Strong bi-homogeneous Bézout theorem and its use in effective real algebraic geometry

Mohab Safey El Din, Philippe Trebuchet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Strong bi-homogeneous Bézout theorem and its use in effective real algebraic geometry

Mohab Safey El Din — Philippe Trébuchet

N° 6001

Octobre 2006

Thème SYM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--6001--FR+ENG

Strong bi-homogeneous Bézout theorem and its use in effective real algebraic geometry

Mohab Safey El Din*, Philippe Trébuchet†

Thème SYM — Systèmes symboliques
Projets SALSA

Rapport de recherche n° 6001 — Octobre 2006 — 43 pages

Abstract: Let $(f_1, \ldots, f_s)$ be a polynomial family in $\mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leqslant n-1$) of degree bounded by D. Suppose that $\langle f_1, \ldots, f_s \rangle$ generates a radical ideal, and defines a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Consider a projection $\pi : \mathbb{C}^n \to \mathbb{C}$. We prove that the degree of the critical locus of π restricted to $\mathcal{V}$ is bounded by $D^s(D-1)^{n-s}\binom{n}{n-s}$. This result is obtained in two steps. First the critical points of π restricted to $\mathcal{V}$ are characterized as projections of the solutions of Lagrange's system for which a bi-homogeneous structure is exhibited. Secondly we prove a bi-homogeneous Bézout Theorem, which bounds the sum of the degrees of the equidimensional components of the radical of an ideal generated by a bi-homogeneous polynomial family. This result is improved when $(f_1, \ldots, f_s)$ is a regular sequence. Moreover, we use Lagrange's system to design an algorithm computing at least one point in each connected component of a smooth real algebraic set. This algorithm generalizes, to the non equidimensional case, the one of Safey El Din and Schost. The evaluation of the output size of this algorithm gives new upper bounds on the first Betti number of a smooth real algebraic set. Finally, we estimate its arithmetic complexity and prove that in the worst cases it is polynomial in n, s, $D^s(D-1)^{n-s}\binom{n}{n-s}$ and the complexity of evaluation of $f_1, \ldots, f_s$.

Key-words: computer algebra, polynomial system solving, effective real algebraic geometry

* `Mohab.Safey@lip6.fr`
† `Philippe.Trebuchet@lip6.fr`

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Théorème de Bézout bi-homogène fort et applications en géométrie algébrique réelle effective

Résumé : Soit $(f_1, \ldots, f_s)$ une famille de polynômes dans $\mathbb{Q}[X_1, \ldots, X_n]$ (où $s \leqslant n-1$) de degré borné par D. On suppose que $\langle f_1, \ldots, f_s \rangle$ engendre un idéal radical, et définit une variété algébrique lisse $\mathcal{V} \subset \mathbb{C}^n$. Considérons une projection $\pi : \mathbb{C}^n \to \mathbb{C}$. On prouve que le degré du lieu critique de π restreinte à $\mathcal{V}$ est borné par $D^s(D-1)^{n-s}\binom{n}{n-s}$. Ce résultat est obtenu en deux temps. Tout d'abord, on caractérise les points critiques de π restreinte à $\mathcal{V}$ comme projections des solutions du système de Lagrange pour lequel on exhibe une structure bi-homogène. Puis, on prouve une théorème de Bézout bi-homogène, qui borne la somme des degrés des composantes équi-dimensionnelles du radical d'un idéal engendré par une famille de polynômes bi-homogènes. Ce résultat est amélioré dans le cas où $(f_1, \ldots, f_s)$ est une suite régulière. De plus, on utilise la formulation Lagrangienne pour décrire un algorithme calculant au moins un point par composante connexe d'une variété algébrique réelle lisse. Cet algorithme généralise au cas non équi-dimensionnel celui de Safey El Din et Schost. L'estimation de la taille de la sortie de notre algorithme donne de nouvelles (et meilleures) bornes sur le premier nombre de Betti d'une variété algébrique réelle lisse. Finalement, on montre qu'une instance probabiliste de notre algorithme est de complexité polynomiale en n, s, $D^s(D-1)^{n-s}\binom{n}{n-s}$ et la complexité d'évaluation de $f_1, \ldots, f_s$.

Mots-clés : calcul formel, résolution de systèmes polynomiaux, géométrie algébrique réelle effective

1 Introduction

Consider polynomials $(f_1, \dots, f_s)$ in $\mathbb{Q}[X_1, \dots, X_n]$ (with $s \leqslant n-1$) of degree bounded by D. Suppose that this polynomial family generates a radical ideal and defines a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. The core of this paper is to give an optimal bound on the degree of the critical locus of a projection $\pi : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathcal{V}$ and to provide an algorithm computing at least one point in each connected component of the real algebraic set $\mathcal{V} \cap \mathbb{R}^n$ whose worst case complexity is polynomial in this bound.

Motivation and description of the problem. Since computing critical points solves the problem of algebraic optimization, it has many applications in chemistry, electronics, financial mathematics (see [15] for a non-exhaustive list of problems and applications). More traditionally, computations of critical points are used in effective real algebraic geometry to compute at least one point in each connected component of a real algebraic set. Indeed, every polynomial mapping restricted to a smooth compact real algebraic set reaches its extrema on each connected component. Thus, computing the critical locus of such a mapping provides at least one point in each connected component of a smooth compact real algebraic set.

In [22, 25, 26, 9, 10], the authors consider the hypersurface defined by $f_1^2 + \cdots + f_s^2 = 0$ to study the real algebraic set $\mathcal{V} \cap \mathbb{R}^n$. The problem is reduced, via several infinitesimal deformations, to a smooth and compact situation. Such techniques yield algorithms returning zero-dimensional algebraic sets encoded by rational parameterizations of degree $\mathcal{O}(D)^n$ (the best bound is obtained in [9, 10] and is $(4D)^n$). Similar techniques based on the use of a distance function to a generic point and a single infinitesimal deformation (see [34]) yield the bound $(2D)^n$ on the output.

More recently, other algorithms, avoiding the sum of squares (and the associated growth of degree) have been proposed (see [2, 35, 7, 4, 5, 38, 37]). The compactness assumption is dropped either by considering distance functions and their critical locus (see [2, 35, 7, 6]) or by ensuring properness properties of some projection functions (see [37]). These algorithms compute the critical loci of polynomial mappings restricted to *equidimensional* and smooth algebraic varieties of dimension d, defined by polynomial systems generating radical ideals. Indeed, under these assumptions, critical points can be algebraically defined as points where the jacobian matrix has rank $n-d$, and thus by the vanishing of some minors of the considered jacobian matrix. On the one hand, some of these algorithms allow us to obtain efficient implementations (see [36]) while the algorithms mentioned in the above paragraph do not permit to obtain usable implementations. On the other hand, applying the classical Bézout bound to the polynomial systems defining the critical locus of a projection provides a degree bound on the output which is equal to $D^{n-d}\left((n-d)(D-1)\right)^d$ (see [37, 7] where such a bound is explicitly mentioned). This bound is worse than the aforementioned bounds, but it has never been reached in the experiments we performed with our implementations.

Remark that these polynomial systems defining critical points are not generic: they are overdetermined, and the extracted minors from the jacobian matrix depend on $f_1, \dots, f_s$, so that one can hope that the classical Bézout bound is pessimistic.

Our aim is twofold:

- providing an optimal bound (in the sense that it can be reached) on the degree of the critical locus of a polynomial mapping restricted to an algebraic variety;
- and designing an algorithm computing at least one point in each connected component of a real algebraic set whose worst case complexity is polynomial in this bound.

Main contributions. Consider a projection π from $\mathbb{C}^n$ onto a line whose base-line vector is $\mathbf{e} \in \mathbb{C}^n$, and the restriction of π to the smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. *Lagrange's characterization* of critical points of π restricted to $\mathcal{V}$ consists in writing that, at a critical point, there exists a linear relation between the vectors $(\mathbf{grad}(f_1), \dots, \mathbf{grad}(f_s), \mathbf{e})$. The resulting polynomial system is called in the sequel *Lagrange's system*. Additional variables are introduced and are classically called *Lagrange multipliers*. Equipped with such a characterization, critical points can be geometrically seen as *projections of the complex solution set of Lagrange's system*. We prove that such a characterization remains valid *even in non equidimensional situations*, contrary to the algebraic characterization of critical points used in [2, 35, 7, 4, 5, 38, 37]. If $f_1, \dots, f_s$ is a regular sequence and if the critical locus of π restricted to $\mathcal{V}$ is zero-dimensional, then Lagrange's system is a zero-dimensional ideal, else it is a positive dimensional ideal.

Since Lagrange multipliers appear with degree 1 in Lagrange's system, bounding the degree of the critical locus of π restricted to $\mathcal{V}$ is equivalent to bounding the sum of the degrees of the isolated primary components of the ideal generated by Lagrange's system. Lagrange's system can be easily transformed into a bi-homogeneous polynomial system by a bi-homogenization process which distinguish the variables $X_1, \dots, X_n$ from the Lagrange multipliers. Thus, bounding the degree of a critical locus is reduced to proving a strong bi-homogeneous Bézout Theorem, i.e. to proving a bound on the *sum of the degrees of all the isolated primary components defining a non-empty bi-projective variety* of an ideal generated by a bi-homogeneous system. In the sequel, this sum is called the strong bi-degree of a bi-homogeneous ideal. Such a result is obtained by using a convenient notion of *bi-degree*, which is given originally in [41]. In [41], a multi-homogeneous Bézout theorem is proved and provides a bound on the sum of the degrees of the isolated primary components of *maximal dimension*, which is not sufficient to reach our goal.

We generalize this result by proving that the same quantity bounds the sum of the degrees of all the isolated primary components defining a bi-projective variety of an ideal defined by a bi-homogeneous polynomial system (see Theorem 1). Additionally, we prove that the strong bi-degree of an ideal generated by a bi-homogeneous polynomial system equals the strong degree of this ideal augmented with two generic homogeneous affine linear forms lying respectively in each block of variables (see Theorem 2).

This allows us to prove that the critical locus of a projection $\pi : \mathbb{C}^n \to \mathbb{C}$ restricted to $\mathcal{V}$ has degree $D^s(D-1)^{n-s}\binom{n}{n-s}$. This bound becomes $D^s(D-1)^{n-s}\binom{n-1}{n-s}$ when $(f_1, \dots, f_s)$ is a regular sequence (see Theorem 3). Some computer simulations show it is a sharp estimation, since the bound is reached on several examples.

Next, we use the aforementioned properties of Lagrange's systems to generalize, to non equidimensional situations, the algorithm due to Safey El Din and Schost (see [37]) which computes at least one point in each connected component of a smooth and equidimensional real algebraic set (see Theorem 5). Then, the estimation of the output size of this generalized algorithm provides some improved upper bounds on the first Betti number of a smooth real algebraic set (see Theorem 6).

The complexity of our algorithm depends on the complexity of the routine used to perform algebraic elimination. We consider the elimination subroutine of [28] which inherits of [17, 18, 19, 20]. The procedure of [28] comnputes generic fibers of each equidimensional components of an algebraic variety defined by a polynomial system provided as input. It is polynomial in the evaluation complexity of the input system, and in an intrinsic geometric degree. This allows us to prove that the worst case complexity of our algorithm is polynomial in n, s, the evaluation complexity L of $(f_1, \ldots, f_s)$ and the bi-homogeneous Bézout bound $D^s(D-1)^{n-s}\binom{n}{n-s}$ (see Theorem 8).

Organization of the paper. The paper is organized as follows. In Section 2, we prove the strong bi-homogeneous Bézout Theorem. Additionally, we prove that the strong bi-degree of an ideal equals the strong degree of the same ideal augmented with two generic affine linear forms lying in each block of variables. In Section 3, we focus on the properties of Lagrange's system and use our Bézout Theorem to prove some bounds on the degree of critical loci of projections on a line. In Section 4, we generalize the algorithm provided in [37] to the non equidimensional case. Moreover, using the results of the preceding sections, we provide some improved upper bounds on the first Betti number of a smooth real algebraic set. The last section is devoted to the complexity estimation of our algorithm.

Acknowledgments. We thank D. Lazard, J. Heintz and the anonymous referees for their helpful remarks which have allowed us to improve and correct some mistakes in the preliminary version of this paper.

2 Strong Bi-homogeneous Bézout theorem

This section is devoted to the proof of the strong bi-homogeneous Bézout Theorem. This one generalizes the statements of [40, 41, 31, 32, 24].

In the first paragraph, we provide some useful properties of bi-homogeneous ideals which allow to define the notions of bi-degree and strong bi-degree of a bi-homogeneous ideal, these notion seems to go back to [40]. In the second paragraph, we relate this bi-degree with some properties of the Hilbert bi-series of a bi-homogeneous ideal in the case when it defines a zero-dimensional bi-projective variety. Such properties are already given in [40]. In the third paragraph, we provide a canonical form of Hilbert bi-series. Our statements generalize slightly those of [31, 32] and allow us to relate the Hilbert bi-series of a bi-homogeneous ideal and the Hilbert series of this ideal, seen as a homogeneous one. In the third paragraph, we investigate how the bi-degree of a bi-homogeneous ideal augmented with a bi-homogeneous polynomial is related to the bi-degree of the first considered ideal. Finally, we prove that, under some assumptions, the classical bi-homogeneous Bézout bound is greater than or equal to the sum of the bi-degrees of the prime ideals associated to the studied bi-homogeneous ideal. This generalizes the results of [40, 31, 32, 24]. Additionally, we prove that this quantity is greater than or equal to the sum of the degrees of the primary ideals associated to the studied bi-homogeneous ideal augmented with two affine linear forms which generalizes the results of [41, 32]. For completeness and readibility, we give full proofs of the required intermediate results.

We denote by R the polynomial ring $\mathbb{Q}[X_0, \dots, X_n, \ell_0, \dots, \ell_k]$.

2.1 Preliminaries and main results

In this paragraph, we introduce some basics on bi-homogeneous ideals. We prove that given a bi-homogeneous ideal $I \subset R$, there exists a minimal primary bi-homogeneous decomposition of I, i.e. a primary decomposition such that each primary component is a bi-homogeneous ideal. We define a notion of *bi-degree* and *strong bi-degree* of bi-homogeneous ideals. Finally, we state the main results (see Theorems 1 and 2 below) of the section which bound the strong bi-degree of a bi-homogeneous ideal I by the classical bi-homogeneous Bézout bound on the one hand, and show that the *strong bi-degree* of I bounds the degree of $I + \langle u-1, v-1\rangle$ (where u and v are homogeneous linear forms respectively chosen in $\mathbb{Q}[X_0, \dots, X_n]$ and $\mathbb{Q}[\ell_0, \dots, \ell_k]$).

Definition 1 *A linear form in R is a polynomial of degree 1 whose support contains only monomials of degree at most 1. A homogeneous linear form is a linear form whose support contains* only *monomials of degree 1.*

A polynomial f in R is said to be bi-homogeneous if and only if there exists a unique *couple of integers (α, β) such that for all $(u, v) \in \mathbb{Q} \times \mathbb{Q}$:*

$$f(uX_0, \dots, uX_n, v\ell_0, \dots, v\ell_k) = u^\alpha v^\beta f(X_0, \dots, X_n, \ell_0, \dots, \ell_n).$$

The couple (α, β) is called the bi-degree *of f.*

Given a polynomial $f \in R$, the bi-homogeneous component of bi-degree (α, β) of f, denoted by $f_{\alpha,\beta}$, is the unique bi-homogeneous polynomial such that the support of $f - f_{\alpha,\beta}$ does not contain any monomial of bi-degree (α, β).

An ideal $I \subset R$ is said to be a bi-homogeneous *ideal if and only if for all $f \in I$, and for all bi-degree (α, β), $f_{\alpha,\beta} \in I$.*

Lemma 1 *Let $I \subset R$ be a* bi-homogeneous *ideal. Then, there exists a finite polynomial family $f_1, \dots, f_s$ generating I such that each f_i (for $i = 1, \dots, s$) is a bi-homogeneous polynomial.*

Proof. Consider a finite set of generators $\mathcal{F}$ of I (there exists one since R is Noetherian). Since I is a bi-homogeneous ideal, the *finite* set $\widetilde{\mathcal{F}}$ of the bi-homogeneous components of all the polynomials in $\mathcal{F}$ generates an ideal J which is contained in I.

Consider now $f \in I$. Since I is generated by $\mathcal{F}$ and since each polynomial of $\mathcal{F}$ is a linear combination of some polynomials in $\widetilde{\mathcal{F}}$, I is contained in J.

□

Lemma 2 *Let $Q_1, \ldots, Q_p$ be a family of bi-homogeneous ideals in R. Then $Q_1 \cap \cdots \cap Q_p$ is a bi-homogeneous ideal.*

Proof. Denote by I the ideal $Q_1 \cap \cdots \cap Q_p$ and consider $f \in I$. For all $i \in \{1, \ldots, p\}$, $f \in Q_i$ and Q_i is, by assumption, bi-homogeneous. Then, for all $i \in \{1, \ldots, p\}$, each bi-homogeneous component of f belongs to Q_i, which implies that each bi-homogeneous component belongs to I.

□

Given a bi-homogeneous ideal $I \subset R$, we now prove that there exists a primary decomposition of I for which each primary component is bi-homogeneous.

Proposition 1 *Let I be a bi-homogeneous ideal, and $Q_1 \cap \cdots \cap Q_p$ be a minimal primary decomposition of I. Then there exist* primary *bi-homogeneous ideals $Q'_1, \ldots, Q'_p$ such that $I = Q'_1 \cap \cdots \cap Q'_p$.*

Proof. For $i = 1, \ldots, p$, consider Q_i, a primary ideal of the above minimal primary decomposition of I and let Q'_i be the ideal generated by the bi-homogeneous polynomials of Q_i. First, remark that Q'_i is non empty since it contains I. We prove now that Q'_i (for $i = 1, \ldots, p$) is a primary ideal and then that $I = Q'_1 \cap \cdots \cap Q'_p$.

Let f and g be two polynomials such that $fg \in Q'_i$ and $g \notin Q'_i$. We show below that this implies there exists an integer N such that $f^N \in Q'_i$ which, in turn, implies Q'_i is a primary ideal. The proof is done by induction on the number h of bi-homogeneous components of f.

If $h = 1$, f is bi-homogeneous, and the result is obvious. Suppose now that for any polynomial $\widetilde{f}$ having h bi-homogeneous components, $\widetilde{f}g \in Q'_i$ with $g \notin Q'_i$ implies there exists an integer N such that $\widetilde{f}^N \in Q'_i$.

Consider f having $h+1$ bi-homogeneous components and such that there exists $g \notin Q'_i$ with $fg \in Q'_i$. Since Q'_i is bi-homogeneous, each bi-homogeneous component of fg belongs to Q'_i. Remark that there exists one bi-homogeneous component of the product fg of maximal degree which can be written as a product $f_{\alpha,\beta} . g_{\alpha',\beta'}$ where $f_{\alpha,\beta}$ (resp. $g_{\alpha',\beta'}$) is a bi-homogeneous component of bi-degree (α, β) (resp. (α', β')) of f (resp. g). Moreover, without loss of generality one can suppose $g_{\alpha',\beta'} \notin Q_i$: if it is not the case, it is sufficient to substitute g by $g - g_{\alpha',\beta'}$.

Since Q_i is a primary ideal, there exists $M \in \mathbb{N}$, such that $f_{\alpha,\beta}^M \in Q_i$. Moreover, since $f_{\alpha,\beta}$ is bi-homogeneous, $f_{\alpha,\beta}^M$ is bi-homogeneous. Thus, since Q'_i is generated by the bi-homogeneous polynomials of Q_i, this implies $f_{\alpha,\beta}^M \in Q'_i$.

