Predicates for Line Transversals in 3D

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Hazel Everett* Sylvian Lazard* Bill Lenhart† Jeremy Redburn† Linqiao Zhang‡*

Abstract

In this paper we study various predicates concerning line transversals to lines and segments in 3D. We compute the degrees of standard methods of evaluating these predicates. The degrees of some of these methods are surprisingly high, which may explain why computing line transversals with finite precision is prone to error. Our results suggest the need to explore alternatives to the standard methods of computing these quantities.

1 Introduction

Computing line transversals to lines or segments is an important operation in solving 3D visibility problems arising in computer graphics [3, 5, 6, 7, 8, 11]. In this paper, we study various predicates and their degrees concerning line transversals to lines and segments in 3D.

A predicate is a function that returns a value from a discrete set. Typically, geometric predicates answer questions of the type "Is a point inside, outside or on the boundary of a set?". We consider predicates that are evaluated by boolean functions of more elementary predicates which are functions that return the sign (−, 0 or +) of a multivariate polynomial whose arguments are a subset of the input variables (see, for instance [1]). By *degree* of a procedure for evaluating a predicate, we mean the maximum degree among all polynomials used in the evaluation of the predicate by the procedure. In what follows we casually refer to this measure as the degree of the predicate. We are interested in the degree because it provides a measure of the number of bits required for an exact evaluation of our predicates; the number of bits required is roughly the product of the degree with the number of bits used in representing each input value.

In this paper, we first study the degree of standard predicates for determining the number of line transversals to four lines or four segments; recall that four lines in 3D admit 0, 1, 2 or an infinite number of line transversals and that four segments admit up to 4 or an infinite number of line transversals [2]. We also consider the predicate of determining whether a minimal (i.e., locally shortest) segment transversal to four line segments is occluded by a triangle. Finally, we study the predicate for ordering planes through two fixed points, each containing a third rational point or a line transversal to four segments or lines. This predicate arises in the rotational plane sweep algorithm of Goaoc [9] that computes the maximal free segments tangent to four among k convex polyhedra in 3D. This algorithm performs n rotational sweeps of a plane, one about each edge in the scene, which we call the reference edge. All transversals to the reference edge and three other edges are computed in one sweep. The events of the sweep correspond to planes that contain a vertex not on the reference line (i.e., the line containing the reference edge) or that contain a line transversal to the reference line and three other segments. This algorithm is the asymptotically fastest known for this problem.

Our study shows that standard procedures for solving these predicates have high degree. In particular, we show that determining whether a minimal segment transversal to four line segments is occluded by a triangle can be evaluated by a degree 90 predicate Also, the predicate for comparing, in a rotational sweep, two planes, each defined by a line transversal, can be evaluated by a degree 168 procedure. These very high degrees may help explain why fixed-precision implementations for solving 3D visibility problems are prone to errors when given real-world data.

In the next section we describe the method we use for computing the line transversals to four lines. In Section 3 we describe the predicates and their degrees.

2 Computing lines through four lines

We describe a method for computing the line transversals to four lines. This method is a variant, suggested by Devillers and Hall-Holt [4] and also described in Redburn [12], of that by Hohmeyer and Teller [10]; the difficulty with the latter method is the use of the singular value decomposition for which we only know of numerical methods.

Each line can be described using Plücker coordinates (see [13], for example, for a review of Plücker coordinates). If line l is represented by a direction vector $\vec{u}$ and a point p then we represent l by the six-tuple $[\vec{u}, \vec{u} \times p]$. The side product $\odot$ of any two six-tuples $A = [a_1, a_2, a_3, a_4, a_5, a_6]$ and $B = [b_1, b_2, b_3, b_4, b_5, b_6]$ is $A \odot B = a_1b_4 + a_2b_5 + a_3b_6 + a_4b_1 + a_5b_2 + a_6b_3$. Recall that two lines intersect if and only if the side product of their Plücker coordinates is 0.

