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# Naming Proofs in Classical Propositional Logic

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**Abstract.** We present a theory of proof denotations in classical propositional logic. The abstract definition is in terms of a semiring of weights, and two concrete instances are explored. With the Boolean semiring we get a theory of classical proof nets, with a geometric correctness criterion, a sequentialization theorem, and a strongly normalizing cut-elimination procedure. With the semiring of natural numbers, we obtain a sound semantics for classical logic, in which fewer proofs are identified. Though a “real” sequentialization theorem is missing, these proof nets have a grip on complexity issues. In both cases the cut elimination procedure is closely related to its equivalent in the calculus of structures, and we get “Boolean” categories which are not posets.

## 1 Introduction

Finding a good way of naming proofs in classical logic—a good theory of proof terms, or proof nets, or whatever—is a notoriously difficult question, and the literature about it is already quite large, and still increasing.

Other logics have been helped enormously by the presence of good semantics, where by semantics we mean mathematical objects that have an independent existence from syntax. Linear logic was found through the observation of the category of coherence spaces and linear maps. For intuitionistic logic, it has been obvious for a long time that all it takes to give an interpretation of formulas and proofs à la Brouwer-Heyting-Kolmogorov-Curry-Howard is a bi-cartesian closed category... and cartesian closed categories abound in nature.

But if we try to extend naively these semantics to classical logic, it is well-known that everything collapses to a poset (a Boolean algebra, naturally) and we are back to the old semantics of provability. Clearly something has to be weakened, if we ever want classical logic to have a meaning beyond syntax. Very recently, it was found [Hyl04] that a class of algebras from geometry permits relevant interpretations of classical proofs; in addition a proposal for an abstract categorical framework has been made [FP04a,FP04b], based on the proof nets in [Rob03], and which uses 2-categorical structure to avoid poset collapse.

Let us describe succinctly the view when this problem is approached from the other end, that of syntax. We use the sequent calculus as our main proof

paradigm, but what we say is presented so as to apply to natural deduction as well. It is well known that problems begin when a proof contains redundancies and has to be normalized. Let us represent this situation in the following manner

$$\text{cut} \frac{\frac{\pi_1}{\vdash \Gamma, A} \quad \frac{\pi_2}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} .$$

Here, we use one-sided *notation* for sequents for the sake of generality, and an expression like  $A$  could be a formula with some polarity information added, instead of just a formula. The  $\pi_1, \pi_2$  represent the proofs that led to the sequents: they could be sequent calculus trees, proof terms or proof nets. Similarly, the expression  $\bar{A}$  is a *formal negation* for  $A$ ; this notation could be used for instance in a natural deduction context like the  $\lambda\mu$  calculus [Par92], where the “cut” inference above would just be a substitution of a term into another, the negated  $\bar{A}$  meaning that it is on the side of the input/premises/ $\lambda$ -variables. But nonetheless the following have for a long time been identified as Desirable Features:

1.  $\bar{A}$  is the *logical* negation of  $A$ ,
2.  $\bar{A}$  is *structurally equivalent (isomorphic)* to  $A$ .

These additional symetries simplify life enormously, allowing things like structural de Morgan duals. The second feature will not happen, for example, when negation is an introduced symbol, as in the case for two-sided sequent calculi or the  $\lambda\mu$  calculus (for which the first feature does not hold either).

The problem of cut-elimination (or normalization) is encapsulated in two cases, called weakening-weakening and contraction-contraction in [Gir91], which hereafters will be written as *weak-weak* and *cont-cont*:

$$\text{weak} \frac{\frac{\pi_1}{\vdash \Gamma} \quad \text{weak} \frac{\frac{\pi_2}{\vdash \Delta}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \text{and} \quad \text{cont} \frac{\frac{\pi_1}{\vdash \Gamma, A, A} \quad \text{cont} \frac{\frac{\pi_2}{\vdash \bar{A}, \bar{A}, \Delta}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} .$$

It is well known [GLT89,Gir91] that both reductions cannot be achieved without choosing a side, and that the outcome is very much dependent on that choice.

The most standard way to resolve these dilemmas is to introduce asymetry in the system (if it’s not already there), by the means of polarity information on the formulas, and using this to dictate the choices. Historically, the first approaches to polarization were closely aligned on the premiss/conclusion duality we have mentioned. One reason for this is that they were derived from double-negation style translations of classical logic into intuitionistic logic. If a classical proof can be turned into an intuitionistic one, then in every sequent in the history of that proof there will be a special formula: the conclusion of the corresponding intuitionistic sequent. This is what is done, for example, *mutatis mutandi*, in the  $\lambda\mu$ -calculus.

In [Gir91] Girard presented System LC, where the sequents have *at most* one special formula. Not only are there both positive and negative formulas—this time an arbitrary number of each—but in addition there is a *stoup*, which

is either empty or contains *one* formula, which has to be positive. Then when a choice has to be made at normalization time, the presence or absence of the positive formula in the stoup is used in addition to the polarity information. This calculus has very nice properties; in particular both Desirable Features above are present. System LC was discovered by the means of a translation into a bi-cartesian closed category  $\mathcal{C}$ , but something much weaker can be used. Our story would be over if everything had been solved by that approach; but semantic cut-elimination (composition of denotations of proofs) is not associative here in general; this is shown by a case analysis on the polarities. Thus normal forms are still dependent somehow on the history of the proof's construction. Viewed from another angle this says that the category  $\mathcal{C}$  cannot *inherently* be a model for classical logic (composition in categories is associative...), it is more a *vehicle for a translation*.

This direction of research has nonetheless been extremely fruitful. It has given a systematic analysis of translations of classical logic into linear logic [DJS97]. Moreover LC's approach to polarities was extended to the formulation of polarized logic LLP [Lau02]. It has the advantage of a much simpler theory of proof nets (e.g., for boxes) and produces particularly perspicuous translations of more traditional logics. This new proof net syntax has been used to represent proofs in LC [Lau02] and the  $\lambda\mu$ -calculus [Lau03].

Let us go back to the *weak-weak* and *cont-cont* problems. It is well-known that one way of solving *weak-weak* is to permit the system to have a mix rule. As for the *cont-cont* problem, the proof formalism of the calculus of structures [GS01,BT01] has permitted the emergence of a novel solution: through the use of deep inference a proof can always be transformed into one whose only contractions are made on *constants and atomic formulas*.

In this paper we exploit these two ideas to construct two systems of proof denotations for classical logic, which are at the border of syntax and semantics, and which possess both Desirable Features as well as one of the main features of proof nets, since a proof of a formula is represented as a graph structure on the set of atoms of that formula, whose edges are directly related to the axioms of a corresponding sequent proof. But other standard feature of non-multiplicative proof nets (boxes and explicit contraction links) are absent. One of these systems benefits from a full completeness theorem—in other words, a correctness criterion and a sequentialization theorem—and so it can be used directly to represent proofs, without recourse to other syntactical formalisms. Moreover strong normalization holds, in the most general sense. The other system conveys both more and less information at the same time: the number of times an axiom link is used in a proof is counted, but since we do not have (as of yet) a sequentialization theorem, it is closer to a semantics, since a “standalone” category can be constructed out of it. Issues of normalization are a little more delicate in this latter system, but it has a definite interest from the point of view of complexity. In both cases we get “Boolean” categories; we will say a bit on the essential properties of these categories that allow them not to collapse.

