Syzygies of points in a projective space

Hans-Christian Von Bothmer, Laurent Busé, Baohua Fu

SYZYGIES OF POINTS IN $\mathbb{P}^N$

HANS-CHRISTIAN VON BOTHMER, LAURENT BUSÉ AND BAOHUA FU

Contents

1. Hilbert Functions 1
2. Minimal Free Resolutions and Graded Betti Numbers 4
3. The Minimal Resolution Conjecture for points in $\mathbb{P}^2$ 8
4. Geometric Syzygies of Points 15
5. Trivial syzygies and residual points in $\mathbb{P}^2$ 20
References 28

1. Hilbert Functions

Let k be an algebraically closed field and $S = k[x_0, \cdots, x_N]$ the homogeneous coordinate ring of the projective space $\mathbb{P}^N$. Consider a closed subvariety X of $\mathbb{P}^N$. Let S_X be the homogeneous coordinate ring of X, which is a finitely generated S-module with a graduation $S_X = \oplus_d (S_X)_d$.

Definition 1.1. The Hilbert function $H_X : \mathbb{N} \to \mathbb{N}$ of X is defined by $H_X(d) = dim_k(S_X)_d$.

Theorem 1.2. *Let n be the dimension of X. Then there exists a polynomial (called Hilbert polynomial) $P_X(t) = \frac{d}{n!}t^n + \cdots$ such that for any $m \in \mathbb{N}$ sufficient big, we have $P_X(m) = H_X(m)$.*

The constant d in the above expression is called the degree of X. Now we will give a cohomological formula for the Hilbert function. Let $\mathcal{I}_X$ be the ideal sheaf defining X.

Proposition 1.3. *For any $d \in \mathbb{Z}$,*

$$H_X(d) = h^0(\mathcal{O}_{\mathbb{P}^N}(d)) - h^0(\mathcal{I}_X(d)) = \binom{N+d}{d} - h^0(\mathcal{I}_X(d)).$$

Received by the editors November 26, 2002.

Proof. ¿From the exact sequence

$$0 \to \mathcal{I}_X(d) \to \mathcal{O}_{\mathbb{P}^N}(d) \to \mathcal{O}_X(d) \to 0,$$

we get the exact sequence $0 \to H^0(\mathcal{I}_X(d)) \to H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \xrightarrow{\phi} H^0(\mathcal{O}_X(d))$. By definition, $H_X(d)$ is nothing but the dimension of the image of ϕ, which is $h^0(\mathcal{O}_{\mathbb{P}^N}(d)) - h^0(\mathcal{I}_X(d))$. □

Note that $h^0(\mathcal{I}_X(d))$ is nothing but the number of hypersurfaces of degree d containing X, which is easy to compute in the case of X being union of points. There is also another way to define the Hilbert polynomial.

Lemma 1.4. *For $d >> 1$, we have $H_X(d) = h^0(\mathcal{O}_X(d))$, thus the Hilbert polynomial is also the asymptotic form of the function $d \mapsto h^0(\mathcal{O}_X(d))$.*

Proof. Since $\mathcal{I}_X$ is coherent, we have $H^1(\mathcal{I}_X(d)) = 0$ for $d >> 1$, which, combining with the above exact sequence, gives that

$$h^0(\mathcal{O}_X(d)) = h^0(\mathcal{O}_{\mathbb{P}^N}(d)) - h^0(\mathcal{I}_X(d)) = H_X(d).$$

□

It should be pointed out that usually the equality $H_X(d) = h^0(\mathcal{O}_X(d))$ does not hold for every d. For example let X be a set of distinct points in $\mathbb{P}^N$, then $\mathcal{O}_X(d)$ is isomorphic to $\mathcal{O}_X$, which gives that $h^0(\mathcal{O}_X(d)) = h^0(\mathcal{O}_X) = deg(X)$ which is a constant function, while in general the Hilbert function H_X is not constant. However in some cases, the two functions are effectively the same, as shown by

Proposition 1.5. *Let X be a closed subvariety of $\dim \geq 1$ in $\mathbb{P}^N$. Suppose that X is a complete intersection, then $H_X(d) = h^0(\mathcal{O}_X(d))$ for all $d \in \mathbb{Z}$.*

Proof. If X is a hypersurface $f = 0$ of degree e, then $\mathcal{I}_X$ is $\mathcal{O}_{\mathbb{P}^N}(-e)$. Thus $H^1(\mathcal{I}_X(d)) = H^1(\mathcal{O}_{\mathbb{P}^N}(d-e))$ which is zero, since $dim(X) \geq 1$ implies $N \geq 2$. Using the exact sequence studied in proposition 1.2, we get immediately that $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\mathcal{O}_X(d))$ is surjective, i.e. $H_X(d) = h^0(\mathcal{O}_X(d))$ for all $d \in \mathbb{Z}$.

For the general case, we have $H^0(\mathcal{O}_X(d)) = (S_X)_d$, as shown by Serre in F.A.C. Now $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) = S_d$ and the map $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\mathcal{O}_X(d))$ is surjective. □

Remark 1.6. In the above situation, we have a free resolution of $\mathcal{O}_X$ by the Koszul complex $\mathfrak{I}_X \leftarrow K_\bullet$, which shows that the cohomology of $\mathfrak{I}_X(d)$ is nothing but the hypercohomology $\mathbb{H}(K_\bullet(d))$ of the complex $K_\bullet(d)$. Using some spectral sequence, we can show $H^1(\mathfrak{I}_X(d)) = 0$ for all $d \geq 1$. See "Complete intersections in projective spaces" by Bas Edixhoven.

Recall that a projective variety $X \subset \mathbb{P}^N$ is called *projectively normal* if the affine cone $V(X) \subset \mathbb{C}^{N+1}$ is normal. As shown in exercise II.5.4 [Har], a projective variety $X \subset \mathbb{P}^N$ is projectively normal if and only if X is normal and $H_X(d) = h^0(\mathcal{O}_X(d))$ for all $d \in \mathbb{Z}$. Now the following corollary is clear.

Corollary 1.7. *Let X be a complete intersection in $\mathbb{P}^N$ of dimension ≥ 1. Then X is projectively normal if and only if X is normal.*

¿From now on, we will suppose that X is a union of distinct points. We introduce the following notion, which measures how complicated the module S_X is.

Definition 1.8. The *Castelnuovo-Mumford regularity $Reg(S_X)$* (or $Reg(X)$) of X is defined to be the least number d such that $P_X(m) = H_X(m)$ for all $m \geq d$.

It should be pointed out that our definition here coincides with the usual one only in the case of X being union of points. By the interpolation formula, one deduces easily the following estimation:

Lemma 1.9. *If X consists of n points, then $Reg(X) \leq n-1$ and we have the equality if all points are on a line.*

Lemma 1.10. *The Hilbert function H_X is never decreasing.*

Proof. We can suppose that X lies in $\mathbb{A}^N \subset \mathbb{P}^N$. Note that $H_X(d)$ is the rank of the map $\phi_d : S_d \longrightarrow k^X$, which is also the dimension of the space of functions that are the restrictions of polynomials of degree $\leq d$ to X. $\square$

Corollary 1.11. *$Reg(X)$ is also the least number d such that $H_X(d) = deg(X)$.*

We will conclude this section by some examples.

Example 1.12. Four points in $\mathbb{P}^2$. There are three cases:

• *the four points lie on a line L.* There is only one line passing the four points, then $h^0(\mathcal{I}_X(1)) = 1$ and $H_X(1) = 2$. Any conic containing the four points is a union of the line L and another line L'. Note that the dimension of L' is $h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$, so $H_X(2) = 3$. When $d \geq 3$, the above lemma gives that $H_X(d) = 4$. From the results, we see that $Reg(X) = 3$.

• *three of the four points lie on a line.* Similar argument gives $H_X(1) = 3$ and $H_X(d) = 4$ for $d \geq 2$. We see $Reg(X) = 2$.

• *no three points on a line.* The same as the second case.

Note that the Hilbert function could not distinct the configurations of the second case and the third case. As we will see later, the minimal free resolution of S_X or the graduated Betti numbers can do this.

Exercise 1.13. Find out all possible Hilbert functions for three points in $\mathbb{P}^2$.

Exercise 1.14. Find out all possible Hilbert functions for four points in $\mathbb{P}^3$.

2. Minimal Free Resolutions and Graded Betti Numbers

Let $S = k[x_0, \cdots, x_N]$ be the polynomial ring of $N+1$ variables. We denote by $S(-d)$ the free S-module of rank 1 generated by an element of degree d, i.e. $S(-d)_k = S_{k-d}$. A free resolution for an S-module M is an exact complex

$$0 \leftarrow M \stackrel{\phi_0}{\longleftarrow} F_0 \stackrel{\phi_1}{\longleftarrow} F_1 \stackrel{\phi_2}{\longleftarrow} F_2 \leftarrow \cdots$$

with F_i being free S-modules. A free resolution $M \leftarrow F_\bullet$ is called *minimal* if for any i, $\phi_i(F_i) \subset \mathfrak{m}F_{i-1}$, where $\mathfrak{m}$ is the ideal $(x_0, \cdots, x_N) \subset S$.