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Our problem then is to find all lines $k = [x_1, x_2, x_3, x_4, x_5, x_6]$ such that $k \odot l_i = 0$ for $1 \leqslant i \leqslant 4$ which can be written in the following form:

$$\begin{bmatrix} a_4 & a_5 & a_6 & a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 & b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 & c_1 & c_2 & c_3 \\ d_4 & d_5 & d_6 & d_1 & d_2 & d_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

where the four rows of the matrix contain the Plücker coordinates of the four lines. This can be rewritten as

$$\begin{bmatrix} a_4 & a_5 & a_6 & a_1 \\ b_4 & b_5 & b_6 & b_1 \\ c_4 & c_5 & c_6 & c_1 \\ d_4 & d_5 & d_6 & d_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} a_2x_5 + a_3x_6 \\ b_2x_5 + b_3x_6 \\ c_2x_5 + c_3x_6 \\ d_2x_5 + d_3x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let δ denote the determinant of the above 4 by 4 matrix. Assuming $\delta \neq 0$, we can solve the system for x_1, x_2, x_3, and x_4 in terms of x_5 and x_6. Applying Cramer's rule, we get:

$$\begin{cases} x_1 &= -(\alpha_1 x_5 + \beta_1 x_6)/\delta \\ x_2 &= -(\alpha_2 x_5 + \beta_2 x_6)/\delta \\ x_3 &= -(\alpha_3 x_5 + \beta_3 x_6)/\delta \\ x_4 &= -(\alpha_4 x_5 + \beta_4 x_6)/\delta \end{cases}$$

where α_i (respectively β_i) is the determinant δ with the i^{th} column replaced by $[a_2, b_2, c_2, d_2]^T$ (respectively $[a_3, b_3, c_3, d_3]^T$). We rewrite this system as

$$\begin{cases} x_1 &= \alpha_1 u + \beta_1 v \\ x_2 &= \alpha_2 u + \beta_2 v \\ x_3 &= \alpha_3 u + \beta_3 v \\ x_4 &= \alpha_4 u + \beta_4 v \\ x_5 &= -u\,\delta \\ x_6 &= -v\,\delta \end{cases} \quad (2)$$

Since k is a line, we have $k \odot k = 0$, that is,

$$x_1x_4 + x_2x_5 + x_3x_6 = 0.$$

Substituting in the expressions for $x_1 \dots x_6$, we get

$$Au^2 + Buv + Cv^2 = 0 \quad (3)$$

where

$$\begin{aligned} A &= \alpha_1\alpha_4 + \alpha_2\delta, \\ B &= \alpha_1\beta_4 + \beta_1\alpha_4 + \beta_2\delta + \alpha_3\delta, \\ C &= \beta_1\beta_4 + \beta_3\delta. \end{aligned}$$

Solving this degree-two equation in (u, v) and replacing in (2), we get (assuming for clarity that $A \neq 0$) that the Plücker coordinates of the transversal lines k are:

$$\begin{cases} x_1 &= -B\,\alpha_1 + 2\,A\,\beta_1 \pm \alpha_1\sqrt{B^2 - 4\,A\,C} \\ x_2 &= -B\,\alpha_2 + 2\,A\,\beta_2 \pm \alpha_2\sqrt{B^2 - 4\,A\,C} \\ x_3 &= -B\,\alpha_3 + 2\,A\,\beta_3 \pm \alpha_3\sqrt{B^2 - 4\,A\,C} \\ x_4 &= -B\,\alpha_4 + 2\,A\,\beta_4 \pm \alpha_4\sqrt{B^2 - 4\,A\,C} \\ x_5 &= B\,\delta \pm \delta\sqrt{B^2 - 4\,A\,C} \\ x_6 &= -2\,A\,\delta \end{cases} \quad (4)$$

Number of line transversals. Note that the four input lines admit infinitely many transversals if $A = B = C = 0$ or the 4 by 6 matrix of Plücker coordinates (in (1)) has rank less than four. Otherwise, if $B^2 - 4\,A\,C$ is negative, zero, or positive, the four input lines admit zero, one, or two line transversals, respectively.