Since $fg \in Q'_i$, $f^M g \in Q'_i$. Moreover $f^M g = (f - f_{\alpha,\beta})^M g + f_{\alpha,\beta}^M g$ while $f_{\alpha,\beta}^M \in Q'_i$. This implies $(f - f_{\alpha,\beta})^M g \in Q'_i$. Suppose now that M is the smallest integer such that $(f - f_{\alpha,\beta})^M g \in Q'_i$ and remark that this implies $(f - f_{\alpha,\beta})^{M-1} g \notin Q'_i$. Thus, the above reasoning can be done using $(f - f_{\alpha,\beta})$ (which has h bi-homogeneous component) instead of

f and $\widetilde{g} = (f - f_{\alpha,\beta})^{M-1} g \notin Q'_i$ instead of g. From the induction hypothesis, this implies there exists an integer M' such that $(f - f_{\alpha,\beta})^{M'} \in Q'_i$, while the integer M is such that $f_{\alpha,\beta}^M \in Q'_i$. Thus, considering the binomial development of $(f - f_{\alpha,\beta} + f_{\alpha,\beta})^N$ where $N > 2M$ shows that $f^N \in Q'_i$.

It remains to prove that $I = Q'_1 \cap \cdots \cap Q'_m$. To this end, remark that for all i, $I \subset Q_i$ and as I is bi-homogeneous, $I \subset Q'_i$ while $Q'_i \subset Q_i$. Thus one has $I \subset Q'_1 \cap \cdots \cap Q'_m \subset Q_1 \cap \cdots \cap Q_m = I$ which ends the proof.

□

From now on, given a bi-homogeneous ideal I, we *only* consider *bi-homogeneous* primary decompositions of I, i.e. decompositions of I such that each primary component is bi-homogeneous. Remark that from such a bi-homogeneous primary decomposition, one can extract a *minimal* primary decomposition. From the uniqueness of the isolated primary components of a minimal primary decomposition (see [12]), one deduces the uniqueness of the isolated primary components of a bi-homogeneous minimal primary decomposition of I. This leads to the following result.

Corollary 1 *Let I be a bi-homogeneous ideal of R. There exists a minimal primary decomposition of I such that each primary component is a bi-homogeneous ideal. The set of isolated bi-homogeneous components is unique.*

Note that primary ideals of R which contain a power of $\langle X_0, \ldots, X_n \rangle$ or a power of $\langle \ell_0, \ldots, \ell_k \rangle$ define an empty bi-projective variety in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C})$.

Definition 2 *A primary bi-homogeneous ideal is said to be* admissible *if and only if it contains neither a power of $\langle X_0, \ldots, X_n \rangle$ nor a power of $\langle \ell_0, \ldots, \ell_k \rangle$.*

A primary component of a bi-homogeneous ideal is said to be admissible *if it is an admissible ideal.*

The set of admissible isolated bi-homogeneous components of a minimal bi-homogeneous primary decomposition of an ideal $I \subset R$ is denoted by $\mathrm{Adm}(I)$.

Let $I \subset R$ be a bi-homogeneous ideal, and (d, e) be a couple in $\mathbb{N} \times \mathbb{N}$. The couple (d, e) is an admissible bi-dimension *of I if and only if $d \leqslant n$, $e \leqslant k$ and $d + e + 2$ equals the maximum of the Krull dimensions of the ideals in* $\mathrm{Adm}(I)$.

Remark 1 *Consider a bi-homogeneous ideal $I \subset R$ and let $J = \cap_{\mathcal{Q} \in \mathrm{Adm}(I)} \mathcal{Q}$. From Lemma 2, J is a bi-homogeneous ideal.*

A bi-homogeneous ideal I in R defines a non-empty bi-projective variety $\mathcal{V}$ in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C})$ if and only if $\mathrm{Adm}(I)$ is not empty. Note also that the maximal dimension of the admissible primary components of I can be less than the Krull dimension of I.

In the homogeneous context, the degree of an ideal of Krull dimension D defining a non-empty projective variety is obtained as the degree of this ideal augmented with D *generic* linear forms (see [11]). Below, we provide a similar process to define a notion of bi-degree of a bi-homogeneous ideal.

In the sequel, a homogeneous linear form $u = \sum_{i=0}^{n} u_i X_i \in \mathbb{Q}[X_0, \ldots, X_n]$ (resp. $v = \sum_{j=0}^{k} v_j \ell_j \in \mathbb{Q}[\ell_0, \ldots, \ell_k]$) is identified to the point $(u_0, \ldots, u_n) \in \mathbb{Q}^{n+1}$ (resp. $(v_0, \ldots, v_k) \in \mathbb{Q}^{k+1}$).

Proposition 2 *Let $I \subset R$ be an ideal of dimension $D \geqslant 2$ and $(d, e) \in \mathbb{N} \times \mathbb{N}$ such that $d + e + 2 = D$.*

- *Either for any choice of homogeneous linear forms $(u_1, \ldots, u_{d+1})$ in $\mathbb{Q}[X_0, \ldots, X_n]$ and $(v_1, \ldots, v_{e+1})$ in $\mathbb{Q}[\ell_0, \ldots, \ell_k]$ the ideal*

$$I + \langle (u_1 - 1), u_2, \ldots, u_{d+1}, (v_1 - 1), v_2, \ldots, v_{e+1} \rangle$$

  equals $\mathbb{Q}[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$;

- *or there exist an integer $\mathfrak{D} \in \mathbb{N}$ and a Zariski closed subset $\mathcal{H} \subsetneq (\mathbb{C}^{n+1})^{d+1} \times (\mathbb{C}^{k+1})^{e+1}$ such that for any choice of homogeneous linear forms $(u_1, \ldots, u_{d+1}, v_1, \ldots, v_{e+1})$ outside $\mathcal{H}$ (where for $i \in \{1, \ldots, d+1\}$, $u_i \in \mathbb{Q}[X_0, \ldots, X_n]$ and for $j \in \{1, \ldots, e+1\}$, $v_j \in \mathbb{Q}[\ell_0, \ldots, \ell_k]$) the ideal*

$$I + \langle (u_1 - 1), u_2, \ldots, u_{d+1}, (v_1 - 1), v_2, \ldots, v_{e+1} \rangle$$

  is zero-dimensional and its degree is $\mathfrak{D}$.

Proof. For $i \in \{1, \ldots, d+1\}$, $p \in \{0, \ldots, n\}$, $j \in \{1, \ldots, e+1\}$, and $q \in \{0, \ldots, k\}$, let $\mathfrak{u}_{i,p}$, and $\mathfrak{v}_{j,q}$ be new indeterminates. Denote by $\mathfrak{U}$ (resp. $\mathfrak{V}$) the set of indeterminates $\{\mathfrak{u}_{1,0}, \ldots, \mathfrak{u}_{d+1,n}\}$ (resp. $\{\mathfrak{v}_{1,0}, \ldots, \mathfrak{v}_{e+1,k}\}$).

Let K be the field of rational fractions $\mathbb{Q}(\mathfrak{U}, \mathfrak{V})$; for $i \in \{1, \ldots, d+1\}$, let $\bar{K}_i$ be the field of rational fractions $\mathbb{Q}(\mathfrak{U} \setminus \{\mathfrak{u}_{i,0}, \ldots, \mathfrak{u}_{i,n}\}, \mathfrak{V})$ and for $j \in \{1, \ldots, e+1\}$, let $\underline{K}_j$ be the field of rational fractions $\mathbb{Q}(\mathfrak{U}, \mathfrak{V} \setminus \{\mathfrak{v}_{j,0}, \ldots, \mathfrak{v}_{j,k}\})$.

Consider for $2 \leqslant i \leqslant d+1$ (resp. $2 \leqslant j \leqslant e+1$) the linear forms $\mathfrak{u}_i = \sum_{p=0}^{n} \mathfrak{u}_{i,p} X_p$ (resp. $\mathfrak{v}_j = \sum_{q=0}^{k} \mathfrak{v}_{j,q} \ell_q$) and $\mathfrak{u}_1 = \sum_{p=0}^{k} \mathfrak{u}_{1,p} X_p - 1$ (resp. $\mathfrak{v}_1 = \sum_{q=0}^{k} \mathfrak{v}_{1,q} \ell_q - 1$).

For $i \in \{1, \ldots, d+1\}$ (resp. $j \in \{1, \ldots, e+1\}$)and $u = (u_0, \ldots, u_n)$ (resp. $v = (v_0, \ldots, v_k)$) a point in $\mathbb{Q}^{n+1}$ (resp. $\mathbb{Q}^{k+1}$), denote by $\varphi_{i,u}$ (resp. $\psi_{j,v}$) the ring homomorphism $\varphi_{i,u} : K[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k] \to \bar{K}_i[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$ (resp. $\psi_{j,v} : K[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k] \to \underline{K}_j[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$) such that:

- for all $p \in \{0, \ldots, n\}$ (resp. $q \in \{0, \ldots, k\}$),$\varphi_{i,u}(\mathfrak{u}_{i,p}) = u_p$ (resp. $\psi_{j,v}(\mathfrak{v}_{j,q}) = v_q$),
- for all $p \in \{0, \ldots, n\}$ (resp. $q \in \{0, \ldots, k\}$), and $r \in \{1, \ldots, d+1\} \setminus \{i\}$ (resp. $s \in \{1, \ldots, e+1\} \setminus \{j\}$), $\varphi_{i,u}(\mathfrak{u}_{r,p}) = \mathfrak{u}_{r,p}$ (resp. $\psi_{j,v}(\mathfrak{v}_{s,q}) = \mathfrak{v}_{s,q}$),
- $\varphi_{i,u}(X_p) = X_p$ (resp. $\psi_{j,v}(X_p) = X_p$) and $\varphi_{i,u}(\ell_q) = \ell_q$ (resp. $\psi_{j,v}(\ell_q) = \ell_q$).

Finally, given a couple of points $u = (u_{1,0}, \ldots, u_{1,n}, \ldots, u_{d+1,0}, \ldots, u_{d+1,n})$ (resp. $v = (v_{1,0}, \ldots, v_{1,k}, \ldots, v_{e+1,0}, \ldots, v_{e+1,k})$) in $(\mathbb{Q}^{n+1})^{d+1}$ (resp. $(\mathbb{Q}^{k+1})^{e+1}$), denote by $\vartheta_{(u,v)}$ the ring homomorphism $\vartheta_{(u,v)} : K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k] \to \mathbb{Q}[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$ such that for all $p=0,\ldots,n$ and $q=0,\ldots,k$), $\vartheta_{u,v}(\mathfrak{u}_{i,p}) = u_{i,p}$, $\vartheta_{u,v}(\mathfrak{v}_{j,q}) = v_{j,q}$, $\vartheta_{u,v}(X_p) = X_p$ and $\vartheta_{u,v}(\ell_q) = \ell_q$.

In the sequel, $J_0 = I$, for $i \in \{1,\ldots,d+1\}$, J_i is the ideal $I + \langle \mathfrak{u}_1,\ldots,\mathfrak{u}_i\rangle$, and for $i \in \{d+2,\ldots,D\}$, J_i is the ideal $I + \langle \mathfrak{u}_1,\ldots,\mathfrak{u}_{d+1},\mathfrak{v}_1,\ldots,\mathfrak{v}_{i-(d+1)}\rangle$.

We first prove that the ideal J_D is either zero-dimensional in $K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$ or equals $K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$. The proof is done by proving that, for $i=1,\ldots,D$, $\dim(J_i) < \dim(J_{i-1})$.

Consider the case $i=1$ and suppose that $\dim(J_1) \geqslant \dim(J_0) > 0$. This implies that there exists an isolated primary component $\mathcal{Q}_0$ of J_0 *of dimension* $\dim(J_0) > 0$ and an integer N such that $\mathfrak{u}_1^N \in \mathcal{Q}_0$.

Since J_0 is generated by a polynomial family which does not involve the indeterminates $\mathfrak{u}_{1,0},\ldots,\mathfrak{u}_{1,n}$, for any isolated primary component $\mathcal{Q}$ of J_0, $\mathcal{Q}$ is generated by a polynomial family which does not involve the indeterminates $\mathfrak{u}_{1,0},\ldots,\mathfrak{u}_{1,n}$. Thus, for any $u \in \mathbb{Q}^{n+1}$, $\varphi_{1,u}(\mathcal{Q}_0) = \mathcal{Q}_0 \cap \bar{K}_1$. Then, there exists a Zariski-closed subset $\mathcal{Z} \subset \mathbb{C}^{n+1}$ such that for any point u in $\mathbb{Q}^{n+1} \setminus \mathcal{Z}$, $\varphi_{1,u}(\mathfrak{u}_1)^N$ belongs to $\mathcal{Q}_0$. Choose now $n+2$ points $u_1,\ldots,u_{n+2}$ in $\mathbb{Q}^{n+2}$ such that $\langle \varphi_{1,u_1}(\mathfrak{u}_1^N),\ldots,\varphi_{1,u_{n+2}}(\mathfrak{u}_1^N)\rangle = \langle 1\rangle$.

Since $\langle \varphi_{1,u_1}(\mathfrak{u}_1^N),\ldots,\varphi_{1,u_{n+2}}(\mathfrak{u}_1^N)\rangle \subset \mathcal{Q}_0$, $\mathcal{Q}_0 = K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$ which contradicts $\dim(\mathcal{Q}_0) = D > 0$. Thus, $\dim(J_1) < \dim(J_0)$.

Consider now the case where $2 \leqslant i \leqslant d+1$ and suppose that $\dim(J_i) \geqslant \dim(J_{i-1}) > 0$. This implies that there exists an isolated primary component $\mathcal{Q}_0$ of J_{i-1} *of positive dimension* and an integer N such that $\mathfrak{u}_i^N \in \mathcal{Q}_0$.

Since J_{i-1} is generated by a polynomial family which does not involve the indeterminates $\mathfrak{u}_{i,0},\ldots,\mathfrak{u}_{i,n}$, $\mathcal{Q}_0$ is generated by a polynomial family which does not involve the indeterminates $\mathfrak{u}_{i,0},\ldots,\mathfrak{u}_{i,n}$ and thus for any $u \in \mathbb{Q}^{n+1}$, $\varphi_{i,u}(\mathcal{Q}_0) = \mathcal{Q}_0 \cap \bar{K}_i$. Then, for any point u in $\mathbb{Q}^{n+1}$, $\varphi_{i,u}(\mathfrak{u}_i)^N$ belongs to $\mathcal{Q}_0$. Choosing $n+2$ points $u_1,\ldots,u_{n+2}$ in $\mathbb{Q}^{n+2}$ such that $\langle \varphi_{i,u_1}(\mathfrak{u}_i^N),\ldots,\varphi_{i,u_{n+2}}(\mathfrak{u}_i^N)\rangle = \langle X_0^N,\ldots,X_n^N\rangle \subset \mathcal{Q}$. Since $\mathfrak{u}_1 \in J_{i-1} \subset \mathcal{Q}_0$ and $\langle \mathfrak{u}_1\rangle + \langle X_0^N,\ldots,X_n^N\rangle = \langle 1\rangle$ this implies that $\mathcal{Q}_0 = K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$ which contradicts the fact that $\mathcal{Q}_0$ has positive dimension. Thus, $\dim(J_i) < \dim(J_{i-1})$.

Proving that $\dim(J_{d+2}) < \dim(J_{d+1})$ is done by the same way as proving that $\dim(J_1) < \dim(J_0)$ using the homomorphism $\psi_{d+2,v}$ instead of $\varphi_{1,u}$ (for $u \in \mathbb{Q}^{n+1}$ and $v \in \mathbb{Q}^{k+1}$). Proving for $i > d+2$ that $\dim(J_i) < \dim(J_{i-1})$ is done following the same arguments as those of the above paragraph (using the homomorphisms $\psi_{i,v}$ for $v \in \mathbb{Q}^{k+1}$).

Thus, if $J_D \neq K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$, one has $\dim(J_D) < \dim(J_{D-1}) < \cdots < \dim(J_i) < \dim(J_{i-1}) < \cdots < \dim(I)$ which implies J is be zero-dimensional.

If $J_D = K[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$, for any $(u,v) \in (\mathbb{Q}^{n+1})^{d+1} \times (\mathbb{Q}^{k+1})^{e+1}$, $\vartheta_{u,v}(J_D)$ equals $\mathbb{Q}[X_0,\ldots,X_n]$.

Suppose now J_D to be zero-dimensional and consider a Gröbner basis G of J_D. Let $\mathcal{H} \subset (\mathbb{C}^{n+1})^{d+1} \times (\mathbb{C}^{k+1})^{e+1}$ be the Zariki-closed set which is the union of zero-sets of the common denominator of each polynomial of G. Thus, for any point $(u,v) \in (\mathbb{Q}^{n+1})^{d+1} \times (\mathbb{Q}^{k+1})^{e+1} \setminus \mathcal{H}$, $\vartheta_{u,v}(G)$ is a Gröbner basis of $\vartheta_{u,v}(J_D)$ (see [16]). Then, $\vartheta_{u,v}(J_D)$ is zero-dimensional and its degree is the one of J_D.

$\square$

Remark 2 *From the proof of the above Proposition, one deduces also the following:*

- *Let $I \subset R$ be a bi-homogeneous ideal which is an intersection of admissible primary ideals. Then, there exists a Zariski-closed subset $\mathcal{H} \subsetneq \mathbb{C}^{n+1} \times \mathbb{C}^{k+1}$ such that for all $(u_1, v_1) \in \mathbb{C}^{n+1} \times \mathbb{C}^{k+1} \setminus \mathcal{H}$, $u_1 - 1$ and $v_1 - 1$ do not divide 0 in R/I and $v_1 - 1$ does not divide 0 in $R/I + \langle u_1 - 1\rangle$.*

  Moreover if I is equidimensional, $I + \langle u_1 - 1, v_1 - 1\rangle$ is equidimensional.

- *Let $I \subset R$ be an equidimensional bi-homogeneous ideal which is an intersection of admissible primary ideals and $J = I + \langle u_1 - 1, v_1 - 1\rangle$. If $J \neq R$, there exists $(d, e) \in \mathbb{N} \times \mathbb{N}$ such that $d + e = \dim(J)$ and a Zariski-closed subset $\mathcal{H} \subsetneq (\mathbb{C}^{n+1})^d \times (\mathbb{C}^{k+1})^{e+1}$ such that for any choice of homogeneous linear forms $(u_2, \ldots, u_{d+1}, v_2, \ldots, v_{e+1}) \in \mathbb{Q}[X_0, \ldots, X_n] \times \mathbb{Q}[\ell_0, \ldots, \ell_k] \setminus \mathcal{H}$, u_2 does not divide 0 in R/J, u_i does not divide 0 in $R/J + \langle u_2, \ldots, u_{i-1}\rangle$ (for $i = 3, \ldots, d+1$), v_2 does not divide zero in $R/J + \langle u_2, \ldots, u_{d+1}\rangle$ and v_j does not divide 0 in $R/J + \langle u_2, \ldots, u_{d+1}, v_1, \ldots, v_{j-1}\rangle$ (for $j \in \{3, \ldots, e+1\}$).*

In the sequel, we shall say that a property $\mathfrak{P}$ is true for a *generic* choice of linear forms, if there exists a Zariski-closed subset in the set of the considered linear forms such that for any choice of forms outside this Zariski-closed subset, the property $\mathfrak{P}$ is satisfied.

Proposition 2 and the uniqueness of $\mathrm{Adm}(I)$ allows us to define the following notion of *bi-degree* of a bi-homogeneous ideal $I \subset R$.

Definition 3 *Let $I \subset R$ be a bi-homogeneous ideal of dimension D. For $(d, e) \in \mathbb{N} \times \mathbb{N}$ such that $d + e + 2 = D$, consider the linear forms $u_1, \ldots, u_{d+1}$ (resp. $v_1, \ldots, v_{e+1}$) which are chosen* generically *in $\mathbb{Q}[X_0, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_0, \ldots, \ell_k]$). $C_{d,e}(I)$ denotes the degree of $I + \langle u_1, \ldots, u_{d+1} - 1, v_1, \ldots, v_{e+1} - 1\rangle$.*

If I is primary, the bi-degree *of I is the sum $\sum_{d+e+2=D} C_{d,e}(I)$.*

If I is not primary, the bi-degree *of I is the sum of the bi-degrees of the ideals of* $\mathrm{Adm}(I)$ *having* maximal Krull dimension*.*

If I is not primary, the strong bi-degree *of I is the sum of the bi-degrees of the ideals of* $\mathrm{Adm}(I)$ *(which are isolated by definition of* $\mathrm{Adm}(I)$*).*

Remark 3 *Let $I \subset R$ be a bi-homogeneous ideal and $J = \cap_{\mathcal{Q} \in \mathrm{Adm}(I)} \mathcal{Q}$ which is bi-homogeneous from Lemma 2. Note that, by definition, the bi-degree (resp. the strong bi-degree) of I equals the bi-degree (resp. strong bi-degree) of J.*

We are now ready to state the main results of this section. The first one bounds the strong bi-degree of a bi-homogeneous ideal $I \subset R$ under some assumptions. This generalizes the statements of [40, 39, 24] which only consider with the admissible primary components of $\mathrm{Adm}(I)$ having maximal dimension.