Theorem 2.15. *Every finitely generated S-module has a minimal free resolution of length $\leq N+1$. The minimal free resolution is unique up to isomorphism.*

Proof. It is easy to construct a minimal free resolution. It is proved by D. Hilbert that the length of the minimal free resolution is less than $N+1$.

□

Proposition 2.16. *Let X be a set of points in $\mathbb{P}^N$, then proj dim$(X) = N$.*

Proof. By definition, the projective dimension of M is the minimal length of a projective resolution of M. In the case of a finitely generated S-module, the projective dimension is equal to the length of the minimal free resolution. Using the Auslander-Buchsbaum theorem, we have *proj dim*$(X) =$ *dim*$(S) -$ *depth*$(X) = N+1-$ *depth*(X) which is equal to N since *depth*$(X) = 1$ in the case of X a set of points. □

¿From now on, we will suppose that the S-module M is finitely generated. As the minimal free resolution is unique up to isomorphism, the following definition makes sense.

Definition 2.17. The *graded Betti numbers* of M are defined to be the numbers $\beta_{i,j}$ such that $F_i = \oplus_j S(-i-j)^{\beta_{i,j}}$, where $M \leftarrow F_\bullet$ is the minimal free resolution of M.

Remark 2.18. Since the resolution is minimal, every F_i should have the form $\oplus_j S(-i-j)^{\beta_{i,j}}$ with $j \geq 0$.

The graded Betti numbers are usually represented by the following Betti diagram:

$$\begin{array}{c|cccc} & 0 & 1 & \dots & i \\ \hline 0 & \beta_{0,0} & \beta_{1,0} & \dots & \beta_{i,0} \\ 1 & \beta_{0,1} & \beta_{1,1} & \dots & \\ \vdots & \vdots & & & \\ j & & \dots & & \beta_{i,j} \end{array}$$

We have the following easy property of the graded Betti numbers.

Proposition 2.19. *If $\beta_{i,j} = 0$ for all $j \leq d$, then $\beta_{i+1,j} = 0$ for all $j \leq d$.*

Proof. Let $M \leftarrow F_\bullet$ be the minimal free resolution of M. The condition $\beta_{i,j} = 0$ for all $j \leq d$ implies that F_i does not contain any element of degree $\leq i + d$, then every element in $\mathfrak{m}F_i$ has degree $\geq i + 1 + d$. This gives that every element in F_{i+1} is of degree $\geq i+1+d$, thus $\beta_{i+1,j} = 0$ for all $j \leq d$. $\square$

Return to the geometry. Let $X \subset \mathbb{P}^N$ be a closed subvariety and I_X the homogeneous ideal of S defining X. Then the homogeneous coordinate ring $S_X = S/I_X$ is finitely generated. The graded Betti number $\beta_{i,j}$ of the S-module S_X is called the graded Betti numbers of X. The minimal free resolution $F_\bullet$ of S_X satisfies $F_0 = S$, thus $\beta_{0,j} = \delta_{0j}$. This minimal free resolution also gives a minimal free resolution of $\mathcal{O}_X$:

$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_{\mathbb{P}^N} \leftarrow \cdots \leftarrow \oplus_j \mathcal{O}_{\mathbb{P}^N}(-i-j)^{\beta_{i,j}} \leftarrow \cdots$$

Proposition 2.20. *The graded Betti numbers determine the Hilbert function by the following formula:*

$$H_X(d) = \sum_k B_k \binom{N+d-k}{N},$$

where $B_k = \sum_{i+j=k} (-1)^i \beta_{i,j}$. Inversely, the Hilbert function determines B_k by the formula

$$B_k = H_X(k) - \sum_{j|j<k} B_j \binom{N+k-j}{N}.$$

The proof is straightforward. Note that the Hilbert function encodes only the information of B_j. As we will see later the Betti numbers contain more information about X. Recall that the Hilbert function determines the Castelnuovo-Mumford regularity of a zero-dimensional scheme.

Theorem 2.21. *Suppose that X is a set of points. Then the Castelnuovo-Mumford regularity $Reg(X)$ is equal to the maximal j (noted J) such that there exists some i with $\beta_{i,j} \neq 0$.*

Proof. Recall that $h^0(\mathcal{O}_{\mathbb{P}^N}(m)) = \binom{m+N}{N} = 0$ if $m < 0$. Thus

$$\binom{m+N}{N} = \frac{(m+N)(m+N-1)\cdots(m+1)}{N!}$$

holds as soon as $m \geq -N$. From the minimal free resolution, we have $H_X(d) = \sum_{i,j}(-1)^i \beta_{i,j}\binom{d-i-j+N}{N}$. As we have noted above, $\binom{d-i-j+N}{N}$ is polynomial as soon as $d \geq i+j-N$. Since the length of the minimal free resolution is the projective dimension of S_X, which is equal to $dim(S) - depth(S_X) = N + 1 - \delta$, then

$$max_{\beta_{i,j}\neq 0}\{i + j - N\} \leq 1 - \delta + J \leq J.$$

This gives that $P_X(d) = H_X(d)$ if $d \geq J$, thus $J \geq Reg(X)$.

The proof of the part $J \leq Reg(X)$ is more involved and is only valid for X a set of points. □

Remark 2.22. For a general projective variety X, the Castelnuovo-Mumford regularity can be defined by the property in the above proposition. Equivalently, it could be defined to be the least number d such that $H^i(\mathbb{P}^N, \mathcal{I}_X(d-i)) = 0$ for all $i \geq 1$.

In this case, $P_X(d) = H_X(d)$ as soon as $d \geq 1 - \delta + Reg(X)$, where δ is the depth of S_X. And $\delta \geq 1$ for X a set of points.

¿From this proposition and our remark 1, we know that $Reg(X)$ (resp. $projdim(S_X)$) is the height (resp. length) of the Betti diagram. To conclude this section, we will continue our examples in section 1. The following proposition is useful to calculate the minimal free resolution.

Proposition 2.23. *Let X be a set of point in $\mathbb{P}^2$ lying on a curve of degree d. Then the defining ideal I_X can be generated by $d+1$ elements.*

Example 2.24. Four points in $\mathbb{P}^2$ (suite). There are three cases:

• *the four points lie on a line L.* We should find out the minimal generators of the defining ideal I_X. Note that I_X contains the degree one element defining the line, so F_1 contains one copy of $S(-1)$. The other element should be degree at least 4. In fact we can take the product of four lines each passing one point. So $F_1 = S(-1) \oplus S(-4)$. For the kernel of $F_0 = S \leftarrow F_1$, it is generated by an element of degree 5. In conclusion, we have the following minimal free resolution

$$0 \leftarrow S_X \leftarrow S \leftarrow S(-1) \oplus S(-4) \leftarrow S(-5) \leftarrow 0.$$

• *three of the four points lie on a line L.* There are two quadrics LL_1 and LL_2, where L_i lines passing the fourth point. Note that $L_2LL_1 =$

L_1LL_2 does not give all cubics, so we need one more cubic $C = F_1L_1 + F_2L_2$ to generate I_X. This gives the minimal free resolution:

$$0 \leftarrow S_X \leftarrow S \xleftarrow{\phi_1} S(-2)^2 \oplus S(-3) \xleftarrow{\phi_2} S(-3) \oplus S(-4) \leftarrow 0,$$

where

$$\phi_1 = (LL_1, LL_2, C) \quad \phi_2 = \begin{pmatrix} L_2 & F_1 \\ -L_1 & F_2 \\ 0 & -A \end{pmatrix}$$

• *no three points on a line.* In this case I_X is generated by two quadrics, each is a product of opposite sides of the quadrilateral formed by the 4 points.

$$0 \leftarrow S_X \leftarrow S \leftarrow S(-2)^2 \leftarrow S(-4) \leftarrow 0.$$

The Betti diagrams are the followings:

$$\begin{matrix} 1 & 1 & - \\ - & - & - \\ - & - & - \\ - & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & - & - \\ - & 2 & 1 \\ - & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \end{matrix}$$

Note that the last two configurations have the same Hilbert function but not the same Betti diagram, so the graded Betti numbers contain more information than the Hilbert function, though it is more involved to calculate.

Now we will use proposition 2.3 to calculate the Hilbert function for the third case. First, we have $B_0 = 1, B_1 = 0, B_2 = -2, B_3 = 0, B_4 = 1$. Then we calculate H_X by the following algorithm, using the same regle as Pascal triangle.

$$\begin{array}{ccccccccccccc}
 & & & & & & \mathbf{1} & & & & & & \\
 & & & & & \mathbf{0} & & 1 & & & & & \\
 & & & & \mathbf{-2} & & 1 & & 1 & & & & \\
 & & & \mathbf{0} & & -1 & & 2 & & \mathbf{1} & & & \\
 & & \mathbf{1} & & -1 & & 1 & & \mathbf{3} & & 1 & & \\
 & 0 & & 0 & & 0 & & \mathbf{4} & & 4 & & 1 & \\
0 & & 0 & & 0 & & \mathbf{4} & & 8 & & 5 & & 1
\end{array}$$

Exercise 2.25. Suppose that there exists some configuration of s points in $\mathbb{P}^2$ with Hilbert function $H_X(d) = min\{\binom{d+2}{2}, s\}$. Calculate the $B_k's$ for $s = 5, 6$ and 7, then construct some possible Betti diagrams.