Remarkably the idea of using axiom links to capture a proof in classical logic (including a correctness criterion!) has been around since Andrews' paper of

1976 [And76], and his use of the term “essence of a proof” clearly shows that he understood what was at stake. But since he was working on proof search, he never explored the possibility of *composing* these “essences”. The idea of summing all axiom links present in a sequent proof is already present in Buss’ *logical flow graphs* [Bus91,Car97].

## 2 Cut free proof nets for classical propositional logic

Let  $\mathcal{A} = \{a, b, \dots\}$  be an arbitrary set of atoms, and let  $\bar{\mathcal{A}} = \{\bar{a}, \bar{b}, \dots\}$  the set of their duals. The set of CL-*formulas* is defined as follows:

$$\mathcal{F} ::= \mathcal{A} \mid \bar{\mathcal{A}} \mid \mathbf{t} \mid \mathbf{f} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \quad .$$

We will use  $A, B, \dots$  to denote formulas. The elements of the set  $\{\mathbf{t}, \mathbf{f}\}$  are called *constants*. A formula in which no constants appear is called a  $\text{CL}_0$ -*formula*. *Sequents*, denoted by  $\Gamma, \Delta, \dots$ , are finite lists of formulas, separated by comma.

In the following, we will consider formulas as binary trees (and sequents as forests), whose leaves are decorated by elements of  $\mathcal{A} \cup \bar{\mathcal{A}} \cup \{\mathbf{t}, \mathbf{f}\}$ , and whose inner nodes are decorated by  $\wedge$  or  $\vee$ . Given a formula  $A$  or a sequent  $\Gamma$ , we write  $\mathcal{L}(A)$  or  $\mathcal{L}(\Gamma)$ , respectively, to denote its set of leaves.

For defining proof nets, we start with a commutative semiring of *weights*  $(W, 0, 1, +, \cdot)$ . That is,  $(W, 0, +)$  is a commutative monoid structure,  $(W, 1, \cdot)$  is another commutative monoid structure, and we have the usual distributivity laws  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$  and  $0 \cdot x = 0$ . This abstraction layer is just there to ensure a uniform treatment for the two cases that we will encounter in practice, namely  $W = \mathbb{N}$  (the semiring of the natural numbers with the usual operations), and  $W = \mathbb{B} = \{0, 1\}$  (the Boolean semiring, where addition is disjunction, i.e.,  $1 + 1 = 1$ , and multiplication is conjunction). There are two additional algebraic properties that we will need:

$$v + w = 0 \text{ implies } v = w = 0 \tag{1}$$

$$v \cdot w = 0 \text{ implies either } v = 0 \text{ or } w = 0 \quad . \tag{2}$$

These are obviously true in both concrete cases. No other structure or property on  $\mathbb{B}$  and  $\mathbb{N}$  is needed, and thus other choices for  $W$  can be made. They all give sound semantics, but completeness (or sequentialization) is another matter.

**2.1 Definition** Given  $W$  and a sequent  $\Gamma$ , a  $W$ -*linking* for  $\Gamma$  is a function  $P: \mathcal{L}(\Gamma) \times \mathcal{L}(\Gamma) \rightarrow W$  which is symmetrical (i.e.,  $P(x, y) = P(y, x)$  always), and such that whenever  $P(x, y) \neq 0$  for two leaves, then one of the following three cases holds:

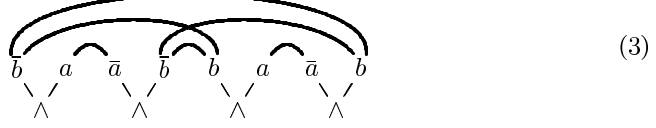
0. one of  $x$  and  $y$  is decorated by an atom  $a$  and the other by its dual  $\bar{a}$ ,
1.  $x = y$  and is decorated by  $\mathbf{t}$ , or
2. one of  $x$  and  $y$  is decorated by  $\mathbf{t}$  and the other by  $\mathbf{f}$ .

A  $W$ -*pre-proof-net*<sup>1</sup> (or shortly  $W$ -*prenet*) consists of a sequent  $\Gamma$  and a linking  $P$  for it. It will be denoted by  $P \triangleright \Gamma$ . If  $W = \mathbb{B}$  we say it is a *simple pre-proof-net* (or shortly *simple prenet*).

In what follows, we will simply say prenet, if no semiring  $W$  is specified.

<sup>1</sup> What we call *pre-proof-net* is in the literature often called a *proof structure*.

If we choose  $W = \mathbb{B}$  a linking is just an ordinary, non-directed graph structure on  $\mathcal{L}(\Gamma)$ , with an edge between  $x$  and  $y$  when  $P(x, y) = 1$ . An example is



To save space that example can also be written as

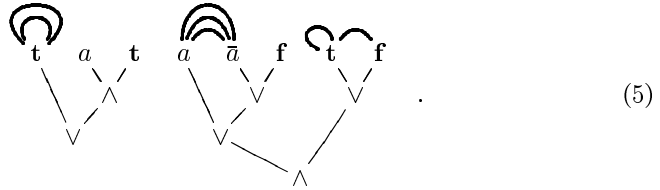
$$\{\widehat{b_1 b_5}, \widehat{b_1 b_8}, \widehat{b_4 b_5}, \widehat{b_4 b_8}, \widehat{a_2 a_3}, \widehat{a_6 a_7}\} \triangleright \bar{b}_1 \wedge a_2, \bar{a}_3 \wedge \bar{b}_4, b_5 \wedge a_6, \bar{a}_7 \wedge b_8,$$

where the set of linked pairs is written explicitly in front of the sequent. Here we use the indices only to distinguish different atom occurrences (i.e.,  $a_3$  and  $a_6$  are not different atoms but different occurrences of the same atom  $a$ ).

For more general cases of  $W$ , we can consider the edges to be decorated with elements of  $W \setminus \{0\}$ . When  $P(x, y) \neq 0$  we say that  $x$  and  $y$  are *linked*. Here is an example of an  $\mathbb{N}$ -prenet:

$$\{\overset{2}{\widehat{t_1 t_1}}, \overset{3}{\widehat{a_4 \bar{a}_5}}, \widehat{t_7 t_7}, \widehat{t_7 f_8}\} \triangleright t_1 \vee (a_2 \wedge t_3), (a_4 \vee (\bar{a}_5 \vee f_6)) \wedge (t_7 \vee f_8). \quad (4)$$

As before, no link means that the value is 0. Furthermore, we use the convention that a link without value means that the value of the linking is 1. When drawing  $\mathbb{N}$ -prenets as graphs (i.e., the sequent forest plus linking), we will draw  $n$  links between two leaves  $x$  and  $y$ , if  $P(x, y) = n$ . Although this might be a little cumbersome, we find it more intuitive. Example (4) is then written as



Now for some more notation: let  $P \triangleright \Gamma$  be a prenet and  $L \subseteq \mathcal{L}(\Gamma)$  an arbitrary subset of leaves. There is always a  $P|_L: L \times L \rightarrow W$  which is obtained by restricting  $P$  on the smaller domain. But  $L$  also determines a subforest  $\Gamma|_L$  of  $\Gamma$ , in the manner that all elements of  $L$  are leaves of  $\Gamma|_L$ , and an inner node  $s$  of  $\Gamma$  is in  $\Gamma|_L$  if *one or two* of its children is in  $\Gamma|_L$ . Thus  $\Gamma|_L$  is something a bit more general than a “subsequent” or “sequent of subformulas”, since some of the connectors are allowed to be unary, although still labeled by  $\wedge$  and  $\vee$ . Let  $\Gamma' = \Gamma|_L$ . Then not only is  $\Gamma'$  determined by  $L$ , but the converse is also true:  $L = \mathcal{L}(\Gamma')$ . We will say that  $P|_L \triangleright \Gamma'$  is a *sub-prenet* of  $P \triangleright \Gamma$ , although it is not *strictu sensu* a prenet. Since this sub-prenet is entirely determined by  $\Gamma'$ , we can also write it as  $P|_{\Gamma'} \triangleright \Gamma'$  without mentioning  $L$  any further.