Computing points on the line transversals. Denote by w_1 (resp. w_2) the vector of the first (resp. last) three coordinates of $(x_1, \dots, x_6)$, and let n denote any vector. Then, if the four-tuple $(w_2 \times n, w_1 \cdot n)$ is not equal to $(0, 0, 0, 0)$, it is a point (in homogeneous coordinates) on the line k [13]. By considering the axis unit vectors for n, we get that the non-zero four-tuples

$$(0, x_6, -x_5, x_1),\ (-x_6, 0, x_4, x_2),\ (x_5, -x_4, 0, x_3) \quad (5)$$

are points on the transversal lines k. Two of these four-tuples are points on k unless $w_2 = 0$ and only one coordinate of w_1 is non-zero, but then k is one of the axis. Hence we have the following lemma:

Lemma 1 *If four lines, defined by pairs of points, admit finitely many transversal lines, we can compute on each transversal two points whose homogeneous coordinates have the form $\phi_i + \varphi_i\sqrt{\Delta}$, $i = 1, \dots, 4$, where ϕ_i, φ_i, and Δ are polynomials of degree at most 20, 7, and 26, respectively, in the coordinates of the input points.*

Proof. The assumption that the four lines admit finitely many transversals ensures that the 4 by 6 matrix of Plücker coordinates (in (1)) has rank four; hence there is a 4 by 4 sub-matrix of rank four which we can use for solving the system as described above. Moreover, if there are finitely many transversals, then A, B, and C are not all zero and Eq. (3) has at most two real solutions.

We compute the degree, in the coordinates of the input points, of the various polynomial terms defining the points (5). For each input line l_i, the first and last three coordinates of its Plücker representation have degree 1 and 2. Hence δ, α_4, and β_4 have degree 7 and α_i and β_i have degree 6 for $i = 1, 2, 3$. Hence, A, B, and C have degree 13 and bounds on the degrees of ϕ_i, φ_i, and Δ follows.

Finally, it should be noted that, when the 4 by 6 matrix of Plücker coordinates (in (1)) has many 4 by 4 sub-matrices of rank four, the choice of such a sub-matrix has an impact on the degree of ϕ_i, φ_i, and Δ. It is straightforward to observe that the 4 by 4 sub-matrix we considered leads to highest degrees for ϕ_i, φ_i, and Δ. This is necessary since the 4 by 6 matrix of plucker coordinates may have only one 4 by 4 sub-matrix of rank four. $\square$

3 Predicates

3.1 Preliminaries

We start by two straightforward lemmas on the degree of predicates for determining the sign of simple algebraic numbers. [1]

Lemma 2 *If $a, b,$ and c are polynomial expressions of (input) rational numbers, the sign of $a + b\sqrt{c}$ can be determined by a predicate of degree*

$$\max\{2\deg(a), 2\deg(b) + \deg(c)\}.$$

Lemma 3 *If $\alpha_i, \beta_i, \delta, \mu,\ i = 1, 2,$ are polynomial expressions of (input) rational numbers, the sign of*

$$\alpha_1 + \beta_1\sqrt{\delta} + (\alpha_2 + \beta_2\sqrt{\delta})\sqrt{\mu}$$

can be obtained by a predicate of degree

$$\begin{aligned}\max\{4\deg(\alpha_1), 4\deg(\beta_1) + 2\deg(\delta),\\ 4\deg(\alpha_2) + 2\deg(\mu),\\ 4\deg(\beta_2) + 2\deg(\delta) + 2\deg(\mu),\\ 2\deg(\alpha_1) + 2\deg(\beta_1) + \deg(\delta),\\ 2\deg(\alpha_2) + 2\deg(\beta_2) + 2\deg(\mu) + \deg(\delta)\}.\end{aligned}$$

3.2 Transversals to four lines

We consider first the predicate of determining whether four lines admit 0, 1, 2, or inifinitely many line transversals. An evaluation of this predicate directly follows from the algorithm described in Section 2 for computing the line transversals. As mentioned there, the number of transversals follows from the sign of the 4 by 4 sub-determinants of the 4 by 6 matrix of Plücker coordinates of the four lines and from the signs of A, B, C, and $B^2 - 4AC$. The degree of the 4 by 4 sub-determinants is 7 or less, and the degree of A, B, and C is 13 or less (see the proof of Lemma 1). Hence the degree of the predicate is 26 in terms of the coordinates of the points defining the lines. To summarize:

Theorem 4 *Given four lines, there is a predicate of degree 26 in the coordinates of the points defining the lines, to determine whether those lines admit 0, 1, 2, or infinitely many line transversals.*

3.3 Transversals to four segments

Because of the lack of space, we only state our result. [2]

Theorem 5 *Given four line segments, there is a predicate of degree 42 in the coordinates of their endpoints to determine whether those lines admit 0, 1, 2, 3, 4, or infinitely many line transversals.*

[1]See Appendix A for details.
[2]See Appendix B for details.

3.4 Transversals to four segments and a triangle

Given a line transversal ℓ to a set S of segments, a triangle T *occludes* ℓ if ℓ intersects T and if there exist two segments in S whose intersections with ℓ lie on opposite sides of T. Because of the lack of space, we only state our result. [3]

Theorem 6 *Let ℓ be a line tranversal to four line segments admitting finitely many transversals and let $T = pqr$ be a triangle. There is a predicate of degree 90 in the coordinates of the points defining the segments and the triangle to determine whether T occludes ℓ.*

3.5 Ordering planes through two fixed points, each containing a third rational point or a line transversal

Let ℓ be a line defined by two points v_1 and v_2, and $\vec{\ell}$ be the line ℓ oriented in the direction $\overrightarrow{v_1 v_2}$.

We define an ordering of all the planes containing ℓ with respect to the oriented line $\vec{\ell}$ and a reference point O (not on ℓ). Let P_0 be the plane containing O and ℓ, and let P_1 and P_2 be two planes containing ℓ.

We say that $P_1 < P_2$ if and only if P_1 is encountered strictly before P_2 when rotating counterclockwise about $\vec{\ell}$ a plane from P_0 (see Figure 1a).

Let p_i be any point on plane P_i but not on ℓ, for $i = 1, 2$, and let $D(p, q)$ denote the determinant of the four points (v_1, v_2, p, q) given in homogeneous coordinates.

Lemma 7 *With $\chi = D(O, p_1) \cdot D(O, p_2) \cdot D(p_1, p_2)$, we have: if $\chi > 0$, then $P_1 > P_2$*
else if $\chi < 0$, then $P_1 < P_2$
else if $D(p_1, p_2) = 0$, then $P_1 = P_2$
else if $D(O, p_1) = 0$, then $P_1 < P_2$
else $P_1 > P_2$.

Proof. Assume first that $D(O, p_1) \cdot D(O, p_2) > 0$, that is that p_1 and p_2 lie strictly on the same side of the plane P_0 (see Figure 1b). Then the order of P_1 and P_2 is determined by the orientation of the four points (v_1, v_2, p_1, p_2), that is by the sign of $D(p_1, p_2)$. It is then straightforward to notice that $P_1 > P_2$ if and only if $D(p_1, p_2) > 0$. Hence, if $\chi > 0$, then $P_1 > P_2$ and, if $\chi < 0$, then $P_1 < P_2$.

Suppose now that $D(O, p_1) \cdot D(O, p_2) < 0$, that is that p_1 and p_2 lie strictly on opposite sides of the plane P_0 that (see Figure 1b). The order of P_1 and P_2 is then still determined by the sign of $D(p_1, p_2)$. However, $P_1 > P_2$ if and only if $D(p_1, p_2) < 0$. Hence, we have in all cases that, if $\chi > 0$, then $P_1 > P_2$ and, if $\chi < 0$, then $P_1 < P_2$.

Suppose finally that $\chi = 0$. If $D(p_1, p_2) = 0$, then p_1 and p_2 are coplanar, and $P_1 = P_2$. Otherwise, if $D(O, p_1) = 0$, then $P_0 = P_1$ thus P_1 is smaller to all

[3]See Appendix C for details.