Exercise 2.26. Find out all possible minimal free resolutions for four points in $\mathbb{P}^3$.

3. The Minimal Resolution Conjecture for points in $\mathbb{P}^2$

Let $X \subset \mathbb{P}^N$ a general set of s points in $\mathbb{P}^N$. What is the minimal free resolution of X?

¿From the last sections we know that for any variety X the betti tableau has the form

$$\begin{array}{ccccl} & \multicolumn{3}{c}{\overbrace{\hphantom{- \quad \cdots \quad -}}^{\operatorname{pd} X}} & \\ 1 & - & \cdots & - & \leftarrow 0 \\ - & - & \cdots & - & \leftarrow 1 \\ \vdots & \vdots & & \vdots & \\ - & - & \cdots & - & \\ - & * & \cdots & * & \leftarrow d_{min}-1 \\ \vdots & \vdots & & \vdots & \\ - & * & \cdots & * & \leftarrow \operatorname{reg}_X \end{array}$$

where d_{min} is the minimal degree of hypersurfaces containing X.

For points we also have $\operatorname{pd}(X) = \operatorname{codim}(X) = N$ (see proposition 2.16). We now determine d_{min}:

Lemma 3.1. *Let $X \subset \mathbb{P}^N$ be a set of s general points over an algebraically closed field. If $s \geq \binom{d+N}{N}$ then there are no hyper surfaces of degree d containing all s points. I.e.*

$$d_{min} = \max_d \{ d \mid \tbinom{d+N}{N} < s \}$$

Proof. There are $h^0(\mathbb{P}^N, \mathcal{O}(d)) = \binom{d+N}{N}$ hyper surfaces of degree d in $\mathbb{P}^N$. Each point poses one condition. Since the points are general over an infinite field, these conditions are independent. □

Now from the last section we also know that for points $\operatorname{reg}(X)$ is the smallest number d such that $H_X(d) = P_X(d)$:

Lemma 3.2. *Let $X \subset \mathbb{P}^N$ be a set of general points, then* $\operatorname{reg} X = \min_d \{ d \mid \binom{d+N}{N} \geq s \}$.

Proof. The Hilbert-function of a general set of points is

$$h_X(d) = h^0(\mathbb{P}^N, \mathcal{O}(d)) - h^0(\mathcal{I}_X(d)) = \min\{ \tbinom{d+N}{N}, s\}.$$

□

So the Betti diagram has the form

$$\begin{array}{cccccl} 1 & - & \cdots & - & \leftarrow 0 \\ - & - & \cdots & - & \\ \vdots & \vdots & & \vdots & \\ - & - & \cdots & - & \\ - & * & \cdots & * & \leftarrow d_{min}-1 = \operatorname{reg} X \end{array} \quad \text{(the columns } - \cdots - \text{ spanning } \overbrace{\quad\quad}^{N}\text{)},$$

if $s = \binom{d_{min}-1+N}{N}$ and

$$\begin{array}{cccccl} 1 & - & \cdots & - & \leftarrow 0 \\ - & - & \cdots & - & \\ \vdots & \vdots & & \vdots & \\ - & - & \cdots & - & \\ - & * & \cdots & * & \leftarrow d_{min}-1 \\ - & * & \cdots & * & \leftarrow d_{min} = \operatorname{reg} X \end{array}$$

otherwise.

Notice that the Hilbert-function $H_X(d)$ determines the diagonal alternating sums

$$B_k = (-1)^i(\beta_{i,d_{min}-1} - \beta_{i-1,d_{min}})$$

with $k = i + d_{min} - 1$. The situation is optimal, if one knows that at least one of the two b_{ij} in each diagonal is zero. Then we can calculate all Betti-numbers from the Hilbert-function.

Conjecture 3.3 (Minimal resolution Conjecture)**.** *Let $X \subset \mathbb{P}^N$ a general set of $s > N$ points. Then we say the minimal resolution conjecture is true for (s, N) if $\beta_{i,d_{min-1}}\beta_{i-1,d_{min}} = 0$ for all i.*

Example 3.4. The MRC predicts the following Betti-numbers of 5, 6 and 7 general points in $\mathbb{P}^2$ are

$$\begin{array}{ccc} 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \end{array} \qquad \begin{array}{ccc} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{array} \qquad \begin{array}{ccc} 1 & - & - \\ - & - & - \\ - & 3 & 1 \\ - & - & 1 \end{array}$$

respectively.

Remark 3.5. The MRC is known for $N = 2, 3$ and $s >> N$. It is known to be false for some small values of (s, N), starting with 11 points in $\mathbb{P}^6$ (see [EP] and [EPSW]).

To prove the MRC for a particular case (s, N) it is enough to give one example of points with these resolutions, since one has the following form of semi-continuity for Betti numbers:

Proposition 3.6. *Let $X \subset \mathbb{P}^N$ be any projective variety and d_{min} the smallest degree of a hypersurface that contains X, then*

$$\beta_{i,d_{min}-1} = h^0\big(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{I}_X(d_{min} + i)\big).$$

In particular the lowest non-zero-row of the Betti-diagram is upper semi continuous.

To prove that the MRC fails for a certain (s, N) is much more involved. In fact candidates for counterexamples were known for a long time, but the failure of the MRC in these cases was only proven in 1999 (see [EP] and [EPSW]).

We will now turn to the case of $\mathbb{P}^2$. We will explicitly give an example of points with the minimal free resolution predicted by the MRC.

For this we use determinantal varieties, since often their minimal free resolution can be explicitly calculated.

Definition 3.7. Let A and B be two vector bundles of ranks $a \geq b$ on $\mathbb{P}^N$, and $\phi\colon A \to B$ a morphism. Then we denote by $X_b(\phi)$ the determinantal locus where ϕ drops rank.

Example 3.8. Consider two quadrics Q_1 and Q_2 in $\mathbb{C}[x_0, x_1, x_2]$ and the morphism

$$\phi\colon \underbrace{\mathcal{O}(-2)^2}_{A} \xrightarrow{\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}} \underbrace{\mathcal{O}_{\mathbb{P}^2}}_{B}$$

then $X_2(\phi)$ is the intersection of Q_1 and Q_2.

Example 3.9. Consider the morphism

$$\phi\colon \underbrace{\mathcal{O}(-1)^3}_{A} \xrightarrow{\begin{pmatrix} l_0 & m_0 \\ l_1 & m_1 \\ l_2 & m_2 \end{pmatrix}} \underbrace{\mathcal{O}^2_{\mathbb{P}^2}}_{B}$$

with l_i, m_i linear forms in $\mathbb{P}^2$. Then $X_2(\phi)$ is cut out by the 2×2 minors $l_i m_j - l_j m_i$ of the above matrix. In general, $X_b(\phi)$ is the zero locaus of $\Lambda^b\phi : \Lambda^b A \to \Lambda^b B$.

Proposition 3.10. *In the situation above the codimension of X_b is bounded by*

$$\operatorname{codim} X_b \leq a - b + 1.$$

If $\operatorname{codim} X_b = a - b + 1$ *we say X_b is of expected codimension. In this case there exist an exact complex*

$$\mathcal{I}_X \leftarrow E_\bullet(\phi)$$

with

$$E_{i+1} = \Lambda^{b+i} A \otimes \Lambda^b B^* \otimes S_i B^*.$$

for $0 \le i \le a-b$. This complex is called the Eagon-Northcott complex induced by ϕ. If A and B are free, ϕ can be represented by an $a \times b$ matrix $M(\phi)$ of polynomials. In this case I_X is generated by the maximal minors of $M(\phi)$.

Example 3.11. If Q_1 and Q_2 of the above example have no common factor, then their complete intersection $X := X_1(\phi)$ consists of $2 \cdot 2 = 4$ points in $\mathbb{P}^2$. In this case the codimension expected, and the Eagon-Northcott-Complex induced by

$$\phi\colon \underbrace{\mathcal{O}(-2)^2}_{A} \xrightarrow{\binom{Q_1}{Q_2}} \underbrace{\mathcal{O}_{\mathbb{P}^2}}_{B}$$

gives a resolution of $\mathcal{I}_X$. More explicitly we have $a = 2$, $b = 1$ and

$$E_1 = \Lambda^b A \otimes \Lambda^b B^* \otimes S_0 B^* = \Lambda^1 A = \mathcal{O}(-2)^2_{\mathbb{P}^2}$$

and

$$E_2 = \Lambda^{b+1} A \otimes \Lambda^b B^* \otimes S_1 B^* = \Lambda^2 A = \mathcal{O}(-4).$$

The minimal free resolution of $\mathcal{I}_X$ is

$$\mathcal{I}_X \xleftarrow{(Q_2,-Q_1)} \mathcal{O}(-2)^2_{\mathbb{P}^2} \xleftarrow{\binom{Q_1}{Q_2}} \mathcal{O}(-4) \leftarrow 0$$

with Betti numbers

$$\begin{matrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \end{matrix}$$

Notice that the last arrow is just the transpose of ϕ.

