Near-Optimal Parameterization of the Intersection of Quadrics: Theory and Implementation

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Abstract

We present an algorithm that computes an exact parametric form of the intersection of two real quadrics in projective three-space given by implicit equations with rational coefficients. This algorithm represents the first complete and robust solution to what is perhaps the most basic problem of solid modeling by implicit curved surfaces.

The parameterizations computed involve only polynomial functions when the curve of intersection has genus 0 and the coefficients of these functions are algebraic numbers with at most one possibly unnecessary square root. Furthermore, for each geometric type of intersection, the number of square roots in the coefficients is always minimal in the worst case.

These results are formalized in the following theorem.

Theorem. *In $\mathbb{P}^3(\mathbb{R})$, given two quadrics by their implicit equations whose coefficients are rational, the algorithm of this paper computes a parameterization of their intersection such that each coordinate in projective space is polynomial over an extension field $\mathbb{K}$ of $\mathbb{Q}$ if such a parameterization exists, or is a polynomial in $\mathbb{K}[\xi, \sqrt{\Delta}]$ otherwise, where $\xi \in \mathbb{P}^1(\mathbb{R})$ is the parameter and $\Delta \in \mathbb{K}[\xi]$ is a polynomial in ξ.*

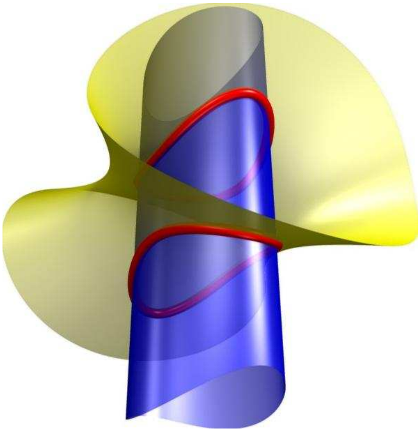
In both cases, the parameterization is either optimal in the degree of the field extension $\mathbb{K}$ needed to represent its coefficients or may involve one (and only one) possibly unnecessary square root. In the latter situation, testing for optimality (i.e., determining whether the extra square root is necessary or not) and finding an optimal parameterization are equivalent to finding rational points on a conic or a degree-eight surface (a long-standing open problem in algebraic geometry).

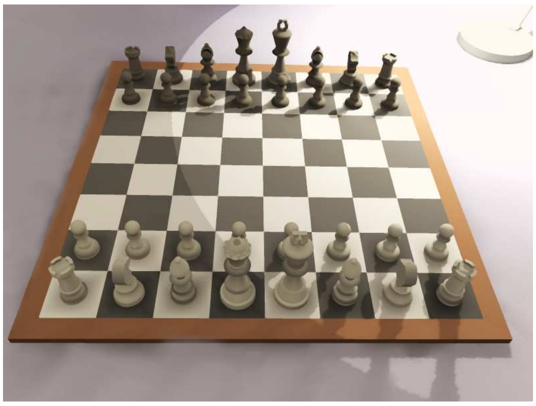
Our work builds upon a large body of literature on intersections of quadrics, dating back to Levin's 1976 seminal paper [4]. Among the main contributions of our approach and the main novelties of this paper are: new parameterizations of ruled projective quadrics, linear in one of their parameters, that are optimal in the worst case in the number of radicals they involve; a complete classification of the type of intersection of two projective quadrics in terms of the number and multiplicity of the multiple roots of some degree-four polynomial and of the inertia of the associated matrices; a proof that any pencil generated by two distinct quadrics with rational coefficients contains at least a ruled quadric with rational coefficients except in one very simple case for which the intersection is reduced to two points.

Our near-optimal parameterization algorithm has been implemented in C++. It is the first exact, robust, efficient and usable implementation of an algorithm for parameterizing the intersection of two arbitrary projective quadrics with integer coefficients (or, equivalently, rational coefficients).

Our implementation builds upon the LiDIA computational number theory library [5] and uses GMP [2] multiprecision integer arithmetic. It consists of more than 17,000 lines of source code and has the following features:

[1]Survey of results that appeared in [1] and [3].





a. b.

Figure 1: Examples of intersection of quadrics. a. Smooth quartic. b. A chess set entirely modelled with quadrics (model courtesy of SGDL Systems, Inc. [6]).

- it computes an exact parameterization of the intersection of two quadrics with integer coefficients of arbitrary size;
- it correctly identifies, separates and parameterizes all the connected or algebraic components of the intersection and gives all the relevant topological information;
- it is fast and efficient and can routinely compute parameterizations of the intersection of quadrics with input coefficients having ten digits in less than 50 milliseconds on a mainstream PC.

