f_0, \ldots, f_m) = 0.$$

PROOF. Fix an integer $d \geq k_1$ and choose $m+1$ global sections $s_0, \ldots, s_m$ of $\mathcal{G}(d)$. For a sufficiently generic choice, they generate the sheaf $\mathcal{G}(d)$ (proof similar as for proposition 1) and thus the global sections $\pi^*(s_0), \ldots, \pi^*(s_m)$ of the invertible sheaf $\pi^*\Big(\mathcal{O}_X(d)\Big)$ generate the invertible subsheaf $\tilde{\mathcal{G}}_d$ (see [Har77] II 7.17.3). So, for each d_i, the invertible sheaf $\tilde{\mathcal{G}}_{d_i}$ on $\tilde{X}$ is generated by its global sections. Let us denote by $\tilde{V}_i$ the vector subspace of $H^0(\tilde{X}, \tilde{\mathcal{G}}_{d_i})$ generated by all $\pi^*(s)$ with $s \in V_i$. We see that $\tilde{V}_i$ generates $\tilde{\mathcal{G}}_{d_i}$, and that $\tilde{V}_i$ and V_i are isomorphic $\mathbb{K}$-vector spaces.
For each $i = 0 \ldots m$, $d_i \geq k_n + 1$. Let S be the variety of $\tilde{X}$ defined by $\pi^*(g_n)$. We will prove that $\tilde{V}_i$ is very ample on $U = \tilde{X} \setminus S$ (for all i), so that the map

$$\begin{aligned} \Gamma_i : U &\longrightarrow \mathbb{P}(\tilde{V}_i^*) \\ x &\mapsto \Gamma_i(x) = \{\pi^*(f) \in \mathbb{P}(\tilde{V}_i) : \pi^*(f)(x) = 0\} \end{aligned}$$

is an embedding. Indeed, taking two different points $x, y \in U$, we can choose a linear form L on X such that $L\Big(\pi(x)\Big) = 0$ and $L\Big(\pi(y)\Big) \neq 0$. The form

$\pi^*(L^{d_i-k_n}g_n)$ is hence in $\Gamma_i(x)$ and not in $\Gamma_i(y)$. This proves the injectivity. For the differential condition, since π is an isomorphism between U and $\mathbb{P}^m \backslash V(g_n)$, it is sufficient to show that for any $x \in \mathbb{P}^m \backslash V(g_n)$ and $v \in T_x(\mathbb{P}^m)$ (the tangent space of $\mathbb{P}^m$ at x) there exists a global section s of $\mathcal{G}(d_i)$ such that $s(x) = 0$ and $ds(x) = v$. Since $d_i - k_n \geq 1$, we can take $s = L\, p\, g_n$ where p is an homogeneous polynomial of degree $d_i - k_n - 1$ such that $p(x) \neq 0$, and L is a linear form on $\mathbb{P}^m$ such that $L(x) = 0$ and $dL(x) = \frac{1}{pg_n(x)}\, v$. This section s satisfies the required conditions $s(x) = 0$ and $ds(x) = pg_n(x)\, dL(x) = v$. This proves that Γ_i is an embedding. By proposition 1, the resultant $\mathrm{Res}_{\tilde{V}_1,\ldots,\tilde{V}_m}$, that we will denote $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$, is well defined. It is a multi-homogeneous function on $\prod_{i=0}^m \tilde{V}_i$, and so on $\prod_{i=0}^m V_i$. As π is an isomorphism between $X \setminus V(G)$ and $\tilde{X} \setminus E$ (where E is the exceptional divisor of the blow-up π), if there exists $x \in X \setminus V(G)$ such that $f_0(x) = \cdots = f_m(x) = 0$, then

$$\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(f_0, \ldots, f_m) = 0.$$

□

Remark 4 The hypothesis $d_m \geq k_n + 1$ is required to have $\tilde{V}_i$ very ample in order to apply proposition 1. For example if $X = \mathbb{P}^2$, $d_0 = d_1 = d_2 = k_1 = k_2 = 2$, g_1 and g_2 are generic conics, G defines four simple points. We look for condition so that three conics f_0, f_1, f_2 pass through these four points and have a solution outside $V(G)$. In such configuration, the conics f_0, f_1, f_2 must be the same. Thus we cannot define a resultant because the fibers of the map π_2 (in the proof of proposition 1) are not 0-dimensional.

Remark 5 If F^{sat} and G^{sat} denote the saturations of the ideals F and G (see definition 11), and if G is a local complete intersection, then the residual resultant satisfies

$$\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(f_0, \ldots, f_m) = 0 \iff F^{sat} \neq G^{sat}.$$

Indeed, if $F^{sat} = G^{sat}$ then the associated ideal sheaves $\mathcal{F}$ and $\mathcal{G}$ are equal and so

$$\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(f_0, \ldots, f_m) \neq 0.$$

Conversely, the exceptional divisor E of π is the projective bundle of the sheaf $\mathcal{G}/\mathcal{G}^2$ which is locally free of rank $d \leq n$ since G is a local complete intersection (see [Ful84], Appendix B.7) and we have $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}^2 \backsimeq \mathcal{O}_E(1)$. Therefore if $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(f_0, \ldots, f_m) \neq 0$, that is, if $\pi^*(f_0), \ldots, \pi^*(f_m)$, do not vanish simultaneously, they generate all the fibers of $\tilde{\mathcal{G}}$ since $d \leq n \leq m + 1$. We deduce that $\mathcal{F} \backsimeq \mathcal{G}$, that is $F^{sat} = G^{sat}$.

Now we will compute the degree of the polynomial $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$ in each V_i (i.e. in the coefficients of the polynomial f_i). For this we suppose now that G

is a local complete intersection. We fix $i = 0$. We have to compute the degree

$$N_0 = \int_{\tilde{X}} \prod_{i=1}^{m} c_1(\tilde{\mathcal{G}}_{d_i}) ,$$

where $c_1(\tilde{\mathcal{G}}_{d_i})$ is the first Chern class of $\tilde{\mathcal{G}}_{d_i}$. According to [Ful84], $c_1(\tilde{\mathcal{G}}_{d_i}) = d_i H - E$, where $H = \pi^*(h)$, h is the class of a generic hyperplane of $\mathbb{P}^m$ and E is the class of the exceptional divisor of the blow-up π. The degree

$$\begin{aligned} N_0 &= \int_{\tilde{X}} \prod_{i=1}^{m} c_1(\tilde{\mathcal{G}}_{d_i}) = \int_X \pi_*\Big((d_1 H - E)(d_2 H - E) \cdots (d_m H - E)\Big) \\ &= \int_X \pi_*\left(\sum_{l=0}^{m} \sigma_{m-l}(\mathbf{d})\, H^{m-l} \cdot E^l\right) = \int_X \sum_{l=0}^{m} \sigma_{m-l}(\mathbf{d})\, h^{m-l} \cdot \pi_*(E^l) \end{aligned}$$

with $\sigma_0(\mathbf{d}) = (-1)^m$, $\sigma_1(\mathbf{d}) = (-1)^{m-1} \sum_{i=1}^{m} d_i$, $\sigma_2(\mathbf{d}) = \sum_{1 \le i < j \le m} d_i d_j$, $\ldots$, $\sigma_m(\mathbf{d}) = \prod_{i=1}^{m} d_i$.