Example 3.12. Consider

$$\phi\colon \underbrace{\mathcal{O}(-1)^3}_{A} \xrightarrow{\begin{pmatrix} l_0 & m_0 \\ l_1 & m_1 \\ l_2 & m_2 \end{pmatrix}} \underbrace{\mathcal{O}^2_{\mathbb{P}^2}}_{B}$$

with l_i, m_i *general* linear forms. Then $X := X_1(\phi)$ is of expected codimension, and the Eagon-Northcott-Complex

$$\mathcal{I}_X \xleftarrow{(Q_{12},-Q_{02},Q_{01})} \mathcal{O}(-2)^3 \xrightarrow{\begin{pmatrix} l_0 & m_0 \\ l_1 & m_1 \\ l_2 & m_2 \end{pmatrix}} \mathcal{O}(-3)^2 \leftarrow 0$$

with $Q_{ij} = l_i m_j - l_j m_i$ gives a free resolution of $\mathcal{I}_X$. The Betti diagram of X is

$$\begin{matrix} 1 & - & - \\ - & 3 & 2 \end{matrix}$$

As indicated by the examples above, the Eagon-Northcott complex is particularly simple in the case $a = b + 1$, i.e codim $X_b = 2$:

Proposition 3.13. *Let $\phi\colon A \to B$ be a morphism of free vector bundles of ranks a and $b = a-1$. If ϕ drops rank in expected codimension $a-b+1 = 2$ then a (not necessarily minimal) free resolution of I_{X_b} if given by*

$$I_{X_b} \xleftarrow{\Lambda^b\phi} A^* \otimes \mathcal{L} \xleftarrow{\phi^t} B^* \otimes \mathcal{L} \leftarrow 0.$$

with $\mathcal{L} = \Lambda^a A \otimes \Lambda^b B^*$ a line bundle. If the matrix representing ϕ has no constant entries, the above complex is minimal.

Proof. Use the Eagon-Northcott complex and the isomorphism $\Lambda^{a-1}A = A^* \otimes \Lambda^a A$. The last claim follows from the definition of minimality. □

There even is a converse to the above proposition

Theorem 3.14 (Hilbert-Burch)**.** *If $X \subset \mathbb{P}^N$ is a codimension 2 arithmetically Cohen-Macaulay scheme, then the minimal free resolution of $\mathcal{I}_X$ is always given by an Eagon Northcott complex induced by a morphism of trivial vector bundles A, and B of ranks a and $b = a-1$.*

Lets now return to the MRC for general points in $\mathbb{P}^2$. By the discussion above, the Betti diagram for s general points in $\mathbb{P}^2$ has the form

$$\begin{array}{cccc}
 & \multicolumn{2}{c}{\overbrace{\qquad\qquad}^{N=2}} & \\
1 & - & - & \leftarrow 0 \\
- & - & - & \\
\vdots & \vdots & \vdots & \\
- & - & - & \\
- & \beta_{1,d-1} & \beta_{2,d-1} & \leftarrow d-1 \\
- & \beta_{1,d} & \beta_{2,d} &
\end{array}$$

If we set $\operatorname{rank} F_1 = a$ and $\operatorname{rank} F_2 = b$ we can also write this as

$$\begin{array}{cccc}
1 & - & - & \leftarrow 0 \\
- & - & - & \\
\vdots & \vdots & \vdots & \\
- & - & - & \\
- & x & b-y & \leftarrow d-1 \\
- & a-x & y &
\end{array}$$

In particular we obtain $B_d = x$, $B_{d+1} = a-x-b+y$ and $B_{d+2} = y$. We now derive some equalities between these unknowns from the Hilbert-function.

The Hilbert-function of s general points in $\mathbb{P}^2$ is given by

$$h_X(d) = \min_i\{\tbinom{d+n}{n}, s\}.$$

Since $X \subset \mathbb{P}^2$ the B_k are the third differences of $h_X(d)$. The corresponding triangle is of the form

$$
\begin{array}{cccccccccccc}
 & & & & & & & & 1 & & & \\
 & & & & & & & 0 & & 1 & & \\
 & & & & & & 0 & & 1 & & 1 & \\
 & & & & & \iddots & & 1 & & 2 & & \mathbf{1} \\
 & & & & 0 & & \iddots & & 3 & & \mathbf{3} & \\
 & & & B_d & & 1 & & \iddots & & \mathbf{6} & & \\
 & & B_{d+1} & & * & & d & & \iddots & & & \\
 & B_{d+2} & & * & & * & & \binom{\mathbf{d+1}}{\mathbf{2}} & & & & \\
0 & & 0 & & 0 & & s & & & & & \\
 & 0 & & 0 & & s & & & & & & \\
 & & 0 & & s & & & & & & & \\
 & & & s & & & & & & & &
\end{array}
$$

Substituting our expressions for B_k we obtain

$$
\begin{array}{ccccccccc}
 & & & & & \iddots & & & \\
 & & & & 0 & & \iddots & & \\
 & & & -x & & 1 & & \iddots & \\
 & & {\scriptstyle -a+x \atop \scriptstyle +b-y} & & 1-x & & d & & \iddots \\
 & y & & {\scriptstyle b-a \atop \scriptstyle +1-y} & & {\scriptstyle 1-x \atop \scriptstyle +d} & & \binom{\mathbf{d+1}}{\mathbf{2}} & \\
0 & & 0 & & 0 & & \mathbf{s} & & \\
 & 0 & & 0 & & \mathbf{s} & & & \\
 & & 0 & & \mathbf{s} & & & & \\
 & & & \mathbf{s} & & & & &
\end{array}
$$

The zeros then imply the equations

$$0 = (y) + (b - a + 1 - y) = b - a + 1$$

and

$$0 = (b - a + 1 - y) + (1 - x + d) = d + 1 - x - y.$$

Taking all of this together, we look for a set of points $X \subset \mathbb{P}^2$ with minimal free resolution

$$\mathcal{I}_X \leftarrow \mathcal{O}(-d)^x \oplus \mathcal{O}(-d-1)^{a-x} \xleftarrow{\phi} \mathcal{O}(-d-1)^{b-y} \oplus \mathcal{O}(-d-2)^y$$

satisfying the above restrictions and also $(a - x)(b - y) = 0$. The question is how to choose ϕ. Fortunately a general matrix will work:

Proposition 3.15. *Consider a general morphism*

$$\psi\colon \underbrace{\mathcal{O}(d)^x \oplus \mathcal{O}(d+1)^{a-x}}_{A} \to \underbrace{\mathcal{O}(d+1)^{b-y} \oplus \mathcal{O}(d+2)^y}_{B}$$

with $b = a+1$ and $d = x+y-1$, then the corresponding Eagon-Northcott complex is exact and of the form

$$\mathcal{I}_X \leftarrow \underbrace{\mathcal{O}(-d)^x \oplus \mathcal{O}(-d-1)^{a-x}}_{A^*} \xleftarrow{\psi^t} \underbrace{\mathcal{O}(-d-1)^{b-y} \oplus \mathcal{O}(-d-2)^y}_{B^*}$$

Proof. Since ψ is general and $a = b+1$ the determinantal locus X_b is of expected codimension $a-b+1 = 2$, i.e. of dimension 0. Consequently the Eagon-Northcott complex

$$\mathcal{I}_X \leftarrow A^* \otimes \mathcal{L} \xleftarrow{\phi^t} B^* \otimes \mathcal{L} \leftarrow 0$$

is exact. Since

$$\begin{aligned}
\mathcal{L} = \Lambda^a A \otimes \Lambda^b B^* &= \mathcal{O}(-dx - (d+1)(a-x)) \otimes \mathcal{O}((d+1)(b-y) + (d+2)y) \\
&= \mathcal{O}(-(d+1)a + x + (d+1)b + y) \\
&= \mathcal{O}(-(d+1) + x + y) \\
&= \mathcal{O}
\end{aligned}$$

this complex is the one claimed. □

Remark 3.16. Notice that the complex constructed above is not necessarily minimal.

Corollary 3.17 (MRC for $\mathbb{P}^2$)**.** *If $(a-x)(b-y) = 0$ then the complex of the previous proposition is minimal.*

Proof. The Betti diagram of the Eagon-Northcott complex above is of the form

$$\begin{matrix} 1 & - & - \\ - & - & - \\ \vdots & \vdots & \vdots \\ - & - & - \\ - & * & * \\ - & - & * \end{matrix} \qquad \text{or} \qquad \begin{matrix} 1 & - & - \\ - & - & - \\ \vdots & \vdots & \vdots \\ - & - & - \\ - & * & - \\ - & * & * \end{matrix}$$

The matrix representing the second arrow has only entries of degree 1 and 2 by the form of the diagram. In particular it has no constant entries. □

Example 3.18. For 5 general points in $\mathbb{P}^2$ we expect a Betti diagram

$$\begin{matrix} 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \end{matrix}$$

we obtain an example by choosing a general 3×2 matrix M with polynomial entries of the following degrees:

$$\deg(M) = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

For 6 general points in $\mathbb{P}^2$ we have

$$\begin{matrix} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{matrix} \qquad \text{and} \qquad \deg(M) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

For 7 general points in $\mathbb{P}^2$ we have

$$\begin{matrix} 1 & - & - \\ - & - & - \\ - & 3 & 1 \\ - & - & 1 \end{matrix} \qquad \text{and} \qquad \deg(M) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Exercise 3.19. Assuming the minimal resolution conjecture for general points in $\mathbb{P}^3$ calculate the Betti-tableau of 9 such points.