Theorem 1 *Let $s \in \{1, \ldots, n+k\}$ and $f_1, \ldots, f_s$ be bi-homogeneous polynomials in R of respective bi-degree (α_i, β_i) generating a bi-homogeneous ideal I. Suppose that there exist at most n f_i such that $\beta_i = 0$ and at most k f_i such that $\alpha_i = 0$, then the sum of the bi-degrees of the bi-homogeneous associated primes of I is bounded by $\mathcal{B}(f_1, \ldots, f_s) = \sum_{\mathcal{I},\mathcal{J}} (\Pi_{i \in \mathcal{I}} \alpha_i) \, . \, (\Pi_{j \in \mathcal{J}} \beta_j)$ where $\mathcal{I}$ and $\mathcal{J}$ are disjoint subsets for which the union is $\{1, \ldots, s\}$ such that the cardinality of $\mathcal{I}$ (resp. $\mathcal{J}$) is bounded by n (resp. k).*

The following result generalizes the one of [41] and states that given an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ its strong degree (in the meaning of [23]) is bounded by the strong bi-degree of a bi-homogeneous ideal constructed from a bi-homogeneization process applied to each polynomial in I.

Theorem 2 *Consider the mapping :*

$$\begin{array}{cccl} \phi : & \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k] & \rightarrow & \mathbb{Q}[X_0, X_1, \ldots, X_n, \ell_0, \ell_1, \ldots, \ell_k] \\ & f & \mapsto & X_0^{\deg_X(f)} \ell_0^{\deg_\ell(f)} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}, \frac{\ell_1}{\ell_0}, \ldots, \frac{\ell_k}{\ell_0}) \end{array}$$

where $\deg_X(f)$ (resp. $\deg_\ell(f)$) denotes the degree of f seen as a polynomial in $\mathbb{Q}(\ell_1, \ldots, \ell_k)[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}(X_1, \ldots, X_n)[\ell_1, \ldots, \ell_k]$).

Given an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$, denote by $\phi(I)$ the ideal $\{\phi(f) \mid f \in I\} \subset \mathbb{Q}[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$.

Then, $\phi(I)$ is a bi-homogeneous ideal and the sum of the degrees of the isolated primary components of I is bounded by the strong bi-degree of $\phi(I)$.

The proof of these results rely on the study of Hilbert bi-series of bi-homogeneous ideals which are introduced in [40].

2.2 Hilbert bi-series: basic properties

We follow here [40] which introduce Hilbert bi-series of bi-homogeneous ideals and study their properties when the considered bi-homogeneous ideal has bi-dimension $(0, 0)$.

Notation. Given a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $R_{i,j}$ the $\mathbb{Q}$-vector space of the *bi-homogeneous* polynomials in R of degree i in the set of variables $X_0, \ldots, X_n$ and j in the set of variables $\ell_0, \ldots, \ell_k$.

Given a *bi-homogeneous* ideal $I \subset R$ and a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $I_{i,j}$ the intersection of I with $R_{i,j}$.

Given a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $R_{\leqslant i, \leqslant j}$ the $\mathbb{Q}$-vector space of polynomials in R of degree less than or equal to i (resp. j) in the set of variables $X_0, \ldots, X_n$ (resp. $\ell_0, \ldots, \ell_k$).

Given an ideal $I \subset R$ and a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $I_{\leqslant i, \leqslant j}$ the intersection of I with $R_{\leqslant i, \leqslant j}$.

Definition 4 *The Hilbert bi-series of a bi-homogeneous ideal* $I \subset R$ *is the series* $\sum_{i,j} \dim(R_{i,j}/I_{i,j})\, t_1^i t_2^j$.
The affine Hilbert bi-series of an ideal $I \subset R$ *is* $\sum_{i,j} \dim(R_{\leqslant i,\leqslant j}/I_{\leqslant i,\leqslant j})\, t_1^i t_2^j$.

Consider a bi-homogeneous ideal $I \subset R$ of Krull dimension 2. This section is devoted to prove that, there exists $(i_0, j_0) \in \mathbb{N} \times \mathbb{N}$ such that for all $i \geqslant i_0$ *and* $j \geqslant j_0$, $\dim(R_{i,j}/I_{i,j}) = \dim(R_{i_0,j_0}/I_{i_0,j_0})$. Such a result already appears in [40].

Denote by R' the polynomial ring $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$. Consider the application $\phi_{i,j} : R'_{\leqslant i,\leqslant j} \to R_{i,j}$ sending a polynomial $f \in R'_{\leqslant i,\leqslant j}$ of degree α (resp. β) in the variables $X_1, \ldots, X_n$ (resp. $\ell_1, \ldots, \ell_k$) to the polynomial $\phi_{i,j}(f) = X_0^i \ell_0^j f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}, \frac{\ell_1}{\ell_0}, \ldots, \frac{\ell_k}{\ell_0})$.

Given an ideal $I \subset R'$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$, $\phi_{i,j}(I)$ denotes the set $\{\phi_{i,j}(f) \mid f \in I_{\leqslant i,\leqslant j}\}$.

Additionally, consider the mapping $\psi_{i,j} : R_{i,j} \to R'_{\leqslant i,\leqslant j}$ (which takes place of a de-homogenization process in a homogeneous context) sending a bi-homogeneous polynomial $f \in R_{i,j}$ to

$$\psi_{i,j}(f) = f(1, X_1, \ldots, X_n, 1, \ell_1, \ldots, \ell_k) \in R'.$$

Given a bi-homogeneous ideal $I \subset R$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$, $\psi_{i,j}(I)$ denotes the set $\{\psi_{i,j}(f) \mid f \in I_{i,j}\}$.

Lemma 3 *Consider two integers* i *and* j, *an ideal* I' *of* R' *and* $\phi_{i,j}$ *the above application from* $R'_{\leqslant i,\leqslant j}$ *to* $R_{i,j}$. *Then*

$$\dim(R'_{\leqslant i,\leqslant j}/I'_{\leqslant i,\leqslant j}) = \dim(R_{i,j}/\phi_{i,j}(I'_{\leqslant i,\leqslant j})).$$

Proof. Remark that $R_{i,j}$, $R'_{\leqslant i,\leqslant j}$, and $I'_{\leqslant i,\leqslant j}$ and $\phi_{i,j}(I'_{\leqslant i,\leqslant j})$ are finite dimensional $\mathbb{Q}$-vector spaces.

Moreover, for all $f \in R'_{\leqslant i,\leqslant j}$ and $(i,j) \in \mathbb{N} \times \mathbb{N}$, $\psi_{i,j}(\phi_{i,j}(f)) = f$. Then, $\psi_{i,j}(R_{i,j}) = R'_{\leqslant i,\leqslant j}$ and $\psi_{i,j}(\phi_{i,j}(I'_{\leqslant i,\leqslant j})) = I'_{\leqslant i,\leqslant j}$. Hence as $\psi_{i,j}$ is an injective morphism, $R_{i,j}$ and $R'_{\leqslant i,\leqslant j}$ on the one hand, and $I'_{\leqslant i,\leqslant j}$ and $\phi_{i,j}(I'_{\leqslant i,\leqslant j})$ on the other hand, are isomorphic finite dimensional $\mathbb{Q}$-vector spaces.

Since for vector spaces E, F with $F \subset E$, $\dim(E) = \dim(F) + \dim(E/F)$, we are done. $\square$

The following lemma is used further.

Lemma 4 *Let* $I \subset R$ *be a bi-homogeneous ideal such that* X_0 *(resp.* ℓ_0*) is not a zero divisor in* R/I. *Denote by* I' *the ideal* $I + \langle X_0 - 1, \ell_0 - 1\rangle \cap R'$. *Then* $\phi_{i,j}(I'_{\leqslant i,\leqslant j}) = I_{i,j}$.

Proof. Consider a bi-homogeneous polynomial $p \in I_{i,j}$. Obviously, $\psi_{i,j}(p) \in I'_{\leqslant i,\leqslant j}$. Since the bi-degree of p is, by definition, the couple (i,j), $\phi_{i,j}(\psi_{i,j}(p)) = p$, this implies $p \in \phi_{i,j}(I'_{\leqslant i,\leqslant j})$. Thus, $I_{i,j} \subset \phi_{i,j}(I'_{\leqslant i,\leqslant j})$; we prove now that $\phi_{i,j}(I'_{\leqslant i,\leqslant j}) \subset I_{i,j}$.

From Lemma 1, since I is bi-homogeneous, there exists a finite family of bi-homogeneous polynomials $p_1, \ldots, p_m$ generating I. Consider $p' \in I'_{\leqslant i,\leqslant j}$. Then, there exist polynomials q_r (for $r \in \{1, \ldots, m\}$), P and Q in R, such that $p' = \sum_{r=1}^m q_r.p_r + Q(X_0 - 1) + P(\ell_0 - 1)$. Denote by p the polynomial $\sum_{r=1}^m q_r p_r$ and remark that $p \in I$. Since I is bi-homogeneous, all the bi-homogeneous components of p belong to I, and one just has to consider the case where p' is such that p is bi-homogenous.

Under this assumption, it is easy to see that $\phi_{i,j}(p')$ belongs to $R_{i,j}$ and that there exists a couple $(\alpha,\beta)\in\mathbb{Z}\times\mathbb{Z}$ such that $\phi_{i,j}(p')=X_0^\alpha.\ell_0^\beta.p$. If both α,β are positive, then $\phi_{i,j}(p')\in I_{i,j}$. If $\alpha<0$ and $\beta\geqslant 0$, then $X_0^{-\alpha}\phi_{i,j}(p')=\ell_0^\beta p\in I$. Suppose $\phi_{i,j}(p')\notin I$. This would imply that X_0 is a zero-divisor in R/I, which is not possible by assumption. Moreover, since $\phi_{i,j}(p')$ has bi-degree (i,j), $\phi_{i,j}(p')$ belongs to $I_{i,j}$. Similar arguments allow us to conclude when $\alpha\geqslant 0$ and $\beta<0$ and when both α and β are negative.

□

Given a polynomial $f\in\mathbb{Q}[X_0,\ldots,X_n,\ell_0,\ldots,\ell_k]$, and $\mathbf{A}\in GL_{n+k+2}(\mathbb{Q})$, we denote by $f^{\mathbf{A}}$ the polynomial obtained by performing the change of variables induced by $\mathbf{A}$ on f. We denote by $I^{\mathbf{A}}$ the ideal generated by $f_1^{\mathbf{A}},\ldots,f_s^{\mathbf{A}}$. In the sequel, we consider exclusively matrices $\mathbf{A}$ such that the action of $\mathbf{A}$ on R is a bi-graded isomorphism of bi-degree $(0,0)$ on R with respect to the variables $X_0,\ldots,X_n$ and $\ell_0,\ldots,\ell_k$, i.e. for all homogeneous linear forms $u\in\mathbb{Q}[X_0,\ldots,X_n]$ (resp. $v\in\mathbb{Q}[\ell_0,\ldots,\ell_k]$), $u^{\mathbf{A}}$ (resp. $v^{\mathbf{A}}$) is a homogeneous linear form in $\mathbb{Q}[X_0,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_0,\ldots,\ell_k]$).

Lemma 5 *Let $I\subset R$ be a bi-homogeneous ideal, then I and $I^{\mathbf{A}}$ have the same Hilbert bi-series.*

Proof. The action of $\mathbf{A}$ on R is an isomorphism of bi-graded ring of bi-degree $(0,0)$, the inverse action of $\mathbf{A}$ is the action of $\mathbf{A}^{-1}$. Thus, $R_{i,j}$ equals $R_{i,j}^{\mathbf{A}}$ and, if $E\subset R$ is a $\mathbb{Q}$-vector space, $\dim(E)=\dim(E^{\mathbf{A}})$. Since for vector spaces E,F with $F\subset E$, $\dim(E)=\dim(F)+\dim(E/F)$, we have $\dim(R_{i,j}/I_{i,j})=\dim(R_{i,j}^{\mathbf{A}}/I_{i,j}^{\mathbf{A}})$ which implies the equality of the Hilbert bi-series of I and $I^{\mathbf{A}}$.

□

Lemma 6 *Let $I\subset R$ be a bi-homogeneous ideal which is an intersection of admissible ideals. Then, for a generic choice of homogeneous linear form u_1 (resp. v_1) in $\mathbb{Q}[X_0,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_0,\ldots,\ell_k]$) the affine Hilbert bi-series of $I+\langle u_1-1,v_1-1\rangle$ equals the Hilbert bi-series of I.*

Proof. From Remark 2, since is u_1 and v_1 are chosen generically they do not divide 0 is R/I. Choose $\mathbf{A}\in GL_{n+k+2}(\mathbb{Q})$ such that the action of $\mathbf{A}$ on R is a bi-graded isomorphism of bi-degree $(0,0)$ and such that $u_1^{\mathbf{A}}=X_0$ and $v_1^{\mathbf{A}}=\ell_0$. Using Lemma 5, the Hilbert series of I equals the one of $I^{\mathbf{A}}$. From Lemma 3 and Lemma 4, the affine Hilbert bi-series of $I^{\mathbf{A}}+\langle X_0-1,\ell_0-1\rangle\cap\mathbb{Q}[X_1,\ldots,X_n,\ell_1,\ldots,\ell_k]$ equals the bi-series of $I^{\mathbf{A}}$ which equals the one of I.

We prove now $\psi'_{i,j}:R^{\mathbf{A}}_{\leqslant i,\leqslant j}\rightarrow R'^{\mathbf{A}}_{\leqslant i,\leqslant j}$ sending $f\in R^{\mathbf{A}}_{\leqslant i,\leqslant j}$ to $f(1,X_1,\ldots,X_n,1,\ell_1,\ldots,\ell_k)$ induces an isomorphism between $R'^{\mathbf{A}}_{\leqslant i,\leqslant j}/I'^{\mathbf{A}}_{\leqslant i,\leqslant j}$ and $R^{\mathbf{A}}_{\leqslant i,\leqslant j}/(I^{\mathbf{A}}+\langle X_0-1,\ell_0-1\rangle_{\leqslant i,\leqslant j})$ which is immediate using the isomorphism between $R'^{\mathbf{A}}_{\leqslant i,\leqslant j}/I'^{\mathbf{A}}_{\leqslant i,\leqslant j}$ and $R^{\mathbf{A}}_{i,j}/I^{\mathbf{A}}_{i,j}$.

□

The following result is used in the proof of Theorem 2.

Proposition 3 *Let I be a bi-homogeneous ideal of R. Then, the Hilbert series of I equals the series obtained by putting $t_1 = t_2$ in the Hilbert bi-series of I.*

Proof. For all $d \in \mathbb{N}$, denote by R_d the $\mathbb{Q}$-vector space of the *homogeneous* polynomials of R of total degree d. For any couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, denote by $R_{i,j}$ the $\mathbb{Q}$-vector space of bi-homogeneous polynomials of bi-degree (i, j). By definition of a bi-homogeneous ideal, if f is a polynomial of I, then the bi-homogeneous components of f also belong to I. Hence the morphism from I_d to $\Pi_{i+j=d} I_{i,j}$ sending a polynomial onto its bi-homogeneous components is defined for any f of I and is invertible, i.e. is an isomorphism of $\mathbb{Q}$-vector spaces. This proves that $\dim(I_d) = \sum_{i+j=d} \dim(I_{i,j})$. With the same arguments, we show that $\dim(R_d) = \sum_{i+j=d} \dim(R_{i,j})$. Consequently

$$\dim(R_d/I_d) = \dim(R_d) - \dim(I_d) = \sum_{i+j=d} \dim(R_{i,j}) - \dim(I_{i,j})$$

On the other hand, for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, $\dim(R_{i,j}) - \dim(I_{i,j}) = \dim(R_{i,j}/I_{i,j})$.

□

Proposition 4 *Let $I \subset R$ be a bi-homogeneous ideal and let $J = \cap_{\mathcal{Q} \in \mathrm{Adm}(I)} \mathcal{Q}$. Suppose that J has Krull dimension 2. There exists $(i_0, j_0) \in \mathbb{N} \times \mathbb{N}$ such that for all $i \geqslant i_0$* and *$j \geqslant j_0$, $\dim(R_{i,j}/J_{i,j}) = \dim(R_{i_0,j_0}/J_{i_0,j_0})$ which equals the bi-degree of I.*

Proof. From Remark 3, the bi-degree of I is the bi-degree of the ideal $J = \cap_{\mathcal{Q} \in \mathrm{Adm}(I)} \mathcal{Q}$. From Proposition 2 and Definition 3, the bi-degree of J is the degree of the ideal $J + \langle u - 1, v - 1\rangle$ where u (resp. v) is a generic homogeneous linear form in $\mathbb{Q}[X_0, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_0, \ldots, \ell_k]$).

Consider now $\mathbf{A} \in GL_{n+k+2}(\mathbb{Q})$ such that $u^{\mathbf{A}} = X_0$ and $v^{\mathbf{A}} = \ell_0$, and such that the canonical action of $\mathbf{A}$ on R is a bi-graded isomorphism on R of bi-degree $(0, 0)$ with respect to the variables $X_0, \ldots, X_n$ and $\ell_0, \ldots, \ell_k$. Note that the bi-degree of J equals the one of $J^{\mathbf{A}}$ which is the degree of $J^{\mathbf{A}} + \langle X_0 - 1, \ell_0 - 1\rangle$. In the sequel, denote by ${J'}^{\mathbf{A}}$ the ideal $\left(J^{\mathbf{A}} + \langle X_0 - 1, \ell_0 - 1\rangle\right) \cap \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$

From Lemma 5, $\dim(R_{i,j}/J_{i,j}) = \dim(R^{\mathbf{A}}_{i,j}/J^{\mathbf{A}}_{i,j})$. Moreover, since by definition of $J^{\mathbf{A}}$, X_0 (resp. ℓ_0) is not a zero-divisor in $R^{\mathbf{A}}/J^{\mathbf{A}}$, from Lemma 4, $\dim(R^{\mathbf{A}}_{i,j}/J^{\mathbf{A}}_{i,j}) = \dim({R'}^{\mathbf{A}}_{i,j}/\phi_{i,j}({J'}^{\mathbf{A}}_{\leqslant i,\leqslant j}))$. Finally, from Lemma 3 $\dim({R'}^{\mathbf{A}}_{i,j}/\phi_{i,j}({J'}^{\mathbf{A}}_{\leqslant i,\leqslant j})) = \dim({R'}^{\mathbf{A}}_{\leqslant i,\leqslant j}/{J'}^{\mathbf{A}}_{\leqslant i,\leqslant j})$. Thus, one has $\dim(R_{i,j}/J_{i,j}) = \dim({R'}^{\mathbf{A}}_{\leqslant i,\leqslant j}/{J'}^{\mathbf{A}}_{\leqslant i,\leqslant j})$. It is sufficient to prove there exists $(i_0, j_0) \in \mathbb{N} \times \mathbb{N}$ such that for all $(i, j) \in \mathbb{N} \times \mathbb{N}$ satisfying $i \geqslant i_0$ and $j \geqslant j_0$, $\dim({R'}^{\mathbf{A}}_{\leqslant i,\leqslant j}/{J'}^{\mathbf{A}}_{\leqslant i,\leqslant j})$
$= \dim({R'}^{\mathbf{A}}_{\leqslant i_0,\leqslant j_0}/{J'}^{\mathbf{A}}_{\leqslant i_0,\leqslant j_0})$ and that $\dim({R'}^{\mathbf{A}}_{\leqslant i_0,\leqslant j_0}/{J'}^{\mathbf{A}}_{\leqslant i_0,\leqslant j_0})$ equals the degree of ${J'}^{\mathbf{A}}$. These are a consequence of $\dim({J'}^{\mathbf{A}}) = 0$ and for i, j and d large enough $\dim({R'}^{\mathbf{A}}_{\leqslant i,\leqslant j}/{J'}^{\mathbf{A}}_{\leqslant i,\leqslant j}) = \dim({R'}^{\mathbf{A}}_{\leqslant d}/{J'}^{\mathbf{A}}_{\leqslant d}) = \deg({J'}^{\mathbf{A}})$, where ${R'}^{\mathbf{A}}_{\leqslant d}$ (resp. ${J'}^{\mathbf{A}}_{\leqslant d}$) denotes the set of polynomials in ${R'}^{\mathbf{A}}$

(resp. $J'^{\mathbf{A}}$) of degree less than or equal to d.