On the set of  $W$ -linkings we define the following operations. Let  $P: L \times L \rightarrow W$  and  $Q: M \times M \rightarrow W$  be two  $W$ -linkings.

- If  $L = M$  we define the *sum*  $P + Q$  to be the pointwise sum  $(P + Q)(x, y) = P(x, y) + Q(x, y)$ . When  $W = \mathbb{B}$  this is just the union of the two graphs.

- If  $L \subseteq M$  we define the *extension*  $P \uparrow^M$  of  $P$  to  $M$  as the binary function on  $M$  which is equal to  $P(x, y)$  when both  $x$  and  $y$  are in  $L$  and zero elsewhere. Most of the times we will write  $P \uparrow^M$  simply as  $P$ .
- If  $L$  and  $M$  are disjoint, we define the *disjoint sum*  $P \oplus Q$  on the disjoint union<sup>2</sup>  $L \uplus M$  as  $P \oplus Q = P \uparrow^{L \uplus M} + Q \uparrow^{L \uplus M}$ .

**2.2 Definition** A *conjunctive resolution* of a prenet  $P \triangleright \Gamma$  is a sub-prenet  $P|_{\Gamma'} \triangleright \Gamma'$  where  $\Gamma'$  has been obtained by deleting one child subformula for every conjunction node of  $\Gamma$  (i.e., in  $P|_{\Gamma'} \triangleright \Gamma'$  every  $\wedge$ -node is unary).

**2.3 Definition** A  $W$ -prenet  $P \triangleright \Gamma$  is said to be *correct* if for every one of its conjunctive resolutions  $P|_{\Gamma'} \triangleright \Gamma'$  the  $W$ -linking  $P|_{\Gamma'}$  is not the zero function. A *W-proof-net* (or shortly *W-net*) is a correct  $W$ -prenet. A correct  $\mathbb{B}$ -prenet is also called a *simple net*.

Both examples shown so far are correct: (3) is a  $\mathbb{B}$ -net as well as an  $\mathbb{N}$ -net; (5) is an  $\mathbb{N}$ -net. Notice that the definition of correctness does not take the exact values of the weights into account, only the presence ( $P(x, y) \neq 0$ ) or absence ( $P(x, y) = 0$ ) of a link. Notice also that correctness is a monotone property because of Axiom (1): if  $P \triangleright \Gamma$  is correct then  $P + Q \triangleright \Gamma$  is also correct, whatever  $Q$  is.

### 3 Sequentialization

Figure 1 shows how sequent proofs are mapped into prenets. The sequent system we use contains the multiplicative versions of the  $\wedge$ - and  $\vee$ -rules, the usual axioms for identity (reduced to atoms and constants) and truth, as well as the rules of exchange, weakening, contraction, and mix. We call that system  $\text{CL}$  (for Classical Logic). We will use three minor variations: first there is  $\text{CL}_0$  where the only axiom available is  $\text{id}_0$ , involving only atoms, and all formulas are  $\text{CL}_0$ -formulas. Then, there is  $\text{CL}_1$  which allows in addition the axiom  $\text{t}_1$ , and finally  $\text{CL}_2$  which allows all three axioms. Clearly, in the case of  $\text{CL}_1$  and  $\text{CL}_2$ , all formulas are  $\text{CL}$ -formulas. When we write just  $\text{CL}$  we mean  $\text{CL}_2$ .

There are two rules that are not strictly necessary from the point of view of provability, namely  $\text{mix}$  and  $\text{id}_2$ . Thus, although we do not get more tautologies by including these rules in the system, we get *more proofs*. These new proofs are needed in order to make cut elimination confluent ( $\text{mix}$ ), and get identity proofs in the logic with units ( $\text{id}_2$ ), thus allowing us to construct a category. It is only one more example of the extreme usefulness of completions in mathematics: adding stuff can make your life simpler.

Let us now explain the translation from  $\text{CL}_i$  sequent proofs to prenets. We should have a notion of  $\text{CL}_i$ -prenet for any one of  $i = 0, 1, 2$ , and Definition 2.1 has been designed to produce them automatically: a  $\text{CL}_i$ -prenet is one where conditions  $0, \dots, i$  hold in that definition.

The construction is done inductively on the size of the proof. In the two rules for disjunction and exchange, nothing happens to the linking. In the case

<sup>2</sup> If  $L$  and  $M$  are not actually disjoint, we can rename one of the sets to ensure that they are.

$$\begin{array}{ccc}
\text{id}_0 \frac{}{\{a \widehat{\bar{a}}\} \triangleright a, \bar{a}} & \text{t}_1 \frac{}{\{\mathbf{t} \widehat{\mathbf{t}}\} \triangleright \mathbf{t}} & \text{id}_2 \frac{}{\{\mathbf{f} \widehat{\mathbf{t}}\} \triangleright \mathbf{f}, \mathbf{t}} \\
\vee \frac{P \triangleright A, B, \Gamma}{P \triangleright A \vee B, \Gamma} & \wedge \frac{P \triangleright \Gamma, A \quad Q \triangleright B, \Delta}{P \oplus Q \triangleright \Gamma, A \wedge B, \Delta} & \text{exch} \frac{P \triangleright \Gamma, A, B, \Delta}{P \triangleright \Gamma, B, A, \Delta} \\
\text{weak} \frac{P \triangleright \Gamma}{P \triangleright A, \Gamma} & \text{cont} \frac{P \triangleright A, A, \Gamma}{P' \triangleright A, \Gamma} & \text{mix} \frac{P \triangleright \Gamma \quad Q \triangleright \Delta}{P \oplus Q \triangleright \Gamma, \Delta}
\end{array}$$

**Fig. 1.** Translation of cut free sequent calculus proofs into prenets

of weakening we apply the extension operation to  $P$ . In the **mix** and  $\wedge$  rules we get the linking of the conclusion by forming the disjoint sum of the linkings of the premises. Therefore, the only rule that deserves further explanation is contraction. Consider the two sequents  $\Delta' = A, \Gamma$  and  $\Delta = A, A, \Gamma$ , where  $\Delta$  is obtained from  $\Delta'$  by duplicating  $A$ . Let  $p: \mathcal{L}(\Delta) \rightarrow \mathcal{L}(\Delta')$  be the function that identifies the two occurrences of  $A$ . We see in particular that for any leaf  $x \in \mathcal{L}(\Delta')$ , the inverse image  $p^{-1}\{x\}$  has either one or two elements. Given  $P \triangleright \Delta$  let us define the linking  $P'$  for  $\Delta'$  as

$$P'(x, y) = \sum_{\substack{z \in p^{-1}\{x\} \\ w \in p^{-1}\{y\}}} P(z, w) .$$

**3.1 Theorem (Soundness)** *Given any  $W$  as above, any  $i \in \{0, 1, 2\}$  and a sequent proof in  $\text{CL}_i$ , the construction above yields a correct  $W$ -prenet.*

**Proof:** The proof is an easy induction and will be left to the reader for the time being. Notice that we need Axiom (1) for the contraction rule, but not (2).  $\square$

We will call a prenet *sequentializable* if it can be obtained from a sequent calculus proof via this translation. Theorem 3.1 says that every sequentializable prenet is a net.