Proposition 6 *For any $r \in \mathbb{Q}[T]$, let*

$$P_r(y_1, \ldots, y_n) = \det \begin{pmatrix} r(y_1) & \cdots & r(y_n) \\ y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{n-1} & \cdots & y_n^{n-1} \end{pmatrix} .$$

Then the degree of $\mathrm{Res}_{\mathcal{G}(d_0), \ldots, \mathcal{G}(d_m)}$ *in each* V_i *is*

$$N_i = \frac{P_{\kappa_i}}{P_1}(k_1, \ldots, k_n),$$

with $\kappa_i(T) = \sigma_m(\mathbf{d}) + \sum_{l=n}^{m} \sigma_{m-l}(\mathbf{d})\, T^l$ *and* $\mathbf{d} = (d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_m)$.

PROOF. Assume that $i = 0$. According to the projection formula and because $V(G)$ is of dimension $m - n$, we have

$$h^k \cdot \pi_*(E^{m-k}) = 0 \qquad \text{for} \qquad m > k > m - n. \tag{3}$$

Therefore

$$N_0 = \int_X \sigma_m(\mathbf{d}) h^m + \sum_{l=n}^{m} \sigma_{m-l}(\mathbf{d})\, h^{m-l} \cdot \pi_*(E^l).$$

In order to compute this value, we use the following relation, which asserts that the strict transforms of $g_1, \ldots, g_n$ do not intersect in the blowing-up $\tilde{X}$:

$$\pi_*\Big((k_1 H - E)(k_2 H - E) \cdots (k_n H - E)\Big) = \sum_{l=0}^{n} \omega_{n-l}(\mathbf{k})\, h^{n-l} \cdot \pi_*(E^l) = 0$$

where $\omega_0(\mathbf{k}) = (-1)^n$, $\omega_1(\mathbf{k}) = (-1)^{n-1} \sum_{i=1}^n k_i$, $\omega_2(\mathbf{k}) = \sum_{1 \le i < j \le n} k_i k_j$, $\dots$, $\omega_n(\mathbf{k}) = \prod_{i=1}^n k_i$. Intersecting this class with $h^{m-n-p} \cdot \pi_*(E^p)$ for $p = 0 \dots m-n$, we get the following relations

$$\sum_{l=p}^{p+n} \omega_{n+p-l}(\mathbf{k})\, h^{m-l} \cdot \pi_*(E^l) = 0 \qquad \text{for} \qquad p = 0 \dots m-n. \tag{4}$$

Any class of dimension 0 is an integer multiple of h^m, in the Chow ring of $\mathbb{P}^m$. So, let us denote by Λ_l the integer such that

$$h^{m-l} \cdot \pi_*(E^l) \equiv \Lambda_l\, h^m.$$

According to the relations (3) and (4), we have

$$\begin{cases} \Lambda_0 = 1 \\ \Lambda_1 = \dots = \Lambda_{n-1} = 0 \\ \sum_{l=p}^{p+n} \omega_{n+p-l}(\mathbf{k})\, \Lambda_l = 0 \ \text{ for } p = 0 \dots m-n. \end{cases} \tag{5}$$

It is a linear system of the form

$$T \begin{pmatrix} \Lambda_0 \\ \Lambda_1 \\ \vdots \\ \Lambda_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where T is a lower triangular matrix with 1 or -1 on the diagonal. In particular, it implies that $\Lambda_0, \dots, \Lambda_m$ are polynomial functions of $k_1, \dots, k_n$. We are going to compute these polynomial functions, assuming for the moment that the $k_i, i = 1 \dots n$ are distinct. Let Λ be the linear form defined on $\mathbb{Q}[T]$ by

$$\Lambda(1) = 1\ ,\ \Lambda(T) = 0\ ,\ \dots\ ,\ \Lambda(T^{n-1}) = 0\ ,\ \Lambda(T^l q) = 0\ ,\ l \ge 0$$

where $q = \sum_{l=0}^n \omega_{n-l}(\mathbf{k})\, T^l$. We remark that $\left(\Lambda(T^i)\right)_{i=0\dots m}$ is the unique solution of the system (5). As Λ is in the orthogonal of the ideal generated by the polynomial q, whose roots are $k_1, \dots, k_n$, it is of the form

$$\Lambda = \alpha_1\, \mathbf{1}_{k_1} + \dots + \alpha_n\, \mathbf{1}_{k_n}$$

with $\mathbf{1}_{k_i} : p \in \mathbb{Q}[T] \mapsto p(k_i) \in \mathbb{Q}$ and $\alpha_i \in \mathbb{Q}$ satisfying the equations

$$\begin{cases} \Lambda(1) = \alpha_1 + \cdots + \alpha_n = 1 \\ \Lambda(T) = \alpha_1 k_1 + \cdots + \alpha_n k_n = 0 \\ \vdots \\ \Lambda(T^{n-1}) = \alpha_1 k_1^{n-1} + \cdots + \alpha_n k_n^{n-1} = 0. \end{cases}$$

Solving this linear system by Cramer's rule, we get

$$\alpha_i P_1 = \det \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ k_1 & \cdots & k_{i-1} & k_i & k_{i+1} & \cdots & k_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k_1^{n-1} & \cdots & k_{i-1}^{n-1} & k_i^{n-1} & k_{i+1}^{n-1} & \cdots & k_n^{n-1} \end{pmatrix}$$

where P_1 is the $n \times n$ Vandermonde determinant of $k_1, \ldots, k_n$, and for any $l \in \mathbb{N}$,

$$\Lambda(T^l) = \Lambda_l = \sum_{i=1}^{n} \alpha_i \, k_i^l = S_{T^l}(k_1, \ldots, k_n)$$

with $S_r = \frac{P_r}{P_1}$ for any $r \in \mathbb{Q}[T]$. The function $y = (y_1, \ldots, y_n) \mapsto P_{T^l}(y)$ vanishes when two coordinates of y are equal and thus it is divisible by the Vandermonde determinant P_1. Therefore the map $y \mapsto S_{T^l}(y)$ is a polynomial function and it is well defined when the coordinates of y are not all distinct. Consequently, the solution of the system (5) for any value of $\mathbf{k}$ is $\left(S_{T^l}(\mathbf{k})\right)_{l=0\ldots m}$. As $N_0 = \Lambda(\kappa_0)$, we deduce by linearity that $N_0 = S_{\kappa_0}(\mathbf{k})$. $\square$

Remark 7 The degree N_i of $\mathrm{Res}_{V_0,\ldots,V_m}$ in V_i is also implicitly computed in the recent work [CEB00], but seems to be more difficult to be recovered explicitly as here.