Exercise 3.20. Calculate the Betti-tableau of 10 general points in $\mathbb{P}^3$. Is there a determinantal set of points with this Betti tableau?

Exercise 3.21. Calculate the Betti-tableau of the determinantal variety given by a general map

$$\phi\colon \mathcal{O}_{\mathbb{P}^3}(-1)^4 \to \mathcal{O}_{\mathbb{P}^3}^2$$

What are the codimension and degree of this variety?

4. Geometric Syzygies of Points

We will start with an extensive example. Consider 10 general points in $\mathbb{P}^3$. Their minimal free resolution is expected to be

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 10 & 15 & 6 \end{matrix}$$

An example of set of points with this resolution is given by the determinantal locus of a general map

$$\varphi\colon \mathcal{O}_{\mathbb{P}^3}^5(-1) \to \mathcal{O}_{\mathbb{P}^3}^3.$$

which corresponds to a general 5×3 matrix with linear entries.

Now forget one of the 10 points. These 9 general points have expected Betti tableau

$$\begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 7 & 12 & 5 \end{matrix}$$

since we know that 9 general points lie on only one quadric, and that every first syzygy involves at least 2 polynomials, this is also the real Betti tableau of 9 general points.

Observe the difference between the two tableaux

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 10 & 15 & 6 \end{matrix} \quad - \quad \begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 7 & 12 & 5 \end{matrix} \quad = \quad \begin{matrix} - & - & - & - \\ - & -1 & - & - \\ - & 3 & 3 & 1 \end{matrix}.$$

What is the explanation for this difference? The -1 is easy to explain. This is just the unique quadric Q going through the 9 points but not through the 10th. Now consider this 10th point p. Without restriction we can assume $I_p = (x, y, z)$, i.e $p = (0 : 0 : 0 : 1) \in \mathbb{P}^3$. But then the cubics $Q \cdot x$, $Q \cdot y$, and $Q \cdot z$ do contain all 10 points. These three cubics are therefore among the 10 cubics that cut out the 10 points, but not among the 7 cubics for 9 points, since they are not minimal there. This explains the first 3 of the difference. The other 3 and the final 1 are the syzygies between these three difference cubics:

$$1 \xleftarrow{(Qx, Qy, Qz)} 3 \xleftarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & z \\ 0 & x & -y \end{pmatrix}} 3 \xleftarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} 1$$

This is just the Koszul complex, tensored with Q. Geometrically each cubic corresponds to a hyperplane through p. Each first syzygy defines a line through p, and the last syzygy finally recovers the point.

Repeating this argument with all 10 points we get 10 subcomplexes

$$\begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 3 & 3 & 1 \end{matrix} \quad \subset \quad \begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 10 & 15 & 6 \end{matrix}$$

To get some more geometry, we projectivize the syzygy spaces. This gives us a configuration of 10 $\mathbb{P}^2$'s in $\mathbb{P}(10) \cong \mathbb{P}^9$, a configuration of 10 $\mathbb{P}^2$'s in $\mathbb{P}(15) \cong \mathbb{P}^{14}$ and 10 points in $\mathbb{P}(6) \cong \mathbb{P}^5$. This set of points is the Gale transform of our original 10 points.

This last correspondence is the most interesting. We have shown, that for each point we get a syzygy. But what about the reverse? For this observe that the second syzygies that we constructed involve exactly 3 first syzygies, each with a linear form. The vanishing set of these linear forms gave back our point. So for any second syzygy that involves only 3 first syzygies we obtain a point. Is this one of our 10 points? The answer is yes. For this dualize the minimal free resolution of the 10 points and consider the maps given by the syzygy above

$$\begin{matrix} 1 & 3 & & \\ \downarrow & \downarrow & & \\ 6 & 15 & 10 & - \\ - & - & - & - \\ - & - & - & 1 \end{matrix}$$

and calculate the Koszul complex associated to the three linear forms of the syzygy:

$$\begin{matrix} 1 & 3 & 3 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 6 & 15 & 10 & - \\ - & - & - & - \\ - & - & - & 1 \end{matrix}$$

Since the Koszul complex is exact and the dual of the resolution is minimal the inclusion of the syzygy lifts to a map of complexes. The last map is of degree 2 and gives a quadric going through 9 of the 10 points. The remaining points is the one defined by the linear forms of the syzygy we started with.

Because of the geometry involved, we call the special syzygies above *geometric syzygies*. To give a more formal definition we need some background:

Definition 4.1. Let A, B and C be vector spaces of dimension a, b and c and

$$\gamma\colon A \otimes B \to C$$

a linear map. We call γ a triple tensor, since $\gamma \in A^* \otimes B^* \otimes C$.

Remark 4.2. After choosing bases we can represent γ as an $a \times b$ matrix whose entries are linear forms in c variables.

Example 4.3. The middle map of the Koszul-complex above defines a triple tensor

$$\gamma\colon A \otimes B \to C$$

with $a = b = c = 3$, if we choose $c_1 = x$, $c_2 = y$ and $c_3 = z$ as basis of C, the rows as basis of A and the columns as basis of B. The matrix

$$\begin{pmatrix} -c_2 & -c_3 & 0 \\ c_1 & 0 & c_3 \\ 0 & c_1 & -c_2 \end{pmatrix}$$

says for example $\gamma(a_1 \otimes b_1) = -c_2$.

Definition 4.4. A linear map $\mathbb{C} \to A$ is called a *generalized row* of γ since it induces a map

$$\mathbb{C} \otimes B \to C$$

which can be interpreted as a $1 \times b$ row vector of linear-forms. The images of such generalized rows $\mathbb{C} \to A$ form a projective space $\mathbb{P}(A^*)$ which we call the *row space* of γ. Similarly $\mathbb{P}(B^*)$ is the *column space* of γ.

Example 4.5. In the example above the map $\mathbb{C} \xrightarrow{a_1} A$ corresponds to the first row of the matrix. The map $\mathbb{C} \xrightarrow{a_1+a_2} A$ corresponds to the sum of the first and second row.

On the row space $\mathbb{P}(A^*)$ the triple tensor γ induces a map of vector bundles

$$\gamma_A\colon \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to C$$

by composing it with the first map of the twisted Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to A \otimes B \to \mathbb{T}_{\mathbb{P}(A^*)}(1) \otimes B \to 0$$

on $\mathbb{P}(A^*)$. Similarly we have

$$\gamma_B \colon A \otimes \mathcal{O}_{\mathbb{P}(B^*)}(-1) \to C$$

on the column space $\mathbb{P}(B^*)$. ¿From now on we will restrict our discussion to the row space $\mathbb{P}(A^*)$, leaving the analogous constructions for the column space $\mathbb{P}(B^*)$ to the reader.

Given a generalized row $\alpha \in \mathbb{P}(A^*)$ the restriction of γ_A to α

$$\gamma_\alpha \colon B \to C$$

is a map of vector spaces.

Definition 4.6. The *rank* of a generalized row α is defined as $\operatorname{rank} \alpha := \operatorname{rank} \gamma_\alpha$. The image $\operatorname{Im}(\gamma_\alpha) \subset C$ is called the *space of linear forms involved in* α.

Example 4.7. For γ as above we obtain

$$\gamma_A \colon \mathcal{O}_{\mathbb{P}(A^*)}(-1) \otimes B \to C$$

which can be written as

$$\begin{pmatrix} a_2 & -a_1 & 0 \\ a_3 & 0 & -a_1 \\ 0 & -a_3 & -a_2 \end{pmatrix}$$

since for example $\gamma(a_2 \otimes b_1) = c_1$.

Remark 4.8. The determinantal varieties associated to γ_A stratify the row space $\mathbb{P}(A^*)$ according to the rank of the rows. In particular the minimal-rank-rows form a closed subscheme $Y_{min} \subset \mathbb{P}(A^*)$.

Remark 4.9. In practice Y_{min} is often not of expected codimension, which makes it hard to control as determinantal variety. Sometimes Y_{min} can be constructed by representation theory.

Definition 4.10. Let

$$\gamma_C \colon \underbrace{\mathcal{O}(-i-j)^a}_{F_{ij}} \otimes \mathcal{O}(-i-j) \to \mathcal{O}(-i+1-j)^b$$

be a linear step in the minimal free resolution of some ideal sheaf $I_X \subset \mathcal{O}_{\mathbb{P}(C)}$. Then γ_C defines a triple tensor

$$\gamma \colon A \otimes B \to C.$$

with generalized rows $\mathbb{C} \to A \cong F_{ij}$ corresponding to ith syzygies of I_X. We define the rank of a syzygy $s \in F_{ij}$ as the rank of the corresponding generalized row. The scheme of minimal rank rows $Y_{min} \in \mathbb{P}(A^*)$ is called the *space of minimal rank ith syzygies*.

In our case above the last syzygies that corresponded to point where exactly the syzygies of rank 3, which are also the minimal rank syzygies in this case. We have shown

Proposition 4.11. *Let X be a general set of 10 points in $\mathbb{P}^3$ then the minimal rank last syzygies are in $1:1$ correspondence with the points of X.*

In general we have

Proposition 4.12. *Let X be a general set of $\binom{d+N}{N}$ points in $\mathbb{P}^N$, then the (projectivized) minimal rank ith syzygies are in $1:1$ correspondence with the codim $i+1$ linear spaces though points of X.*

In particular the last minimal rank syzygies correspond to the points themselves. The scheme $Y_{min} \subset \mathbb{P}(\text{last syzygies})$ is the Gale-Transform of X.