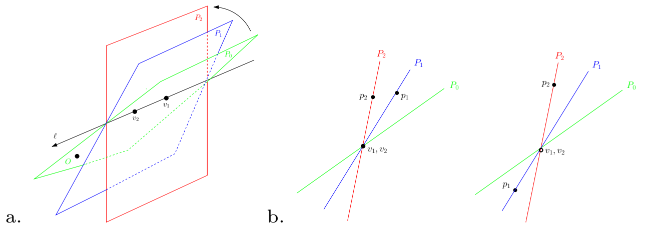

Figure 1. $P_1 < P_2$

other planes (containing $\vec{\ell}$), and in particular $P_1 \leqslant P_2$. Furthermore, since $D(p_1, p_2) \neq 0$, $P_1 \neq P_2$ and thus $P_1 < P_2$. Otherwise, $D(O, p_2) = 0$ and we get similarly that $P_2 < P_1$. □

Comparing two planes. We want to order planes P_i that are defined by either line ℓ and another (input) rational point not on ℓ, or by line ℓ and a line transversal to ℓ and three other lines.

By Lemma 7, ordering such planes about ℓ amounts to computing the sign of determinants of four points (in homogeneous coordinates). Two of these points are input (non-homogeneous) rational points on ℓ (v_1 and v_2) and each of the two other points is either an input (non-homogeneous) rational point r_i, $i = 1, 2$, or is, by Lemma 1, a point of the form $p_i + q_i\sqrt{\Delta_i}$, $i = 1, 2$, where the Δ_i have degree 26 and where the p_i and q_i are points with homogeneous coordinates of degree 20 and 7 (in the coordinates of the input points). If the four points are all input rational points, then the determinant of the four points has degree 3. If only three of the four points are input rational points, then the determinant of the four points can be expanded into

$$D(p_1, r_1) + D(q_1, r_1)\sqrt{\Delta_1}$$

where the degrees of the $D()$ are 23 and 10, respectively. Hence, by Lemma 2, the sign of this expression can be determined with a predicate of degree 46. Finally, if only two of the four points are input rational points, then the determinant can be expanded into

$$\begin{aligned} &D(p_1, p_2) + D(q_1, p_2)\sqrt{\Delta_1} \\ &\qquad + \left(D(p_1, q_2) + D(q_1, q_2)\sqrt{\Delta_1}\right)\sqrt{\Delta_2}, \end{aligned}$$

where the degrees of the $D()$ are, in order, 42, 29, 29, and 16. Hence, by Lemma 3, the sign of this expression can be determined with a predicate of degree 168. We thus get the following result

Theorem 8 *Let l be an oriented line specified by two rational points, let p_0 be a rational point not on l, and let P_0 be the plane determined by l and p_0. Given two planes P_1, P_2 containing l there is a predicate which determines the relative order of P_1 and P_2 about l with respect to P_0 having degree*

- *3 if $P_i, i = 1, 2$ are each specified by l and some rational point p_i;*
- *46 if P_1 is specified by l and some rational point p_1 and P_2 is specified by a line transversal to l along with three other lines l_1, l_2, l_3, each specified by two rational points;*
- *168 if $P_i, i = 1, 2$ are each specified by a line transversal to l along with three other lines $l_{i,1}, l_{i,2}, l_{i,3}$, each specified by two rational points.*

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A Determining the sign of simple algebraic numbers

In this section we prove Lemmas 2 and 3 by computing the degree of standard predicates for computing the signs of quantities of the form $a + b\sqrt{c}$, where c is rational, but a and b may be of the form $\alpha + \beta\sqrt{\delta}$, where α and β are rationals (and $\sqrt{\delta}$ is not necessarily a rational multiple of $\sqrt{c}$). We start by two straightforward lemmas which yield Lemma 2.

Lemma 9 *Let a, b, c be real numbers with $c \geqslant 0$, and let $E = a + b\sqrt{c}$. Then $sign(E)$ can be determined by the rule:*
if $sign(a)sign(b) = 1$ then $sign(E) = sign(a)$
else if $sign(a^2 - b^2c) = 0$ then $sign(E) = 0$
else if $sign(a^2 - b^2c) = 1$ then $sign(E) = sign(a)$
else $sign(E) = sign(b)$.