In addition to being optimal from the point of view of the functions involved and near-optimal from the point of view of the extension on which the coefficients of the parameterizations are defined, we show how our implementation was carefully designed to keep the size of the integer coefficients small. These coefficients are polynomials in the coefficients of the input quadrics. The degree of these polynomials (often called the *height* of the output) is a direct measure of the required arithmetic bit length and thus of the computational efficiency. We show that the parameterizations we compute involve coefficients that have degree at most 38 asymptotically in the coefficients of the input quadrics; this bound is reached when the intersection is a smooth quartic, an example of which is displayed in Figure 1.a. When the intersection is singular, we prove that using whatever structural and geometric information is available to drive the parameterization process can drastically reduce the asymptotic degree of the output coefficients.

We also give various experimental results illustrating the efficiency of the implemented algorithms. For instance, we show that in case the intersection is a smooth quartic (the generic and computationally most demanding case) our implementation computes a parameterization of the intersection in at most 50 milliseconds when the input quadrics have coefficients with 10 digits - see Figure 2. We also show that degenerate intersections, for which a lot of additional geometric information can be obtained *a priori*, are parameterized a lot faster on average. On the chess set example (Figure 1.b), roughly 1,000 intersections of quadrics with coefficients with between 2 and 7 (a few with 15) digits are computed in 3.4 milliseconds on average. The height of the coefficients in the parameterizations never exceeds 8, far below the asymptotic bound, illustrating how well our algorithm behaves on real data.

We conclude by giving various examples of parameterizations computed by our algorithm, illustrating both its efficiency and its completeness. Consider for instance the example depicted in Figure 1.a, where the intersection is given in implicit form by the equations:

$$\begin{cases} 4\,x^2 + z^2 - w^2 = 0, \\ x^2 + 4\,y^2 - z^2 - w^2 = 0. \end{cases}$$

Our implementation outputs the following exact and simple parameterization of the intersec-

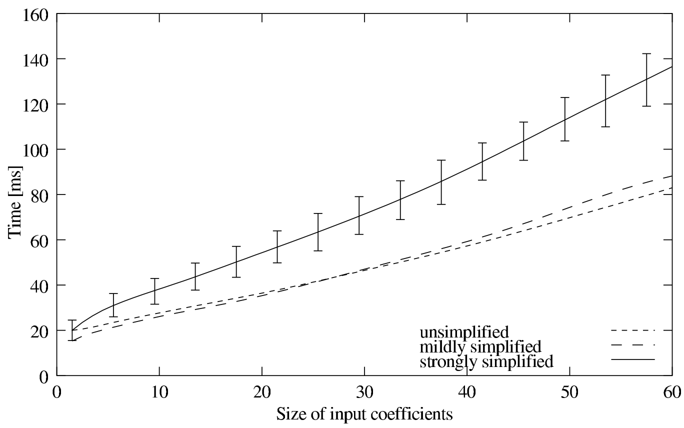


Figure 2: Evolution of computation time in the smooth quartic case.

tion in less than 10 ms:

$$\mathbf{X}(u,v) = \begin{pmatrix} 2\,u^3 - 6\,uv^2 \\ 7\,u^2v + 3\,v^3 \\ 10\,u^2v - 6\,v^3 \\ 2\,u^3 + 18\,uv^2 \end{pmatrix} \pm \begin{pmatrix} -2\,v \\ u \\ 2\,u \\ 2\,v \end{pmatrix} \sqrt{-3\,u^4 + 26\,u^2v^2 - 3\,v^4}, \quad (u,v) \in \mathbb{P}^1(\mathbb{R}).$$

This result is optimal both in terms of the functions involved (the square root is necessary since the intersection is a smooth quartic, with genus 1) and the field extension on which it is defined. The coefficients are also remarkably simple.

Other, large-size examples can be tested by querying our online parameterization server at the following URL: `http://www.loria.fr/isa/qi`. The source code can be downloaded at the same address.

References

[1] L. Dupont, D. Lazard, S. Lazard, and S. Petitjean. Near-optimal parameterization of the intersection of quadrics. In *Proc. of SoCG (ACM Symposium on Computational Geometry), San Diego*, pages 246–255, 2003.

[2] GMP: The GNU MP Bignum Library. The Free Software Foundation. `http://www.swox.com/gmp`.

[3] S. Lazard, L. M. Peñaranda, and S. Petitjean. Intersecting quadrics: An efficient and exact implementation. In *Proc. of SoCG (ACM Symposium on Computational Geometry), Brooklyn, NY*, 2004.

[4] J. Levin. A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces. *Communications of the ACM*, 19(10):555–563, 1976.

[5] LiDIA: A C++ Library for Computational Number Theory. Darmstadt University of Technology. `http://www.informatik.tu-darmstadt.de/TI/LiDIA`.

[6] The SGDL Platform: A Software for Interactive Modeling, Simulation and Visualization of Complex 3D Scenes. SGDL Systems, Inc. `http://www.sgdl-sys.com`.

Keywords: Intersection of surfaces, quadrics, curve parameterization, experimental analysis