What about other numbers of points? We return to $\mathbb{P}^3$ and give all Betti tableaux from $s = 4$ to $s = 10$

$$\text{4 points} \quad \begin{matrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{matrix}$$

$$\text{5 points} \quad \begin{matrix} 1 & - & - & - \\ - & 5 & 5 & - \\ - & - & - & 1 \end{matrix}$$

$$\text{6 points} \quad \begin{matrix} 1 & - & - & - \\ - & 4 & 2 & - \\ - & - & 3 & 2 \end{matrix}$$

$$\text{7 points} \quad \begin{matrix} 1 & - & - & - \\ - & 3 & - & - \\ - & 1 & 6 & 3 \end{matrix}$$

$$\text{8 points} \quad \begin{matrix} 1 & - & - & - \\ - & 2 & - & - \\ - & 4 & 9 & 4 \end{matrix}$$

$$\text{9 points} \quad \begin{matrix} 1 & - & - & - \\ - & 1 & - & - \\ - & 7 & 12 & 5 \end{matrix}$$

$$\text{10 points} \quad \begin{matrix} 1 & - & - & - \\ - & - & - & - \\ - & 10 & 15 & 6 \end{matrix}$$

Notice that in going from 10 points to 6 points we subtract

$$\begin{matrix} 1 & - & - & - \\ - & -1 & - & - \\ - & 3 & 3 & 1 \end{matrix}$$

in each step, and make some changes in the diagonal if the result is negative. Geometrically we project in each step from the geometric syzygies of the point that we remove.

Remark 4.13. This is true independently from the MRC.

Because of the rich geometry associated with minimal rank syzygies we also call them *geometric syzygies*.

Definition 4.14. We say the geometric syzygy conjecture is true for the ith syzygies of a scheme X, if the ith step $F_i \to F_{i-1}$ in the minimal free resolution is linear and the space of geometric ith syzyzgies Y_{min} is non-degenerate. In this case each syzygy can be written as a sum of geometric syzygies.

Corollary 4.15. *The geometric syzygy conjecture is true for the top row of s general points in $\mathbb{P}^N$.*

Proof. Since we obtain the top syzygies of any number of points by successively projecting from geometric syzygies, it is enough to show that the geometric syzygies span in the case of $s = \binom{d+N}{N}$ which has a linear resolution. Now consider all the projections down to $s' = \binom{d-1+N}{N}$ points. These have again a linear resolution but of one degree lower. This means that while projecting from geometric syzygies we have lost all syzygies of the top row. Consequently these geometric syzygies must span. □

Corollary 4.16. *If for a set of $\binom{d+N}{N}$ points with linear resolution we find a subset of k points, such that their linear spans of ith geometric syzygies are either independent or span the whole space of ith syzygies, then the MRC is true of $\binom{d+N}{N} - k$ points.*

Proof. Projecting form these syzygies gives the expected numbers for the minimal free resolution of the remaining $\binom{d+N}{N} - k$ points. □

5. Trivial syzygies and residual points in $\mathbb{P}^2$

The aim of this section is to show how (again) syzygies and geometry are related.

Hereafter we will always work on $\mathbb{P}^2$, $R = k[x, y, z]$ will denote its coordinate ring, k being an algebraically closed field.

Let f_0, f_1, f_2 be three homogeneous polynomials in $\mathbb{P}^2$ of respective positive degree d_0, d_1, d_2. We denote by I the ideal (f_0, f_1, f_2). It is well-known that for a sufficiently generic choice of f_0, f_1, f_2, then $\sqrt{I} = R$, that is f_0, f_1, f_2 have no common root in $\mathbb{P}^2$. In fact the existence of a common root can be traduced in term of non exactness of the Koszul complex associated to f_0, f_1, f_2 in R. This complex, denoted $K_\bullet(f_0, f_1, f_2)$, is

$$R(-d_0 - d_1 - d_2) \xrightarrow{\partial_3} \bigoplus_{0 \le i < j \le 2} R(-d_i - d_j) \xrightarrow{\partial_2} \oplus_{i=0}^2 R(-d_i) \xrightarrow{\partial_1} R,$$

where

$$\partial_1 = \begin{pmatrix} f_0 & f_1 & f_2 \end{pmatrix}, \partial_2 = \begin{pmatrix} f_1 & f_2 & 0 \\ -f_0 & 0 & f_2 \\ 0 & -f_0 & -f_1 \end{pmatrix}, \partial_3 = \begin{pmatrix} f_2 \\ -f_1 \\ f_0 \end{pmatrix}.$$

Notice that this complex is built from the *trivial* syzygies of f_0, f_1, f_2, that is syzygies which are always available whatever f_0, f_1, f_2 are. They are for instance of the form $f_2 f_1 - f_1 f_2$, $f_1 f_0 - f_0 f_1$, and so on .... The first trivial syzygies are the (there are 3) 2×2 minors (which are identically 0) of the matrix

$$\begin{pmatrix} f_0 & f_1 & f_2 \\ f_0 & f_1 & f_2 \end{pmatrix},$$

and the third syzygy (there is only one) is the (identically zero) determinant of the matrix

$$\begin{pmatrix} f_0 & f_1 & f_2 \\ f_0 & f_1 & f_2 \\ f_0 & f_1 & f_2 \end{pmatrix}.$$

Proposition 5.17. *The following properties are equivalent :*

- f_0, f_1, f_2 *have no common root in* $\mathbb{P}^2$
- $\sqrt{I} = R$
- $I^{sat} = R$
- $\mathrm{codim}(I) = 3$
- $K_\bullet(f_0, f_1, f_2)$ *is acyclic*
- I *has only trivial syzygies*

Proof. All is standard. See [Eis94]. □

Remark 5.18. This proposition is the starting point of elimination theory, and more particularly of resultant theory. For instance, in the previous proposition we can add $\mathrm{Res}(f_0, f_1, f_2) = 0$, where Res denotes what is called the resultant of f_0, f_1, f_2 (a direct generalization of the well-known Sylvester's resultant).

If our ideal I has only trivial syzygies, its associated Koszul complex $K_\bullet(f_0, f_1, f_2)$ is then a free resolution of R-modules of R/I, with betti diagram of the form

$$\begin{array}{r|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & ? & ? & ? \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_0+d_1+d_2-3 & - & ? & ? & 1 \\ d_0+d_1+d_2-2 & - & - & - & - \end{array}.$$

It follows that I is $(d_0+d_1+d_2-2)$-regular, and we have the nice property :

Corollary 5.19. *Let ν be any integer such that $\nu \geq d_0+d_1+d_2-2$, then polynomials f_0, f_1, f_2 have a common root in $\mathbb{P}^2$ if and only if the map $\partial_{1\nu}$ is not surjective, that is not of full rank $\binom{\nu+2}{2}$.*

Proof. First remark that for all integer ν we have an exact sequence of vector spaces

$$R_{\nu-d_0-d_1} \oplus R_{\nu-d_0-d_2} \oplus R_{\nu-d_1-d_2} \xrightarrow{\partial_{1\nu}} R_\nu \rightarrow R_\nu/I_\nu.$$

It is clear that if f_0, f_1, f_2 have a common root , then $\dim(R_\nu/I_\nu) \geq 1$ for all integer ν, and hence $\partial_{1\nu}$ is not surjective.

Now recall that, for any integer ν, if I is ν-regular then it is ν-saturated (i.e. $I_d = I_d^{sat}$ for all $d \geq \nu$). If f_0, f_1, f_2 have no common root then $K_\bullet(f_0, f_1, f_2)$ is a free resolution of R-modules of I and hence I is ν-regular, and hence ν-saturated, for all $\nu \geq d_0+d_1+d_2-2$. Since $I^{sat} = R$, it follows that $R_\nu/I_\nu = 0$ for all $\nu \geq d_0+d_1+d_2-2$, and hence $\partial_{1\nu}$ is surjective. □

Remark 5.20. Notice that the previous corollary can be seen as a generalization of the well-known Sylvester's matrix of two polynomials in two homogeneous variables.

Exercise 1. *What is the result of corollary 5.19 in case f_0, f_1, f_2 are linear forms and ν is the lowest possible ?*

We have seen that the acyclicity of the Koszul complex $K_\bullet(f_0, f_1, f_2)$, that is f_0, f_1, f_2 have only trivial syzygies, implies that f_0, f_1, f_2 have no common root in $\mathbb{P}^2$, and inversely. Thus, for instance, being given any three conics in $\mathbb{P}^2$, we can compute (with Macaulay2, Singular, ...) if they have a common root or not.