Lemma 10 *If $a, b,$ and c are polynomial expressions of (input) rational numbers, the degree in these rational numbers of the predicate of Lemma 9 is*

$$\max\{2\deg(a), 2\deg(b) + \deg(c)\}.$$

For computing the sign of an expression of the form $a + b\sqrt{\mu}$, where, $a = \alpha_1 + \beta_1\sqrt{\delta}$, $b = \alpha_2 + \beta_2\sqrt{\delta}$, and $\alpha_i, \beta_i, \mu, \delta$ are rational, we can repeatedly apply Lemma 9: we need to evaluate the signs of a, b, and $a^2 - b^2\mu$. The first two signs can be obtained by directly applying the lemma; to determine the sign of $a^2 - b^2\mu$, we first substitute and expand:

$$\begin{aligned} a^2 - b^2\mu &= (\alpha_1 + \beta_1\sqrt{\delta})^2 - (\alpha_2 + \beta_2\sqrt{\delta})^2\mu \\ &= A + B\sqrt{\delta}, \end{aligned}$$

where

$$\begin{aligned} A &= \alpha_1^2 + \beta_1^2\delta - \alpha_2^2\mu - \beta_2^2\mu\delta \quad \text{and} \\ B &= 2\alpha_1\beta_1 - 2\alpha_2\beta_2\mu. \end{aligned}$$

The sign of $A + B\sqrt{\delta}$ can be determined by another application of Lemma 9. The degree of this predicate in terms of the degree of $\alpha_i, \beta_i, \delta, \mu$, which follows from Lemma 10, is

$$\begin{aligned} \max\{&4\deg(\alpha_1), 4\deg(\beta_1) + 2\deg(\delta), \\ &4\deg(\alpha_2) + 2\deg(\mu), \\ &4\deg(\beta_2) + 2\deg(\delta) + 2\deg(\mu), \\ &2\deg(\alpha_1) + 2\deg(\beta_1) + \deg(\delta), \\ &2\deg(\alpha_2) + 2\deg(\beta_2) + 2\deg(\mu) + \deg(\delta)\}. \end{aligned}$$

This concludes the proof of Lemma 3.

B Transversals to four segments

We prove here Theorem 5 which states:

> *Given four line segments, there is a predicate of degree 42 in the coordinates of their endpoints to determine whether those lines admit 0, 1, 2, 3, 4, or infinitely many line transversals.*

Recall that four segments may admit up to 4 or infinitely many line transversals by [2].

We first consider the case where the four lines containing the four segments admit finitely many transversals; this can be determined by a predicate of degree 26 by Theorem 4. Let ℓ denote one of the at most two transversals to the four lines. We consider the predicate of determining whether ℓ intersects each of the four segments, in turn. Assume that all four segments are defined by their endpoints, and let p and q denote the endpoints of one of these segments.

Let $\bar{\ell}$ denote the Plücker coordinates of the line ℓ oriented arbitrarily. For any two distinct points p and q, denote $\overline{pq}$ the Plücker coordinates of the line pq oriented along $\overrightarrow{pq}$.

If a point o does not lie in the plane containing line ℓ and segment pq, then line ℓ intersects segment pq if and only the (oriented) line ℓ is on the same side of the two oriented lines from p to o and from o to q, that is if $(\bar{\ell} \odot \bar{p}\bar{o})\ (\bar{\ell} \odot \bar{o}\bar{q}) \geqslant 0$.

On the other hand, point o lies in the plane containing line ℓ and segment pq if and only if ℓ intersects both lines op and oq, that is both side operators $\bar{\ell} \odot \bar{o}\bar{p}$ and $\bar{\ell} \odot \bar{o}\bar{q}$ are zero. By choosing point o to be for instance $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$, we ensure that one of the points will not be coplanar with ℓ and segment pq.

Hence the predicate follows from the sign of side operators of the line transversal and of a line defined by two points, one of which with coordinates equal to 0 or 1. The degree of the Plücker coordinates of the line through these two points is thus 1 (in the coordinates of the input points). Hence, by Lemma 1, the predicate resolves to computing the sign of expressions of the form $a + b\sqrt{c}$ where a, b and c have degree 21, 8, and 26, respectively. By Lemma 9, the predicate thus have degree 42. We thus get the following lemma.