$\square$

2.3 Canonical form of the Hilbert bi-series

In this paragraph, we provide a canonical form of the Hilbert bi-series of a bi-homogeneous ideal $I \subset R$ with respect to the integers $C_{d,e}(I)$ where (d,e) lies in the set of admissible bi-dimensions of I.

Lemma 7 *Let $I \subset R$ be a bi-homogeneous ideal. For $i = 1, \ldots, s$ let f_i be a bi-homogeneous polynomial in R of bi-degree (α_i, β_i). For $i = 1, \ldots, s$, denote by I_i the ideal $I + \langle f_1, \ldots, f_i \rangle$ and suppose that for all $i \in \{1, \ldots, s-1\}$, f_{i+1} is not a divisor of zero in R/I_i.*

Then the Hilbert bi-series of I_s equals $\left(\Pi_{i=1}^{s}(1 - t_1^{\alpha_i} t_2^{\beta_i})\right)\mathcal{H}(I)$.

Proof. We proceed by induction on s. Suppose first $s = 1$.

Denote by $\mathrm{ann}_{R/I}(f_1)$ the annihilator of f_1 in R/I. The sequence below

$$0 \to \mathrm{ann}_{R/I}(f_1) \longrightarrow R/I \xrightarrow{f_1} R/I \longrightarrow R/(I + \langle f_1 \rangle) \to 0$$

is exact. Since f_1 is not a zero divisor in R/I, $\mathrm{ann}_{R/I}(f_1)$ is 0. Remark that, since f_1 is bi-homogeneous, and $I_{i,j}$ and $R_{i,j}$ contain only bi-homogeneous polynomials, the following one

$$0 \longrightarrow R_{i,j}/I_{i,j} \xrightarrow{f_1} R_{i+\alpha_1, j+\beta_1}/I_{i+\alpha_1, j+\beta_1} \longrightarrow R_{i,j}/(I + \langle f_1 \rangle)_{i,j} \to 0$$

is also exact. Thus, classically, the alternate sum of the dimension of the vector spaces of this exact sequence is null. Adding these sums for all $(i,j) \in \mathbb{N} \times \mathbb{N}$, one obtains:

$$(1 - t_1^{\alpha_1} t_2^{\beta_1})\mathcal{H}(I) = \mathcal{H}(I_1)$$

where (α_1, β_1) is the bi-degree of f_1. Suppose now the result to be true for $s-1$, i.e. the Hilbert bi-series of I_{s-1}, $\mathcal{H}(I_{s-1})$, equals $\left(\prod_{i=1}^{s-1}(1 - t_1^{\alpha_i} t_2^{\beta_i})\right)\mathcal{H}(I)$. Since, by assumption, f_s is not a divisor of 0 in R/I_{s-1}, the following sequence is exact:

$$0 \to \mathrm{ann}_{R/I_{s-1}}(f_s) \longrightarrow R/I_{s-1} \xrightarrow{f_s} R/I_{s-1} \longrightarrow R/(I_{s-1} + \langle f_s \rangle) \to 0$$

which implies, as above, that $\mathcal{H}(I_s) = (1 - t_1^{\alpha_s} t_2^{\beta_s})\mathcal{H}(I_{s-1})$ and then, using the induction hypothesis, $\mathcal{H}(I_s) = \left(\Pi_{i=1}^{s}(1 - t_1^{\alpha_i} t_2^{\beta_i})\right)\mathcal{H}(I)$

$\square$

Lemma 8 *The Hilbert bi-series of R is $\frac{1}{(1-t_1)^{n+1}(1-t_2)^{k+1}}$*

Proof. This is an application of Lemma 7, considering the sequence $X_0, \ldots, X_n, \ell_0, \ldots, \ell_k$, which is a regular one.

□

Proposition 5 *Let $I \subset R$ be a bi-homogeneous ideal. Then the Hilbert bi-series of I has the form :*

$$\mathcal{H}(I) = \frac{P(t_1, t_2)}{(1-t_1)^{n+1}(1-t_2)^{k+1}}$$

where $P(t_1, t_2)$ is a polynomial of $\mathbb{Z}[t_1, t_2]$.

Proof. We first exhibit a free finite bi-graded resolution of I. Next, considering each bi-homogeneous part of such a resolution, we conclude by using the alternate summation of the exhibited bi-graded resolution.

From Lemma 1, since I is bi-homogeneous, it is generated by a finite family of bi-homogeneous polynomials $f_1, \ldots, f_s$, and denote by (α_i, β_i) the bi-degree of f_i for $i = 1, \ldots, s$. Thus, I is an R-module of finite type. From Hilbert's syzygies theorem (see [30, p. 208]), I admits a finite free graded resolution :

$$0 \longrightarrow L_k = \bigoplus_{i=1}^{d_k} R(\gamma_{k,i})) \xrightarrow{\phi_k} \cdots \xrightarrow{\phi_2} L_1 = \bigoplus_{i=1}^{d_1} R(\gamma_{1,i}) \xrightarrow{\phi_1} I \longrightarrow 0$$

The quantities γ_i are integer shift of the usual graduation of R making the morphisms ϕ_i homogeneous and of degree 0. The morphism ϕ_1 from L_1 to I, sends the element $(p_1, \ldots, p_s)$ of L_1 to the element $\Sigma_{i=1}^s p_i f_i$ of I.

In order to make ϕ_1 a bi-homogeneous morphism of bi-degree $(0,0)$, we shift the bi-graduation on each R by the bi-degree of the f_i's : the R-module L_1 becomes $L_1 = \bigoplus_{i=1}^s R((\alpha_i, \beta_i))$ instead of $\bigoplus_{i=1}^s R(\gamma_{1,i})$ (where for $i = 1, \ldots, s$, $\gamma_{1,i} = \alpha_i + \beta_i$).

Considering the system of generators of the syzygies between the f_i's which are used in the above resolution, say $r_1, \ldots, r_{d_1} \in L_1$, one sees that, as the f_i's are bi-homogeneous, all bi-homogeneous component of the r_j are also some syzygies. This means that we can restrict ourselves to consider that the r_i's are bi-homogeneous.

As above, in order to make ϕ_2 bi-homogeneous of bi-degree $(0,0)$ we shift the bi-graduation of L_2 accordingly to the bi-degrees of the r_j. We apply the same process to the other syzygy modules to finally get a bi-graded free resolution of I.

This free bi-graded resolution is an exact sequence, and the morphisms are bi-homogeneous of bi-degree $(0,0)$. Thus, the alternate sum of the dimensions of the bi-graded parts of degree (α, β) of the L_i and of I is null.

Remark now that from Lemma 8 the Hilbert bi-series of R with the bi-graduation shifted of (α, β) is $\frac{t_1^\alpha t_2^\beta}{(1-t_1)^{n+1}(1-t_2)^{k+1}}$.

Consequently, the expression of the Hilbert bi-series of I is obtained by summing these Hilbert bi-series, as the denominators of these fractions are always the same, the sum can be performed only on the numerators to finally get a polynomial $P(t_1, t_2) \in \mathbb{Z}[t_1, t_2]$ such that the Hilbert bi-series of I equals $P(t_1,t_2)/(1-t_1)^{n+1}(1-t_2)^{k+1}$.

□

Lemma 9 *Let $I \subset R$ be a bi-homogeneous ideal and $J = \cap_{\mathcal{Q}\in \mathrm{Adm}(I)}\mathcal{Q}$. There exists a couple $(i_0,j_0) \in \mathbb{N}\times\mathbb{N}$ such that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ satisfying $i \geqslant i_0$ and $j \geqslant j_0$:*

$$\dim\left(\frac{R_{i,j}}{I_{i,j}}\right) = \dim\left(\frac{R_{i,j}}{J_{i,j}}\right)$$

Proof. Denote by $\mathcal{X}$ (resp. $\mathcal{L}$) the intersection of the isolated primary ideals $\mathcal{Q}$ belonging to a minimal primary decomposition of I such that there exists $N\in\mathbb{N}$ such that $\langle X_0,\dots,X_n\rangle^N \subset \mathcal{Q}$ (resp. $\langle \ell_0,\dots,\ell_k\rangle^N \subset \mathcal{Q}$). One has $I = J\cap\mathcal{X}\cap\mathcal{L}$. We show below that there exists $(i_0,j_0)\in\mathbb{N}\times\mathbb{N}$ such that for all $i\geqslant i_0$ and all $j\geqslant j_0$, $I_{i,j}=J_{i,j}$.

Let i_0 be the smallest integer such that $\langle X_0,\dots,X_n\rangle^{i_0}\subset\mathcal{X}$, (resp. j_0 the smallest integer such that $\langle \ell_0,\dots,\ell_k\rangle^{j_0}\subset\mathcal{L}$). Then for all $i\geqslant i_0$ and all $j\in\mathbb{N}$, $\mathcal{X}_{i,j}=R_{i,j}$. Similarly, for all $j\geqslant j_0$ and all $i\in\mathbb{N}$, $\mathcal{L}_{i,j}=R_{i,j}$.

Remark that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$, $I_{i,j}=(J\cap\mathcal{X}\cap\mathcal{L})_{i,j}$ equals $J_{i,j}\cap\mathcal{X}_{i,j}\cap\mathcal{L}_{i,j}$. Then, for all $i\geqslant i_0$ and all $j\geqslant j_0$, $I_{i,j}$ equals $J_{i,j}$. This allows to conclude.

□

Lemma 10 *Let $I\subset R$ be a bi-homogeneous ideal, $J=\cap_{\mathcal{Q}\in\mathrm{Adm}(I)}\mathcal{Q}$ and $f\in R$ be a bi-homogeneous polynomial of bi-degree (α,β) which does not divide 0 in R/J. Denote by $\sum_{i,j}a_{i,j}t_1^it_2^j$ the Hilbert bi-series of $I+\langle f\rangle$ and by $\sum_{i,j}b_{i,j}t_1^it_2^j$ the Hilbert bi-series of I times $(1-t_1^\alpha t_2^\beta)$.*

There exists a couple $(i_0,j_0)\in\mathbb{N}\times\mathbb{N}$ such that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ satisfying $i\geqslant i_0$ and $j\geqslant j_0$, $a_{i,j}=b_{i,j}$.

Proof. We denote by $\sum_{i,j}c_{i,j}t_1^it_2^j$ the Hilbert bi-series of $J+\langle f\rangle$ and by $\sum_{i,j}d_{i,j}t_1^it_2^j$ the Hilbert bi-series of J times $(1-t_1^\alpha t_2^\beta)$. From Lemma 7, $H(J+\langle f\rangle)$ equals $(1-t_1^\alpha t_2^\beta)H(J)$. This implies that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ $d_{i,j}=c_{i,j}$. In the sequel, we prove that there exists a couple $(i_0,j_0)\in\mathbb{N}\times\mathbb{N}$ such that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ satisfying $i\geqslant i_0$ and $j\geqslant j_0$, $c_{i,j}=a_{i,j}$ and $b_{i,j}=d_{i,j}$.

Remark now that $\mathrm{Adm}(J+\langle f\rangle)=\mathrm{Adm}(I+\langle f\rangle)$. Thus, from Lemma 9, there exists a couple $(i_1,j_1)\in\mathbb{N}\times\mathbb{N}$ such that for $(i,j)\in\mathbb{N}\times\mathbb{N}$ satisfying $i\geqslant i_1$ and $j\geqslant j_1$, $\dim\left(\frac{R_{i,j}}{(\cap_{\mathcal{Q}\in\mathrm{Adm}(I+\langle f\rangle)}\mathcal{Q})_{i,j}}\right)=c_{i,j}=a_{i,j}$. Similarly, applying Lemma 9 to I and J implies the existence of a couple $(i_2,j_2)\in\mathbb{N}\times\mathbb{N}$ such that for all (i,j) satisfying $i\geqslant i_2$ and $j\geqslant j_2$, $b_{i,j}=d_{i,j}$. Choosing $i_0=\max(i_1,i_2)$ and $j_0=\max(j_1,j_2)$ allows us to conclude.

□

The following result provides a canonical form of the Hilbert bi-series of a bi-homogeneous ideal. It generalizes the results of [31] which yields a similar result in the case of a prime admissible bi-homogeneous ideal.

Proposition 6 *Let $I \subset R$ be a bi-homogeneous ideal, D be the maximal Krull-dimension of its isolated admissible primary components and $\mathcal{D} = \{(d,e) \in \mathbb{N}\times\mathbb{N} \mid d+e+2 = D\}$. For $(i,j) \in \mathbb{N}\times\mathbb{N}$ such that $i+j \leqslant D-3$, there exist $c_{i,j} \in \mathbb{Z}$ and a polynomial $Q \in \mathbb{Z}[t_1,t_2]$ such that the Hilbert bi-series of I equals:*

$$\left(\sum_{(d,e)\in\mathcal{D}} \frac{C_{d,e}(I)}{(1-t_1)^{d+1}(1-t_2)^{e+1}}\right) + \left(\sum_{-1\leqslant i+j\leqslant D-3} \frac{c_{i,j}}{(1-t_1)^{i+1}(1-t_2)^{j+1}}\right) + Q(t_1,t_2)$$

Proof. From Proposition 5, there exists a polynomial $P \in \mathbb{Z}[t_1,t_2]$ such that the Hilbert bi-series of I we denote by $\mathcal{H}(I)$ equals:

$$\frac{P(t_1,t_2)}{(1-t_1)^{n+1}(1-t_2)^{k+1}}$$

Writing the polynomial P on the basis $\{(1-t_1)^i(1-t_2)^j \mid (i,j)\in\mathbb{N}\times\mathbb{N}\}$, one obtains the existence of a polynomial $Q \in \mathbb{Z}[t_1,t_2]$ and of integers $c_{i,j}\in\mathbb{Z}$ such that the Hilbert bi-series of I equals:

$$\sum_{0\leqslant i+j\leqslant n+k+2} \frac{c_{i,j}}{(1-t_1)^i(1-t_2)^j} + Q. \tag{1}$$

Given a couple $(d,e)\in\mathcal{D}$, for a generic choice of homogeneous linear forms $u_1,\dots,u_d$ (resp. $v_1,\dots,v_e$) in $\mathbb{Q}[X_0,\dots,X_n]$ (resp. $\mathbb{Q}[\ell_0,\dots,\ell_k]$), one has:

- from Proposition 2, $I^{(d,e)} = I + \langle u_1,\dots,u_d,v_1,\dots,v_e\rangle$ is a bi-homogeneous ideal of bi-dimension $(0,0)$,
- from Remark 2, u_i (resp. v_j) does not divide zero in $\frac{R}{\left(\cap_{\mathcal{Q}\in\mathrm{Adm}(I+\langle u_1,\dots,u_{i-1}\rangle)}\mathcal{Q}\right)}$,
- from Proposition 4 and Lemma 9, there exists $(i_0,j_0)\in\mathbb{N}\times\mathbb{N}$ such that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ such that $i\geqslant i_0$ and $j\geqslant j_0$, the term of index (i,j) in the Hilbert bi-series of $I^{(d,e)}$ equals the bi-degree of $I^{(d,e)}$,
- from Lemma 7 and Lemma 9, there exists $(i_1,j_1)\in\mathbb{N}\times\mathbb{N}$ such that for all $(i,j)\in\mathbb{N}\times\mathbb{N}$ such that $i\geqslant i_1$ and $j\geqslant j_1$, the term of index (i,j) in the Hilbert bi-series of $I^{(d,e)}$ equals the term of index (i,j) in the bi-series $(1-t_1)^d(1-t_2)^e\mathcal{H}(I)$.

Remark now that there exists a polynomial $\widetilde{Q}\in\mathbb{Z}[t_1,t_2]$ such that:

$$\begin{aligned}(1-t_1)^d(1-t_2)^e\mathcal{H}(I) &= \sum_{i,j}\frac{c_{i,j}}{(1-t_2)^{i-d}(1-t_2)^{j-e}} + (1-t_1)^d(1-t_2)^eQ\\ &= \frac{c_{d+1,e+1}}{(1-t_1)(1-t_2)} + \sum_{\substack{i\geqslant d,\\ j\geqslant e}}\frac{c_{i,j}}{(1-t_1)^{i-d}(1-t_2)^{j-e}} + \widetilde{Q}\end{aligned}$$

which implies that $c_{d+1,e+1} = C_{d,e}(I)$ and $c_{i,j}=0$ if $i\geqslant d$ and $j\leqslant e$.

□

2.4 Properties of the bi-degree of a bi-homogeneous ideal

In this paragraph, we study the bi-degree of $I+\langle f\rangle$ when I (resp. f) is a bi-homogeneous ideal (resp. polynomial) of R and exhibit how it is related to the one of I.

The following result appears in a slightly different form in [40] and [31, Lemma 2.11] in the case of a prime ideal.

Proposition 7 *Let $I \subset R$ be a bi-homogeneous ideal, $J = \cap_{\mathcal{Q}\in \mathrm{Adm}(I)}\mathcal{Q}$, D be the dimension of J, and $f \in R$ a non-divisor of zero in R/J. Suppose that J is equidimensional.*

Then, the bi-degree of $I+\langle f\rangle$ is equal to:

$$\alpha\left(\sum_{(d,e)|d+e+3=D} C_{d+1,e}(I)\right)+\beta\left(\sum_{(d,e)|d+e+3=D} C_{d,e+1}(I)\right)$$

Proof. From Proposition 6, the Hilbert bi-series of J can be written as:

$$\sum_{d+e+2=D}\frac{C_{d,e}(J)}{(1-t_1)^{d+1}(1-t_2)^{e+1}}+\sum_{i+j+2\leqslant D-1}\frac{c_{i,j}}{(1-t_1)^{i+1}(1-t_2)^{j+1}}+Q(t_1,t_2)$$

From Lemma 7, since f does not divide zero in R/J, the Hilbert bi-series of $J+\langle f\rangle$ equals the Hilbert bi-series of J multiplied by $(1-t_1^{\alpha}t_2^{\beta})$. Since

$$\begin{aligned} 1-t_1^{\alpha}t_2^{\beta} &= 1-t_1^{\alpha}+t_1^{\alpha}(1-t_2)^{\beta} \\ &= (1-t_1)\sum_{p=0}^{\alpha-1}t_1^p+t_1^{\alpha}(1-t_2)\sum_{q=0}^{\beta-1}t_2^q \end{aligned}$$

some easy computations show that the Hilbert bi-series of $J+\langle f\rangle$ has the form:

$$\sum_{d+e+2=D-1}\frac{\alpha C_{d+1,e}(J)+\beta C_{d,e+1}(J)}{(1-t_1)^{d+1}(1-t_2)^{e+1}}+ \sum_{i+j+2\leqslant D-2}\frac{\widetilde{c}_{i,j}}{(1-t_1)^{i+1}(1-t_2)^{j+1}}+\widetilde{Q}(t_1,t_2)$$

for some $\widetilde{c}_{i,j}\in\mathbb{Z}$ and $Q\in\mathbb{Z}[t_1,t_2]$. Thus, from Proposition 6, the bi-degree of $J+\langle f\rangle$, which equals the bi-degree of $I+\langle f\rangle$ is

$$\alpha\sum_{d+e+2=D}C_{d+1,e}(I)+\beta\sum_{d+e+2=D}C_{d,e+1}(I).$$

□

The following lemma extends to the bi-projective case a result of [23].