**3.2 Theorem (Sequentialization)** *For any  $i \in \{0, 1, 2\}$ , a simple  $\text{CL}_i$ -net (i.e.,  $W = \mathbb{B}$ ) is sequentializable in  $\text{CL}_i$ .*

**Proof:** We will proceed by induction on the size of the sequent (i.e., the number of  $\wedge$ -nodes,  $\vee$ -nodes, and leaves in the sequent forest). Consider a simple net  $P \triangleright \Delta$ . We have the following cases:

- If  $\Delta$  contains a formula  $A \vee B$ , then we can apply the  $\vee$ -rule, and proceed by induction hypothesis.
- If  $\Delta$  contains a formula  $A \wedge B$ , i.e.,  $\Delta = \Gamma, A \wedge B$ , then we can form the three simple nets  $P' \triangleright \Gamma, A$  and  $P'' \triangleright \Gamma, B$  and  $P \triangleright \Gamma, A, B$ , where  $P' = P|_{\Gamma, A}$  and  $P'' = P|_{\Gamma, B}$ . All three of them are correct, quite obviously. Therefore, we can apply the induction hypothesis to them. Now we apply the  $\wedge$ -rule twice to get  $P' \oplus P'' \oplus P \triangleright \Gamma, \Gamma, \Gamma, A \wedge B, A \wedge B$ . To finally get  $P \triangleright \Gamma, A \wedge B$ , we only need a sufficient number of contractions (and exchanges).

Let us make two remarks about that case:



$$\begin{array}{c}
\text{id}^4 \\
\hline
\text{s}^4 \frac{\overline{[b, b] \wedge [\bar{b}, b] \wedge [\bar{b}, b] \wedge [\bar{b}, b]}}{\overline{[\bar{b} \wedge \bar{b}, b, b] \wedge [\bar{b} \wedge \bar{b}, b, b]}} \\
\hline
\text{s}^2 \frac{\overline{[\bar{b} \wedge \bar{b}, \bar{b} \wedge \bar{b}, [b, b] \wedge [b, b]]}}{\overline{[[\bar{b}, \bar{b}] \wedge [\bar{b}, \bar{b}], [b, b] \wedge [b, b]]}} \\
\hline
\text{m} \\
\hline
\text{cont}^4 \frac{\overline{[[\bar{b}, \bar{b}] \wedge [\bar{b}, \bar{b}], [b, b] \wedge [b, b]]}}{\overline{[\bar{b} \wedge \bar{b}, b \wedge b]}} \\
\hline
\text{id}^2 \\
\hline
\text{s}^4 \frac{\overline{[\bar{b} \wedge [a, \bar{a}] \wedge \bar{b}, b \wedge [a, \bar{a}] \wedge b]}}{\overline{[\bar{b} \wedge a, \bar{a} \wedge \bar{b}, b \wedge a, \bar{a} \wedge b]}}
\end{array}
\qquad
\begin{array}{c}
\text{id} \frac{\overline{\quad}}{\vdash \bar{a}, a} \qquad \text{id} \frac{\overline{\quad}}{\vdash \bar{a}, a} \qquad \text{weak} \frac{\overline{\vdash \bar{a}, a}}{\vdash \bar{a}, \bar{a}, a} \\
\text{weak} \frac{\overline{\vdash \bar{a}, a, a}}{\vdash \bar{a}, a, a} \qquad \wedge \frac{\overline{\vdash \bar{a}, a} \quad \overline{\vdash \bar{a}, a}}{\vdash \bar{a}, a \wedge \bar{a}, a} \\
\wedge \frac{\overline{\vdash \bar{a}, a, a} \quad \overline{\vdash \bar{a}, a \wedge \bar{a}, a}}{\vdash \bar{a}, a, a \wedge \bar{a}, a} \\
\text{exch}^5 \frac{\overline{\vdash \bar{a}, a, a \wedge \bar{a}, a \wedge \bar{a}, a}}{\vdash \bar{a}, \bar{a}, a \wedge \bar{a}, a \wedge \bar{a}, a} \\
\text{cont}^3 \frac{\overline{\vdash \bar{a}, a, a \wedge \bar{a}, a}}{\vdash \bar{a}, a \wedge \bar{a}, a}
\end{array}$$

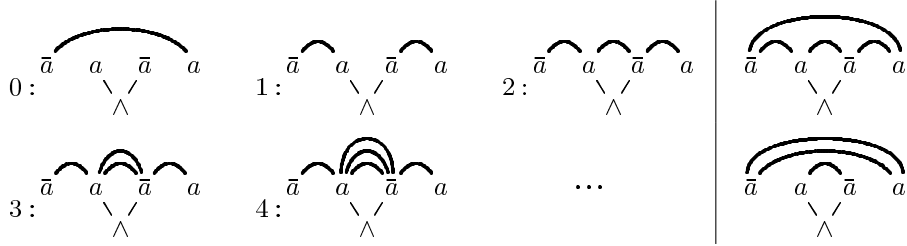
**Fig. 2. Left:** (3) in the calculus of structures      **Right:** A “fake” church numeral

- If there is in  $P \triangleright \Gamma, A \wedge B$  no link between  $A$  and  $B$ , i.e.,  $P|_{A,B} = P|_A \oplus P|_B$ , then we do not need the net  $P \triangleright \Gamma, A, B$ , and can instead proceed by a single use of the  $\wedge$ -rule, followed by contractions.
- This is the only case where the fact that  $W = \mathbb{B}$  is needed.
- If  $\Delta = \Gamma, A$ , such that  $P = P|_\Gamma$ , i.e., the formula  $A$  does not take part in the linking  $P$ , then we can apply the weakening rule and proceed by induction hypothesis.
- The only remaining case is where all formulas in  $\Delta$  are atoms, negated atoms, or constants. Then the sequent proof is obtained by a sufficient number of instances of the axioms  $\text{id}_0$ ,  $\text{t}_1$ ,  $\text{id}_2$ , and the rules  $\text{cont}$ ,  $\text{exch}$ , and  $\text{mix}$ .  $\square$

Note that the particular system CL is only needed for obtaining the completeness, i.e., sequentialization, of  $\mathbb{B}$ -nets. For obtaining the soundness result, any sequent system for classical propositional logic (with the identity axiom reduced to atomic form) can be used. Moreover, this is not restricted to sequent calculus. We can also start from resolution proofs (as done in [And76]), tableau proofs, Hilbert style proofs, etc.