- *Blow-up of points:* If $n = m$, we have N_i of $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$ with respect to V_i is
$$N_i = \frac{d_0 \cdots d_m}{d_i} - k_1 \cdots k_m.$$
- *Blow-up of curves:* If $n = m - 1$, the degree
$$N_i = \frac{d_0 \cdots d_m}{d_i} - k_1 \cdots k_{m-1}(d_0 + \cdots + d_m - d_i - k_1 - \cdots - k_{m-1}).$$
- *Blow-up of hypersurfaces:* If $n = 1$, $N_i = \prod_{j \neq i}(d_j - k_1)$, which is the expected degree since it corresponds to the degree of the projective resultant of the polynomials $h_{0,1}(x), \ldots, h_{m,1}(x)$ (see [Jou91]).

4 Matrix for the residual resultant

In this section, we move to algebra and describe a matrix construction which yields multiples of the residual resultant $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$. It combines a Sylvester (or Macaulay) part involving the monomial multiples of $f_0, \ldots, f_m$ and a Bezout (or Dixon) part involving the minors of the matrix H defined in (1). The gcd of these multiples is exactly the residual resultant. Our construction is based on the resolution of the ideal $(F : G) = \{f \in R : f.G \subset F\}$ in the case of a regular sequence $g_1, \ldots, g_n$ [BKM90]. A generalization of this resolution in the case of a d-sequence is given in [CU00].

4.1 The resolution of a residual intersection

In this subsection, we consider two ideals $A = (a_1, \ldots, a_p)$ and $B = (b_1, \ldots, b_n)$ of R with $\deg(a_i) = \alpha_i$, $\deg(b_j) = \beta_j$, $\alpha_1 \geq \cdots \geq \alpha_p$ and $\beta_1 \geq \cdots \geq \beta_n$. We assume that $A \subset B$ and we denote by P the $n \times p$ matrix such that $(a_1, \ldots, a_p) = (b_1, \ldots, b_n)\, P$.

We need to introduce the definition of a residual intersection as stated in [HU98].

Definition 8 *Let B and $A = (a_1, \ldots, a_p)$ be two ideals of R such that $A \subsetneq B$. Set $J = (A : B)$. If $ht(J) \geqslant p \geqslant ht(B)$, then J is said to be a p-residual intersection of B (with respect to A). If furthermore $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(B)$ with $ht(\mathfrak{p}) \leqslant p$, then we say that J is a geometric p-residual intersection of B.*

In particular, if $V(A : B) = \emptyset$ then $(A : B)$ is a p-residual intersection of B. Indeed, for any $\mathfrak{p} \in V(B)$, $(A : B)_{\mathfrak{p}} = (1) = (A_{\mathfrak{p}} : B_{\mathfrak{p}})$ so that $A_{\mathfrak{p}} = B_{\mathfrak{p}}$.

We recall briefly the construction of the complex used to resolve $(A : B)$ (see [BKM90]). We denote by $\mathfrak{A}$ (resp. $\mathfrak{B}$) the free R-module R^p (resp. R^n) of canonical basis $(\mathbf{a}_i)_{i=1\ldots p}$ (resp. $(\mathbf{b}_j)_{j=1\ldots n}$). These modules are graded as follows: $\deg(\mathbf{a}_i) = \deg(a_i) = \alpha_i$, $i = 1 \ldots p$, $\deg(\mathbf{b}_i) = \deg(b_i) = \beta_i$, $i = 1 \ldots n$. We assume that $n \leq p$. The complex associated with $J = (A : B)$ is

$$0 \to C_p \to \cdots \to C_n \to \cdots \to C_1 \to C_0 \to R/J \to 0 \tag{6}$$

where

$$C_i = \oplus_{j+l=i-1, 0 \leq j \leq p-n, 0 \leq l \leq n-1} \left(K_j^l(\mathfrak{B}) \otimes \wedge^{n+j} \mathfrak{A} \right) \bigoplus \wedge^i \mathfrak{A}\ , \quad i = 0 \ldots n-1,$$
$$C_i = \oplus_{j+l=i-1, 0 \leq j \leq p-n, 0 \leq l \leq n-1} \left(K_j^l(\mathfrak{B}) \otimes \wedge^{n+j} \mathfrak{A} \right)\ , \quad i = n \ldots p\ ,$$

where $K_i^j(\mathfrak{B})$ is the kernel of the Eagon-Northcott map $D_i(\mathfrak{B}) \otimes \wedge^j \mathfrak{B}^* \to D_{i-1}(\mathfrak{B}) \otimes \wedge^{j-1} \mathfrak{B}^*$ induced by the identity map of $\mathfrak{B}^*$, and $D(\mathfrak{B})$ is the graded dual of the symmetric algebra $S(\mathfrak{B}^*)$ over R. This complex comes from the bi-complex of free R-modules $(C_{a,b})_{0 \leq a \leq p-n, -1 \leq b \leq n-1}$ where

$$\begin{array}{l} C_{a,b} = K_a^b(\mathfrak{B}) \otimes \wedge^{n+a} \mathfrak{A} \ \ , \quad a = 0 \ldots p-n \ , \quad b = 0 \ldots n-1 \ , \\ C_{a,-1} = \wedge^a \mathfrak{A} \ \ , \ a = 0 \ldots n-1. \end{array}$$

In particular, $C_0 = R$ and the first map $\partial_1 : C_1 \to C_0$ of the resolution (6) is

$$\begin{array}{rcl} \partial_1 : \left(\oplus_{1 \leqslant i_1 < \cdots < i_n \leqslant p} \ R\,\mathbf{a}_{i_1} \wedge \ldots \wedge \mathbf{a}_{i_n} \right) \oplus \left(\oplus_{i=1}^p \ R\,\mathbf{a}_i \right) & \to & R \\ \mathbf{a}_{i_1} \wedge \ldots \wedge \mathbf{a}_{i_n} & \mapsto & D_{i_1 \ldots i_n} \\ \mathbf{a}_i & \mapsto & a_i \end{array}$$

where $D_{i_1 \ldots i_n}$ is the determinant of the $n \times n$ submatrix of P corresponding to the columns $i_1, \ldots, i_n$. The main result that we will use is the following:

Theorem 9 *[BKM90] Let $b_1, \ldots, b_n$ be a regular sequence. If $J = (A : B)$ is a geometric p-residual intersection, then the complex (6) is exact.*

In this case, $J = A + I_n(P)$, where $I_n(P)$ is the ideal of R generated by all the $n \times n$ minors of the matrix P.

Let us recall the notions of regularity of Castelnuovo-Mumford and saturation of an ideal (see [Eis94], [BS87] for more details). Let $d \in \mathbb{N}$. We denote by $R(-d)$ the graded algebra R, where the degrees are shifted by $-d$. For any ideal I of R, $I_{[d]}$ is the component of I of degree d.

Definition 10 *A homogeneous ideal J is said to be r-regular if there exists a free resolution of J*

$$0 \to \oplus_j R(-e_{r,j}) \to \cdots \to \oplus_j R(-e_{1,j}) \to \oplus_j R(-e_{0,j}) \to J \to 0$$

with $e_{i,j} \leqslant r + i$ for all i, j. The Castelnuovo-Mumford regularity (or simply the regularity) of J is the smallest integer r for which J is r-regular.