Lemma 11 *Let I be a bi-homogeneous ideal, and denote by J the ideal, $J=\cap_{\mathcal{Q}\in\mathrm{Adm}(I)}\mathcal{Q}$. Suppose that J is equidimensional and $\dim(J)\geqslant 3$. Let f be a bi-homogeneous polynomial of bi-degree (α,β) dividing zero in R/J and such that $\dim(J+\langle f\rangle)=\dim(J)$. Then, the bi-degree of $I+\langle f\rangle$ is less than or equal to the bi-degree of I.*

Suppose now that there exists $\tilde{f}$ a bi-homogeneous polynomial of bi-degree (α,β) which does not divide 0 in R/J. Then, denoting by D the dimension of J:

- *if* $\alpha \neq 0$ *and* $\beta \neq 0$, $\mathrm{bideg}(I + \langle f \rangle) \leqslant \mathrm{bideg}(I + \langle \tilde{f} \rangle)$;
- *if* $\alpha = 0$ *and* $C_{0,D-2}(J) = 0$, $\mathrm{bideg}(I + \langle f \rangle) \leqslant \mathrm{bideg}(I + \langle \tilde{f} \rangle)$;
- *if* $\beta = 0$ *and* $C_{D-2,0}(J) = 0$, $\mathrm{bideg}(I + \langle f \rangle) \leqslant \mathrm{bideg}(I + \langle \tilde{f} \rangle)$.

Proof. Let $\mathcal{D} \subset \mathbb{N} \times \mathbb{N}$ be the set of admissible dimensions of J. Since $\dim(J + \langle f \rangle) = \dim(J)$, $\mathrm{bideg}(J+\langle f \rangle) = \sum_{(d,e)\in\mathcal{D}} C_{d,e}(J+\langle f \rangle)$. From Proposition 2 there exist $\mathcal{H}_1$ and $\mathcal{H}_2$, two Zariski closed subset of $(\mathbb{C}^{n+1})^{d+1} \times (\mathbb{C}^{k+1})^{e+1}$ such that if we choose $u_1, \ldots, u_d, u_{d+1}$, $v_1, \ldots, v_e, v_{e+1}$ outside $\mathcal{H}_1 \cup \mathcal{H}_2$, $J+\langle (u_1-1), \ldots, u_d, u_{d+1}, (v_1-1), \ldots, v_e, v_{e+1} \rangle$ and $J+\langle (u_1-1), \ldots, u_d, u_{d+1}, v_1, \ldots, v_e,$ $v_{e+1} \rangle + \langle f \rangle$ are zero-dimensional ideals. Hence as $J+\langle (u_1-1), \ldots, u_d, u_{d+1}, (v_1-1), \ldots, v_e, v_{e+1} \rangle$ is included in $J + \langle (u_1 - 1), \ldots, u_d, u_{d+1}, (v_1 - 1), \ldots, v_e,$ $v_{e+1}, f \rangle$, $\deg(J+\langle (u_1-1), \ldots, u_d, u_{d+1}, (v_1-1), \ldots, v_e, v_{e+1} \rangle + \langle f \rangle) \leqslant \deg(J+\langle u_1, \ldots, u_d, v_1, \ldots, v_e \rangle)$. This proves that $C_{d,e}(I + \langle f \rangle) = C_{d,e}(J + \langle f \rangle) \leqslant C_{d,e}(J) = C_{d,e}(I)$. Summing these equality for all admissible bi-dimension one obtains that $\mathrm{bideg}(I + \langle f \rangle) \leqslant \mathrm{bideg}(I)$ which proves the first part of the result.

Consider now $\widetilde{f}$ a bi-homogeneous polynomial of bidegree (α, β) which does not divide zero in R/J. From the formula given by Proposition 7, the bi-degree of $J + \langle \widetilde{f} \rangle$ is obviously greater than the one of J if

- $\alpha \neq 0$ and $\beta \neq 0$,
- or $\alpha = 0$ and $C_{D-2,0}(J) = 0$ where $D = \dim(J)$,
- or $\beta = 0$ and $C_{0,D-2}(J) = 0$ where $D = \dim(J)$,

which ends the proof.

□

2.5 Proofs of Theorems 1 and 2

We can now prove Theorem 1, which we now restate.

Theorem 1 *Let* $s \in \{1, \ldots, n + k\}$ *and* $f_1, \ldots, f_s$ *be bi-homogeneous polynomials in* R *of respective bi-degree* (α_i, β_i) *generating a bi-homogeneous ideal* I*. Suppose that there exist at most* n f_i *such that* $\beta_i = 0$ *and at most* k f_i *such that* $\alpha_i = 0$*, then the sum of the bi-degrees of the bi-homogeneous associated primes of* I *is bounded by* $\mathcal{B}(f_1, \ldots, f_s) = \sum_{\mathcal{I},\mathcal{J}} (\Pi_{i\in\mathcal{I}}\alpha_i) . (\Pi_{j\in\mathcal{J}}\beta_j)$ *where* $\mathcal{I}$ *and* $\mathcal{J}$ *are disjoint subsets for which the union is* $\{1, \ldots, s\}$ *such that the cardinality of* $\mathcal{I}$ *(resp.* $\mathcal{J}$*) is bounded by* n *(resp.* k*).*

Proof of Theorem 1. For $i \in \{1, \ldots, s\}$ we denote by I_i the ideal generated by $\langle f_1, \ldots, f_i \rangle$. Given an ideal I, we denote by $\mathrm{Ass}(I)$ the set of primes associated to I. Given $(d, e) \in \mathbb{N} \times \mathbb{N}$, we identify the cartesian product of the set of d linear homogeneous forms in $\mathbb{Q}[X_0, \ldots, X_n]$ and e linear homogeneous forms in $\mathbb{Q}[\ell_0, \ldots, \ell_k]$ to $\mathbb{Q}^{d(n+1)} \times \mathbb{Q}^{e(k+1)}$.

Let $\mathfrak{P}(i)$ be the property: for every couple $(d,e) \in \mathbb{N} \times \mathbb{N}$ such that $d+e = n+k-i$, there exists a Zariski-closed subset $\mathcal{H} \subsetneq \mathbb{Q}^{d(n+1)} \times \mathbb{Q}^{e(k+1)}$ such that for every choice of

- d linear homogeneous forms $u_1, \ldots, u_d$ in $\mathbb{Q}[X_0, \ldots, X_n]$ generating the ideal denoted by $\mathcal{U}_d$
- e linear homogeneous forms $v_1, \ldots, v_e$ in $\mathbb{Q}[\ell_0, \ldots, \ell_k]$ generating the ideal denoted by $\mathcal{V}_e$
- such that $(u_1, \ldots, u_d, v_1, \ldots, v_e) \in \left(\mathbb{Q}^{d(n+1)} \times \mathbb{Q}^{e(k+1)}\right) \setminus \mathcal{H}$

One has :

$$\sum_{\mathcal{P} \in \mathrm{Ass}(\sqrt{I_i})} \mathrm{bideg}\,(\mathcal{P} + \mathcal{U}_d + \mathcal{V}_e) \leq \sum_{\substack{|\mathcal{A}|=n-d, |\mathcal{B}|=k-e \\ \mathcal{A}\cap\mathcal{B}=\emptyset, \mathcal{A}\cup\mathcal{B}=\{1,\ldots,i\}}} \left(\prod_{p\in\mathcal{A}} \alpha_p \prod_{q\in\mathcal{B}} \beta_q\right)$$

By definition 3 of the bi-degree of a bi-homogeneous ideal $\mathfrak{P}(1)$ is true. Let us show now that $\mathfrak{P}(i)$ implies $\mathfrak{P}(i+1)$

Remark that

$$\Big(\bigcap_{\mathcal{P}\in\mathrm{Ass}(\sqrt{I_i})} \mathcal{P}\Big) + \langle f_{i+1}\rangle \subset \bigcap_{\mathcal{P}\in\mathrm{Ass}(\sqrt{I_i})} (\mathcal{P} + \langle f_{i+1}\rangle) \subset \bigcap_{\mathcal{P}\in\mathrm{Ass}(\sqrt{I_i})} \left(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle}\right).$$

Note also that all the above ideals have the same radical which is $\sqrt{I_{i+1}}$. Let $\mathcal{Q}_0 \in \mathrm{Ass}(\sqrt{I_{i+1}})$ be a bihomogeneous admissible prime ideal. We show now that there exists $\mathcal{P}_0 \in \mathrm{Ass}(\sqrt{I_i})$ such that $\mathcal{Q}_0 \in \mathrm{Ass}(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle})$.

Notice that the ideal $J = \bigcap_{\mathcal{P}\in\mathrm{Ass}(\sqrt{I_i})} \sqrt{\mathcal{P} + \langle f_{i+1}\rangle}$ equals the ideal $\sqrt{I_{i+1}} = \cap_{\mathcal{Q}\in\mathrm{Ass}(\sqrt{I_{i+1}})} \mathcal{Q}$. Consider $h \in \bigcap_{\mathcal{Q}\in\mathrm{Ass}(\sqrt{I_i})\setminus\{\mathcal{Q}_0\}} \mathcal{Q}$. The saturation of J by h, $J : h^\infty = \{f \in R \mid \exists p \in \mathbb{N} \mid h^p f \in J\}$ equals $\mathcal{Q}_0$ ($\mathcal{Q}_0$ is prime) which implies that $\bigcap_{\mathcal{P}\in\mathrm{Ass}(\sqrt{I_i})} \left(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle} : h^\infty\right) = \mathcal{Q}_0$. For all $\mathcal{P} \in \mathrm{Ass}(\sqrt{I_i})$, $\sqrt{\mathcal{P} + \langle f_{i+1}\rangle} : h^\infty$ is the intersection of the ideals in $\mathrm{Ass}(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle})$ which do not contain h. Thus, there exists $\mathcal{P} \in \mathrm{Ass}(\sqrt{I_i})$ such that $\sqrt{\mathcal{P} + \langle f_{i+1}\rangle} : h^\infty = \mathcal{Q}_0$ which implies that $\mathcal{Q}_0 \in \mathrm{Ass}(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle})$.

Consider a prime ideal $\mathcal{P}$ of $\mathrm{Ass}(\sqrt{I_i})$, let $(d,e) \in \mathbb{N}\times\mathbb{N}$ be such that $d+e = n+k-i-1$, and let $u_1, \ldots, u_d$ (resp. $v_1, \ldots, v_e$) be linear homogeneous forms in $\mathbb{Q}[X_0, \ldots, X_n]$ (resp. in $\mathbb{Q}[\ell_0, \ldots, \ell_k]$), then denoting by $\mathcal{U}_d$ the ideal generated by $u_1, \ldots, u_d$ (resp. $\mathcal{V}_e$ the one generated by $v_1, \ldots, v_e$), one has :

$$\begin{array}{lll} \mathcal{P} + \langle f_{i+1}\rangle + \mathcal{U}_d + \mathcal{V}_e & \subset \sqrt{\mathcal{P} + \langle f_{i+1}\rangle} + \mathcal{U}_d + \mathcal{V}_e & \subset \\ \Big(\bigcap_{\mathcal{Q}\in\mathrm{Ass}(\mathcal{P}+\langle f_{i+1}\rangle)} \mathcal{Q}\Big) + \mathcal{U}_d + \mathcal{V}_e & \subset \bigcap_{\mathcal{Q}\in\mathrm{Ass}(\sqrt{\mathcal{P}+\langle f_{i+1}\rangle})} (\mathcal{Q} + \mathcal{U}_d + \mathcal{V}_e) & \end{array}$$

Since $\mathcal{P}$ is prime, $\mathcal{P}+\langle f_{i+1}\rangle$ is equidimensionnal which implies that for all $\mathcal{Q} \in \mathrm{Ass}(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle})$, $\dim(\mathcal{Q}) = \dim(\mathcal{P} + \langle f_{i+1}\rangle)$.

So from the above four inclusions, one deduces that:

$$\mathrm{bideg}(\mathcal{P} + \langle f_{i+1}\rangle + \mathcal{U}_d + \mathcal{V}_e) \geq \sum_{\mathcal{R}\in \mathrm{Ass}(\sqrt{\mathcal{P}+\langle f_{i+1}\rangle})} \mathrm{bideg}(\mathcal{R} + \mathcal{U}_d + \mathcal{V}_e)$$

As shown above, for all $\mathcal{Q} \in \mathrm{Ass}(\sqrt{I_{i+1}})$ there exists at least one ideal $\mathcal{P} \in \mathrm{Ass}(\sqrt{I_i})$ such that $\mathcal{Q} \in \mathrm{Ass}(\sqrt{\mathcal{P} + \langle f_{i+1}\rangle})$. Thus, the above inequality implies that:

$$\sum_{\mathcal{Q}\in \mathrm{Ass}(\sqrt{I_{i+1}})} \mathrm{bideg}\left(\mathcal{Q} + \mathcal{U}_d + \mathcal{V}_e\right) \leq \sum_{\mathcal{P}\in \mathrm{Ass}(\sqrt{I_i})} \mathrm{bideg}\left(\mathcal{P} + \langle f_{i+1}\rangle + \mathcal{U}_d + \mathcal{V}_e\right)$$

From the theorem's assumptions, I_{i+1} is generated by $f_1, \ldots, f_{i+1}$, with $i \leq n+k-1$. Thus, since the couple (d, e) is such that $d + e = n + k - i - 1$, for all $\mathcal{P} \in \mathrm{Ass}(\sqrt{I_i})$, one has $\dim(\mathcal{P} + \mathcal{U}_d + \mathcal{V}_e) \geq 2$ which allows us to use Lemma 11 and Proposition 7 to prove the existence of a Zariski-closed subset $\mathcal{A} \subsetneq \mathbb{C}^{n+1} \times \mathbb{C}^{k+1}$ such that if u and v are homogeneous linear forms of $\mathbb{Q}[X_0, \ldots, X_n]$ and $\mathbb{Q}[\ell_0, \ldots, \ell_k]$ and (u, v) is chosen outside the Zariski-closed subset $\mathcal{A}$

$$\sum_{\mathcal{Q}\in \mathrm{Ass}(\sqrt{I_{i+1}})} \mathrm{bideg}\left(\mathcal{Q} + \mathcal{U}_d + \mathcal{V}_e\right)$$

is bounded by the sum :

$$\alpha_{i+1} \sum_{\mathcal{P}\in \mathrm{Ass}(\sqrt{I_i})} \mathrm{bideg}\left(\mathcal{P} + \langle u\rangle + \mathcal{U}_d + \mathcal{V}_e\right) + \beta_{i+1} \sum_{\mathcal{P}\in \mathrm{Ass}(\sqrt{I_i})} \mathrm{bideg}\left(\mathcal{P} + \langle v\rangle + \mathcal{U}_d + \mathcal{V}_e\right)$$

Then, using $\mathfrak{P}(i)$ on the preceding formulation allows us to state $\mathfrak{P}(i+1)$.

Thus, under the assumptions of Theorem 1, for $i < s$, the property $\mathfrak{P}(i)$ implies $\mathfrak{P}(i+1)$ and $\mathfrak{P}(1)$ is true.

We now bound the strong bi-degree of I using $\mathfrak{P}(s)$. Let (d, e) be a couple of integers such that $d + e \leq n + k$, and $\mathcal{H} \subsetneq \mathbb{C}^{d(n+1)} \times \mathbb{C}^{e(k+1)}$ be a Zarikisi-closed subset such that chosing homogeneous linear forms $u_1, \ldots, u_d$ (resp. $v_1, \ldots, v_e$) in $\mathbb{Q}[X_0, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_0, \ldots, \ell_k]$) outisde $\mathcal{H}$ and denoting by $\mathcal{U}_d$ (resp. $\mathcal{V}_e$) be the ideal generated by $u_1, \ldots, u_d$ (resp. $v_1, \ldots, v_e$), by definition of the bi-degree, for any prime ideal $\mathcal{P}$ such that $\dim(\mathcal{P}) > d + e$, one has :

$$\mathrm{bideg}(\mathcal{P} + \mathcal{U}_d + \mathcal{V}_e) = \sum_{\substack{(d',e')\in\mathbb{N}\times\mathbb{N} \\ d+e+d'+e'=\dim(\mathcal{P})-2}} C_{d+d',e+e'}(\mathcal{P}).$$

We end the proof looking at the hilbert bi-series of these ideals and remarking that for any associated prime $\mathcal{P}$ of $\sqrt{I}$

$$\mathrm{bideg}(\mathcal{P}) \leq \sum_{\substack{(d,e)\in\mathbb{N}\times\mathbb{N} \\ d+e=n+k-s}} \mathrm{bideg}(\mathcal{P} + \mathcal{U}_d + \mathcal{V}_e).$$

Hence $\mathfrak{P}(s)$ allows us to bound the strong bi-degree of I by $\mathcal{B}(f_1, \ldots, f_s)$.

We prove now Theorem 2 which we now restate:

Consider the mapping :

$$\begin{array}{rccc} \phi : & \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k] & \to & \mathbb{Q}[X_0, X_1, \ldots, X_n, \ell_0, \ell_1, \ldots, \ell_k] \\ & f & \mapsto & X_0^{\deg_X(f)} \ell_0^{\deg_\ell(f)} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}, \frac{\ell_1}{\ell_0}, \ldots, \frac{\ell_k}{\ell_0}) \end{array}$$

where $\deg_X(f)$ *(resp.* $\deg_\ell(f)$*) denotes the degree of* f *seen as a polynomial in* $\mathbb{Q}(\ell_1, \ldots, \ell_k)[X_1, \ldots, X_n]$ *(resp.* $\mathbb{Q}(X_1, \ldots, X_n)[\ell_1, \ldots, \ell_k]$*).*

Given an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$*, denote by* $\phi(I)$ *the ideal generated by* $\{\phi(f) \mid f \in I\} \subset \mathbb{Q}[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$*.*

Then, $\phi(I)$ *is a bi-homogeneous ideal and the sum of the degrees of the isolated primary components of* I *is bounded by the strong bi-degree of* $\phi(I)$*.*

Proof of Theorem 2. The first assertion is obvious. We focus on the second one. Denote by ψ the mapping:

$$\begin{array}{rccc} \psi : & \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k] & \to & \mathbb{Q}[X_0, X_1, \ldots, X_n, \ell_1, \ldots, \ell_k] \\ & f & \mapsto & X_0^{\deg(f)} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}, \frac{\ell_1}{X_0}, \ldots, \frac{\ell_k}{X_0}) \end{array}$$

Given an ideal $I \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$, we denote by $\psi(I)$ the homogeneous ideal generated by $\{\psi(f) \mid f \in I\} \subset \mathbb{Q}[X_0, X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$.

Following [11], if $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$ are the primary ideals of a minimal primary decomposition of I, $\phi(I) = \cap_{i=1}^r \phi(\mathcal{Q}_i)$ and $\psi(I) = \cap_{i=1}^r \psi(\mathcal{Q}_i)$.

Thus, if I is equidimensional, $\phi(I)$ and $\psi(I)$ are equidimensional. Additionnally, if I is radical, $\phi(I)$ and $\psi(I)$ are radical.

Suppose now that I is an equidimensional ideal. Given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$, remark that $\phi(f)$ divided by $X_0 - \ell_0$ with respect to ℓ_0 equals

$$X_0^{\min(\deg_X(f), \deg_\ell(f))} \psi(f).$$

This proves that the ideal $(\phi(I) + \langle X_0 - \ell_0 \rangle) \cap \mathbb{Q}[X_0, \ldots, X_n, \ell_1, \ldots, \ell_k]$ saturated by X_0, denoted by J in the sequel, equals $\psi(I)$. This implies that the degree of J equals the one of $\psi(I)$. Thus the degree of $\psi(I)$ is bounded by the degree of $(\phi(I) + \langle X_0 - \ell_0 \rangle) \cap \mathbb{Q}[X_0, \ldots, X_n, \ell_1, \ldots, \ell_k]$ which is itsself bounded by the degree of $\phi(I) + \langle X_0 - \ell_0 \rangle$. From Bézout's theorem (see [, Fulton]), the sum of the degrees of the isolated primary components of $\phi(I) + \langle X_0 - \ell_0 \rangle$ is bounded by the degree of $\phi(I)$, since $\phi(I)$ is equidimensional.