Whereas  $\mathbb{B}$ -nets only take into account *whether* a certain axiom link is used in the proof,  $\mathbb{N}$ -nets also count *how often* it is used. Therefore,  $\mathbb{N}$ -nets can be used for investigating certain complexity issues related to the size of proofs. These are not visible for  $\mathbb{B}$ -nets, where size of a proof net is always quadratic in the size of the sequent. It should not come as a surprise that finding a fully complete correctness criterion for  $\mathbb{N}$ -nets is much harder. One reason is the close connection to the NP vs. co-NP problem [CR79]. Moreover, there are correct  $\mathbb{N}$ -nets for which no corresponding sequent proof exists, for example (3) seen as  $\mathbb{N}$ -net, but which can be represented in other formalisms, for example the calculus of structures [GS01,BT01]). Because of its conflation of sequents and formulas into a single kind of syntactic expression, we can write the proof that is shown on the left of Figure 2, and whose translation into  $\mathbb{N}$ -nets is exactly (3)—To save space we did in the figure several steps in one. For example  $\text{id}^4$  stands for four applications of the identity rule.

To give some more examples, consider the sequent  $\vdash \bar{a}, a \wedge \bar{a}, a$ . This is equivalent to the formula  $(a \rightarrow a) \rightarrow (a \rightarrow a)$  modulo some applications of asso-



**Fig. 3. Left:** Church numerals as N-nets      **Right:** “Fake” Church numerals

ciativity and commutativity (here  $\rightarrow$  stands for implication). Hence, the proofs of that sequent can be used to encode the Church numerals. Figure 3 shows on the left the encodings of the numbers 0 to 4 as N-nets. Observe that using B-nets, we can distinguish only the numbers 0, 1, and 2, because all numbers  $\geq 2$  are collapsed. Note that there are also proofs of that sequent that do not encode a numeral. There are two examples on the right of Figure 3. The top one is obtained by simply mixing together the two proofs 0 and 2. One of the arguments for not having the mix rule in a system is that it causes types (resp. formulas) to be inhabited by more terms (resp. proofs) than the intended ones. However, we would like to stress the (well-known) fact, that this phenomenon is by no means caused by the mix rule, as the bottom “fake” numeral in Figure 3 shows, which comes from the mix-free sequent proof on the right of Figure 2.

#### 4 Proof nets with cuts

A *cut* is a formula  $A \diamond \bar{A}$ , where  $\diamond$  is called the *cut connective*, and where the function  $(-)$  is defined on formulas as follows (with a trivial abuse of notation):

$$\bar{\bar{a}} = a, \quad \bar{a} = \bar{a}, \quad \bar{\bar{\mathbf{t}}} = \mathbf{t}, \quad \bar{\mathbf{f}} = \mathbf{f}, \quad \overline{(A \wedge B)} = \bar{A} \vee \bar{B}, \quad \overline{(A \vee B)} = \bar{A} \wedge \bar{B}.$$

A *sequent with cuts* is a sequent where some of the formulas are cuts. But cuts are not allowed to occur inside formulas, i.e., all  $\diamond$ -nodes are roots. A *prenet with cuts* is a prenet  $P \triangleright \Gamma$ , where  $\Gamma$  may contain cuts. The  $\diamond$ -nodes have the same geometric behavior as the  $\wedge$ -nodes. Therefore the correctness criterion has to be adapted only slightly:

**4.1 Definition** A *conjunctive resolution* of a prenet  $P \triangleright \Gamma$  with cuts is a sub-prenet  $P|_{\Gamma'} \triangleright \Gamma'$  where  $\Gamma'$  has been obtained by deleting one child subformula for every  $\wedge$ -node and every  $\diamond$ -node of  $\Gamma$ .

**4.2 Definition** A *W-prenet*  $P \triangleright \Gamma$  with cuts is said to be *correct* if for every one of its conjunctive resolutions  $P|_{\Gamma'} \triangleright \Gamma'$  the *W-linking*  $P|_{\Gamma'}$  is not the zero function. A *W-net with cuts* is a correct *W-prenet* with cuts.

An example of a correct net with cuts (taken from [Gir91]):



In the translation from sequent proofs containing the cut rule into prenets with cuts, the cut is treated as follows:

$$\text{cut} \frac{\Gamma, A \quad \bar{A}, \Delta}{\Gamma, \Delta} \rightsquigarrow \text{cut} \frac{P \triangleright \Gamma, A \quad Q \triangleright \bar{A}, \Delta}{P \oplus Q \triangleright \Gamma, A \diamond \bar{A}, \Delta} .$$

In order to simplify the presentation and maintain the similarity between cut and conjunction, our sequent calculus allows contraction to be applied on cut formulas. This slightly unconventional rule is used only for obtaining a generic proof of sequentialization; in no way does it affect the other results (statements or proofs) in the rest of this paper.

The generalization of soundness and completeness is now immediate:

**4.3 Theorem (Soundness)** *For any  $W$  and any  $i \in \{0, 1, 2\}$ , a sequentializable  $W$ -prenet in  $\text{CL}_i$  with cuts is correct.*

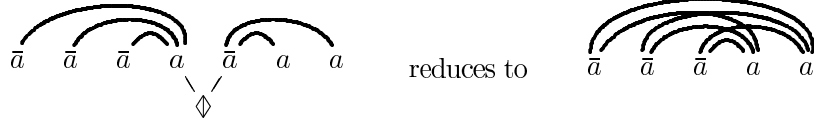
**4.4 Theorem (Sequentialization)** *For any  $i \in \{0, 1, 2\}$ , a  $\mathbb{B}$ -net in  $\text{CL}_i$  with cuts is sequentializable in  $\text{CL}_i + \text{cut}$ .*

## 5 Cut elimination

Cut elimination in CL-nets has much in common with multiplicative proof nets. The cut-reduction step on a compound formula is exactly the same:

$$P \triangleright (A \wedge B) \diamond (\bar{A} \vee \bar{B}), \Gamma \rightarrow P \triangleright A \diamond \bar{A}, B \diamond \bar{B}, \Gamma$$

and so it does not affect the linking itself (although we have to show it preserves correctness). The really interesting things happen in the atomic case, which this time splits in two: atomic formulas or constants. Here cut elimination means “counting paths through the cuts”. Let us illustrate the idea by an example:



More generally, if some weights are different from 1, we multiply them:

$$\begin{aligned} \{ \bar{a}_1 \overset{p}{a_4}, \bar{a}_2 \overset{q}{a_4}, \bar{a}_3 \overset{r}{a_4}, \bar{a}_5 \overset{m}{a_6}, \bar{a}_5 \overset{n}{a_7} \} \triangleright \bar{a}_1, \bar{a}_2, \bar{a}_3, a_4 \diamond \bar{a}_5, a_6, a_7 \\ \rightarrow \{ \bar{a}_1 \overset{pm}{a_6}, \bar{a}_1 \overset{q}{a_7}, \bar{a}_2 \overset{r}{a_6}, \bar{a}_2 \overset{qn}{a_7}, \bar{a}_3 \overset{rm}{a_6}, \bar{a}_3 \overset{rn}{a_7} \} \triangleright \bar{a}_1, \bar{a}_2, \bar{a}_3, a_6, a_7 . \end{aligned}$$