Definition 11 *Two homogeneous ideals I and J have the same saturation if $I_{[s]} = J_{[s]}$ for $s \gg 0$. The saturation of a homogeneous ideal I, denoted I^{sat}, is the largest ideal with the same saturation than I.*

Two homogeneous ideals I and J have the same saturation if and only if they define the same closed subscheme of $\mathrm{Proj}(R)$. If an ideal J is r-regular, then it is r-saturated (i.e. $I_{[s]} = {I^{sat}}_{[s]}$ for $s \geq r$.)

Remark 12 $V(F:G)=\emptyset$ is equivalent to $(F:G)^{sat}=(F^{sat}:G^{sat})=R$ that is to $F^{sat}=G^{sat}$.

Proposition 13 *If $b_1,\ldots,b_n$ is a regular sequence and J is a p-residual intersection, then J is ν-regular, for*

$$\nu \geq \alpha_1+\cdots+\alpha_p-p+1-(p-n+1)\beta_n.$$

PROOF. The rows of the bi-complex $(C_{a,b})$ are Eagon-Northcott type complexes and the columns are Koszul type complexes. The first column corresponds to the n first modules of the Koszul complex associated to $a_1,\ldots,a_p$. The other columns correspond to the Koszul complex of $b_1,\ldots,b_n$. The last row is

$$0\to C_{p-n,0}\to\cdots\to C_{1,0}\to\wedge^n\mathfrak{A}\to R$$

where $C_{i,0}=S_i(\mathfrak{B}^*)^*\otimes\wedge^{n+i}\mathfrak{A}$ (see [BKM90]). The degree of the generators of $\wedge^n\mathfrak{A}$ is at most $\sum_{i=1}^n\alpha_i-\sum_{i=1}^n\beta_i$, for the map $\wedge^n\mathfrak{A}\to R$ which associates to $\mathbf{a}_{i_1}\wedge\ldots\wedge\mathbf{a}_{i_n}$ the minor $D_{i_1\ldots i_n}$ of P. We deduce that the generators of $C_{i,0}$ are at most of degree

$$\alpha_{i,0}:=\sum_{j=1}^{n+i}\alpha_j-\sum_{j=1}^{n}\beta_j-i\,\beta_n \quad\text{for}\quad i=1\ldots p-n.$$

The last column of the bi-complex is

$$K_{p-n}^{n-1}(\mathfrak{B})\otimes\wedge^p\mathfrak{A}\to\cdots\to K_{p-n}^{1}(\mathfrak{B})\otimes\wedge^p\mathfrak{A}\to K_{p-n}^{0}(\mathfrak{B})\otimes\wedge^p\mathfrak{A}.$$

As the generators of $C_{p-n,0}=K_{p-n}^0(\mathfrak{B})\otimes\wedge^p\mathfrak{A}$ are at most of degree $\alpha_{p-n,0}$, we deduce that the generators of $C_{p-n,j}=K_{p-n}^j(\mathfrak{B})\otimes\wedge^p\mathfrak{A}$ are at most of degree $\alpha_{p-n,0}+\sum_{i=1}^j\beta_i$. In particular, the generators of the last module are at most of degree

$$\sum_{i=1}^p\alpha_i-(p-n+1)\,\beta_n.$$

By definition 10, the Castelnuovo-Mumford regularity of J is bounded by

$$\sum_{i=1}^p\alpha_i-(p-n+1)\,\beta_n-p+1.$$

□

4.2 Computing the residual resultant

We recall that $f_0,\ldots,f_m,g_1,\ldots,g_n$ are homogeneous polynomials of degree $d_0,\ldots,d_m,k_1,\ldots,k_n$ such that $n\leq m+1$, $d_0\geq\cdots\geq d_m\geq k_1\geq\cdots\geq k_n$

and $d_m \geq k_n + 1$. F and G are the ideals of R generated respectively by $f_0, \ldots, f_m$ and $g_1, \ldots, g_n$. The $n \times (m+1)$ matrix H such that $(f_0, \ldots, f_m) = (g_1, \ldots, g_n)\, H$ is a matrix of polynomials $h_{i,j} = \sum_{|\alpha|=d_j-k_i} c_\alpha^{i,j}\, \mathbf{x}^\alpha$. We say that $\mathbf{f} = \{f_0, \ldots, f_m\}$ is a generic system of type (1), if all the coefficients $\mathbf{c} = (c_\alpha^{i,j})$ are chosen generically in the field $\mathbb{K}$.

Proposition 14 *If $\mathbf{f}$ is a generic system of type (1), then $J = (F : G)$ is a geometric $(m+1)$-residual intersection of G with respect to F.*

PROOF. For a generic system $\mathbf{f}$ of type (1), $V(F) \setminus V(G)$ is empty (see proposition 3). Moreover, we can prove by induction that for every $\mathfrak{p} \in V(G)$, $G_\mathfrak{p} = F_\mathfrak{p}$. Indeed we choose f_1 such that $\{\mathfrak{p} \in V(G) : f_{0,\mathfrak{p}} = f_{1,\mathfrak{p}} \in G_\mathfrak{p}\}$ is of codimension 1 in $V(G)$. Then we choose f_2 such that $\{\mathfrak{p} \in V(G) : f_{0,\mathfrak{p}} = f_{2,\mathfrak{p}}$ or $f_{1,\mathfrak{p}} = f_{2,\mathfrak{p}} \in G_\mathfrak{p}\}$ is of codimension 1 in $V(G)$ and $\{\mathfrak{p} \in V(G) : f_{0,\mathfrak{p}} = f_{1,\mathfrak{p}} = f_{2,\mathfrak{p}} \in G_\mathfrak{p}\}$ is of codimension 2 in $V(G)$. We construct in the same way $f_3, \ldots, f_n$ and obtain a dense open subset $U \subset V(G)$ where $G_\mathfrak{p} = F_\mathfrak{p}$. Now we can choose $f_{n+1}, \ldots, f_m$ such that $G_\mathfrak{p} = F_\mathfrak{p}$ with $\mathfrak{p} \in V(G) \setminus U$ since $n \leq m+1$. Thus, $G_\mathfrak{p} = F_\mathfrak{p}$ for all $\mathfrak{p}$ and $J_\mathfrak{p} = (F_\mathfrak{p} : G_\mathfrak{p}) = R_\mathfrak{p}$. We deduce that $V(J)$ is empty and that J is a geometric $(m+1)$-residual intersection. $\square$

Hereafter we will concentrate on the the map ∂_1 of the resolution (6).