Thus, the degree of $\psi(I)$ is bounded by the one of $\phi(I)$.

On the one hand, the degree of I equals the one of $\psi(I)$ (see [11, a expliciter]). On the other hand, from Proposition 6 and Proposition 3, the degree of $\phi(I)$ equals the bi-degree of $\phi(I)$. Finally, if I is equidimensional, its degree is bounded by the bi-degree of $\phi(I)$.

If I is not equidimensional, it is sufficient to apply the above to each isolated primary component of I since if $\mathcal{Q}_1, \ldots, \mathcal{Q}_r$ are the primary ideals of a minimal primary decomposition of I, $\psi(I) = \cap_{i=1}^r \psi(\mathcal{Q}_i)$ and $\phi(I) = \cap_{i=1}^r \phi(\mathcal{Q}_i)$.

Corollary 2 *Let S be a finite polynomial family in $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ and I be the ideal generated by S which is supposed to be radical. Consider the ideal J of $\mathbb{Q}[X_0, \ldots, X_n, \ell_0, \ldots, \ell_k]$ generated by $\{\phi(f) \mid f \in S\}$.*

Then, the sum of the degrees of the irreducible components of I is bounded by the strong bi-degree of $\sqrt{J}$.

Proof. From Theorem 2, the degree of I is bounded by the strong bi-degree of $\phi(I)$. Thus, it is sufficient to prove that the strong bi-degree of $\phi(I)$ is bounded by the one of $\sqrt{J}$.

Since, from [11], for all $f \in \phi(I)$, there exists p and q in $\mathbb{N}$ such that $X_0^p \ell_0^q f$ belongs to J, $\phi(I)$ equals the ideal obtained by saturating J by X_0 and ℓ_0. Since I is radical, following the proof of Theorem 2, $\phi(I)$ is radical. Then, each prime component $\mathcal{Q}$ of $\phi(I)$ is an isolated primary component of J. Hence is a prime component of $\sqrt{J}$. This implies that the strong bi-degree of $\sqrt{J}$ (being the sum of the degree of the prime components of $\sqrt{J}$) bounds the one of $\phi(I)$ as $\phi(I)$ is radical and as all the prime components of $\phi(I)$ can be found among the ones of $\sqrt{J}$.

□

3 Degree bounds on the critical locus of a projection

Consider a polynomial family $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal such that the algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ defined by $f_1 = \cdots = f_s = 0$ is smooth, $f_{s+1} \in \mathbb{Q}[X_1, \ldots, X_n]$ and the polynomial mapping $\widetilde{f}_{s+1} : y \in \mathcal{V} \to f_{s+1}(y)$. For $i \in \{1, \ldots, s\}$, we denote by D_i the degree of f_i and by $D = \max(D_i, i = 1, \ldots, s+1)$.

We prove in this section that the sum of the degrees of the equidimensional components of the critical locus of the polynomial mapping $\widetilde{f}_{s+1}$ is bounded by $D_1 \cdots D_s (D-1)^{n-s} \binom{n}{n-s}$.

Definition 5 *Consider an algebraic variety $\mathcal{V} \subset \mathbb{C}^n$, and denote by $I(\mathcal{V}) \subset \mathbb{Q}[X_1, \ldots, X_n]$ the ideal associated to $\mathcal{V}$.*

- *If f is a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$, the* linear part *of f at a point $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$, denoted by $d_p(f)$, is defined to be: $d_p(f) = \frac{\partial f}{\partial X_1}(X_1 - p_1) + \ldots + \frac{\partial f}{\partial X_n}(X_n - p_n)$.*
- *The tangent space of $\mathcal{V}$ at p, denoted by $T_p(\mathcal{V})$, is the set of common zeroes of $d_p(f)$ for $f \in I(\mathcal{V})$.*
- *For $p \in \mathcal{V}$, the dimension of $\mathcal{V}$ at p, denoted by $\dim_p(\mathcal{V})$, is the maximum dimension of an irreducible component of $\mathcal{V}$ containing p.*
- *A point $p \in \mathcal{V}$ is said to be* smooth *(or* nonsingular*) if $\dim(T_p(\mathcal{V})) = \dim_p(\mathcal{V})$.*
- *An algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ is* smooth *if and only if all points $p \in \mathcal{V}$ are smooth points.*

Lemma 12 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a* smooth *algebraic variety defined by s polynomials $f_1, \dots, f_s$ in $\mathbb{Q}[X_1, \dots, X_n]$. Suppose $\langle f_1, \dots, f_s \rangle$ to be radical, and let f_{s+1} be a polynomial in $\mathbb{Q}[X_1, \dots, X_n]$ and $\widetilde{f}_{s+1}$ be the mapping:*

$$\begin{array}{cccc} \widetilde{f}_{s+1}: & \mathcal{V} \subset \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & (x_1, \dots, x_n) & \mapsto & f_{s+1}(x_1, \dots, x_n) \end{array}$$

Given $p \in \mathcal{V}$, the point p is a critical point of $\widetilde{f}_{s+1}$ if and only if there exists a point $(\lambda_1, \dots, \lambda_s)$ in $\mathbb{C}^s$ such that $(\lambda_1, \dots, \lambda_s, p) \in \mathbb{C}^s \times \mathbb{C}^n$ is a solution of the polynomial system in $\mathbb{Q}[\ell_1, \dots, \ell_s, X_1, \dots, X_n]$:

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0 \\ \ell_1 \frac{\partial f_1}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s}{\partial X_1} = \frac{\partial f_{s+1}}{\partial X_1} \\ \ell_1 \frac{\partial f_1}{\partial X_2} + \cdots + \ell_s \frac{\partial f_s}{\partial X_2} = \frac{\partial f_{s+1}}{\partial X_2} \\ \qquad\qquad \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s}{\partial X_n} = \frac{\partial f_{s+1}}{\partial X_n} \end{array} \right.$$

where $\ell_1, \dots, \ell_s$, are new variables.

Proof. By definition, a point $p \in \mathcal{V}$ is a critical point of $\widetilde{f}_{s+1}$ restricted to $\mathcal{V}$ if and only if the differential of $\widetilde{f}_{s+1}$ at p, denoted by $d_p(\widetilde{f}_{s+1})$ is not surjective. This is equivalent to say that the gradient $\mathbf{grad}_p(f_{s+1})$ is orthogonal to $T_p(\mathcal{V})$. On the other hand, from the second item of Definition 5 and since $\langle f_1, \dots, f_s \rangle$ is radical, the vector space $\mathrm{Span}(\mathbf{grad}_p(f_1), \dots, \mathbf{grad}_p(f_s))$ is supplementar with $T_p(\mathcal{V})$.

Thus, p is a critical point of $\widetilde{f}_{s+1}$ restricted to $\mathcal{V}$ if and only if the gradient $\mathbf{grad}_p(f_{s+1})$ belongs to $\mathrm{Span}(\mathbf{grad}_p(f_1), \dots, \mathbf{grad}_p(f_s))$.

In other words, there exist complex numbers $\lambda_1, \dots, \lambda_s$ such that:

$$\begin{array}{l} f_1(p) = \cdots = f_s(p) = 0 \\ \lambda_1 \frac{\partial f_1}{\partial X_1} + \cdots + \lambda_s \frac{\partial f_s}{\partial X_1} = \frac{\partial f_{s+1}}{\partial X_1} \\ \lambda_1 \frac{\partial f_1}{\partial X_2} + \ldots + \lambda_s \frac{\partial f_s}{\partial X_2} = \frac{\partial f_{s+1}}{\partial X_2} \\ \qquad\qquad \vdots \\ \lambda_1 \frac{\partial f_1}{\partial X_n} + \cdots + \lambda_s \frac{\partial f_s}{\partial X_n} = \frac{\partial f_{s+1}}{\partial X_n} \end{array}$$

which ends the proof.

$\square$

Remark 4 *This algebraic characterization is well known as* Lagrange's characterization *(or* Lagrange's system*).*

Note that the above Lemma defines critical points of a polynomial mapping restricted to an algebraic variety as roots of an elimination ideal. Remark that the considered algebraic variety is not supposed to be equidimensional contrarily to the algebraic characterization of critical points which is used in [2, 35, 37, 7, 4, 5].

This is a key point to generalize Safey-Schost's algorithm [37] computing at least one point in each connected component of a real algebraic variety to the non equidimensional case.

We are now ready to state the main result of this section.

Theorem 3 *Let $f_1, \ldots, f_s$ in $\mathbb{Q}[X_1, \ldots, X_n]$ be s polynomials (with $s \leqslant n-1$). Suppose $\langle f_1, \ldots, f_s \rangle$ is a radical ideal and defines a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Let $f_{s+1} \in \mathbb{Q}[X_1, \ldots, X_n]$ and consider the mapping $\widetilde{f}_{s+1}$ sending $x \in \mathbb{C}^n$ to $f_{s+1}(x)$. Denote by $D_1, \ldots D_s, D_{s+1}$ the respective degrees of $f_1, \ldots, f_s, f_{s+1}$, and by D the maximum of $D_1, \ldots, D_s, D_{s+1}$. Then, the sum of the degrees of the equidimensional components of the critical locus of $\widetilde{f}_{s+1}$ restricted to $\mathcal{V}$ is bounded by:*

$$D_1 \cdots D_s (D-1)^{n-s} \binom{n}{n-s}$$

Proof. Let X_0 and ℓ_0 be new variables and denote, as in the proof of Theorem 2, by ϕ the mapping which associates to $f \in \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ the polynomial

$$\phi(f) = X_0^{\deg_X(f)} \ell_0^{\deg_\ell(f)} f(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}, \frac{\ell_1}{\ell_0}, \ldots, \frac{\ell_k}{\ell_0})$$

where $\deg_X(f)$ (resp. $\deg_\ell(f)$) denotes the degree of f when it is seen as a polynomial in $\mathbb{Q}(\ell_1, \ldots, \ell_k)[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}(X_1, \ldots, X_n)[\ell_1, \ldots, \ell_k]$).

Denote by J the ideal generated by Lagrange's system S given in Lemma 12. Bounding the sum of the geometric degrees of the equidimensional components of the critical locus of $\widetilde{f}_{s+1}$ is equivalent to bounding the sum of the degrees of the equidimensional components of $\sqrt{J}$. From Corollary 2, this sum is bounded by the strong bi-degree of $\sqrt{\phi(J)}$. Now, applying Theorem 1 to $\langle \phi(f), f \in S \rangle$ ends the proof.

□

Remark 5 *Suppose $\mathcal{V} \subset \mathbb{C}^n$ is an equidimensional algebraic variety of dimension d. In this case, the critical points of a mapping can be characterized by the vanishing of some minors of a jacobian matrix. Then, applying the classical Bézout theorem to the obtained polynomial system yields the degree bound:*

$$D^{n-d} \left((n-d)(D-1)\right)^d .$$

This quantity, which is greater than the one obtained in Theorem 3, is used in [5, 4] to bound, in the worst case, the number of critical points computed by the algorithms proposed in these papers. A similar approach is used in [35, 7, 6].

Our bound, given in Theorem 3, shows that the previous ones were not sharp, in particular when $d = n/2$.

The following corollary is intensively used in the next section.

Corollary 3 *Let $f_1, \dots, f_s$ in $\mathbb{Q}[X_1, \dots, X_n]$ be s polynomials (with $s \leqslant n-1$). Suppose they generate a radical ideal and define a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Denote by $D_1, \dots D_s$ the respective degrees of $f_1, \dots, f_s$, and by D the maximum of $D_1, \dots, D_s$. Consider $\pi : \mathbb{C}^n \to \mathbb{C}$ the canonical projection on the first coordinate and suppose the critical locus of its restriction to $\mathcal{V}$ to be zero-dimensional. Then, the degree of the critical locus of π restricted to $\mathcal{V}$ is bounded by:*

$$D_1 \cdots D_s (D-1)^{n-s} \binom{n}{n-s}$$

Moreover, if $(f_1, \dots, f_s)$ is a regular sequence, the degree of the critical locus of π restricted to $\mathcal{V}$ is bounded by:

$$D_1 \cdots D_s (D-1)^{n-s} \binom{n-1}{n-s}$$

Proof. The first item is a direct application of Theorem 3.

We focus on the case where $(f_1, \dots, f_s)$ is a regular sequence. From Lemma 12, the critical locus of the restriction of π to $\mathcal{V}$ is the projection on $X_1, \dots, X_n$ of the zero-set of

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0, \\ \ell_1 \frac{\partial f_1}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s}{\partial X_1} = 1 \\ \\ \ell_1 \frac{\partial f_1}{\partial X_2} + \cdots + \ell_s \frac{\partial f_s}{\partial X_2} = 0 \\ \qquad \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s}{\partial X_n} = 0 \end{array} \right. \tag{2}$$

Since the critical locus of the restricition of π to $\mathcal{V}$ is supposed to be zero-dimensional, there exists a Zariski-closed subset $\mathcal{A} \subset \mathbb{C}^{s-1}$ such that chosing $(a_1, \dots, a_{s-1}) \subset \mathbb{C}^{s-1} \setminus \mathcal{A}$ and substituting f_s by $f_s + a_1 f_1 + \cdots + a_{s-1} f_{s-1}$ in (2) yields a polynomial system such that for any of its solution $(x_1, \dots, x_n, \lambda_1, \dots, \lambda_s)$, $\lambda_s \neq 0$. Thus, one can suppose that all solutions of (2) satisfy $\ell_s \neq 0$.

Then, the set of critical points of π restricted to $\mathcal{V}$ is contained in the projection on $X_1, \dots, X_n$ of the zero-set of:

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0 \\ m_1 \frac{\partial f_1}{\partial X_2} + \cdots + m_{s-1} \frac{\partial f_{s-1}}{\partial X_2} + \frac{\partial f_s}{\partial X_2} = 0 \\ \\ \qquad \vdots \\ m_1 \frac{\partial f_1}{\partial X_n} + \cdots + m_{s-1} \frac{\partial f_{s-1}}{\partial X_n} + \frac{\partial f_s}{\partial X_n} = 0 \end{array} \right. \tag{3}$$

We prove now the reverse inclusion.

Consider a solution $p = (x_1, \ldots, x_n, \mu_1, \ldots, \mu_{s-1})$ of (3). Since $(f_1, \ldots, f_s)$ is regular sequence defining a radical ideal and $\mathcal{V}$ is smooth, the jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_s)$ has full rank at $(x_1, \ldots, x_n)$. This implies the polynomial $m_1 \frac{\partial f_1}{\partial X_1} + \cdots + m_{s-1} \frac{\partial f}{\partial X_1} + \frac{\partial f_s}{\partial X_1}$ to be not null at p.

Thus, for any solution $p = (x_1, \ldots, x_n, \mu_1, \ldots, \mu_{s-1})$ there exists $\mu \neq 0$ such that $(x_1, \ldots, x_n, \mu_1, \ldots, \mu_{s-1}, \mu)$ is a solution of

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0 \\ m_1 \frac{\partial f_1}{\partial X_2} + \cdots + m_{s-1} \frac{\partial f_{s-1}}{\partial X_2} + \frac{\partial f_s}{\partial X_2} = m_s \\ m_1 \frac{\partial f_1}{\partial X_2} + \cdots + m_{s-1} \frac{\partial f_{s-1}}{\partial X_2} + \frac{\partial f_s}{\partial X_2} = 0 \\ \quad \vdots \\ m_1 \frac{\partial f_1}{\partial X_n} + \cdots + m_{s-1} \frac{\partial f_{s-1}}{\partial X_n} + \frac{\partial f_s}{\partial X_n} = 0 \end{array} \right. \tag{4}$$

Conversly, since $\mathrm{Jac}(f_1, \ldots, f_s)$ has full rank at any point of $\mathcal{V}$, for any solution $(x_1, \ldots, x_n, \mu_1, \ldots, \mu_s)$ of (4) $\mu_s \neq 0$. Then, one can divide each equation in (4) by m_s and put $\ell_1 = m_1/m_s, \ldots, \ell_{s-1} = m_{s-1}/m_s$ and $\ell_s = 1/m_s$ to recover (2). This allows us to conclude that the projection of the zero-set of (3) is the critical locus of π restricted to $\mathcal{V}$.

Now, applying Corollary 2 and Theorem 1 to the system obtained by bihomogeneizing the system (3) ends the proof.

□

We show in the following section how our bound can be used to improve the already known bounds on the first Betti number of a smooth real algebraic variety defined by a polynomial system generating a radical ideal.

4 Generalization of Safey/Schost's Algorithm

In this section, we first generalize to the non equidimensional case, the algorithm provided in [37] computing at least one point in each connected component of the real counterpart of a smooth equidimensional algebraic variety. Then, we estimate the number of points computed by the algorithm we propose using Theorem 3 to bound the first Betti number of a smooth real algebraic set defined by a polynomial system generating a radical ideal.

Given a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d, we denote by Π_i (for i in $\{1, \ldots, d\}$) the canonical projection:

$$\begin{array}{cccc} \Pi_i : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ & (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_i) \end{array}$$

and by $W_{n-(i-1)}(\mathcal{V})$ the critical locus of the restriction of Π_i to $\mathcal{V}$, i.e. **the union of the critical points of the restrictions of Π_i to each equidimensional component of $\mathcal{V}$.**

Following [37], we set $W_{n-d}(\mathcal{V}) = \mathcal{V}$ and we have:

$$W_n(\mathcal{V}) \subset W_{n-1}(\mathcal{V}) \subset \ldots \subset W_{n-d+1}(\mathcal{V}) \subset W_{n-d}(\mathcal{V})$$

In the equidimensional case, the algorithm provided in [37] is based on the following geometric result:

Theorem 4 *[37] Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth equidimensional algebraic variety of dimension d. Up to a generic linear change of variables, given an* arbitrary *point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$, $W_{n-(i-1)}(\mathcal{V}) \cap \Pi_{i-1}^{-1}(p_1, \ldots, p_{i-1})$ is zero-dimensional for i in $\{1, \ldots, d+1\}$, and the union of the finite algebraic sets:*

$$W_{n-d}(\mathcal{V}) \cap \Pi_d^{-1}(p_1, \ldots, p_d), \ldots, W_{n-(i-1)}(\mathcal{V}) \cap \Pi_{i-1}^{-1}(p_1, \ldots, p_{i-1}), \ldots, W_n(\mathcal{V})$$

intersects each connected component of $\mathcal{V} \cap \mathbb{R}^n$.

A naive way of using this result in non equidimensional situations is to compute an equidimensional decomposition of the ideal $\langle f_1, \ldots, f_s \rangle$ and to apply Theorem 4 to each computed equidimensional component. This technique is underlying in many recent algorithms computing at least one point in each connected component of a real algebraic set (see [2, 38, 35]) and does not allow us to prove satisfactory complexity results neither on the output of the algorithms nor on the arithmetic complexity, since the degree of the polynomials defining each equidimensional component is not well controlled.

Lemma 13 *Let $(f_1, \ldots, f_s)$ be a polynomial family in $\mathbb{Q}[X_1, \ldots, X_n]$. Suppose it generates a radical ideal of dimension d and defines a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Given a point $(p_1, \ldots, p_d)$ in $\mathbb{Q}^d$, consider the polynomial system in $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_s]$:*

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0, \\ X_1 - p_1 = \cdots = X_i - p_i = 0 \\ \ell_1 \frac{\partial f_1}{\partial X_{i+1}} + \cdots + \ell_s \frac{\partial f_s}{\partial X_{i+1}} = 1 \\ \ell_1 \frac{\partial f_1}{\partial X_{i+2}} + \cdots + \ell_s \frac{\partial f_s}{\partial X_{i+2}} = 0 \\ \qquad\qquad \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s}{\partial X_n} = 0 \end{array} \right. \tag{5}$$

The projection of its complex solution set on $X_1, \ldots, X_n$ is $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(\mathcal{V})$.