Lemma 11 *Given four segments by their endpoints, if their supporting lines admit at most two line transversals, then determining whether these transversals intersect each of the four segments can be done by a predicate of degree 42.*

In the case where the four lines admit infinitely many transversals, either the four segments also admit infinitely many transversals or each of the at most four transversals go through two endpoints of the segments [2]. A straightforward case analysis following the proof in [2] shows that determining whether each of the lines through two segment endpoints is an isolated transversal to the four segments can be done with a predicate of degree no more than 42.

C Transversals to four segments and a triangle

We prove here Theorem 6 which states:

> *Let ℓ be a line tranversal to four line segments admitting finitely many transversals and let $T = pqr$ be a triangle. There is a predicate of degree 90 in the coordinates of the points defining the segments and the triangle to determine whether T occludes ℓ.*

Recall that given a line transversal ℓ to a set S of segments, a triangle T *occludes* ℓ if ℓ intersects T and if there exist two segments in S whose intersections with ℓ lie on opposite sides of T.

We describe a method for evaluating the predicate of determining whether a triangle with rational vertices occludes a transversal to a given set of rational line segments and establish its degree.

Let ℓ be a line transversal to rational segments $s_i = e_i f_i, i = 1, \dots, 4$ and let $T = \triangle pqr$ be a triangle with rational vertices. We study the degree of the predicate in the case where the four lines containing segments s_i have finitely many transversals because, otherwise, if the four segments admit finitely many transversals, then each transversal goes through two distinct endpoints of four segments [2] and the degree of the predicate is then smaller.

We first determine whether ℓ intersects T by taking the side product of ℓ with each supporting line of T; ℓ intersects T iff no two side products have *opposite* signs (i.e. ± 1). Similarly as in Section 3.3, there is a predicate of degree 44 for determining the sign of these side operators.

Assuming that ℓ intersects T, we next find the point of intersection. Assuming that ℓ is represented parametrically in the form $\pi + \rho t$, we need to find the value of t for which the determinant of $p, q, r, \pi + \rho t$ equals 0. Denote this value of t by t_0. This determinant has the form $a_0 + b_0 t_0$, where a_0 and b_0 each have the form $\alpha + \beta\sqrt{\Delta}$ where α, β, and Δ have degree 23, 10, and 26, respectively, in the coordinates of p, q, r, e_i, f_i, by Lemma 1.

Now for each segment s_i, we compute the point of intersection of s_i with ℓ in terms of the parameter t using the method similar to that of the previous section: Choose a rational point o_i not in the plane determined by s_i and ℓ and compute the value t for which the determinant of $e_i, f_i, o_i, \pi + \rho t$ equals 0. Denote this value by t_i. As in the previous paragraph, each of these determinants have the form $a_i + b_i t_i$ where a_i and b_i also have the form $\alpha + \beta\sqrt{\Delta}$. Here, however, α, β, and Δ have degree 22, 9, and 26, respectively, because o_i can be chosen with all coordinates equal to 0 and 1.

Determining whether T occludes ℓ is now only a matter of determining whether t_0 lies between two of the values $t_i, i = 1, \dots, 4$, which requires only that we be able to compare t-values, that is, compute $\operatorname{sign}(t_i - t_j)$. Observe that $t_i - t_j < 0$ iff $\frac{b_j a_i - b_i a_j}{a_i a_j} < 0$, so $\operatorname{sign}(t_i - t_j) = \operatorname{sign}(b_j a_i - b_i a_j)\operatorname{sign}(a_i)\operatorname{sign}(a_j)$. As mentioned above, each of the a_i and b_i is of the form $\alpha + \beta\sqrt{\Delta}$, where α has degree 22 or 23, β has degree 9 or 10, and Δ has degree 26. A product of two such values also has form $\alpha + \beta\sqrt{\Delta}$, but now α has degree 45, β has degree 32, and Δ has degree 26. Applying Lemma 2 we get degree 90 for computing the necessary signs. We thus get the result.