Definition 15 *For any $s \in \mathbb{N}$, we denote by $\partial_{1,s}$ the map ∂_1 in degree s:*

$$\partial_{1,s} \,:\, \left(\prod_{I, 0 \leq i_1 < \ldots < i_n \leq m} R_{[s - d_{i_1} - \cdots - d_{i_n} + \sum_{i=1}^n k_i]} \right) \times R_{[s-d_0]} \times \cdots \times R_{[s-d_m]} \longrightarrow R_{[s]}$$

such that

$$\partial_{1,s}\Big((q_I)_I, (q_0, \ldots, q_m)\Big) = \sum_I q_I \Delta_I + q_0 f_0 + \cdots + q_m f_m.$$

Its matrix in the corresponding monomial bases is denoted by $\mathtt{M}_{1,s}$.

We recall that $\Delta_{i_1 \ldots i_n}$ is the $n \times n$ minor of the matrix H corresponding to the columns $i_1, \ldots, i_n$. It is a bihomogeneous polynomial of degree $d_{i_1} + \cdots + d_{i_n} - \sum_{i=1}^n k_i$ in the variables $x_0, \ldots, x_m$ and of degree n in the coefficients $\mathbf{c}$.

Proposition 16 *Let $g_1, \ldots, g_n$ be a regular sequence. The map $\partial_{1,\nu}$ is surjective for $\nu \geq \nu_{\mathbf{d},\mathbf{k}} := \sum_{i=0}^m d_i - m - (m-n+2)\, k_n$ if and only if the variety $V(F : G)$ is empty.*

PROOF. If $V(F : G) \neq \emptyset$, there exists a point $\zeta \in \mathbb{P}^m$ such that $\Delta_I(\zeta) = f_j(\zeta) = 0$ (the minors of H are in $(F : G)$ by Cramer's rule) so that any polynomial in the image of $\partial_{1,\nu}$ vanishes at ζ and $\partial_{1,\nu}$ is not surjective.

Conversely, if $V(F:G)$ is empty, then $(F:G)$ is a geometric $(m+1)$-residual intersection of G and the complex (6) is exact. By proposition 13, we deduce that the regularity of $(F:G)$ is bounded by $\nu_{\mathbf{d},\mathbf{k}}$. So the image of $\partial_{1,\nu}$ is $R_{[\nu]}$ for $\nu \geq \nu_{\mathbf{d},\mathbf{k}}$. $\square$

Theorem 17 *Any nonzero minor (of size* $\dim_{\mathbb{K}}(R_{[\nu]})$*) of the matrix* $\mathtt{M}_{1,\nu}$ *of* $\partial_{1,\nu}$ *is a multi-homogeneous polynomial in the coefficients of* $f_0,\ldots,f_m$*, and a multiple of* $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$.

PROOF. Let us denote by ρ a nonzero minor of size $\dim_{\mathbb{K}}(R_{[\nu]})$ of $\mathtt{M}_{1,\nu}$. It is clear that ρ is a homogeneous polynomial in the coefficients $\mathbf{c}$.

We recall that $X=\mathbb{P}^m$ and $\tilde{X}$ is the blow-up of X along the sheaf of ideals associated to G. If $\tilde{X}^0$ is the dense open subset of $\tilde{X}$ defined as the complement of the exceptional divisor in $\tilde{X}$, let $Z^0=\{\mathbf{c}:\pi^*(f_0),\ldots,\pi^*(f_m)$ have a common root in $\tilde{X}^0\}$. Assume that there exists $\mathbf{c}_0 \in Z^0$ such that $\rho(\mathbf{c}_0)\neq 0$. For this specialization $\mathbf{c}_0$, $\partial_{1,\nu}$ is then surjective, and $R_{[\nu]}=F_{[\nu]}+I_n(H)_{[\nu]}$. Let $\gamma\in\tilde{X}^0$ be a common root of $\pi^*(f_0),\ldots,\pi^*(f_m)$ and $\xi=\pi(\gamma)\in X\setminus V(G)$ its projection. We have $f_0(\xi)=\cdots=f_m(\xi)=0$ and all the $n\times n$ minors of H vanish on ξ, since $\big(g_1(\xi),\ldots,g_n(\xi)\big)$ is a nonzero vector which satisfies $\big((g_1(\xi),\ldots,g_n(\xi)\big)H(\xi)=0$. Hence for any element p in the image of $\partial_{1,\nu}$, $p(\xi)=0$. In particular, $\xi^\alpha=0$ for every $\alpha\in\mathbb{N}^n$ such that $|\alpha|=\nu$. But this is impossible in $\mathbb{P}^m$, which implies that ρ vanishes on Z^0. As $\tilde{X}^0$ is dense in $\tilde{X}$, Z^0 is also dense in $Z=\{\mathbf{c}:\pi^*(f_0),\ldots,\pi^*(f_m)$ have a common root in $\tilde{X}\}$. ρ vanishes on Z^0, and also on Z. The theorem follows from proposition 3. $\square$

Proposition 18 *For any* $i=0\ldots m$*, there exists a nonzero maximal minor of* $\mathtt{M}_{1,\nu}$ *of degree* N_i *(see proposition 6) in the coefficients* $\mathbf{c}$ *of* $f_0,\ldots,f_m$.

PROOF. Let us denote by F' the ideal $(f_1,\ldots,f_m)$, where f_i are generic polynomials of type (1). The variety $V(F':G)$ is of dimension 0 and of degree N_0 (see proof of proposition 1). By proposition 14 and theorem 9, $(F':G)$ is a m-residual intersection and the complex (6) associated to the m polynomials $f_1,\ldots,f_m$ and $g_1,\ldots,g_n$ is exact. The regularity ν' of $(F':G)$ is

$$\nu' \leq d_1+\cdots+d_m-(m-n+1)\,k_n-m+1=\nu_{\mathbf{d},\mathbf{k}}-d_0+k_n+1\leq\nu_{\mathbf{d},\mathbf{k}}$$

(by hypothesis $d_i\geq k_n+1$, $i=0\ldots m$). Since the Castelnuovo-Mumford regularity bounds the regularity of the Hilbert function, for any $s\geq\nu'$, we have

$$\dim_{\mathbb{K}}(F':G)_{[s]}=\dim_{\mathbb{K}}(R_{[s]})-N_0. \tag{7}$$

Let us denote by $(\Delta_{I'})_{I'}$ the set of minors of the matrix H associated with $f_1, \ldots, f_m$.

Consider now the ideal $(F:G)$ in degree ν which is generated by the multiples of degree ν of $f_1, \ldots, f_m$, the minors $\Delta_{I'}$, f_0 and the minors $\Delta^0_{I''}$ involving the first column of H indexed by 0 and $n-1$ other columns of H. The multiples of $f_0, \Delta^0_{I''}$ are of degree 1 in the coefficients of f_0.

By (7), the monomial multiples of degree ν of the polynomials $f_1, \ldots, f_m, \Delta_{I'}$ generate a vector space L_1 of degree $\dim_{\mathbb{K}}(R_{[\nu]}) - N_0$. Let L_0 be the vector space generated by the monomial multiples of degree ν of the polynomials $f_0, \Delta^0_{I''}$. As $(F:G)_{[\nu]} = R_{[\nu]}$, we have $L_0 + L_1 = R_{[\nu]}$. Thus we can complete a basis of L_1 by N_0 monomial multiples of $f_0, \Delta^0_{I''}$ in order to obtain a basis of $R_{[\nu]}$.