In the case of constants we have for example (here two cuts are reduced):

$$\{ \mathbf{t}_1 \overset{p}{t_1}, \mathbf{f}_2 \overset{q}{t_3}, \mathbf{f}_4 \overset{r}{t_5} \} \triangleright \mathbf{t}_1 \diamond \mathbf{f}_2, \mathbf{t}_3 \diamond \mathbf{f}_4, \mathbf{t}_5 \rightarrow \{ \mathbf{t}_5 \overset{pqr}{t_5} \} \triangleright \mathbf{t}_5 .$$

To understand certain subtleties let us consider

$$\{ \bar{a}_1 \overset{p}{a_4}, \bar{a}_1 \overset{q}{a_2}, \bar{a}_2 \overset{r}{a_3}, \bar{a}_3 \overset{s}{a_4} \} \triangleright \bar{a}_1, a_2 \diamond \bar{a}_3, a_4 \rightarrow \{ \bar{a}_1 \overset{z}{a_4} \} \triangleright \bar{a}_1, a_4 .$$

What is the value of  $z$ ? We certainly cannot just take  $q \cdot s$ , we also have to add  $p$ . But the question is what happens to  $r$ , i.e., is the result  $z = p + qs$  or

$z = p + q \cdot (1 + r) \cdot s = p + qs + qrs$ ? Both choices lead to a sensible theory of cut elimination, but here we only treat the first, simpler case: we drop  $r$ .

Let us now introduce some notation. Given  $P \triangleright \Gamma$  and  $x, y, u_1, u_2, \dots, u_n \in \mathcal{L}(\Gamma)$ , with  $n$  even, we write  $P(\widehat{x u_1 \cdot u_2 \cdot u_3 \cdot \dots \cdot u_n} y)$  as an abbreviation for  $P(x, u_1) \cdot P(u_2, u_3) \cdot \dots \cdot P(u_n, y)$ . In addition we define

$$P(x \uparrow u_1 \cdot u_2 \cdot u_3 \cdot \dots \cdot u_n \uparrow y) = \begin{cases} P(\widehat{x u_1 \cdot u_2 \cdot u_3 \cdot \dots \cdot u_n} y) + P(\widehat{x u_n \cdot \dots \cdot u_3 \cdot u_2 \cdot u_1} y) & \text{if } y \neq x \\ P(\widehat{x u_1 \cdot u_2 \cdot u_3 \cdot \dots \cdot u_n} u_n) + P(\widehat{x u_n \cdot \dots \cdot u_3 \cdot u_2 \cdot u_1} u_1) & \text{if } y = x \end{cases}$$

Notice that in both cases one of the summands is always zero, and that the second case applies only when  $x$  is a  $\mathbf{t}$  and a  $\mathbf{t}_1$ -axiom is involved. We now can define cut reduction formally for a single atomic cut:

$$P \triangleright u \diamond v, \Gamma \rightarrow Q \triangleright \Gamma, \text{ where } Q(x, y) = P(x, y) + P(x \uparrow u \cdot v \uparrow y)$$

for all  $x, y \in \mathcal{L}(\Gamma)$ , and where  $u$  is labelled by an arbitrary atom or constant, and  $v$  by its dual. But we can go further and do *simultaneous reduction* on a set of atomic cuts:

$$P \triangleright u_1 \diamond v_1, u_2 \diamond v_2, \dots, u_n \diamond v_n, \Gamma \rightarrow Q \triangleright \Gamma, \quad (7)$$

where each  $u_i$  labelled by an arbitrary atom or constant, and  $v_i$  by its dual. For defining  $Q$ , we need the following notion:

A *cut-path between  $x$  and  $y$*  in a net  $P \triangleright \Delta$  with  $x, y \in \mathcal{L}(\Delta)$  is an expression of the form  $\widehat{x w_1 \cdot z_1 w_2 \cdot z_2 w_3 \cdot \dots \cdot z_k} y$  where  $w_i \diamond z_i$  are all *distinct* atomic cuts in  $\Delta$ , and such that  $P(x \uparrow w_1 \cdot z_1 \uparrow w_2 \cdot z_2 \uparrow w_3 \cdot \dots \cdot z_k \uparrow y) \neq 0$ . For a set  $S$  of atomic cuts in  $\Delta$ , the cut-path is *covered by  $S$*  if all the  $w_i \diamond z_i$  are in  $S$ , and it *touches  $S$*  if at least one of the  $w_i \diamond z_i$  is in  $S$ . The  $Q$  in (7) is now given by

$$Q(x, y) = P(x, y) + \sum_{\{\widehat{x w_1 \cdot z_1 w_2 \cdot \dots \cdot z_k} y\}} P(x \uparrow w_1 \cdot z_1 \uparrow w_2 \cdot \dots \cdot z_k \uparrow y),$$

where the sum ranges over all cut-paths covered by  $\{u_1 \diamond v_1, u_2 \diamond v_2, \dots, u_n \diamond v_n\}$ .

**5.1 Lemma** *Let  $P \triangleright \Delta$  be a  $W$ -prenet in  $\mathbf{CL}_i$ , and let  $P \triangleright \Delta \rightarrow P' \triangleright \Delta'$ . Then  $P' \triangleright \Delta'$  is also in  $\mathbf{CL}_i$ . Furthermore, if  $P \triangleright \Delta$  is correct, then  $P' \triangleright \Delta'$  is also correct.*

The proof is an ordinary case analysis, and it is the only place where Axiom (2) is used. The next observation is that there is no infinite sequence  $P \triangleright \Gamma \rightarrow P' \triangleright \Gamma' \rightarrow P'' \triangleright \Gamma'' \rightarrow \dots$ , because in each reduction step the size of the sequent (i.e., the number of  $\wedge, \vee$  and  $\diamond$ -nodes) is reduced. Therefore:

**5.2 Lemma** *The cut reduction relation  $\rightarrow$  is terminating.*

Let us now attack the issue of confluence. Obviously we only have to consider atomic cuts; let us begin when two singleton cuts:  $P \triangleright a_i \diamond \bar{a}_j, a_h \diamond \bar{a}_k, \Gamma$  are reduced. If the first cut is reduced, we get  $P' \triangleright a_h \diamond \bar{a}_k, \Gamma$ , where  $P'(x, y) = P(x, y) + P(x \uparrow a_i \cdot \bar{a}_j \uparrow y)$ . Then reducing the second cut gives us  $Q_1 \triangleright \Gamma$ , where  $Q_1(x, y) = P'(x, y) + P'(x \uparrow a_h \cdot \bar{a}_k \uparrow y)$ . An easy computation shows that

$$Q_1(x, y) = P(x, y) + P(x \uparrow a_i \cdot \bar{a}_j \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_k \uparrow y) + P(x \uparrow a_i \cdot \bar{a}_j \cdot a_k \cdot \bar{a}_h \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_k \cdot a_i \cdot \bar{a}_j \uparrow y) \quad (8)$$