Consider now the following example of the intersection of three circles in $\mathbb{P}^2$:

$$\left\{ \begin{array}{l} f_0 = a_0z^2 + a_1xz + a_2yz + a_3(x^2+y^2) \\ f_1 = b_0z^2 + b_1xz + b_2yz + b_3(x^2+y^2) \\ f_2 = c_0z^2 + c_1xz + c_2yz + c_3(x^2+y^2) \end{array} \right. , \tag{5.1}$$

where the a_i's, b_i's and c_i's are any element of k. We would like to know if these three circles intersect in $\mathbb{P}^2$ (this of course depend on the parameters a_i, b_i and c_i). But these three circles *always* intersect at infinity along both points $P_1 = (1 : i : 0)$ and $P_2 = (1 : -i : 0)$ (defined by the ideal $(z, x^2 + y^2)$). Consequently the Koszul complex $K_\bullet(f_0, f_1, f_2)$ will never be acyclic in this case. Since the points P_1 and P_2 are always present, we can reformulate our question : "Are our three circles intersect in $\mathbb{P}^2$ *outside* the points P_1 and P_2 ? ". In what follows we describe how we can answer this question, always with some *trivial* syzygies, and give a sens to the word "outside".

Let $G = (g_1, \ldots, g_n)$ be a homogeneous ideal of R. We denote by $k_1 \geq \ldots \geq k_n$ the respective degrees of the homogeneous polynomials $g_1, \ldots, g_n$. We consider three homogeneous polynomials f_0, f_1, f_2 of respective degree $d_0 \geq d_1 \geq d_2 \geq k_1$ in the ideal G, that is we can write :

$$\left\{ \begin{array}{lcl} f_0(x) & = & \sum_{i=1}^n h_{i,0}(x)\, g_i(x) \\ f_1(x) & = & \sum_{i=1}^n h_{i,1}(x)\, g_i(x) \\ f_2(x) & = & \sum_{i=1}^n h_{i,2}(x)\, g_i(x) \end{array} \right. ,$$

or equivalently

$$(5.2) \qquad \begin{pmatrix} f_0 & f_1 & f_2 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \end{pmatrix} \begin{pmatrix} h_{1,0} & h_{1,1} & h_{1,2} \\ h_{2,0} & h_{2,1} & h_{2,2} \\ \vdots & \vdots & \vdots \\ h_{n,0} & h_{n,1} & h_{n,2} \end{pmatrix},$$

where $h_{i,j} = \sum_{\alpha_1+\alpha_2+\alpha_3=d_j-k_i} c_\alpha^{i,j}\, x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$ are homogeneous polynomials of degree $d_j - k_i$.

The geometric situation goes as follows : the ideal G defines a closed subscheme of $\mathbb{P}^2$ (which is $\mathrm{Proj}(R/G)$); we denote $\mathcal{G}$ its associated ideal sheaf. Polynomials f_0, f_1, f_2 are respectively global sections of $\mathcal{G}(d_0), \mathcal{G}(d_1), \mathcal{G}(d_2)$ (and hence vanish along $\mathrm{Proj}(R/G)$) and we would like to know if they have a common root "outside" the subscheme defined by $\mathcal{G}$. Denoting by $\mathcal{I}$ the ideal sheaf associated to the ideal $I = (f_0, f_1, f_2)$, we give a sens to the word "outside" by asking that the subscheme defined by $\mathcal{I}$ is strictly bigger than the subscheme defined by $\mathcal{G}$. This is equivalent to ask $(\mathcal{I} : \mathcal{G}) \subsetneq \mathcal{O}_{\mathbb{P}^2}$, or $V(I : G) \neq \emptyset$, or also $I^{sat} \subsetneq G^{sat}$, where the exponent *sat* stands for the saturation by the maximal ideal (x, y, z) of R. Our aim is now to construct a complex of *trivial* syzygies for ideals of the form $(I : G)$ with G fixed; such an ideal is often called a *residual intersection* ideal.

¿From now and on we assume that the ideal $G = (g_1, \ldots, g_n)$ is fixed and is *saturated of codim 2*. We begin with the

Proposition 5.21. (Hilbert-Burch) *Every minimal graded resolution of R-modules of the ideal G is of the form :*

$$0 \to \bigoplus_{i=1}^{n-1} R(-l_i) \xrightarrow{\psi} \bigoplus_{i=1}^{n} R(-k_i) \xrightarrow{\gamma=(g_1,\ldots,g_n)} G \to 0,$$

where $\sum_{i=1}^{n-1} l_i = \sum_{i=1}^{n} k_i$ *and* $aI_{n-1}(\psi) = \gamma$*, with* $a \in k \setminus \{0\}$.

Proof. This proposition follows from the so-called Hilbert-Burch theorem (see [Eis94], theorem 20.15) which applies for ideals having projective dimension 1. In fact, in $\mathbb{P}^2$, an ideal is arithmetically Cohen-Macaulay of codimension 2 if and only if it is saturated of codimension 2, and if and only if it has projective dimension 1. To be complete, we recall here how we can prove that G (which is supposed to be saturated of codimension 2) has projective dimension 1 "by hand"':

By the Auslander-Buchsbaum formula we have :

$$\mathrm{pd}(G) = \mathrm{depth}_{\mathfrak{m}}(R) - \mathrm{depth}_{\mathfrak{m}}(G) = 3 - \mathrm{depth}_{\mathfrak{m}}(G),$$

where $\mathfrak{m} = (x, y, z)$ is the irrelevant ideal of R. We can suppose that no points of G are on the line $\{x = 0\}$, and hence we have :

$$x.f \in G \Longrightarrow f \in G.$$

We deduce that x is not a zero divisor in G and moreover that y is not a zero divisor in G/xG. Indeed, if $y.h + x.q = 0$ with $h, q \in G$, then h divides x and hence $h = xh'$. But since $h \in G$, $h' \in G$ and consequently h is zero in G/xG. Finally, we have $\mathrm{depth}_{\mathfrak{m}}(G) \geq 2$. Since $\mathrm{pd}(G) \neq 0$ (otherwise G would be free and so would not define points), we have $\mathrm{pd}(G) = 1$. □

Remark 5.22. In fact this theorem applies in a more general setting : every ideal Q of a commutative ring A of codimension 2 such that A/Q is Cohen-Macaulay has projective dimension 1.

¿From this last description of the ideal G, we can answer the question "how many points are we trying to remove ? "

Corollary 5.23. *The ideal G of proposition 5.21 defines exactly*

$$\frac{\sum_{i=1}^{n-1} l_i^2 - \sum_{i=1}^{n} k_i^2}{2}$$

points (counted with multiplicity).

Proof. Since G defines isolated points, we just have to compute the Euler characteristic of any graded part of its Hilbert-Burch complex, say t. In this way the number N (independent of t) of points is given by :

$$N = \binom{t+2}{2} - \sum_{i=1}^{n} \binom{t-k_i+2}{2} + \sum_{i=1}^{n-1} \binom{t-l_i+2}{2}.$$

By a straightforward computation we find the desired result. $\square$

Exercise 2. *In case G is a complete intersection $G = (g_1, g_2)$, observe how we recover the Bézout theorem.*

¿From proposition 5.21 we deduce the following graded presentation of R-modules of the ideal G/I (recall that $I \subset G$) :

$$\bigoplus_{i=1}^{n-1} R(-l_i) \bigoplus_{i=0}^{2} R(-d_i) \xrightarrow{\psi \oplus \phi} \bigoplus_{i=1}^{n} R(-k_i) \xrightarrow{\gamma} G/I \to 0, \tag{5.3}$$

where ϕ is the $n \times 2$ matrix $(h_{i,j})_{1 \leq i \leq n, 0 \leq j \leq 2}$ appearing in (5.2).
Our interest in the ideal G/I is due to the easy equality

$$\mathrm{ann}_R(\mathrm{coker}(\psi \oplus \phi)) = \mathrm{ann}_R(G/I) = (I : G),$$

which leads us to the following theorem of Buchsbaum-Eisenbud (see [BE]) :

Proposition 5.24. *Let S be a noetherian ring and $\alpha : S^m \to S^n$ be a morphism with $m \geq n$. Then*

$$\mathrm{ann}_S(coker(\alpha))^n \subseteq I_n(\alpha) \subseteq \mathrm{ann}_S(coker(\alpha)),$$

where $I_n(\alpha)$ denotes the ideal generated by the $n \times n$ minors of the matrix of α. Moreover, if $\mathrm{depth}(I_n(\alpha)) = m - n + 1$*, then $I_n(\alpha) = \mathrm{ann}_S(coker(\alpha))$.*

In our context this proposition tells us that if $I_n(\psi \oplus \phi)$ has the expected codimension $(n+2) - n + 1 = 3$, then it equals the ideal $(I : G)$. Moreover, if it is the case, we would know a free resolution of $(I : G)$:

Proposition 5.25. *Let S be a noetherian ring and $\alpha : S^m \to S^n$ be a morphism with $m \geq n$. The Eagon-Northcott complex* $\mathrm{EN}(\alpha)$ *of the map α is exact (and thus gives a free resolution of $S/I_n(\alpha)$) if and only if* $\mathrm{depth}(I_n(\alpha)) = m - n + 1$*, the expected one (that is the greatest possible).*

Let us stop a moment on this well-known complex. Notice that if $n = 1$, it is just the Koszul complex associated to the sequence formed with the elements of the row-matrix $S^m \to S$. In fact, as the Koszul complex, the Eagon-Northcott complex of a given map α is built from the *trivial* syzygies of α. To illustrate this, suppose that α is the $n \times (n+1)$ matrix

$$\alpha = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n+1} \end{pmatrix}.$$