Let us write the coefficient matrix of these polynomials. It is a square matrix of size $\dim_{\mathbb{K}}(R_{[\nu]})$ with N_0 columns representing the N_0 monomial multiples of degree ν of $f_0, \Delta^0_{I''}$. Consequently, its determinant is a nonzero polynomial in $\mathbf{c}$, and of degree N_0 in the coefficients of f_0.

A similar proof applies by symmetry for $i = 1 \ldots m$. $\square$

Theorem 19 *The gcd of all maximal minors of the matrix* $\mathtt{M}_{1,\nu}$ *is exactly* $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$.

PROOF. According to theorem 17, the gcd of all maximal minors of $\mathtt{M}_{1,\nu}$ is divisible by $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$. By proposition 18, this gcd is at most of degree N_i in the coefficients of f_i. As the resultant $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}$ is also of degree N_i in the coefficients of f_i (proposition 6), we deduce that the two polynomials are equal up to an invertible constant. $\square$

Combining proposition 16 and remark 5, we obtain the equivalent of Macaulay's theorem for the projective resultant:

Theorem 20 *The following statements are equivalent:*

- $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(\mathbf{f}) \neq 0$,
- $\partial_{1,\nu}$ *is surjective,*
- $V(F:G) = \emptyset$ *or* $F^{sat} = G^{sat}$.

It implies in particular, that the variety defined by all the minors Δ_I of $\mathtt{M}_{1,\nu}$ (in the space of coefficients $\mathbf{c}$) is the one defined the $\mathrm{Res}_{\mathcal{G}(d_0),\ldots,\mathcal{G}(d_m)}(\mathbf{f}) = 0$.

A direct approach for computing a square submatrix of $\mathtt{M}_{1,\nu}$ whose determinant is not generically zero would be to add incrementally to a subset of the columns of $\mathtt{M}_{1,\nu}$, a new column and to check generically the linear independence (by Gaussian elimination) until we get a square matrix. The way to choose a new column at each step may induce many non-useful tests. We present now an algorithm, which avoids such redundant tests and produces a square submatrix of $\mathtt{M}_{1,\nu}$ whose determinant is not generically zero and of degree N_0 in the coefficients of f_0. It is based on incremental Gröbner basis computations up to the degree ν, for a specialization of the system (1) over a field $\mathbb{F}$ (for instance a prime field). This computation can be seen as an economic way to triangularize the matrix of $\mathtt{M}_{1,\nu}$ and thus is less expensive than a global Gaussian elimination process. It follows the same idea as in Macaulay's construction for projective resultants with the specialization $f_i = x_i^{d_i}$.

We denote by $J = F + I_n(H)$ the ideal generated by $f_0, \ldots, f_m$ and by the $n \times n$ minors of the matrix H. Similarly, we denote by H_k the submatrix $(h_{i,j})_{0 \le i \le n, k \le j \le m}$ corresponding to the polynomials $f_k, \ldots, f_m$ and by J_k the ideal generated by $f_k, \ldots, f_m$ and the $n \times n$ minors of the matrix H_k (if $m - k + 1 \ge n$). With these notations, $J_0 = J$.

Algorithm: A non-zero minor of the matrix $\mathtt{M}_{1,\nu}$ of degree N_0 in f_0

1. *Choose a random specialization* $\mathbf{f}$ *of type (1), with coefficients in* $\mathbb{F}$.
2. *For $i = m, \ldots, 0$, compute a Gröbner basis of J_i (using the Gröbner basis of J_{i+1}) and define L_i as a list of polynomials p such that*
   a. *p is a monomial multiple of degree $\nu := \nu_{\mathbf{d},\mathbf{k}}$ of f_i or of a $n \times n$ minor of H_i and of degree 1 in the coefficients of f_i,*
   b. *$J_{i_{[\nu]}} = J_{i+1_{[\nu]}} \oplus \langle L_i \rangle$.*
3. *Check that L_0 has N_0 elements and that $J_{0_{[\nu]}} = R_{[\nu]}$ (by computing its Hilbert function in degree ν, from the Gröbner basis of J_0). Otherwise, go to 1.*

Then, the coefficient matrix of the polynomials of the list $L = \cup_{i=0}^{m-1} L_i$ is a matrix whose determinant is not zero and of degree N_0 in the coefficient of f_0.

The fact that L_0 has generically N_0 elements (which are of degree 1 in the coefficients of f_0) is a consequence of (7). The point 2.b of the algorithm is achieved by keeping the trace of the multiples of the generators of J_i which are not in J_{i+1}, and which are used during the computation of the Gröbner basis of J_i.

Notice that this algorithm has a probabilistic step and may go into an infinite loop. Once this submatrix has been constructed, it can be used for any field. Thus, even if we used a prime field for easier computation of Gröbner bases, the constructed matrix can then be used with floating point or polynomial coefficients. It yields a non-singular matrix for generic systems of type (1).

We consider now some special cases, for which we can be more specific.

4.3 The residual of an hypersurface

In the case $n = 1$, $f_0, \ldots, f_m$ are all divisible by g_1, so that the residual resultant is the projective resultant of the polynomials $h_{1,0}, \ldots, h_{1,m}$. The block of the matrix $\mathtt{M}_{1,\nu}$ corresponding to the minors Δ_I is the usual Macaulay matrix of $h_{1,0}, \ldots, h_{1,n}$, which yields the usual resultant of these polynomials.

4.4 The residual of the empty set

In the case $n = m + 1$, the polynomials $f_0, \ldots, f_m$ have generically no common zeros ($V(G) = \emptyset$). The residual resultant is thus just the condition that they have a common zero in $\mathbb{P}^m$. The previous construction can be used to obtain a smaller matrix for the projective resultant than Macaulay's one, taking into account the support of the polynomials f_i. Suppose that we can find positive integers k_i such that $G = (x_0^{k_0}, \ldots, x_m^{k_m})$ contains $F = (f_0, \ldots, f_m)$. By applying our construction, we get a matrix $\mathtt{M}_{1,\nu}$, from which we compute the projective resultant as the gcd of its maximal minors. In this case, only one determinant $\Delta := \Delta_{1,\ldots,n}$ of the matrix H is involved. The well-known Macaulay matrix is of size the number of homogeneous monomials in the variables $x_0, \cdots, x_m$ of degree $\delta = \sum_{i=0}^{m} d_i - m$ whereas the size of a square matrix extracted from $\mathtt{M}_{1,\nu}$ is the number of homogeneous monomials in the variables $x_0, \cdots, x_m$ of degree $\nu = \delta - \min(k_i)$. The regularity of $(F : G)$ is bounded by ν. In the particular case $k_0 = \cdots = k_m = 1$, the determinant Δ (of degree $\nu - 1$) is the first component of the Bezoutian and is equivalent to the Jacobian modulo $(f_0, \ldots, f_m)$. This yields another construction of the projective resultant in degree $\nu - 1$ (see [Jou97]).