Reducing the two cuts in the other order yields  $Q_2 \triangleright \Gamma$ , where

$$Q_2(x, y) = P(x, y) + P(x \uparrow a_i \cdot \bar{a}_j \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_h \uparrow y) + P(x \uparrow a_i \cdot \bar{a}_j \widehat{a_k \cdot \bar{a}_h} \uparrow y) + \\ P(x \uparrow a_h \cdot \bar{a}_k \widehat{a_i \cdot \bar{a}_j} \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_k \widehat{a_i \cdot \bar{a}_j} a_h \cdot \bar{a}_k \uparrow y) \quad . \quad (9)$$

We see that the last summand is different in the two results. But if we reduce both cuts simultaneously, we get  $Q \triangleright \Gamma$ , where

$$Q(x, y) = P(x, y) + P(x \uparrow a_i \cdot \bar{a}_j \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_h \uparrow y) + \\ P(x \uparrow a_i \cdot \bar{a}_j \widehat{a_k \cdot \bar{a}_h} \uparrow y) + P(x \uparrow a_h \cdot \bar{a}_k \widehat{a_i \cdot \bar{a}_j} \uparrow y) \quad . \quad (10)$$

Now the troublesome summand is absent. There are also some good news: In the case of  $\mathbb{B}$ -nets we have that  $Q = Q_1 = Q_2$ . The reason is that if in  $Q_1$  or  $Q_2$  the last summand is 1, then at least one of the other summands is also 1, which means that the whole sum is 1. Therefore we have:

**5.3 Lemma** *On  $\mathbb{B}$ -prenets the cut reduction relation  $\rightarrow$  is locally confluent.*

Lemmas 5.1–5.3 together give us immediately:

**5.4 Theorem** *On  $\mathbb{B}$ -nets cut elimination via  $\rightarrow$  is strongly normalizing. The normal forms are cut free  $\mathbb{B}$ -nets.*

If we want things to work for general  $W$  we have to introduce a restriction, which is always met if our proof nets are used as in denotational semantics.

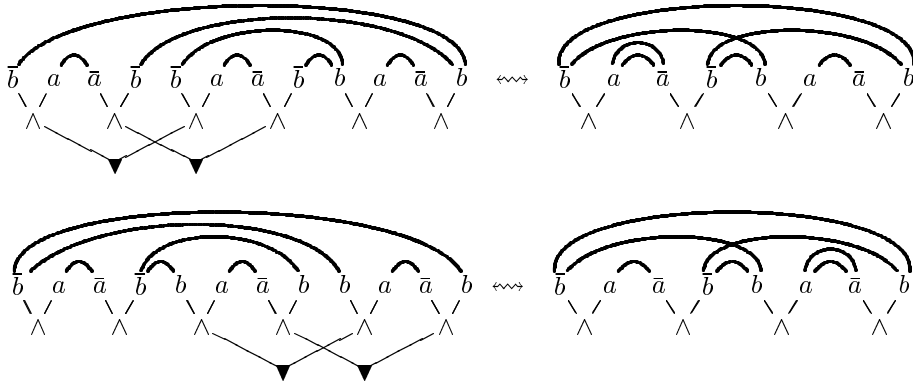
Given an arbitrary proof net  $P \triangleright \Gamma$ , a *virtual cut* is a pair of atoms/constants  $u, v$  in  $\mathcal{L}(\Gamma)$  such that there would be an atomic cut  $u \diamond v$  if all the *compound formula* cuts in  $\Gamma$  were reduced; an ordinary atomic cut is obviously virtual. A *virtual cut-path* is an ordinary cut-path were virtual atomic cuts are allowed. A set  $S$  of (virtual or ordinary) atomic cuts is said to be *safe* if for any two atoms  $x, y$  not in  $S$  either every virtual cut-path between  $x$  and  $y$  touches  $S$ , or there is no virtual path between  $x$  and  $y$  that touches  $S$ .

**5.5 Theorem** *For any  $W$ -net  $P \triangleright \Gamma$ , given two disjoint safe sets of ordinary atomic cuts  $S$  and  $T$  in  $\Gamma$ , the result of applying two simultaneous reductions on  $S$  and  $T$  is independent of the order of application, and is the same as as single simultaneous reduction on  $S \cup T$ .*

Now let  $P \triangleright \Gamma$  be a  $W$ -net that is obtained from a sequent calculus proof that ends by a sequence of cuts (via the translation given in Sections 3 and 4). Then every one of these cuts defines a set of virtual cuts by eliminating only the connectors on that cut and taking the atomic cuts that are left. It is easy to see that every one of these sets is safe. Therefore, we have the following:

**5.6 Corollary** *Let  $P \triangleright \Gamma$  be a  $W$ -net that is obtained from a sequent calculus proof in which all cuts appear at the bottom of the derivation. The final result of applying cut elimination to  $P \triangleright \Gamma$  is independent of the order in which the cuts are eliminated, provided that simultaneous reduction is applied on every one of them.*

Let us compare our cut elimination with other (syntactic) cut elimination procedures. In the case of  $\mathbb{B}$ -nets, the situation is quite similar to the sequent



**Fig. 4.** The two different results of applying *sequent calculus cut elimination* to the proof (6). **Left:** Written as Girard/Robinson proof-net. **Right:** Written as N-proof-net.

calculus: the main difference is that we do not lose any information in the weak-weak case, although we lose some information of a numeric nature in the cont-cont case.

In the case of N-nets, the situation is very different. Let us use (6) for an example. The sequent calculus cut elimination needs to duplicate either the right-hand side proof or the left-hand side proof. The two possible outcomes, together with their presentation as N-nets, are shown in Figure 4, where the  $\blacktriangledown$  stand for contractions<sup>3</sup>. However, in our setting, the result of eliminating the cut in (6) is always (3), no matter whether we are in B-nets or in N-nets.

Although our N-net cut elimination has no close relationship to the sequent calculus, it corresponds well to the cut elimination in the calculus of structures, when done via splitting [Gug02].

## 6 A bit of abstract nonsense

For  $i \in \{0, 2\}$ , let  $\mathbf{CL}_i^W$  denote the category whose objects are the  $\mathbf{CL}_i$ -formulas and whose arrows are the cut-free  $W$ -nets (in  $\mathbf{CL}_i$ ) that have two (ordered) conclusions:  $P \triangleright \bar{A}, B$  is a map from  $A$  to  $B$ . The composition of two maps  $P \triangleright \bar{A}, B$  and  $Q \triangleright \bar{B}, C$  is given by eliminating the cut on  $P \oplus Q \triangleright \bar{A}, B \diamond \bar{B}, C$ . The associativity of that operation is a direct consequence of Corollary 5.6. In  $\mathbf{CL}_0^W$  and  $\mathbf{CL}_2^W$  we have an obvious identity map for every object.  $\mathbf{CL}_1^W$  is not a category because it does not have identities: this is easy to check for  $\mathbf{t}$ .

In what follows we will sketch the axiomatic structure that our categorical models possess, allowing comparison with the one proposed in [FP04a, FP04b]; a full treatment deserves another paper. It is immediate that  $-\wedge-$  is a bifunctor, equipped with the standard associativity and symmetry isomorphisms. We also

<sup>3</sup> This idea of using explicit contraction nodes was sketched in [Gir91] and is carried out in detail in [Rob03].

have a contravariant involution  $\overline{(-)}$ , giving us the usual de Morgan dual  $-\vee-$ , and the standard natural isomorphism  $\text{Hom}(A \wedge B, C) \cong \text{Hom}(A, \overline{B} \vee C)$ . Thus in general we get a “\*-autonomous category, but not necessarily with units”.