The first map of $\mathrm{EN}(\alpha)$ is $\wedge^n \alpha : \wedge^n(S^{n+1}) \to \wedge^n(S^n)$ which, in terms of basis, sends the element $e_{i_1} \wedge \ldots \wedge e_{i_n}$ to the determinant $\Delta_{i_1,\ldots,i_n}$ of the submatrix of α corresponding to the columns $i_1, \ldots, i_n$. Now if we look for the trivial syzygies of these $\binom{m}{n}$ determinants Δ's, we have to choose one row of α (we have n possibilities), say the row number i, and write that the determinant of the matrix (which is α plus the row i)

$$\begin{vmatrix} a_{i,1} & \cdots & a_{i,n+1} \\ a_{1,1} & \cdots & a_{1,n+1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n+1} \end{vmatrix} = a_{i,1}\Delta_{2,\ldots,n+1} - a_{i,2}\Delta_{1,3,\ldots,n+1} + \ldots = 0.$$

In this way we obtain n trivial syzygies of the Δ's and hence a map $S_1(S^{n*}) \to \wedge^n(S^{n+1})$ corresponding to choose a line in α (this explains the *) and to associate it the trivial syzygy we just describe. If we put signs on the Δ's, then it is easy to see that this last map is here nothing but the transpose of α. We have thus constructed the Eagon-Northcott complex, which is in this case often called the Hilbert-Burch complex.

If now we suppose that α is $n \times (n+2)$, we then have to add a line of α (so we have $S_1(S^{n*})$) and then choose $n+1$ columns in the new matrix obtained from α by adding this chosen row, which is $n \times (n+1)$ (this gives $\wedge^{n+1}(S^{n+2})$). We hence obtain the complex

$$S_1(S^{n*}) \otimes \wedge^{n+1}(S^{n+2}) \to \wedge^n(S^{n+2}) \xrightarrow{\wedge^n \alpha} \wedge^n(S^n).$$

The last step is to add another row of α to this last matrix, which corresponds to a choice in $S_2(S^{n*})$. Finally the Eagon-Northcott complex is here given by :

$$0 \to S_2(S^{n*}) \to S_1(S^{n*}) \otimes \wedge^{n+1}(S^{n+2}) \to \wedge^n(S^{n+2}) \xrightarrow{\wedge^n \alpha} \wedge^n(S^n).$$

Returning to our situation. We have :

Theorem 5.26. *The following are equivalent :*

- $\sqrt{(I:G)} = R$
- $I^{sat} = G^{sat}$ *(that is $\mathcal{I} = \mathcal{G}$)*
- $\operatorname{codim}((I:G)) = 3$
- $\mathrm{EN}(\psi \oplus \phi)$ *is acyclic*
- $(I:G)$ *has only trivial syzygies*

We thus have that $\mathrm{EN}(\psi \oplus \phi)$ is not acyclic if and only if $\mathcal{I} \subsetneq \mathcal{G}$, that is the polynomials f_0, f_1, f_2 defines scheme-theoretically a point which is not defined by $\mathcal{G}$.

Suppose that we have an ideal I such that $\operatorname{codim}(I:G) = 3$. By the previous theorem, $\mathrm{EN}(\psi \oplus \phi)$ gives a free graded resolution of R-modules of $R/(I:G)$. It follows that we can bound the regularity of $(I:G)$:

Corollary 5.27. *Suppose that* $\operatorname{codim}(I : G) = 3$*, then* $(I : G)$ *is* ν*-regular for all* $\nu \geq d_0 + d_1 + d_2 - 2(k_n + 1)$ *(recall* $k_n = \min k_i$*).*

Proof. We just have to write the shifting degrees of the Eagon-Northcott complex $\mathrm{EN}(\psi \oplus \phi)$. The considered map is

$$\psi \oplus \phi : E := \bigoplus_{i=1}^{n-1} R(-l_i) \bigoplus_{i=0}^{2} R(-d_i) \longrightarrow F := \bigoplus_{i=1}^{n} R(-k_i),$$

and $\mathrm{EN}(\psi \oplus \phi)$ is the complex :

$$0 \to \wedge^{n+2} E \otimes S_2(F^*) \otimes \wedge^n F^* \to \wedge^{n+1} E \otimes S_1(F^*) \otimes \wedge^n F^*$$
$$\to \wedge^n E \otimes S_0(F^*) \otimes \wedge^n F^* \to R \to R/(I : G) \to 0.$$

The term on the far left has shifting degree $-d_0 - d_1 - d_2 - \sum l_i$ from $\wedge^{n+2} E$, $\sum k_i$ from $\wedge^n F^*$, and finally $k_i k_j$ from $S_2(F^*)$. It comes that $R/(I : G)$ is $(d_0 + d_1 + d_2 - 2k_n - 3)$-regular. □

We can now state the generalization of corollary 5.19 :

Corollary 5.28. *Let* ν *be any integer such that* $\nu \geq d_0 + d_1 + d_2 - 2(k_n + 1)$*, then* $\operatorname{codim}(I : G) \leq 2$ *if and only if the map* $\wedge^n(\psi \oplus \phi)_\nu$ *is not surjective, that is not of full rank* $\binom{\nu+2}{2}$*.*

Proof. Similar proof to corollary 5.19. □

Remark 5.29. This corollary generalizes the result of corollary 5.19 by taking $G = R$ (and hence $k_n = 0$).

This corollary is the starting point of the definition and computation of a resultant with assigned base points in the plane. To do this we have to assume that we can find a sufficiently generic ideal I (that is a generic matrix $(h_{i,j})$ defining polynomials f_0, f_1, f_2) such that $\operatorname{codim}(I : G) \geq 3$. The restriction is on the (local) number of generators of G: indeed, as I is generated by three polynomials, it can not generate an ideal G with more than three generators locally. For instance, with Macaulay2 you can test it :

```
R=QQ[x,y,z]
G=(ideal(x,y))^3
saturate G == G --true
F=ideal(
     random(1,R)*y^3+random(1,R)*x*y^2+random(1,R)*x^2*y+random(1,R)*x^3,
     random(1,R)*y^3+random(1,R)*x*y^2+random(1,R)*x^2*y+random(1,R)*x^3,
     random(1,R)*y^3+random(1,R)*x*y^2+random(1,R)*x^2*y+random(1,R)*x^3)
codim(F:G) --it always returns 2.
```

The usual hypothesis made to be use corollary 5.28 is to suppose that G is a *projective local complete intersection* saturated of codimension 2 (hypothesis which is in fact also made to compare our presentation with another presentation involving blowing-up constructions).

We end with the example of the three circles (5.1) :

$$\left\{\begin{array}{l} f_0 = a_0z^2 + a_1xz + a_2yz + a_3(x^2+y^2) \\ f_1 = b_0z^2 + b_1xz + b_2yz + b_3(x^2+y^2) \\ f_2 = c_0z^2 + c_1xz + c_2yz + c_3(x^2+y^2) \end{array}\right. .$$

Our ideal G is here $G = (z, x^2+y^2)$, two complete intersection points. The resolution of G is the classical Koszul complex and hence $\psi = \begin{pmatrix} x^2+y^2 \\ -z \end{pmatrix}$. The matrix $\psi \oplus \phi$ is hence

$$\begin{pmatrix} x^2+y^2 & a_0z+a_1x+a_2y & b_0z+b_1x+b_2y & c_0z+c_1x+c_2y \\ -z & a_3 & b_3 & c_3 \end{pmatrix}$$

The regularity bound is here $6-4=2$, and hence the matrix $\wedge^2(\psi\oplus\phi)_2$ is a 6×12 matrix of the form :

$$\begin{pmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & -b_1c_3+c_1b_3 & -b_2c_3+c_2b_3 & -c_1a_3+a_1c_3 & \cdots \\ a_1 & b_1 & c_1 & 0 & -c_3b_0+b_3c_0 & 0 & \\ c_2 & b_2 & c_2 & -c_3b_0+b_3c_0 & 0 & a_0c_3-c_0a_3 & \cdots \\ a_3 & b_3 & c_3 & 0 & -b_1c_3+c_1b_3 & 0 & \\ a_3 & b_3 & c_3 & -b_2c_3+c_2b_3 & 0 & -c_2a_3+a_2c_3 & \end{pmatrix}.$$

References

[BE] A. Buchsbaum and David Eisenbud. *What annihilates a module?* J. of Alg. **47**, 231–243, 1977.

[Eis94] D. Eisenbud. *Commutative Algebra with a view toward Algebraic Geometry*, GTM **150**, Springer-Verlag, 1994.

[Eis] D. Eisenbud, *The Geometry of Syzygies*, downloadable from the link http://www.msri.org/people/staff/de/ready.pdf.

[EP] D. Eisenbud and S. Popescu, *Gale Duality and Free Resolutions of Ideals of Points*, preprint math.AG/960618.

[EPSW] D. Eisenbud, S. Popescu, F. Schreyer and C. Walter, *Exterior algebra methods for the Minimal Resolution Conjecture*, preprint, math.AG/0011236.

[Har] R. Hartshorne, *Algebraic Geometry*, GTM **52**, Springer-Verlag, 1977.