Proof. Denote by $\mathcal{W}$ the projection of the complex solution set of (5) on $X_1, \ldots, X_n$. We first prove it contains $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(\mathcal{V})$, then we prove the reverse inclusion.

Consider $(g_1, \ldots, g_k) \in \mathbb{Q}[X_1, \ldots, X_n]$ a polynomial family generating a radical ideal whose associated algebraic variety, denoted by C_q, is an equidimensional component of

$\mathcal{V} \subset \mathbb{C}^n$ of dimension $q \leqslant d$. Let y be a point in $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(C_q)$. We first prove that it belongs to $\mathcal{W}$.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_{i+1}$ be the gradient vectors of $X_1, \ldots, X_{i+1}$. Since $y \in W_{n-i}(C_q)$:

$$\dim(\mathrm{Span}(\mathbf{e}_1, \ldots, \mathbf{e}_{i+1}) + \mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k))) \leqslant n - q + i.$$

and this implies that:

$$\mathbf{e}_{i+1} \in \mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i).$$

Since $\langle f_1, \ldots, f_s \rangle$ is radical,

$$\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k)) = \mathrm{Span}(\mathbf{grad}_y(f_1), \ldots, \mathbf{grad}_y(f_s))$$

which implies that:

$$\mathbf{e}_{i+1} \in \mathrm{Span}(\mathbf{grad}_y(f_1), \ldots, \mathbf{grad}_y(f_s), \mathbf{e}_1, \ldots, \mathbf{e}_i).$$

Hence, there exists $(\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s$ such that $(y, \lambda) \in \mathbb{C}^n \times \mathbb{C}^s$ belongs to the solution set of (5).

Consider now $y \in \mathcal{V}$ such that there exists $\lambda \in \mathbb{C}^s$ for which (y, λ) is a solution of the system (5). We prove in the sequel that there exists an equidimensional component C of $\mathcal{V}$ such that y belongs to $W_{n-i}(C)$.

Since $y \in \mathcal{V}$ there exists an equidimensional component C of $\mathcal{V}$ such that $y \in C$ and let q be the dimension of C. We prove in the following that $y \in W_{n-i}(C)$ which is sufficient to conclude since y already belongs to $\Pi_i^{-1}(p_1, \ldots, p_i)$. Consider a set of generators $g_1, \ldots, g_k$ of the ideal associated to C and remark that since $\langle f_1, \ldots, f_s \rangle$ is radical

$$\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k)) = \mathrm{Span}(\mathbf{grad}_y(f_1), \ldots, \mathbf{grad}_y(f_s)).$$

Note also that since $y \in \Pi_i^{-1}(p_1, \ldots, p_i)$, the vector-space

$$\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i)$$

is the co-tangent space of $C \cap \Pi_i^{-1}(p_1, \ldots, p_i)$ which has dimension at most $n - q + i$. Since there exists $\lambda \in \mathbb{C}^s$ such that $(y, \lambda) \in \mathcal{W}$

$$\mathbf{e}_{i+1} \in \mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i)$$

which implies that

$$\dim(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i, \mathbf{e}_{i+1}) \leqslant n - q + i$$

By definition, this proves y belongs to $W_{n-i}(C)$.

□

Given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$, and $\mathbf{A} \in GL_n(\mathbb{Q})$, we denote by $f^{\mathbf{A}}$ the polynomial obtained by performing the change of variables induced by $\mathbf{A}$ on f.

Lemma 14 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth algebraic variety defined by s polynomials $f_1, \ldots, f_s$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a* radical *ideal. Given $\mathbf{A} \in GL_n(\mathbb{Q})$, consider $I_0^{\mathbf{A}} \subset \mathbb{Q}[\ell_1, \ldots, \ell_s, X_1, \ldots, X_n]$ the ideal generated by the polynomial system:*

$$\left\{\begin{array}{l} f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_2} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_2} = 0 \\ \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

There exists a proper Zariski-closed subset $\mathcal{H} \subset GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}$, $I_0^{\mathbf{A}}$ is radical and the elimination ideal $I_0^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ is zero-dimensional *or equal to $\langle 1 \rangle$.*

Suppose additionally that $f_1, \ldots, f_s$ is a regular sequence. Then, there exists a proper Zariski-closed subset $\mathcal{H}' \subset GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}'$, the ideal $I_0^{\mathbf{A}}$ is radical and is either zero-dimensional *or equal to $\langle 1 \rangle$.*

Proof. Consider a polynomials family $(g_1, \ldots, g_k)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical equidimensional ideal whose associated algebraic variety is an equidimensional component C_d of $\mathcal{V} \subset \mathbb{C}^n$. Let $\mathcal{P}^{\mathbf{A}} \subset \mathbb{C}^n$ be the algebraic variety associated to $I_0^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{P}_d^{\mathbf{A}}$ a subset of $\mathcal{P}^{\mathbf{A}}$ such that $\mathcal{P}_d^{\mathbf{A}}$ is the intersection of $\mathcal{P}^{\mathbf{A}}$ and $C_d^{\mathbf{A}}$, the complex solution set of:

$$g_1^{\mathbf{A}} = \cdots = g_k^{\mathbf{A}} = 0.$$

From [37, Theorem 2], there exists a proper Zariski closed subset of $GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}_d$, the critical locus of the restriction of the canonical projection π:

$$\begin{array}{cccc} \pi : & \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & (x_1, \ldots, x_n) & \mapsto & x_1 \end{array}$$

to $C_d^{\mathbf{A}}$ is zero-dimensional or empty. Now, consider the polynomial system in $\mathbb{Q}[X_1, \ldots, X_n, m_1, \ldots, m_k]$:

$$\left\{\begin{array}{l} g_1^{\mathbf{A}} = \cdots = g_k^{\mathbf{A}} = 0 \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_1} + \cdots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_1} = 1 \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_2} + \cdots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_2} = 0 \\ \vdots \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_n} + \cdots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

and $J^{\mathbf{A}}$ the ideal it generates. From Lemma 12, the critical locus of the restricition of π to $C_d^{\mathbf{A}}$ is the algebraic variety associated to $J^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, which is consequently zero-dimensional if $\mathbf{A} \notin \mathcal{H}_d$.

Since $\mathcal{V}$ is smooth, the tangent space to $\mathcal{V}$ at any point p in $C_d \subset \mathcal{V}$ is equal to the tangent space to C_d at p, and since $\langle g_1, \ldots, g_k \rangle$ is a radical ideal, the following holds:

$$\mathrm{Span}(\mathbf{grad}_p(g_1), \ldots, \mathbf{grad}_p(g_k)) = \mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))$$

Moreover, for $\mathbf{A}.p \in C_d^{\mathbf{A}} \subset \mathcal{V}^{\mathbf{A}}$:

$$\mathbf{A}.\left(\mathrm{Span}(\mathbf{grad}_p(g_1), \ldots, \mathbf{grad}_p(g_k))\right) = \mathrm{Span}(\mathbf{grad}_{\mathbf{A}.p}(g_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{\mathbf{A}.p}(g_k^{\mathbf{A}}))$$

and

$$\mathbf{A}\left(\mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))\right) = \mathrm{Span}(\mathbf{grad}_{\mathbf{A}.p}(f_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{\mathbf{A}.p}(f_s^{\mathbf{A}}))$$

Thus, at each point $p_{\mathbf{A}} \in \mathcal{P}_d^{\mathbf{A}}$,

$$\mathrm{Span}(\mathbf{grad}_{p_{\mathbf{A}}}(g_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{p_{\mathbf{A}}}(g_k^{\mathbf{A}})) = \mathrm{Span}(\mathbf{grad}_{p_{\mathbf{A}}}(f_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{p_{\mathbf{A}}}(f_s^{\mathbf{A}}))$$

Hence, $\mathcal{P}_d^{\mathbf{A}}$ is exactly the algebraic variety associated to $J^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, which is zero-dimensional if $\mathbf{A} \notin \mathcal{H}_d$. Iterating the above on each equidimensional component of $\mathcal{V}$ proves that there exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that if $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$, $I_0^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ is either zero-dimensional or equal to $\langle 1 \rangle$.

We prove now that there exists a Zariski-closed subset $\mathcal{H}' \subsetneq GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{H}'$, $I_0^{\mathbf{A}}$ is radical. Consider the mapping

$$\begin{array}{cccc} \phi: & \mathbb{C}^n \times \mathbb{C}^s & \to & \mathbb{C}^n \\ & (y, \lambda_1, \ldots, \lambda_s) & \to & \left(\sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial X_1}(y), \ldots, \sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial X_n}(y)\right) \end{array}$$

From Sard's theorem , for each equidimensional component C of $\mathcal{V}$, the set of critical values $K(\phi, C)$ of the restriction of ϕ to $C \times \mathbb{C}^s$ is Zariski-closed in $\mathbb{C}^n$. Let $\mathcal{B}$ be the union of the sets $K(\phi, C)$ for all equidimensional components C of $\mathcal{V}$ and $(a_1, \ldots, a_n) \in \mathbb{Q}^n \setminus \mathcal{A}$. We prove now that the ideal J generated by:

$$\left\{ \begin{array}{l} f_1 = \cdots = f_s = 0 \\ \ell_1 \frac{\partial f_1}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s}{\partial X_1} = a_1 \\ \qquad\qquad \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s}{\partial X_n} = a_n \end{array} \right. \tag{6}$$

is radical. Remark first that from the above paragraph, there exists a Zariski-closed subset such that if $(a_1, \ldots, a_n) \in \mathbb{Q}^n \setminus \mathcal{C}$, $J \cap \mathbb{Q}[X_1, \ldots, X_n]$ is zero-dimensional.

Let $(y, \lambda) \in \mathbb{C}^n \times \mathbb{C}^s$ be a solution of the above polynomial system, C be the equidimensional component of $\mathcal{V}$ containing y; let also $g_1, \ldots, g_k$ be a set of generators of the ideal associated to C and d be the dimension of C. Since $\mathcal{V}$ is smooth and $\langle f_1, \ldots, f_s \rangle$ is radical, the jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_s)$ has rank $n - d$ at y. Since $J \cap \mathbb{Q}[X_1, \ldots, X_n]$

is zero-dimensional, and since $\mathrm{Jac}(f_1, \ldots, f_s)$ has rank $n-d$ at y, the set $\Lambda_y \subset \mathbb{C}^s$ defined such that $\lambda \in \Lambda_y$ if and only if (y, λ) belongs to the complex solution set of J has dimension $d-(n-s)$. Thus, the irreducible component of the algebraic variety defined by J containing (y, λ) has dimension $d-(n-s)$. In order to prove that J is radical, it is sufficient to prove that the jacobian matrix associated to the above polynomial system has rank $n+s-(d-(n-s)) = 2n-d$ at (y, λ).

Since, by definition, $\langle g_1, \ldots, g_k \rangle$ is radical and C is smooth, there exists a subset $\{g_{i_1}, \ldots, g_{i_{n-d}}\} \subset \{g_1, \ldots, g_k\}$ such that the rank of the jacobian matrix $\mathrm{Jac}(g_{i_1}, \ldots, g_{i_{n-d}})$ at y equals the rank of $\mathrm{Jac}(g_1, \ldots, g_k)$ which is $n-d$.

Since $(a_1, \ldots, a_n)$ is not a critical value of the restriction of ϕ to $C \times \Lambda_y$, the rank of the jacobian matrix associated to the polynomial family

$$(g_{i_1}, \ldots, g_{i_{n-d}}, \sum_{i=1}^{s} \ell_i \frac{\partial f_i}{\partial X_1}, \ldots, \sum_{i=1}^{s} \ell_i \frac{\partial f_i}{\partial X_n})$$

is maximal and then is $2n-d$ at (y, λ) which implies that the jacobian matrix associated to $\sum_{i=1}^{s} \ell_i \frac{\partial f_i}{\partial X_1}, \ldots, \sum_{i=1}^{s} \ell_i \frac{\partial f_i}{\partial X_n})$ with respect to the variables $\ell_1, \ldots, \ell_s$ has rank n at (y, λ). Since $\mathrm{Jac}(f_1, \ldots, f_s)$ has rank $n-d$ at y, this implies that the rank of the jacobian matrix associated to the polynomial system (6) equals $2n-d$ at (y, λ) which ends to prove that there exists a Zariski-closed subset $\mathcal{B}$ such that if $(a_1, \ldots, a_n) \in \mathbb{Q}^n \setminus \mathcal{B}$, J is radical.

Suppose now $f_1, \ldots, f_s$ to be a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal. Then the ideal $\langle f_1, \ldots, f_s \rangle$ is equidimensional of dimension $n-s$. Thus, at any point of $\mathcal{V}$ the jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_s)$ has rank s. Consequently, at any point p of the algebraic variety associated to $I_0^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, the jacobian matrix $\mathrm{Jac}(f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}})$ has rank s since it equals $\mathbf{A}^{-1}.\mathrm{Jac}(f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}})$. Moreover, there exists a proper Zariski closed subset $\mathcal{H} \subsetneq GL_n(\mathbb{C})$ such that if $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{H}$, the algebraic variety $\mathcal{P}^{\mathbf{A}}$ associated to $I_0^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ is zero-dimensional . Thus, for any point $p \in \mathcal{P}^{\mathbf{A}}$ there exists at most a finite set of points $(\lambda_1, \ldots, \lambda_s)$ in $\mathbb{P}(\mathbb{C})^s$ which is a solution of the linear system:

$$\lambda_1.\mathbf{grad}_p(f_1^{\mathbf{A}}) + \ldots + \lambda_s.\mathbf{grad}_p(f_s^{\mathbf{A}}) = \mathbf{u}$$

(where all, but the first, coordinates of $\mathbf{u}$ are zero), which ends the proof.

□

Given a polynomial family $(f_1, \ldots, f_s) \subset \mathbb{Q}[X_1, \ldots, X_n]$, d the dimension of the ideal $\langle f_1, \ldots, f_s \rangle$, $\mathbf{A} \in GL_n(\mathbb{Q})$, and $p = (p_1, \ldots, p_d)$ an arbitrary point of $\mathbb{Q}^d$ we denote by $I_i^{\mathbf{A},p}$ (for $i \in \{1, \ldots, d-1\}$) the ideal in $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ generated by:

$$\left\{ \begin{array}{l} f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = 0, \\ X_1 - p_1 = 0, \ldots, X_i - p_i = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_{i+1}} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_{i+1}} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_{i+2}} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_{i+2}} = 0 \\ \qquad \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array} \right.$$

and by $I_d^{\mathbf{A},p}$ the ideal in $\mathbb{Q}[X_1,\ldots,X_n,\ell_1,\ldots,\ell_k]$ generated by $f_1^{\mathbf{A}}=\cdots=f_s^{\mathbf{A}}=X_1-p_1=\cdots=X_d-p_d=0$. Remember that $I_0^{\mathbf{A},p}$ denotes the ideal generated by the polynomial system:

$$\begin{cases} f_1^{\mathbf{A}}=\cdots=f_s^{\mathbf{A}}=0, \\ \ell_1\frac{\partial f_1^{\mathbf{A}}}{\partial X_1}+\cdots+\ell_1\frac{\partial f_s^{\mathbf{A}}}{\partial X_1}=1 \\ \ell_1\frac{\partial f_1^{\mathbf{A}}}{\partial X_2}+\cdots+\ell_1\frac{\partial f_s^{\mathbf{A}}}{\partial X_2}=0 \\ \qquad\vdots \\ \ell_1\frac{\partial f_1^{\mathbf{A}}}{\partial X_n}+\cdots+\ell_1\frac{\partial f_s^{\mathbf{A}}}{\partial X_n}=0 \end{cases}$$

The following result allows us to generalize the algorithm provided in [37] to the non equidimensional case.

Theorem 5 *Let $(f_1,\ldots,f_s)\subset\mathbb{Q}[X_1,\ldots,X_n]$ be a polynomial family. Suppose it generates a radical ideal and defines a smooth algebraic variety $\mathcal{V}\subset\mathbb{C}^n$ of dimension d. Then, there exists hypersurfaces $\mathcal{H}\subset GL_n(\mathbb{Q})$ and $\mathcal{P}\subsetneq\mathbb{C}^d$ such that if $\mathbf{A}\notin\mathcal{H}$ and $p\in\mathbb{Q}^d\setminus\mathcal{P}$,*

- *the ideals $I_i^{\mathbf{A},p}$ (for all $i\in\{0,\ldots,d\}$) are radical;*
- *the ideals $I_i^{\mathbf{A},p}\cap\mathbb{Q}[X_1,\ldots,X_n]$ (for all $i\in\{0,\ldots,d\}$) are either zero-dimensional or equal to $\langle 1\rangle$;*
- *the set of their real roots has a non-empty intersection with each connected component of $\mathcal{V}\cap\mathbb{R}^n$.*

Proof. Since the ideal $I=\langle f_1,\ldots,f_s\rangle$ is radical and has dimension d, there exists Zariski-closed subsets $\mathcal{A}\subsetneq GL_n(\mathbb{C})$ and $\mathcal{P}\subsetneq\mathbb{C}^d$ such that if $\mathbf{A}\in GL_n(\mathbb{Q})\setminus\mathcal{A}$ and $p=(p_1,\ldots,p_d)\in\mathbb{Q}\setminus\mathcal{P}$, then the ideals $\langle f_1^{\mathbf{A}},\ldots,f_s^{\mathbf{A}},X_1-p_1,\ldots,X_i-p_i\rangle$ are radical.

For $i=1,\ldots,d-1$, denote by $J_i^{\mathbf{A},p}\subset\mathbb{Q}[X_{i+1},\ldots,X_n]$ the ideal obtained by substituting $X_1,\ldots,X_i$ by $p_1,\ldots,p_i$ in the given set of generators of $I_i^{\mathbf{A},p}$. Remark that $I_i^{\mathbf{A},p}$ is radical if and only if $J_i^{\mathbf{A},p}$ is radical. From Lemma 14, there exists a Zariski-closed subset $\mathcal{A}\subsetneq GL_{n-i}(\mathbb{C})$ such that if $\mathbf{A}\in GL_{n-i}(\mathbb{Q})\setminus\mathcal{A}$, $J_i^{\mathbf{A},p}$ is radical. This proves the first item.

The second item is a direct consequence of Lemma 13.

We prove now the third item. Let C_d be an equidimensional component of $\mathcal{V}\subset\mathbb{C}^n$ of dimension d. From Theorem 4, given an arbitrary point $(p_1,\ldots,p_d)\in\mathbb{Q}^d$, there exists a proper Zariski closed subset $\mathcal{H}_d\subset GL_n(\mathbb{Q})$ such that if $\mathbf{A}\notin\mathcal{H}_d$, then the union of the sets $\Pi_i^{-1}(p_1,\ldots,p_i)\cap W_{n-i}(C_d^{\mathbf{A}})$ for $i=1,\ldots,d$ and $W_n^{\mathbf{A}}(C_d)$ has a non-empty intersection with each connected component of the real counterpart of $\mathcal{V}$. The conclusion follows by applying again Lemma 13.

$\square$

Following the above result, after a generic choice of $\mathbf{A}\in GL_n(\mathbb{Q})$, the elimination ideals $I_i^{\mathbf{A}}\cap\mathbb{Q}[X_1,\ldots,X_n]$ are zero-dimensional or $\langle 1\rangle$ and encode at least one point in each connected component of $\mathcal{V}\cap\mathbb{R}^n$. To obtain new bounds on the first Betti number of a smooth real algebraic variety, it is sufficient to sum the bounds on the number of the critical points which are computed by applying Corollary 3 to each polynomial system defining the ideals $I_i^{\mathbf{A}}$ once the variables $X_1, \ldots, X_i$ have been substituted by $p_1, \ldots, p_i$. This proves the following result.