In the case of  $\text{CL}_2$ , the  $\wedge$  does have a unit  $\mathbf{t}$  of some sort, but the standard unit laws are not exactly true: there is a natural map  $\lambda_A: \mathbf{t} \wedge A \rightarrow A$ , (and thus, because of the symmetry, a  $\rho_A: A \wedge \mathbf{t} \rightarrow A$ ) and they obey the “standard diagram” [ML71, p. 159], *but they are not isomorphisms*. Instead  $\lambda$  only has a right inverse  $\lambda_A^*: A \rightarrow \mathbf{t} \wedge A$ , with  $\lambda_A \circ \lambda_A^* = 1_A$ . It is not hard to generalize the notion of symmetric monoidal category to accomodate that situation: just plug  $\lambda_A^*$  in the “standard diagram” and reverse the arrows. A suitable coherence theorem can then be formulated and proved.

But naturally our categories have more: first there is a natural diagonal map  $\Delta_A: A \rightarrow A \wedge A$ , which obeys the expected co-associativity and co-commutativity laws; in addition the following holds and should be considered an axiom:  $(1_A \wedge \text{twist} \wedge 1_A) \circ \Delta_{A \wedge A} = \Delta_A \wedge \Delta_A$ . By duality there is an associative, commutative codiagonal  $\nabla_A: A \vee A \rightarrow A$ . Also, every object  $A$  in  $\text{CL}_2^W$  is equipped with a map  $!_A: A \rightarrow \mathbf{t}$ , with the dual  $\mathbf{i}_A: \mathbf{f} \rightarrow A$ . Thus the composite  $\lambda_X \circ (!_A \wedge 1_X)$  defines a weakening map  $\pi_X^A: A \wedge X \rightarrow X$ , natural in  $X$ . Moreover in our models the weakening verifies the counit law:  $\pi_A^A \circ \Delta_A = 1_A$ . Thus every object in  $\text{CL}_2^W$  has a comonoid structure; in the case of  $\text{CL}_0^W$  the weakening map can be used as the primitive and the usual notion of comonoid can be adapted.

Very standard results in category theory hint strongly that all that structure brings us dangerously close to a collapse to a Boolean algebra. The reader might think that our having a category and not a poset is due to a weaker notion than usual for the unit in the monoidal structure. The answer is no: *if  $\mathcal{C}$  has all the structure described above, and if the family  $(!_A)_A$  is natural, then  $\mathcal{C}$  is a poset*.

Thus the real reason that  $\text{CL}_2^W$  does not collapse is that *the family  $(!_A)_A$  is not natural*. In other words, every object of  $\text{CL}_2^W$  is equipped with a comonoid structure, and every map preserves comultiplication, *but not necessarily the counit*. A suitable version of these statements also holds for  $\text{CL}_0^W$ .

The map  $\mathbf{f} \xrightarrow{\mathbf{i}_t} \mathbf{t} \xrightarrow{!_t} \mathbf{t}$  obeys the required conditions [CS97] to permit the construction of a natural map  $\text{mix}_{A,B}: A \wedge B \rightarrow A \vee B$ . Thus given any pair of parallel maps  $f, g: A \rightarrow B$  we can construct  $f+g = \nabla_B \circ (f \vee g) \circ \text{mix}_{A,A} \circ \Delta_A$ . It is then easy to check that this operation is commutative and associative, defining a natural enrichment of  $\text{CL}_i^W$  over the category of commutative semigroups<sup>4</sup>. Thus the sum of linkings (see Section 2) can be recovered through the categorical axioms. When  $W = \mathbb{B}$  the additional axiom  $f + f = f$  of idempotency holds; thus in this case the enrichment is over sup-semilattices; this is to be compared with [FP04a,FP04b], where the enrichment is over posets, but is extra structure, not deducible from the 1-categorical structure. In that model the preservation of *both* weakening and diagonal by arbitrary maps is not on the nose, but lax, and this is what prevents poset collapse.

<sup>4</sup> This observation has also been made independently by C. Führmann, M. Hasegawa, and D. Pym (private communication).

## References

- [And76] Peter B. Andrews. Refutations by matings. *IEEE Transactions on Computers*, C-25:801–807, 1976.
- [BT01] Kai Brännler and Alwen Fernanto Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.
- [Bus91] Samuel R. Buss. The undecidability of  $k$ -provability. *Annals of Pure and Applied Logic*, 53:72–102, 1991.
- [Car97] Alessandra Carbone. Interpolants, cut elimination and flow graphs for the propositional calculus. *Annals of Pure and Applied Logic*, 83:249–299, 1997.
- [CR79] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *The Journal of Symbolic Logic*, 44(1):36–50, 1979.
- [CS97] J.R.B. Cockett and R.A.G. Seely. Proof theory for full intuitionistic linear logic, bilinear logic, and mix categories. *Theory and Applications of Categories*, 3(5):85–131, 1997.
- [DJS97] V. Danos, J.-B. Joinet, and H. Schellinx. A new deconstructive logic: Linear logic. *The Journal of Symbolic Logic*, 62(3):755–807, 1997.
- [FP04a] Carsten Führmann and David Pym. On the geometry of interaction for classical logic (extended abstract). In *19th IEEE Symposium on Logic in Computer Science (LICS 2004)*, pages 211–220, 2004.
- [FP04b] Carsten Führmann and David Pym. Order-enriched categorical models of the classical sequent calculus. 2004.
- [Gir91] Jean-Yves Girard. A new constructive logic: Classical logic. *Mathematical Structures in Computer Science*, 1:255–296, 1991.
- [GLT89] Jean-Yves Girard, Yves Lafont, and Paul Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989.
- [GS01] Alessio Guglielmi and Lutz Straßburger. Non-commutativity and MELL in the calculus of structures. In Laurent Fribourg, editor, *Computer Science Logic, CSL 2001*, volume 2142 of *LNCS*, pages 54–68. Springer-Verlag, 2001.
- [Gug02] Alessio Guglielmi. A system of interaction and structure, 2002. To appear in *ACM Transactions on Computational Logic*. On the web at: <http://www.ki.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf>.
- [Hyl04] J. Martin E. Hyland. Abstract interpretation of proofs: Classical propositional calculus. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *LNCS*, pages 6–21. Springer-Verlag, 2004.
- [Lau02] Olivier Laurent. *Etude de la Polarisation en Logique*. PhD thesis, Univ. Aix-Marseille II, 2002.
- [Lau03] Olivier Laurent. Polarized proof-nets and  $\lambda\mu$ -calculus. *Theoretical Computer Science*, 290(1):161–188, 2003.
- [ML71] Saunders Mac Lane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.
- [Par92] Michel Parigot.  $\lambda\mu$ -calculus: An algorithmic interpretation of classical natural deduction. In *Logic Programming and Automated Reasoning, LPAR 1992*, volume 624 of *LNAI*, pages 190–201. Springer-Verlag, 1992.
- [Rob03] Edmund P. Robinson. Proof nets for classical logic. *Journal of Logic and Computation*, 13:777–797, 2003.