4.5 The residual of points

We consider here the special case where $V(G)$ is of dimension 0 (ie. $m = n$). We describe an explicit construction of a submatrix of $\mathtt{M}_{1,\nu}$ whose determinant is not zero and of minimal degree $N_0 := \prod_{i=1}^{m} d_i - \prod_{i=1}^{n} k_i$ in the coefficients of f_0. For that, we consider the following specialization

$$\begin{cases} g_1 = \prod_{j=1}^{k_1} (x_1 - a_{1,j}\, x_0) \\ \quad \vdots \\ g_n = \prod_{j=1}^{k_n} (x_n - a_{n,j}\, x_0) \end{cases}$$

where $a_{i,j}$ are distinct non-zero elements of $\mathbb{K}$. We take for the f_i the following polynomials

$$\begin{cases} f_1 &= \prod_{j=1}^{d_1-k_1} (x_1 - b_{1,j} x_0)\, g_1 \\ &\vdots \\ f_n &= \prod_{j=1}^{d_n-k_n} (x_n - b_{n,j} x_0)\, g_n \\ f_0 &= \sum_{i=1}^{n} \prod_{j=1}^{d_0-k_i} (x_n - c_{n,j} x_0)\, g_j \end{cases}$$

such that all $a_{i,j}, b_{i,j}, c_{n,j}$ are distinct elements of $\mathbb{K}$. We have $H_{0,i} = \prod_{j=1}^{d_i-k_i}(x_i - c_{i,j}x_0)$ for $i = 1, \dots, n$, $H_{i,i} = \prod_{j=1}^{d_i-k_i}(x_i - b_{i,j}x_0)$ for $i = 1, \dots, n$, and $H_{i,j} = 0$ otherwise. The $n+1$ minors of the matrix H are

$$\delta_i = \prod_{j \neq i} H_{j,j}(x_0, x_i) H_{0,i}(x_0, x_i), i = 1, \dots, n$$

and $\delta_0 = \prod_{i=1}^{n} H_{j,j}(x_0, x_i)$.

Proposition 21 *The ideal $J = (F : G)$ associated with this specialization is a geometric $m + 1$-residual intersection.*

PROOF. Let K be the ideal generated by $f_0, \dots, f_n, \delta_0, \dots, \delta_n$. Its is a subset of $(F : G)$ (the minors of H are in $(F : G)$ by Cramer's rule). We are going to prove that $V(K)$ is empty which implies that $V(F : G)$ is empty and that $(F : G)$ is a geometric $m + 1$-residual intersection.

We consider first the affine part $x_0 = 1$. Remark, that if $\zeta = (1, \zeta_1, \dots, \zeta_n)$ is a point of $V(f_1, \dots, f_n) - V(G)$ then its coordinates ζ_i are among the $a_{i,l}, b_{i,l}$ and at least one of the polynomials g_i is not zero. Thus choosing $c_{0,l}$ conveniently, we can assume that $f_0(\zeta) \neq 0$ so that $V(F) - V(G)$ is empty in the affine space.

If $\zeta = (0 : \zeta_1 : \cdots : \zeta_n)$ and $f_1(\zeta) = \cdots = f_n(\zeta) = 0$, then we must also have $\zeta_1 = \cdots = \zeta_n = 0$ so that there is no point at infinity in $V(F) - V(G)$.

Now let us consider $V(K) = V(K) \cap V(G)$. As the $a_{i,j}, b_{i,j'}$ are distinct, for any point $\zeta \in V(G)$ and $i = 1, \dots, n$, we have $H_{i,i}(\zeta) \neq 0$ so that $\delta_0(\zeta) \neq 0$ and $V(K) = \emptyset$. $\square$

By theorem 9, we deduce that the complex (6) is exact. In particular, the map ∂_1 is surjective in degree $\nu \geq \nu_{\mathbf{d},\mathbf{k}}$.

Lemma 22 *The polynomials $f_0, \dots, f_n, \delta_0$ form a Gröbner basis of J_1 for the lexicographic ordering such that $x_1 > \cdots > x_m > x_0$.*

PROOF. We apply Buchberger criterion 2 to the S-polynomial of f_i and δ_0. $\square$

A square matrix with a non-zero determinant of degree N_0 in f_0 can now be constructed by applying the algorithm of the previous section. It yields the following definitions:

- $\nu = \sum_{i=0}^{m} d_m - m - \min_i k_i$.
- Let $M_m = \{\mathbf{x}^{\alpha_m};\ |\alpha_m| = \nu - d_m\}$.
- For $i = m-1, \ldots, 1$, let $M_i = \{\mathbf{x}^{\alpha_i};\ |\alpha_i| = \nu - d_i,\ x_i^{d_i}\mathbf{x}^{\alpha_i} \notin (x_{i+1}^{d_{i+1}}, \ldots, x_m^{d_m})\}$,
- Let $N_0 = \{\mathbf{x}^{\beta_1};\ |\beta_1| = \nu - \sum_{i=1}^{m} d_i + \sum_{i=1}^{m} k_i,\ x_1^{d_1-k_1} \cdots x_m^{d_m-k_m}\mathbf{x}^{\beta_1} \notin (x_1^{d_1}, \ldots, x_m^{d_m})\}$,
- Let $M_0 = \{\mathbf{x}^{\alpha_0};\ |\alpha_1| = \nu - d_0, \mathbf{x}^{\alpha_0} \notin (x_2^{d_2}, \ldots, x_m^{d_m}, x_1^{d_1-k_1} \cdots x_m^{d_m-k_m})\}$,
- Let $\tilde{\mathtt{M}}_{1,\nu}$ be the matrix of the map

$$\begin{aligned} \tilde{\partial}_{1,\nu} : \langle M_0 \rangle \times \cdots \times \langle M_m \rangle \times \langle N_0 \rangle &\to R_{[\nu]} \\ (q_0, \ldots, q_m, r_0) &\mapsto \sum_{i=0}^{m} q_i f_i + r_0 \delta_0 \end{aligned}$$

Proposition 23 *The determinant of $\tilde{\mathtt{M}}_{1,\nu}$ is not zero and of degree $N_0 := \prod_{i=1}^{m} d_i - \prod_{i=1}^{n} k_i$ in f_0.*

PROOF. We check that the product of the elements of the diagonal of $\tilde{\mathtt{M}}_{1,\nu}$ yields the component of maximal degree in the $a_{i,j}, b_{i,j}, c_{i,k}$ of $\det(\tilde{\mathtt{M}})$, which thus cannot be identically 0 for a convenient choice of these parameters. The degree of $\det(M_0)$ in f_0 is the number of monomials in M_0, that is $\prod_{i=1}^{m} d_i - \prod_{i=1}^{n} k_i$. $\square$

5 Examples

We illustrate our construction on some examples. The computations have been performed in MAPLE. A package, called `multires`, implements this residual resultant formulation among other more classical resultant matrix constructions. It is available at `http://www.inria.fr/saga/logiciels/multires`.