Theorem 6 *Let $(f_1, \ldots, f_s) \subset \mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leqslant n-1$) generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d. Denote by $D_1, \ldots, D_s$ the respective degrees of $f_1, \ldots, f_s$ and by D the maximum of $D_1, \ldots, D_s$. The number of connected components of $\mathcal{V} \cap \mathbb{R}^n$ is bounded by:*

$$D_1 \cdots D_s \sum_{i=0}^{d} (D-1)^{n-s-i} \binom{n-i}{n-i-s}$$

Moreover, if $(f_1, \ldots, f_s)$ is a regular sequence, the number of connected components of $\mathcal{V} \cap \mathbb{R}^n$ is bounded by:

$$D_1 \cdots D_s \sum_{i=0}^{n-s} (D-1)^{n-s-i} \binom{n-1-i}{n-i-s}$$

The worst case is the case when $D_1 = \ldots = D_s = D$. In this case, it is easy to prove that $D^s \sum_{i=0}^{n-s} (D-1)^{n-s-i} \binom{n-1-i}{n-i-s}$ is less or equal to the Thom-Milnor bound $D.(2D-1)^{n-1}$ which is the best known bound on the first Betti number. Computer simulations show that $D^s \sum_{i=0}^{d} (D-1)^{n-s-i} \binom{n-i}{n-i-s}$ is less or equal to $D.(2D-1)^{n-1}$ for values D, n and s between 2 and 200.

At last, remark that Theorem 3 can also be used to bound the output of the algorithm provided in [2, 35, 7, 6] by the quantity $D^s(D-1)^{n-s}\binom{n}{n-s}$. A simple application of Pascal's triangle formula shows that this bound is greater than the one of Theorem 6 in the case of a regular sequence.

5 Algorithmic issues

As above, consider a polynomial family $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d. Let $p = (p_1, \ldots, p_d)$ be an arbitrary point of $\mathbb{C}^d$. The algorithm relying on Theorem 5 consists in choosing generically a matrix $\mathbf{A} \in GL_n(\mathbb{Q})$ and

- solving the polynomial systems generating the ideals $I_i^{\mathbf{A}}$ (for $i = 1, \ldots, d-1$):

$$\left\{\begin{array}{l} f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = 0, \\ X_1 - p_1 = 0, \cdots, X_i - p_i = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_{i+1}} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_{i+1}} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_{i+2}} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_{i+2}} = 0 \\ \qquad \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \cdots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

- solving the polynomial system

$$\left\{\begin{array}{l} f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = 0, \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_2} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_2} = 0 \\ \qquad \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

  generating the ideal $I_0^{\mathbf{A}}$;

- and solving the polynomial system $f_1^{\mathbf{A}} = \cdots = f_s^{\mathbf{A}} = X_1 - p_1 = \cdots = X_d - p_d = 0$ generating the ideal $I_d^{\mathbf{A}}$.

Note that by solving a zero-dimensional polynomial system in a polynomial ring $\mathbb{Q}[v_1, \ldots, v_{n+s}]$, we mean computing a rational parameterization of its solutions.

$$\left\{\begin{array}{l} v_{n+s} = \frac{q_{n+s}(T)}{q_0(T)} \\ \qquad \vdots \\ v_1 = \frac{q_1(T)}{q_0(T)} \\ f(T) = 0 \end{array}\right.$$

where $f, q_0, q_1, \ldots, q_{n+s}$ are univariate polynomials with coefficients in $\mathbb{Q}$.

Here, the polynomial systems we want to solve generate positive dimensional ideals $I_i^{\mathbf{A}} \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_s]$ whom intersections with $\mathbb{Q}[X_1, \ldots, X_n]$ are zero-dimensional.

Thus, in order to retrieve the zero-sets of $I_i^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ (for $i = 0, \ldots, d$), it is enough to compute rational parameterizations of each equidimensional component C_p of dimension p of the complex solution set of $I_i^{\mathbf{A}}$ intersected with p generic linear forms in $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$. This is equivalent to

- perform a generic linear change of variables $\mathbf{B} \subset GL_{n+s}(\mathbb{Q})$ sending the vector of coordinates $[X_1, \ldots, X_n, \ell_1, \ldots, \ell_s]$ to a new vector of coordinate $[v_1, \ldots, v_{n+s}]$

- compute a rational parameterization of each equidimensional component C_p of dimension p of the complex solution set of $I_i^{\mathbf{A}}$ intersected with the linear subspace defined by $v_1 = \cdots = v_p = 0$
- retrieve the complex solution set of $I_i^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ by multiplying $\mathbf{B}^{-1}$ with the vector $(q_1/q_0, \ldots, q_{n+s}/q_0)$ for each computed parameterization and keep only the first n coordinates of the computed vector.

Once this computation is performed, one obtains rational parametrizations of at least one point in each connected component of $\mathcal{V} \cap \mathbb{R}^n$ expressed in the coordinates obtained after the linear change of variables induced by $\mathbf{A}$. Retrieving their coordinates in the original system of coordinates is done by multiplying $\mathbf{A}^{-1}$ the previously computed parameterizations. Thus, the cost of this operation is polynomial in n and linear in the degree of each computed rational parametrization. We see below that this cost is negligible compared to the rest of the computation.

Computing rational parameterizations of the complex solution set of a zero-dimensional ideal can be done from a Gröbner basis (see [13, 14]) using linear algebra methods (see [1, 33] and references therein). Other methods based on the representation of polynomials by straight-line programs are provided in [17, 18, 19, 20, 28]. The arithmetic complexity of our algorithm depends on the arithmetic complexity of the chosen routine performing algebraic elimination.

Without additional algebraic informations on the systems generating $I_i^{\mathbf{A}}$ such as regularity or semi-regularity (see [8]), Hilbert-regularity can not be satisfactorily bounded. Thus, at the time being, when using Gröbner bases in the solving process, one can not give a better upper bound than one which is doubly exponential in the number of variables [29] for our algorithm. Investigating algebraic properties of $I_i^{\mathbf{A}}$ could yield better bounds and this is left to a further work in the spirit of [8].

In [28], the author follows the ideas of [17, 18, 19, 20] and extends them to provide a probabilistic incremental algorithm computing generic fibers of the equidimensional components of an algebraic variety; these fibers are encoded by geometric resolutions.

In the sequel, $g_1^{\mathbf{A}}$ denotes the polynomial $\ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_1} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_1} - 1$ and g_i denotes $\ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_i} + \cdots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_i}$ (for $i = 2, \ldots, n$). Below, $\mathcal{O}_{\log}(p)$ denotes the quantity $\mathcal{O}(p(\log p)^a)$ (for some constant a) and $M(p)$ denotes the cost of multiplying two univariate polynomials of degree p. The following result provides the arithmetic complexity of this algorithm:

Theorem 7 *[28] Consider $F_1, \ldots, F_p$ polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by D, represented by a Straight-Line Program of length $\mathcal{L}$ and defining a zero-dimensional variety. There exists an algorithm computing a geometric resolution of $V(F_1, \ldots, F_p)$ whose arithmetic complexity is:*

$$\mathcal{O}_{\log}(pn^4(n\mathcal{L} + n^3)M(D\mathfrak{d})^3)$$

where $\mathfrak{d}$ is the maximum of the sums of the algebraic degrees of the irreducible components of the intermediate varieties defined by $F_1, \ldots, F_i$ for i in $1, \ldots, p$.

From Theorem 5 and Corollary 3, the maximum of the algebraic degrees of the irreducible components of the intermediate varieties defined by $f_1^{\mathbf{A}}, \ldots, f_i^{\mathbf{A}}$ (for $1 \leqslant i \leqslant s$) and $f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}}, g_1^{\mathbf{A}}, \ldots, g_i^{\mathbf{A}}$ (for $1 \leqslant i \leqslant n$) is bounded by $D^s(D-1)^{n-s}\binom{n}{n-s}$.

The polynomial system defining $I_0^{\mathbf{A}}$ has $n+s$ variables and contains $n+s$ polynomials.

Moreover, given a straight-line program of length $\mathcal{L}$ evaluating the system $(f_1, \ldots, f_s)$, using the result of [3], one can construct a straight-line program of length $\mathcal{O}((\mathcal{L}+n^2))$ evaluating the polynomial system defining $I_0^{\mathbf{A}}$.

This discussion allows us to state the following result:

Theorem 8 *Let $(f_1, \ldots, f_s)$ be a polynomial family of $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Denote by D be the maximal degree of f_i (for $i = 1, \ldots, s$) and by L be the length of a straight-line program evaluating $(f_1, \ldots, f_s)$. There exists a probabilistic algorithm computing at least one point in each connected component of $\mathcal{V} \cap \mathbb{R}^n$ in:*

$$\mathcal{O}_{\log}\left((n+s)^5((n+s)(L+n^2)+(n+s)^3)M(D\mathfrak{d})^3\right)$$

operations in $\mathbb{Q}$ where $\mathfrak{d}$ is dominated by $D^s(D-1)^{n-s}\binom{n}{n-s}$.

Remark 6 *Remark that when $(f_1, \ldots, f_s)$ is a regular sequence, $\mathfrak{d}$ is dominated by $D^s(D-1)^{n-s}\binom{n-1}{n-s}$.*

Moreover, the above discussion allows us to state that the algorithm of [28] has a satisfactory complexity in the cases obtained by dehomogenizing a bi-homogeneous system since the degree of intermediate varieties it studies is bounded by bi-homogeneeous Bézout bounds from Theorem 2.

Now, we discuss how this result improves the preceding ones. First remark that our algorithm is *probabilistic*: this is first due that we have to avoid Zariski closed subsets for the choices of the matrix $\mathbf{A}$ on one hand and the point p on the other hand; and the algorithm provided in [28] computing geometric resolutions is also probabilistic. The algorithm provided by [10] is *deterministic* and has a complexity which is dominated by $(4D)^{\mathcal{O}(n)}$, when the number of operations is counted in a *Puiseux series field*. We emphasize that the zero-dimensional polynomial system studied by this algorithm has always a degree equal to $(4D)^n$ regardless of the original structure of the studied variety. Thus, if the random choices performed during our algorithm are correct, our algorithm improves the one of [10] since our worst case complexity involves a degree bound which is $D^s(D-1)^{n-s}\binom{n}{n-s}$. Nevertheless, remark that the algorithm of [10] stands for any case, while ours requires to deal with a reduced polynomial family defining a smooth algebraic variety.

The strategies developed in [2, 35, 7, 6, 37], which are still probabailistic are only valid in the equidimensional case and relies on characterizing the critical points by the vanishing of some minors of the jacobian matrix of the input polynomial system. Remark that this assumption is no more required by our algorithm.

The complexity estimations of the probabilistic algorithms provided in [7, 6] involve a combinatorial quantity $\binom{n}{n-d}$ and a geometric degree which is dominated by $D^{(n-d)}((n-$

$d)(D-1)+1)^d$ where d denotes the dimension of the studied algebraic variety. These algorithms study $\binom{n}{n-d}$ regular sequences defining critical points. These sequences are formed of the input polynomials and of some extracted minors of their jacobian matrix after localization. Our work allows us to avoid the combinatorial quantity $\binom{n}{n-d}$ and to bound the number of computed critical points. Nevertheless, up to our knowledge, ensuring that the *intermediate* degrees appearing in these algorithms do not exceed the bi-homogeneous bound is not immediate and can be the subject of a further study.

Remark also that if the polynomials defining the studied variety are quadratic, our algorithm has a complexity which is polynomial in the number of variables and exponential in the number of equations. In [21], the authors provide an algorithm which is dedicated to the quadratic case having a similar complexity and which uses also Lagrange's system to characterize critical points. The algorithm of [21] deals also with singular situations.

Finally, our algorithm generalizes the one of [37] since the equidimensional assumption is dropped. We emphasize that some computer experiments show that the quantity $D^{(n-d)}((n-d)(D-1))^d$ bounding the maximal degree of *intermediate* varieties appearing in [37] is reached in some cases, at intermediate steps of the algorithm. This problem is solved by our contribution.

Perspectives. Obtaining an efficient implementation from this work is not an easy task. The algorithm designed in [28] performs a linear change of variables which does not take into account the bi-homogeneous structure of Lagrange's system. Thus, some improvements could be brought in a further study.

Using Gröbner bases inside our algorithm has also to be investigated. Implementations of [13] and [14] are at the time being the most efficient for algebraic elimination: the doubly exponential behavior of Gröbner bases is exceptional and is restricted to very particular polynomial systems.

Finally, dropping the assumptions of our algorithm is a work in progress, whose aim is to provide an algorithm working in any case with a complexity which is polynomial in n, s, the bi-homogeneous Bézout bound $D^s(D-1)^{n-s}\binom{n}{n-d}$ and the complexity of evaluation of the input system.

References

[1] M. E. Alonso, E. Becker, M.-F. Roy, and T. Wörmann. Zeroes, multiplicities and idempotents for zerodimensional systems. In *Algorithms in Algebraic Geometry and Applications. Proceedings MEGA'94*, Volume 142 of *Progress in Mathematics*, pages 1–15. Birkhäuser, 1996.

[2] P. Aubry, F. Rouillier, and M. Safey El Din. Real solving for positive dimensional systems. *Journal of Symbolic Computation*, Volume 34(6), pages 543–560, 2002.

[3] W. Baur, V. Strassen. The complexity of partial derivatives, *Theoret. Comput. Science*, Volume 22, pages 317–330, 1982.

[4] B. Bank, M. Giusti, J. Heintz, and G.-M. Mbakop. Polar varieties and efficient real equation solving: the hypersurface case. *Journal of Complexity*, Volume 13(1), pages 5–27, 1997.

[5] B. Bank, M. Giusti, J. Heintz, and G.-M. Mbakop. Polar varieties and efficient real elimination. *Mathematische Zeitschrift*, Volume 238(1), pages 115–144, 2001.

[6] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. Generalized polar varieties: geometry and algorithms. Technical report, Humboldt Universität, 2003.

[7] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. La luz es polar: Projective geometry and real polynomial equation solving. In *Proceedings of WAIT 2003 (Workshop Argentino de Informática Teorética) as part of 32 JAIIO (32-nd Argentinian Conference on Informatics and Operations Research)*, 2003.

[8] M. Bardet, J.C. Faugère and B. Salvy. On the complexity of a Gröbner algorithm Proceedings of the International Conference on Polynomial System Solving, pages 71–75, 2004

[9] S. Basu, R. Pollack, and M.-F. Roy. On the combinatorial and algebraic complexity of quantifier elimination. *Journal of ACM*, Volume 43(6), pages 1002–1045, 1996.

[10] S. Basu, R. Pollack, and M.-F. Roy. A new algorithm to find a point in every cell defined by a family of polynomials. In *Quantifier elimination and cylindrical algebraic decomposition*. Springer-Verlag, 1998.

[11] D. Cox and J. Little and D. O'Shea. Ideals, Varieties, and Algorithms, *Springer Verlag*, New-York, 1992.

[12] D. Eisenbud. Commutative Algebra: with a view toward algebraic geometry, *Springer Verlag*, New-York, 1995.

[13] J.-C. Faugère. A New Efficient Algorithm for Computing Gröbner bases (F4), *Journal of Pure and Applied Algebra*, 139 (1-3), pages 61–88, 1999.

[14] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero F5, *Proceedings of International Symposium on Symbolic and Algebraic Computation Symposium - ISSAC 2002*, Villeneuve d'Ascq, France, 2002.

[15] C. A. Floudas, P. M. Pardalos, C. S. Adjiman, W. R. Esposito, Z. H. Gumus, S. T. Harding, J. L. Klepeis, C. A. Meyer, and C. A. Schweiger. *Handbook of test problems in local and global optimization*. Kluwer Academic Publishers, 1999.

[16] P. Gianni, Properties of Gröbner Basis under Specializations, Lecture Notes in Computer Science, Volume 378, pages 293–297, 1987

[17] M. Giusti, J. Heintz,and J,-E. Morais, and L.-M. Pardo. When Polynomial Equation Systems can be solved fast?,*Proceedings of AAECC-11*, LNCS, Volume 948, pages 205–231,1995.

[18] M. Giusti, K. Hägele, J. Heintz, J.-E Morais, J.-L. Montaña and L.-M. Pardo. Lower Bounds for Diophantine Approximation, Proceedings of MEGA'96, Journal of Pure and Applied Algebra, Volume 117, pages 277-317, 1997.

[19] M. Giusti, J. Heintz, J.-E. Morais, J. Morgenstern and L.-M. Pardo. Straight-Line Programs In Geometric Elimination Theory, *Journal of Pure and Applied Algebra*, Volume 124, pages 101–146,1998.

[20] M. Giusti, G. Lecerf and B. Salvy. A Gröbner Free Alternative for Polynomial System Solving *Journal of Complexity*, Volume 17(1), pages 154–211, 2001.

[21] D. Grigoriev and D. Pasechnik, Polynomial-time computing over quadratic maps I. Sampling in real algebraic sets. *Computational Complexity*, Volume 14, pages 20–52, 2005.

[22] D. Grigoriev and N. Vorobjov. Solving systems of polynomials inequalities in subexponential time. *Journal of Symbolic Computation*, Volume 5, pages 37–64, 1988.

[23] J. Heintz, Definability and fast quantifier elimination in algebraically closed fields *Theoret. Comput. Science*, Volume 24, pages 239–277 1983.

[24] J. Heintz, G. Jerónimo, J. Sabia, J. San Martin, and P. Solerno. Intersection theory and deformation algorithm. the multi-homogeneous case. Manuscript, 2002.

[25] J. Heintz, M.-F. Roy, and P. Solernò. On the complexity of semi-algebraic sets. In *Proceedings IFIP'89 San Francisco, North-Holland*, 1989.

[26] J. Heintz, M.-F. Roy, and P. Solernò. On the theoretical and practical complexity of the existential theory of the reals. *The Computer Journal*, Volume 36(5), pages 427–431, 1993.

[27] D. Lazard. On the Bézout theorem. Unpublished manuscript.

[28] G. Lecerf. Computing the Equidimensional Decomposition of an Algebraic Closed Set by means of Lifting Fibers *Journal of complexity*, Volume 4, pages 564–596 2002.

[29] E. Mayr and A. Meyer. The complexity of the word problem for commutative semigroups and polynomial ideals In *Advance in Maths*, Volume 127, pages 305–329, 1998.

[30] D. Perrin, Géométrie algébrique: une introduction., Inter Editions, 1995.

[31] G. Rémond. Élimination multi-homogène, Chapitre 5 de *Introduction to algebraic independence theory*, Lecture Notes in Math, Volume 1752, pages 53–81, 2001.

[32] G. Rémond. Géométrie diophantienne multi-projective, Chapitre 7 de *Introduction to algebraic independence theory,* Lecture Notes in Math, Volume 1752, pages 95–131, 2001.

[33] F. Rouillier. Solving zero-dimensional systems through the rational univariate representation, *Journal of Applicable Algebra in Engineering, Communication and Computing*, Volume 9 (5), pages 433–461, 1999.

[34] F. Rouillier, M.-F. Roy, and M. Safey El Din. Finding at least one point in each connected component of a real algebraic set defined by a single equation. *Journal of Complexity*, Volume 16, pages 716–750, 2000.

[35] M. Safey El Din. *Résolution réelle des systèmes polynomiaux de dimension positive.* PhD thesis, Université Paris 6, January 2001.

[36] M. Safey El Din. RAGLib (Real Algebraic Geometry Library). available at the URL `http://www-calfor.lip6.fr/~safey/RAGLib`, 2003.

[37] M. Safey El Din and É. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In *Proceedings of the 2003 international symposium on Symbolic and algebraic computation*, pages 224–231. ACM Press, 2003.

[38] M. Safey El Din and É. Schost. Properness defects of projections and computation of one point in each connected component of a real algebraic set. *Journal of Discrete and Computational Geometry*, Vol.32 (3), pages 417–430, 2004.

[39] I. Shafarevich. *Basic Algebraic Geometry 1.* Springer Verlag, 1977.

[40] B.-L. van der Waerden. On hilbert's function, series of composition of ideals and a generalization of a theorem of bezout. In *Proc. Roy. Acad. Amsterdam*, Volume 31, pages 749–770, 1929.

[41] B.-L. van der Waerden. On varieties in multiple projective spaces. *Nederl. Akad. Wetensch. Indag. Math.*, Volume 40(2), pages 303–312, 1978.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399