5.1 The residual of a point in $\mathbb{P}^2$

We consider the following system in $\mathbb{P}^2$:

$$\begin{cases} f_0 = a_0 z^2 + a_1 zx + a_2 zy + a_3 xy + a_4 x^2 \\ f_1 = b_0 z^2 + b_1 zx + b_2 zy + b_3 xy + b_4 x^2 \\ f_2 = c_0 z^2 + c_1 zx + c_2 zy + c_3 xy + c_4 x^2 \end{cases}$$

We set $G = (x, z)$ and apply our construction of the residual resultant. We have $\nu_{\mathbf{d},\mathbf{k}} = 2$ and the 6×6 matrix $\mathtt{M}_{1,\nu}$ is

$$\begin{bmatrix} a_4 & b_4 & c_4 & -b_1a_4 + a_1b_4 & -c_1a_4 + a_1c_4 & -c_1b_4 + b_1c_4 \\ a_2 & b_2 & c_2 & -a_3b_0 + b_3a_0 & a_0c_3 - c_0a_3 & b_0c_3 - c_0b_3 \\ 0 & 0 & 0 & -b_2a_3 + a_2b_3 & -c_2a_3 + a_2c_3 & c_3b_2 - b_3c_2 \\ a_1 & b_1 & c_1 & a_0b_4 - b_0a_4 & a_0c_4 - c_0a_4 & b_0c_4 - c_0b_4 \\ a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & -b_1a_3 + a_1b_3 + a_2b_4 - b_2a_4 & -c_1a_3 + a_1c_3 + a_2c_4 - c_2a_4 & -c_1b_3 + b_1c_3 + b_2c_4 - c_2b_4 \end{bmatrix}$$

The degrees of the resultant $N_0 = N_1 = N_2 = 3$ and thus the determinant of this matrix is exactly this residual resultant. The projective resultant vanishes identically, for $(0:1:0)$ is a root of the generic system. If we compare the residual resultant with the toric one, we obtain the larger 9×9 matrix

$$\begin{bmatrix} a_3 & a_2 & a_4 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_2 & 0 & a_1 & a_0 & 0 & a_3 & a_4 & 0 & 0 \\ 0 & 0 & a_0 & 0 & 0 & a_2 & a_1 & a_3 & a_4 \\ b_3 & b_2 & b_4 & b_1 & b_0 & 0 & 0 & 0 & 0 \\ b_2 & 0 & b_1 & b_0 & 0 & b_3 & b_4 & 0 & 0 \\ 0 & 0 & b_0 & 0 & 0 & b_2 & b_1 & b_3 & b_4 \\ c_3 & c_2 & c_4 & c_1 & c_0 & 0 & 0 & 0 & 0 \\ c_2 & 0 & c_1 & c_0 & 0 & c_3 & c_4 & 0 & 0 \\ 0 & 0 & c_0 & 0 & 0 & c_2 & c_1 & c_3 & c_4 \end{bmatrix}.$$

Its determinant (which is the toric resultant) is equal to the residual resultant.

5.2 The residual of two points in $\mathbb{P}^2$

We consider the following system in $\mathbb{P}^2$:

$$\begin{cases} f_0 = a_0z^2 + a_1zx + a_2zy + a_3x^2 + a_3y^2 \\ f_1 = \ b_0z^2 + b_1zx + b_2zy + b_3x^2 + b_3y^2 \\ f_2 = \ c_0z^2 + c_1zx + c_2zy + c_3x^2 + c_3y^2 \end{cases}$$

We set $G = (z, x^2 + y^2)$. We have $\nu_{\mathbf{d},\mathbf{k}} = 2$ and a nonzero maximal minor of the matrix $\mathtt{M}_{1,\nu}$ is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_1c_3 + c_1b_3 & -b_2c_3 + c_2b_3 & -c_1a_3 + a_1c_3 \\ a_1 & b_1 & c_1 & 0 & -c_3b_0 + b_3c_0 & 0 \\ c_2 & b_2 & c_2 & -c_3b_0 + b_3c_0 & 0 & a_0c_3 - c_0a_3 \\ a_3 & b_3 & c_3 & 0 & -b_1c_3 + c_1b_3 & 0 \\ a_3 & b_3 & c_3 & -b_2c_3 + c_2b_3 & 0 & -c_2a_3 + a_2c_3 \end{vmatrix}.$$

The formula for the degree gives $N_0 = N_1 = N_2 = 2$ and we check that the determinant of this matrix is the residual resultant times $c_3(c_1b_3 - c_3b_1)$. It has the minimal degree N_0 in the coefficients of f_0. Here also the projective and toric resultants vanish identically.

5.3 The residual of a curve in $\mathbb{P}^3$

We consider the following system of cubics of $\mathbb{P}^3$ containing the umbilic:

$$\begin{cases} f_0 = (a_0x + a_1y + a_2z + a_3t)(x^2 + y^2 + z^2) \\ \qquad +(a_4x^2 + a_5y^2 + a_6z^2 + a_7t^2 + a_8xy + a_9xz + a_{10}xt + a_{11}yz + a_{12}yt + a_{13}zt)t \\ f_1 = (b_0x + b_1y + b_2z + b_3t)(x^2 + y^2 + z^2) \\ \qquad +(b_4x^2 + b_5y^2 + b_6z^2 + b_7t^2 + b_8xy + b_9xz + b_{10}xt + b_{11}yz + b_{12}yt + b_{13}zt)t \\ f_2 = (c_0x + c_1y + c_2z + c_3t)(x^2 + y^2 + z^2) \\ \qquad +(c_4x^2 + c_5y^2 + c_6z^2 + c_7t^2 + c_8xy + c_9xz + c_{10}xt + c_{11}yz + c_{12}yt + c_{13}zt)t \\ f_3 = (d_0x + d_1y + d_2z + d_3t)(x^2 + y^2 + z^2) \\ \qquad +(d_4x^2 + d_5y^2 + d_6z^2 + d_7t^2 + d_8xy + d_9xz + d_{10}xt + d_{11}yz + d_{12}yt + d_{13}zt)t \end{cases}$$

We set $G = (t, x^2 + y^2 + z^2)$ and apply the construction. We obtain $\nu_{\mathbf{d},\mathbf{k}} = 6$, $N_0 = N_1 = N_2 = N_3 = 15$. The matrix $\mathtt{M}_{1,\nu}$ is a 84×200 matrix. A maximal minor of rank 84 whose determinant has degree 15 in the coefficients of f_0 has been constructed as follows. We consider the 84×120 matrix of the map ∂_1, associated to the polynomials f_1, f_2, f_3 and we extract 69 independent columns (by considering a random specialization). We add to this matrix the columns of $\mathtt{M}_{1,\nu}$ depending on the coefficients of f_0 and independent of the 69 columns.

Notice that $\nu_{\mathbf{d},\mathbf{k}} = 6$ is here exactly the regularity. If we compute the matrix of $\partial_{1,5}$, we obtain a matrix of size 56×55. Notice also that the projective and toric resultants are identically 0.

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